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LAWS OF LARGE NUMBERS FOR QUADRATIC FORMS, MAXIMA OF PRODUCTS AND TRUNCATED SUMS OF I.I.D. RANDOM VARIABLES

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Let X, X_i be i.i.d. real random variables with $EX^2 = \infty$. Necessary and sufficient conditions in terms of the law of X are given for $(1/\gamma_n)\max_{1\le i<j\le n}|X_iX_j| \to 0$ a.s. in general and for $(1/\gamma_n)\sum_{1\le i\ne j\le n} X_iX_j \to 0$ a.s. when the variables X_i are symmetric or regular and the normalizing sequence $\{\gamma_n\}$ is (mildly) regular. The rates of a.s. convergence of sums and maxima of products turn out to be different in general but to coincide under mild regularity conditions on both the law of X and the sequence $\{\gamma_n\}$. Strong laws are also established for $X_{1:n}X_{k:n}$, where $X_{j:n}$ is the jth largest in absolute value among $X_1, \dots, X_n$, and it is found that, under some regularity, the rate is the same for all $k \ge 3$. Sharp asymptotic bounds for $b_n^{-1}\sum_{i=1}^n X_i I_{|X_i|<b_n}$, for b_n relatively small, are also obtained.

1. Introduction. In contrast to the situation for sums of independent identically distributed (i.i.d.) random variables, the law of large numbers for U-statistics is not equivalent to finiteness of moments of the defining function h: Let X, X_i, $i \in \mathbb{N}$, be i.i.d. and let h be a measurable function of two variables; the weakest possible general moment condition on h implying $(1/n^{2/\alpha})\sum_{1\le i\ne j\le n} h(X_i, X_j) \to 0$ a.s. is $E|h|^\alpha < \infty$, $0 < \alpha < 2$, assuming $Eh = 0$ if $\alpha = 1$ and $E[h(X, x) + h(x, X)] = 0$ for almost all x (i.e., h degenerate) if $1 < \alpha < 2$. However, the following example shows the converse is not true [Giné and Zinn (1992a)]: Let X satisfy

$$\lim_{t\to\infty} t^\alpha(\log t)P\{|X| > t\} = c, \tag{1.1}$$

for some $0 < \alpha < 2$ and $c > 0$, and assume X is symmetric for $1 \le \alpha < 2$. Then $E|X|^\alpha = \infty$, but

$$\frac{1}{n^{2/\alpha}} \sum_{1\le i\ne j\le n} X_iX_j \to 0 \quad \text{a.s.} \tag{1.2}$$

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As a first step toward understanding the law of large numbers for U-statistics, and also for its intrinsic interest, we shall restrict attention to the U-statistic defined by $h(x, y) = xy$, that is, to quadratic forms in the X_i's.

In the above example, parity is restored if we include the *diagonal* in (1.2) as the expression now is the square of a sum of independent random variables normalized by $n^{1/\alpha}$, which tends to zero a.s. if and only if $E|X|^\alpha < \infty$ by the Marzinkiewicz law of large numbers. Thus, the diagonal has an effect on almost sure convergence to zero of quadratic forms such as in (1.2).

This is not the case for convergence in probability [Giné and Zinn (1992)]. By decomposing the double sum in (1.2) into four sums with (i, j) both even, both odd or one even and one odd, convergence of (1.2) in probability (or a.s.) implies $\xi_n \xi_n' \to 0$ in probability (or a.s.), where $\xi_n = n^{-1/\alpha} \sum_{i=1}^n X_i$ and ξ_n' is defined equivalently for an independent copy $\{X_i'\}$ of $\{X_i\}$. Now, if $\xi_n \xi_n' \to 0$ in probability, then also $\xi_n \to 0$ in probability since $\mathrm{P}\{|\xi_n| > \sqrt{\varepsilon}\}^2 \le \mathrm{P}\{|\xi_n \xi_n'| > \varepsilon\}$ (this is not true for a.s. convergence). Conversely, by the weak law of large numbers, $\xi_n \to 0$ in probability implies $n\mathrm{P}\{|X| > n^{1/\alpha}\} \to 0$ and therefore also $(1/n^{2/\alpha}) \sum_{i=1}^n X_i^2 \to 0$ in pr, yielding $(1/n^{2/\alpha}) \sum_{1 \le i \ne j \le n} X_i X_j = ((1/n^{1/\alpha}) \sum_{i=1}^n X_i)^2 - (1/n^{2/\alpha}) \sum_{i=1}^n X_i^2 \to 0$ in pr.

The diagonal is also irrelevant when $EX^2 < \infty$ and $EX = 0$ since we can write the sum in (1.2) as

$$\left(\sum_{i=1}^n X_i\right)^2 - \sum_{i=1}^n X_i^2 \tag{1.3}$$

and $(1/n) \sum_{i=1}^n X_i^2 \to EX^2$ a.s. by the law of large numbers, but $\limsup[1/(n \log\log n)](\sum_{i=1}^n X_i)^2 = 2EX^2$ a.s. by the law of the iterated logarithm, so that the first term of (1.3) dominates.

However, when $EX^2 = \infty$ (and $EX = 0$ if $E|X| < \infty$), the lim sup behavior of each term in (1.3) is the same as that for $\max_{1 \le i \le n} X_i^2$ under weak regularity conditions, and these terms cancel, offering the possibility of a more rapid convergence to zero.

These observations determine the main object of this article, which is to find when the law of large numbers

$$\frac{1}{\gamma_n} \sum_{1 \le i < j \le n} X_i X_j \to 0 \quad \text{a.s.} \tag{1.4}$$

holds for a general nondecreasing sequence $\{\gamma_n\}$ of positive numbers tending to infinity. We obtain purely analytic necessary and sufficient conditions for (1.4) to hold under (mild) regularity conditions on the normalizing sequence $\{\gamma_n\}$ in two general instances, namely, when X is symmetric and when the tail probability function of X is (mildly) regular. Along the way, we obtain interesting results of two kinds. Letting $X_{j:n}$ denote the jth largest in magnitude among $X_1, \dots, X_n$, we give necessary and sufficient conditions for

$$\frac{1}{\gamma_n} \max_{1 \le i < j \le n} |X_i X_j| \equiv \frac{1}{\gamma_n} |X_{1:n} X_{2:n}| \to 0 \quad \text{a.s.} \tag{1.5}$$

and more generally for

$$\frac{1}{\gamma_n}|X_{1:n}X_{k:n}| \to 0 \quad \text{a.s.} \tag{1.6}$$

(without any restrictions on the normalizing sequence $\gamma_n \nearrow \infty$). We also obtain sharp a.s. asymptotic bounds for truncated sums, $|\sum_{i=1}^n X_i I_{|X_i|<b_n}|/b_n$, which in particular imply a result of Mori (1977) on almost sure convergence to zero of normalized lightly trimmed sums of independent random variables.

Section 2 contains analytic necessary and sufficient conditions for the law of large numbers for maxima, (1.5) and (1.6). For instance, it is shown that (1.5) holds if and only if

$$E\left[\gamma^{-1}\left(\frac{|XY|}{\varepsilon}\right) \wedge \frac{1}{G(|X|)} \wedge \frac{1}{G(|Y|)}\right]^2 < \infty \tag{1.7}$$

and

$$\sum 2^k P\{|X| > \varepsilon v_k\} < \infty \tag{1.8}$$

for all $\varepsilon > 0$, where Y is an independent copy of X, $G(x) = P\{|X| \geq x\}$, $u_k = G^{-1}(2^{-k})$, $v_k = (\gamma(2^k)/u_k)$ and $\gamma(t)$ is a nondecreasing continuous function such that $\gamma(n) = \gamma_n$. These conditions, unlike those for maxima of i.i.d. random variables, are difficult to work with; however, they admit simplifications under reasonable regularity hypotheses on the distribution of X and/or the normalizing sequence $\{\gamma_n\}$. We state a few instances of this, leaving some of the proofs for the Appendix. For example, if X has a continuous distribution (or if its jumps are not too large), then (1.5) holds if and only if (1.7) does. Under further regularity (1.5) holds if and only if

$$\sum n^{-1}(na_n)^2 \log_+ na_n < \infty, \tag{1.9}$$

where $a_n = G(\gamma_n^{1/2})$ and $\log_+ x := |\log x| \vee 1$. Maxima of *decoupled* products are also considered.

Section 3 is devoted to the study of truncated and trimmed sums of independent random variables. Assuming centerings do not matter and $n^{-\beta}b_n \nearrow$ for some $\beta > \frac{1}{2}$, it follows from Feller's (1946) law of large numbers that if $P\{|X_{1:n}| > b_n \text{ i.o.}\} = 0$, then $(1/b_n)\sum_{i=n}^n X_i I_{|X_i|<b_n} \to 0$ a.s. This is generalized in this section to: If $P\{|X_{k:n}| > b_n \text{ i.o.}\} = 0$ and the centerings do not matter, then

$$\limsup \frac{1}{b_n}\left|\sum_{i=1}^n X_i I_{|X_i|<b_n}\right| \leq k-1 \quad \text{a.s.} \tag{1.10}$$

and the bound is sharp. In particular this provides a short proof of the fact that, under the same hypothesis,

$$\frac{1}{b_n}\sum_{j=k}^n X_{j:n} \to 0 \quad \text{a.s.,} \tag{1.11}$$

a result previously obtained, with a different proof, by Mori (1977). Basic in this section is the following result: Let k be a positive integer and let $b(t)$ satisfy $t^{-\beta}b(t) \nearrow \infty$ for some $\beta > \frac{1}{2}$. Then, without further restrictions on the distribution of $X, \sum_{n=1}^{\infty}(2^n G(b(2^n)))^k < \infty$ implies $\sum_{n=1}^{\infty}(2^n b(2^n)^{-2} EX^2 I_{|X|<b(2^n)})^k < \infty$. For $k = 2$ and $b(t) = t^{1/\alpha}$, $0 < \alpha < 2$, this result shows that if X and Y are i.i.d., then $E(|X| \wedge |Y|)^{2\alpha} < \infty$ implies $E[(|X| \wedge |Y|)^2(|X| \vee |Y|)^{2(\alpha-1)}] < \infty$, which is quite surprising for $1 < \alpha < 2$.

We study the law of large numbers for quadratic forms, (1.4), in Section 4. Whereas for sums of i.i.d. variables, symmetry of X and regularity of its tail distribution does not play a role (once some mild regularity for the norming sequence is assumed), these two factors seem to have some influence in the case of products (at least in the present study). For symmetric variables in general (i.e., without regularity assumptions), we obtain two sets of necessary and sufficient conditions (nasc) for the law of large numbers (1.4) to hold: one of an analytic character; the other one related to maxima. The analytic nasc's for (1.4) to hold are condition (1.7) together with

$$\sum 2^k P\{|X| > \varepsilon w_k\} < \infty \tag{1.12}$$

for all $\varepsilon > 0$, where $w_k = \gamma(2^k)/[2^k E(X^2 \wedge u_k^2)]^{1/2}$. In order to compare conditions (1.12) and (1.8), note that w_k is in general of a smaller order of magnitude than v_k, but that they are comparable if the law of X is regular. In connection with maxima, we show that (1.4) is equivalent to

$$\frac{1}{\gamma_n} X_{1:n} \sum_{j=2}^{n} X_{j:n} \to 0 \quad \text{a.s.}, \tag{1.13}$$

that is, one of the sums in $\sum_{j=1}^{n}\sum_{i=1}^{j-1} X_i X_j$ can be replaced by a maximum and still obtain an equivalent statement. These results seem to indicate that, even for $\{\gamma_n\}$ regular, the laws of large numbers for sums and for maxima of products (i.e., replacing the two sums by maxima) may not be equivalent (compare with sums and maxima of i.i.d. random variables); however, at present we have no examples to fully justify this claim. Finally, we prove that if the tail of X satisfies some mild regularity conditions, even if X is not symmetric, then the laws of large numbers for sums and maxima of products are indeed equivalent. We also present analogous results for randomized and decoupled sums and maxima. The results from Sections 2 and 3 are extensively used in the proofs of the theorems in Section 4.

Regarding (1.1), we anticipate that the results obtained below show that if $P\{|X| > t\} \simeq 1/(t^\alpha(\log t)^\beta)$, then (1.2) holds if and only if $\beta > \frac{1}{2}$.

2. Maxima of products. In this section we study the almost sure convergence to zero of $(1/\gamma_n)\max_{1\le i<j\le n} X_i X_j$ and, in more generality, of $(1/\gamma_n)X_{1:n}X_{k:n}$, where $\{X_i\}$ is an i.i.d. sequence of nonnegative random variables, $\{\gamma_n\}$ is a nondecreasing unbounded sequence of positive numbers and

$X_{k:n}$ is the kth largest in absolute value among $X_1, \dots, X_n$, $k \le n < \infty$. Theorems 2.1 (or 2.1′) and 2.10 are the main results. Under regularity of the distribution of X and/or the norming sequence $\{\gamma_n\}$ the necessary and sufficient conditions of these theorems simplify; we present some results of this type. *Decoupled* maxima $(1/\gamma_n)\max_{1\le i,j\le n} X_i X'_j = (1/\gamma_n)(\max_{i\le n} X_i)(\max_{i\le n} X'_i)$, where $\{X_i\}$ and $\{X'_i\}$ are independent, are also considered. Convention: for nonincreasing left continuous functions with right limits, $G(x)$, $G^{-1}(x)$ is defined as $G^{-1}(x) = \sup\{y: G(y) \ge x\}$; then, if $u = G^{-1}(v)$ we have $G(u+) \le v \le G(u)$.

2.1. *The general result for* $\max_{1\le i<j\le n} X_i X_j$. Of course the problem reduces to finding necessary and/or sufficient conditions for $P\{\max_{1\le i<j\le n} X_i X_j > \varepsilon\gamma_n \text{ i.o.}\} = 0$ for all $\varepsilon > 0$. This is done in the following theorem.

THEOREM 2.1. *Let X, Y be nonnegative, independent random variables having the same distribution, characterized by $G(x) = P\{X \ge x\}$ and let $u_k = G^{-1}(2^{-k})$, $k \in \mathbb{N}$. Let $\{\gamma_n\}$ be a nondecreasing sequence of positive numbers tending to infinity and let $\gamma_k^* = \gamma(2^k)$, $k \in \mathbb{N}$. Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. with the same distribution as X. Then*

$$P\Big\{\max_{1\le i<j\le n} X_i X_j > \gamma_n \text{ i.o.}\Big\} = 0 \tag{2.1}$$

if and only if both

$$\sum_{k=1}^{\infty} 2^{2k} P\{XY > \gamma_k^*;\ X > u_k, Y > u_k\} < \infty \tag{2.2}$$

and

$$\sum_{k=1}^{\infty} 2^k P\Big\{X > \frac{\gamma_k^*}{u_k}\Big\} < \infty. \tag{2.3}$$

Condition (2.2) can be written in integral form: if $\gamma(t)$ interpolates $\gamma(n)$ linearly and if $\gamma^{-1}(t)$ denotes its left continuous inverse, then condition (2.2) is equivalent to

$$E\left[\gamma^{-1}(XY) \wedge \frac{1}{G(X)} \wedge \frac{1}{G(Y)}\right]^2 < \infty. \tag{2.4}$$

Also, if the function $v(t) := \gamma(t)/G^{-1}(t^{-1})$ is *monotone* and $v^{-1}(t)$ denotes its left continuous inverse, then condition (2.3) is equivalent to

$$Ev^{-1}(X) < \infty. \tag{2.5}$$

PROOF. We assume X unbounded; otherwise there is nothing to prove. Suppose (2.2) and (2.3) hold. Since $2^k P\{X \ge u_k\} \ge 1$, condition (2.3) implies

$$\gamma_k^* \ge u_k^2 \text{ eventually.} \tag{2.6}$$

In order to prove (2.1) it suffices to show

$$P\Big\{\max_{1\le i<j\le 2^k} X_iX_j > \gamma_{k-1}^* \text{ i.o.}\Big\} = 0. \tag{2.7}$$

To prove this, we first observe

$$\begin{aligned}\max_{1\le i<j\le 2^k} X_iX_j &\\ = [\max X_iI_{X_i\le u_{k-1}}X_jI_{X_j\le u_{k-1}}] &\vee [\max X_iI_{X_i\le u_{k-1}}X_jI_{X_j>u_{k-1}}]\\ \vee [\max X_iI_{X_i>u_{k-1}}X_jI_{X_j\le u_{k-1}}] &\vee [\max X_iI_{X_i>u_{k-1}}X_jI_{X_j>u_{k-1}}].\end{aligned}$$

Then (2.7) will hold if the probability that each of these max's is larger than γ_{k-1}^* infinitely often is 0. This is trivial for the first max since, by (2.6),

$$\max_{1\le i<j\le 2^k} X_iI_{X_i\le u_{k-1}}X_jI_{X_j\le u_{k-1}} \le \gamma_{k-1}^* \text{ eventually.}$$

Condition (2.2) implies control of the fourth max since

$$\begin{aligned}\sum P\Big\{\max_{1\le i<j\le 2^k} X_iI_{X_i>u_{k-1}}X_jI_{X_j>u_{k-1}} > \gamma_{k-1}^*\Big\}&\\ \le \sum 2^{2k}P\{XY > \gamma_{k-1}^*;\ X,Y > u_{k-1}\} < \infty.&\end{aligned}$$

The second and third max's are similar, so we just work with the second. For n large we have

$$\begin{aligned}\sum_{k=n}^{\infty} P\Big\{\max_{1\le i<j\le 2^k} X_iI_{X_i\le u_{k-1}}X_jI_{X_j>u_{k-1}} > \gamma_{k-1}^*\Big\}&\\ \le \sum_{k=n}^{\infty} P\Big\{u_{k-1}\max_{j\le 2^k} X_j > \gamma_{k-1}^*\Big\}&\\ \le \sum_{k=n}^{\infty} 2^kP\{X > \gamma_{k-1}^*/u_{k-1}\},&\end{aligned}$$

which is finite by (2.3). Hence $P\{\max_{1\le i<j\le 2^k} X_iI_{X_i\le u_{k-1}}X_jI_{X_j>u_{k-1}} > \gamma_{k-1}^* \text{ i.o.}\} = 0$. (2.7) is proved.

We now assume (2.1) holds. Then $P\{\max_{2^{k-1}<i<j\le 2^k} X_iX_j > \gamma_k^* \text{ i.o.}\} = 0$, and it follows, by independence of the blocks and Borel–Cantelli, that

$$\sum P\Big\{\max_{1\le i<j\le 2^{k-1}} X_iX_j > \gamma_k^*\Big\} < \infty.$$

Since $\{(i,j): 1 \le i < j \le 2^{k-1}\} \supset \{(i,j): 1 \le i \le 2^{k-2} < j \le 2^{k-1}\}$, letting $X_{2^{k-2}+r} = X_r'$, we obtain

$$\sum_{k=2}^{\infty} P\Big\{\max_{1\le i,j\le 2^{k-2}} X_iX_j' > \gamma_k^*\Big\} < \infty. \tag{2.8}$$

The following estimates show that (2.8) implies (2.3):

$$
\begin{aligned}
P\Big\{\max_{1\le i,j\le 2^{k-2}} X_iX'_j > \gamma_k^*\Big\} &\ge P\Big\{\max_{i\le 2^{k-2}} X_i > \frac{\gamma_k^*}{u_k}\Big\}P\Big\{\max_{i\le 2^{k-2}} X_i \ge u_k\Big\}\\
&\ge \frac{2^{k-2}P\{X>\gamma_k^*/u_k\}}{1+2^{k-2}P\{X>\gamma_k^*/u_k\}}\cdot\frac{2^{k-2}P\{X\ge u_k\}}{1+2^{k-2}P\{X\ge u_k\}}\\
&\ge \frac{1}{5}\frac{2^{k-2}P\{X>\gamma_k^*/u_k\}}{1+2^{k-2}P\{X>\gamma_k^*/u_k\}}.
\end{aligned}
$$

Finally, we show that (2.2) also follows from (2.8). Let $M_i = \max_{1\le s<i} X_s$, $i \le 2^{k-2}$, and $\tau_k = \inf\{i \le 2^{k-2}: X_i > u_k\}$, with $\inf\varnothing = \infty$, and define M'_i and τ'_k by analogy. We then have

$$
\begin{aligned}
P\Big\{\max_{1\le i,j\le 2^{k-2}} X_iX'_j > \gamma_k^*\Big\} &\ge P\{\tau_k<\infty, \tau'_k<\infty, X_{\tau_k}X'_{\tau'_k} > \gamma_k^*\}\\
&= \sum_{i,j\le 2^{k-2}} P\{X_iX'_j>\gamma_k^*;\ X_i, X'_j > u_k;\ M_i, M'_j \le u_k\} \qquad (2.9)\\
&= P\{XY>\gamma_k^*;\ X,Y>u_k\}\Big(\sum_{i\le 2^{k-2}} P\{M_i\le u_k\}\Big)^2.
\end{aligned}
$$

Since

$$P\{M_i \le u_k\} = [1-P\{X>u_k\}]^{i-1} \ge (1-2^{-k})^{i-1} \ge (1-2^{-k})^{2^k} \ge \tfrac{1}{4},$$

(2.9) gives

$$2^{-8}\sum_{k=1}^{\infty} 2^{2k}P\{XY>\gamma_k^*;\ X>u_k, Y>u_k\} \le \sum_{k=2}^{\infty} P\Big\{\max_{1\le i,j\le 2^{k-2}} X_iX'_j > \gamma_k^*\Big\} < \infty,$$

that is, (2.2), concluding the proof of the theorem. □

In all that follows the sequence $\{\gamma_n\}$ is nondecreasing and tends to ∞, and γ_k^*, $\{X_i\}$, X, Y, G and u_k are as defined in Theorem 2.1.

Let us consider the condition

$$\sum_{k=1}^{\infty} 2^{2k}P\{XY>\gamma_k^*;\ X\ge u_k, Y\ge u_k\} < \infty. \qquad (2.10)$$

Inequality (2.10) is obviously stronger than (2.2). It also implies (2.3). To see this we observe first that it implies (2.6). Otherwise, there is a sequence $\{k(\ell)\}$ such that $\{X,Y \ge u_{k(\ell)}\} = \{XY > \gamma^*_{k(\ell)};\ X,Y\ge u_{k(\ell)}\}$, hence, by (2.10), $\sum(2^{k(\ell)}P\{X\ge u_{k(\ell)}\})^2 < \infty$, in contradiction with $2^kP\{X\ge u_k\}\ge 1$. Now, (2.6) and (2.10) give that for some $k_0 < \infty$,

$$\begin{aligned}\sum_{k\geq k_0} 2^k P\{X > \gamma_k^*/u_k\} &\leq \sum_{k\geq k_0} 2^k P\{X > \gamma_k^*/u_k\} 2^k P\{Y \geq u_k\} \\ &= \sum_{k\geq k_0} 2^{2k} P\{X > \gamma_k^*/u_k, Y \geq u_k\} \\ &\leq \sum_{k\geq k_0} 2^{2k} P\{XY > \gamma_k^*;\ X > u_k, Y \geq u_k\} < \infty.\end{aligned}$$

We have thus proved the following corollary.

COROLLARY 2.2. (2.10) $\Longrightarrow$ (2.1).

We may ask whether the converse to Corollary 2.2 holds, and whether condition (2.3) is redundant. The following example answers these two questions in the negative.

EXAMPLE 2.3. Let $b_n > 0$, $n \in \mathbb{N}$, be such that $b_{n+1}/b_n \nearrow \infty$ strictly (so that, in particular, $b_{n+1}^2 < b_n b_{n+2}$) and let $a_t = t^{\alpha t}$ for some $\alpha > 1$ and all $t > 1$. Let X be a random variable concentrated on $\{b_n\}$ and such that $P\{X \geq b_n\} = 1/a_n$. Note that a_n grows fast enough so that $P\{X = b_n\} \simeq 1/a_n$. Then $u_k = G^{-1}(2^{-k}) = b_n$ for k such that $a_n \leq 2^k < a_{n+1}$. For a sequence of positive numbers $\delta_n \to 0$ with $\delta_n < b_n b_{n+2} - b_{n+1}^2$ and for $k \geq 1$, we let

$$\gamma_k^* = \begin{cases} b_n b_{n+1} - \delta_n, & \text{if } a_n \leq 2^k < a_{n+1/2}, \\ b_n b_{n+2} - \delta_n, & \text{if } a_{n+1/2} \leq 2^k < a_{n+1}. \end{cases}$$

Then, at least for n large,

$$P\{XY > \gamma_k^*;\ X, Y \geq u_k\} \simeq \begin{cases} \dfrac{1}{a_n a_{n+1}}, & \text{for } a_n \leq 2^k < a_{n+1/2}, \\ \dfrac{1}{a_n a_{n+2}}, & \text{for } a_{n+1/2} \leq 2^k < a_{n+1}. \end{cases}$$

So, the series in (2.10) is convergence equivalent to the series

$$\sum \frac{a_{n+1/2}^2}{a_n a_{n+1}} + \sum \frac{a_{n+1}^2}{a_n a_{n+2}},$$

which is divergent. Similarly, the series in (2.2) is convergence equivalent to the series

$$\sum \frac{a_{n+1/2}^2}{a_{n+1}^2} + \sum \frac{a_{n+1}^2}{a_{n+1} a_{n+2}} \simeq \sum \frac{1}{n^\alpha},$$

which is convergent, and the series in (2.3) is convergence equivalent to

$$\sum \frac{a_{n+1/2}}{a_{n+1}} + \sum \frac{a_{n+1}}{a_{n+2}} \simeq \sum \frac{1}{n^{\alpha/2}} + \sum \frac{1}{n^{\alpha}},$$

so that (2.3) holds iff $\alpha > 2$. Then, by Theorem 2.1, (2.1) holds iff $\alpha > 2$. Hence, in this example (2.1) is equivalent to (2.3), which is strictly between (2.2) and (2.10).

Theorem 2.1 translates directly into a result on a.s. convergence to zero of normalized maxima:

THEOREM 2.1′. *In order that*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \max_{1\le i<j\le n} X_i X_j = 0 \quad a.s. \tag{2.11}$$

hold, it is necessary and sufficient that

$$\sum_{k=1}^{\infty} 2^{2k} P\{XY > \varepsilon\gamma_k^*;\ X > u_k, Y > u_k\} < \infty \tag{2.2′}$$

and

$$\sum_{k=1}^{\infty} 2^{k} P\left\{X > \frac{\varepsilon\gamma_k^*}{u_k}\right\} < \infty \tag{2.3′}$$

for all $\varepsilon > 0$.

Theorem 2.1′ is not redundant: we may have conditions (2.1) and (2.2) satisfied and yet the lim sup of the normalized maxima be different from zero, as in Example 4.4 below.

It is worthwhile to observe that the above results also apply to decoupled maxima. In the following corollary, we let $\{X_i'\}$ denote a sequence of i.i.d. random variables also with the distribution of X, independent of $\{X_i\}$.

COROLLARY 2.4. *Theorems* 2.1 *and* 2.1′ *also hold if* (2.1) *and* (2.11) *are replaced, respectively, by*

$$P\left\{\max_{1\le i,j\le n} X_i X_j' > \gamma_n \ i.o.\right\} = 0 \tag{2.1′}$$

and

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \max_{1\le i,j\le n} X_i X_j' = 0 \quad a.s. \tag{2.11′}$$

PROOF. If (2.1′) holds, then we obtain (2.8) by blocking and Borel–Cantelli, as in the proof of Theorem 2.1, and the second part of the proof of this theorem shows that (2.8) implies (2.2) and (2.3). The first part of the proof of

Theorem 2.1, with obvious trivial changes, shows that (2.2) and (2.3) imply (2.1′). □

2.2. *Maxima of products under regularity conditions.* Conditions (2.2′) and (2.3′) are difficult to verify. Here we present simplifications under increasing degrees of regularity for the tail of X. The proofs of Corollaries 2.5 and 2.7 are omitted. The proof of Corollary 2.8 is given in the Appendix since this corollary is used in the next subsection and is handiest for the computations that produce the examples.

COROLLARY 2.5. *If the distribution of X satisfies the regularity condition*

$$\sup 2^k P\{X \geq u_k\} < \infty, \tag{2.12}$$

then (2.1), (2.2) *and* (2.10) *are all equivalent.*

REMARK 2.6. Note that (2.12) is satisfied if X has a continuous distribution or if the tail distribution G of X is regularly varying. The stronger condition (2.13) below is also satisfied by these two types of distributions.

Condition (2.2) or, equivalently, (2.4), requires double integration with respect to P. Under extra, but mild, regularity conditions on γ and G it can be simplified. Here are two instances.

COROLLARY 2.7. *Suppose that*

$$\liminf_{k\to\infty} 2^k P\{u_{k-1} < X \leq u_k\} > 0, \tag{2.13}$$

that there exists $0 < c_1 < c_2 < 1$ such that $c_1\gamma_{2n} \leq \gamma_n \leq c_2\gamma_{2n}$ for all $n \in \mathbb{N}$ and that the sequence $v_k := v(2^k) = \gamma_k^*/u_k$ is eventually nondecreasing. Then (2.1) *holds if and only if both*

$$\lim_{k\to\infty} \frac{u_k}{v_k} = 0 \tag{2.14}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{2^k} E[\gamma^{-1}(u_k X)]^2 I_{u_k < X \leq v_k} < \infty. \tag{2.15}$$

Note that if $\gamma(t) = t^{2/\alpha}$, then condition (2.15) becomes

$$\sum_{k=1}^{\infty} \frac{u_k^\alpha}{2^k} EX^\alpha I_{u_k < X \leq v_k} < \infty.$$

COROLLARY 2.8. *Suppose G and $\{\gamma_n\}$ satisfy the following conditions:*

(a) *G is regularly varying with exponent $-\alpha$, $\alpha > 0$, and there exist $p \in (1/2\alpha, \infty)$, $x_0 < \infty$ and $0 < K_1 < K_2 < \infty$ such that the slowly varying factor*

L of G satisfies

$$K_1 L^2(x) \le L(ax)L\left(\frac{x}{a}\right) \le K_2 L^2(x) \tag{2.16}$$

for $x > x_0$ *and* $1 \le a \le (\log x)^p$.

(b) $\gamma_{2n} \le C\gamma_n$ *for some* $C < \infty$ *and from some* n_0 *on.*

Then (2.1) *holds if and only if*

$$\sum_{n=1}^{\infty} n^{-1}(na_n)^2 \log_+ na_n < \infty, \tag{2.17}$$

where $a_n := G(\gamma_n^{1/2})$.

Note that condition (2.21) holds for many slowly varying functions. For instance, it holds for $L(x) \sim \log^\gamma x$ for any γ as well as for $L(x) \sim \exp(\alpha \log^\beta x)$ for any α and $0 \le \beta < 1$.

Deheuvels and Mason [(1988), Corollary 2] have a criterion for $P\{(U_{1:n} \cdots U_{k:n})^{1/k} < (na_n)^{-1} \text{ i.o.}\}$ to be 0 or 1, where $U_{j:n}$ are the order statistics associated to a sequence of i.i.d. random variables uniform on $[0,1]$. Translation into a result for $\max_{1\le i<j\le n} X_i X_j$ requires G to satisfy $G(X_{1:n})G(X_{2:n}) \simeq [G((X_{1:n}X_{2:n})^{1/2})]^2$ a.s. The hypotheses on G in Corollary 2.8 give this relationship for $(X_{1:n}/X_{2:n})^{1/2} < (\log n)^p$, $p > 1/2\alpha$, and can also be used along with Kiefer's theorem to check that $(X_{1:n}/X_{2:n})^{1/2} = o(\log n)^p$ a.s., $p > 1/2\alpha$; therefore this corollary can be seen as a translation of the Deheuvel–Mason result to nonuniform random variables. However, their approach does not seem to yield any of the other results in this section, since they are too general for reduction to the uniform case.

Theorem 2.1′ can be simplified if we require some extra, mild regularity on $\{\gamma_n\}$:

COROLLARIES 2.5′, 2.7′, 2.8′. *Suppose there exists* $0 < c < 1$ *such that* $\gamma_n \le c\gamma_{2n}$, $n \in \mathbb{N}$. *Then:*

(a) *If* X *satisfies* (2.12), *then* (2.1), (2.1′), (2.2), (2.10), (2.11) *and* (2.11′) *are all equivalent.*

(b) *If* X *and* $\{\gamma_n\}$ *satisfy the hypotheses of Corollary* 2.7, *then the conditions in part* (a) *are also equivalent to* (2.14) *and* (2.15).

(c) *If* X *and* $\{\gamma_n\}$ *satisfy the hypotheses of Corollary* 2.8, *then these conditions are also equivalent to* (2.18).

EXAMPLE 2.9. The following can be easily verified using, for example, Corollary 2.8: Let $\alpha > 0$ and let the law of X have tails

$$G(x) \sim \frac{1}{x^\alpha (\log x)^\beta},$$

$$G(x) \sim \frac{1}{x^\alpha (\log x)^{1/2} (\log_2 x)^{1/2+\beta}}$$

or

$$G(x) \sim \frac{1}{x^\alpha (\log x)^{1/2} (\log_2 x)(\log_3 x)^{1/2} \cdots (\log_{k-1} x)^{1/2} (\log_k x)^\beta}, \qquad k \geq 3,$$

where $\log_k x := \log_+(\log_{k-1} x)$, $k > 1$. Then, $(1/n^{2/\alpha}) \max_{i<j\leq n} X_i X_j \to 0$ a.s. if and only if $\beta > \frac{1}{2}$.

2.3. *Other products of order statistics.* The expression $\max_{i<j\leq n} |X_i X_j|$ can also be written as $|X_{1:n} X_{2:n}|$, where $X_{j:n}$ is the jth largest in magnitude among $X_1, \ldots, X_n$ (more precisely, $X_{j:n} = X_\ell$ if and only if there are exactly $j-1$ X_i's, $i \leq n$, such that either $|X_i| > |X_\ell|$ or $|X_i| = |X_\ell|$ and $i < \ell$). This gives another interpretation of the results in this section as *strong laws for the product of the first two order statistics*. It is also of interest to examine the lim sup behavior of products of other order statistics, in particular of $X_{1:n} X_{\ell:n}$. The following approach provides an alternate way of developing the material in this section and also yields a surprising result for $\ell \geq 3$.

THEOREM 2.10. *Under the conditions of Theorem* 2.1 *and for* $\ell \geq 2$,

$$P\{X_{1:n} X_{\ell:n} > \gamma_n \quad i.o.\} = 0 \tag{2.18}$$

if and only if, letting $F(x) = 1 - G(x)$,

$$\sum_{k=1}^{\infty} 2^{k\ell} \int_{u_k+}^{\infty} G\left(\frac{\gamma_k^*}{x}+\right) G^{\ell-2}(x)\, dF(x) < \infty \tag{2.19}$$

and

$$\sum_{k=1}^{\infty} 2^k G\left(\frac{\gamma_k^*}{u_k}+\right) < \infty. \tag{2.20}$$

Under the assumptions of Corollary 2.8, (2.18) *holds for* $\ell \geq 3$ *if and only if*

$$\sum_{n=1}^{\infty} n^{-1} (n a_n)^2 < \infty, \tag{2.21}$$

where $a_n := G(\gamma_n^{1/2})$.

When $\ell = 2$, (2.19) and (2.20) reduce to the conditions for Theorem 2.1. Condition (2.3) is retained for $\ell > 2$, but (2.2) is strengthened to (2.19). Under the conditions of Corollary 2.8, (2.19) does not depend on ℓ for $\ell > 2$ and reduces to (2.21). Condition (2.21) is Kiefer's (1972) necessary and sufficient condition for $P\{X_{2:n} > \gamma_n^{1/2} \quad \text{i.o.}\} = 0$, so that in that case $(1/\gamma_n) X_{1:n} X_{\ell:n} \to 0$ a.s., $\ell \geq 3$, iff

$$\frac{1}{\gamma_n^{1/2}} X_{2:n} \to 0 \quad \text{a.s.}$$

PROOF. When no confusion arises we write $X_{(j)}$ for $X_{j:2^k}$. We also let $F(x) = 1 - G(x)$. As above, (2.20) implies $\gamma_k^* \geq u_k^2$ eventually and

$$P\left\{X_{(1)} > \frac{\gamma_{k-1}^*}{u_{k-1}} \text{ i.o.}\right\} = 0;$$

in fact, $P\{X_{(1)} > \gamma_{k-r}^*/u_{k-r} \text{ i.o.}\} = 0$ for all $r < \infty$. Also, (2.19) implies

$$\begin{aligned}\sum_k 2^{k\ell} G(b_k^*) \int_{b_k^*+}^{\infty} G^{\ell-2}(x)\, dF(x) \\ \leq \sum_k 2^{k\ell} \int_{b_k^*+}^{\infty} G\left(\frac{\gamma_k^*}{x}+\right) G^{\ell-2}(x)\, dF(x) < \infty,\end{aligned}$$

where $b_k^* = (\gamma_k^*)^{1/2}$. Now, since G is left continuous and decreasing,

$$\int_{b_k^*+}^{\infty} G^{\ell-2}(x)\, dF(x) \geq (\ell-1)^{-1} G^{\ell-1}(b_k^*+)$$

so that

$$\sum_k (2^k G(b_k^*+))^{\ell} < \infty,$$

implying [Kiefer (1972)]

$$P\{X_{(\ell)} > b_{k-1}^* \text{ i.o.}\} = 0$$

(in fact, $P\{X_{(\ell)} > b_{k-r}^* \text{ i.o.}\} = 0$ for all $r < \infty$). Thus, it is enough to show $P\{X_{(1)}X_{(\ell)} > \gamma_{k-1}^*, u_{k-1} \leq X_{(\ell)} \leq b_{k-1}^*, \text{ i.o.}\} = 0$ or, by Borel–Cantelli, that $\sum_k P\{X_{(1)}X_{(\ell)} > \gamma_{k-1}^*, u_{k-1} \leq X_{(\ell)} \leq b_{k-1}^*\} < \infty$, which can be rewritten as

$$\sum_k \int_{u_{k-1}}^{b_{k-1}^*} P\left\{X_{(1)} > \frac{\gamma_{k-1}^*}{x}, X_{(\ell)} \in dx\right\} < \infty. \tag{2.22}$$

Now, in general, by counting the ways in which one X_i is in $(x, x+dx)$, another is greater than γ_{k-1}^*/x, another $\ell - 2$ are greater than or equal to x and the rest are less than or equal to x, we have

$$\begin{aligned}P\left\{X_{(1)} > \frac{\gamma_{k-1}^*}{x}, X_{(\ell)} \in dx\right\} \\ \leq 2^k(2^k - 1)\binom{2^k - 2}{\ell - 2} G\left(\frac{\gamma_{k-1}^*}{x}+\right) G^{\ell-2}(x)(1 - G(x+))^{2^k-\ell}\, dF(x).\end{aligned}$$

Likewise, by requiring all the remaining $2^k - \ell$ variables to be strictly less than x gives

$$(2.23)\quad \begin{aligned} &P\left\{X_{(1)} > \frac{\gamma^*_{k-1}}{x}, X_{(\ell)} \in dx\right\} \\ &\quad \geq (\ell-1)^{-2} 2^k (2^k - 1)\binom{2^k-2}{\ell-2} G\left(\frac{\gamma^*_{k-1}}{x}+\right) \\ &\quad\quad \times G^{\ell-2}(x)(1-G(x))^{2^k-\ell}\, dF(x), \end{aligned}$$

where the factor $(\ell-1)^{-2}$ is included to account for the possibility that as many as $\ell - 1$ variables could be greater than γ^*_{k-1}/x and as many as $\ell - 1$ could equal x. Now on the set for which either $x > u_{k-1}$ or $x = u_{k-1}$, but $2^k \Delta F(u_{k-1}) \leq \frac{1}{2}$, these bounds are of the same order of magnitude and the right-hand sides are convergence equivalent to

$$2^{\ell k} G\left(\frac{\gamma^*_{k-1}}{x}+\right) G^{\ell-2}(x)\, dF(x).$$

In general, when $x = u_{k-1}$ the left-hand sides are less than or equal to $P\{X_{(1)} > \gamma^*_{k-1}/u_{k-1}\} \simeq 2^k G((\gamma^*_{k-1}/u_{k-1})+)$. When $x = u_{k-1}$ and $2^k \Delta F(u_{k-1}) > \frac{1}{2}$, by considering the ways in which one X_i is greater than γ^*_{k-1}/u_{k-1}, at least $(\ell-1)$ of them are equal to u_{k-1} and the rest are less than u_{k-1}, we have

$$\begin{aligned} &P\left\{X_{(1)} > \frac{\gamma^*_{k-1}}{u_{k-1}}, X_{(\ell)} = u_{k-1}\right\} \\ &\quad \geq 2^k G\left(\frac{\gamma^*_{k-1}}{u_{k-1}}+\right) \sum_{m=\ell-1}^{2^k-1} \binom{2^k-1}{m} \Delta^m (1 - \Delta - G(u_{k-1}+))^{2^k-m-1} \\ &\quad = 2^k G\left(\frac{\gamma^*_{k-1}}{u_{k-1}}+\right)(1 - G(u_{k-1}+))^{2^k-1} \\ &\quad\quad \times P\left\{B\left(\frac{\Delta}{1 - G(u_{k-1}+)}, 2^k - 1\right) \geq \ell - 1\right\}, \end{aligned}$$

where $\Delta = \Delta F(u_{k-1})$ and $B(p, n)$ is a binomial (p, n) random variable. Since this expression is increasing in Δ we can replace Δ by 2^{-k-1}. The binomial probability is then seen to be bounded below as $k \to \infty$ by a positive constant, and since $G(u_{k-1}+) < 2^{-k+1}$, the whole expression is bounded below by a positive constant times $2^k G((\gamma^*_{k-1}/u_{k-1})+)$. Thus (2.18) follows from (2.19) and (2.20). To prove the converse, (2.18) implies (2.22) (with k replacing $k-1$) by the usual exponential blocking and Borel–Cantelli lemma. Equation (2.18) also implies $P\{X_{(\ell)} > (\gamma^*_k)^{1/2} \text{ i.o.}\} = 0$, so (2.22) (with $k+1$ replacing $k-1$) also holds when the upper limit of integration is changed from b^*_k to ∞. The previously established lower bounds on the integrand can now be used to verify that (2.19) and (2.20) hold. We defer the proof of the last statement of the theorem to the Appendix. □

EXAMPLE 2.11. If we modify the definitions of G in Example 2.9 by replacing $(\log_2 x)^{1/2+\beta}$ by $(\log_2 x)^{\beta}$ and $\log_2 x$ by $(\log_2 x)^{1/2}$, then Theorem 2.10 gives that, for $\ell \geq 3$, $\frac{1}{n^{2/\alpha}} X_{1:n} X_{\ell:n} \to 0$ a.s. if and only if $\beta > \frac{1}{2}$.

3. Truncated and trimmed sums. Kiefer [(1972) Theorem 1] observed that, for $b_n \nearrow \infty$,

$$P\{|X_{k:n}| \geq b_n \text{ i.o.}\} = 0 \tag{3.1}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n}(nG(b_n))^k < \infty, \tag{3.2}$$

and Mori (1977) proved that, under mild regularity on the sequence $\{b_n\}$ and if $n^{-\beta} b_n \nearrow \infty$ for some $\beta > \frac{1}{2}$, this condition is also necessary and sufficient for the existence of a numerical sequence $\{c_n\}$ such that

$$\frac{1}{b_n} \sum_{j=k}^{n} X_{j:n} - c_n \to 0 \quad \text{a.s.}$$

and that c_n can be taken to be $(n/b_n) EXI_{|X|<b_n}$. The sufficiency part of Mori's theorem can be obtained as a corollary of the main result of this section, which is a sharp a.s. bound for truncated sums of i.i.d. random variables whose distribution satisfies condition (3.2) for some $k \geq 1$. For $k = 1$ it is essentially Feller's (1946) law of large numbers, whereas for $k > 1$ the levels of truncation b_n are smaller than the usual in proofs of laws of large numbers. The result, Theorem 3.2, is just a consequence of a simple exponential inequality of Klass and Teicher (1977) if G is regularly varying. However, in the general case it also relies on the surprising fact that condition (3.2), which can also be written as

$$\sum_{n=1}^{\infty} [2^n G(b(2^n))]^k < \infty, \tag{3.2$'$}$$

implies

$$\sum_{n=1}^{\infty} \left(\frac{2^n EX^2 I_{|X|<b(2^n)}}{b^2(2^n)} \right)^k < \infty \tag{3.3}$$

for b_n as above and for *any* random variable X (Theorem 3.1). This is an integrated one-sided analogue of the equivalence $x^2 G(x) \simeq EX^2 I_{|X|<x}$ (as $x \to \infty$), valid only for regularly varying functions G with exponent $-\alpha$, $0 < \alpha < 2$. Although the law of large numbers for quadratic forms in Section 4 will only be proved under some (mild) regularity on G, Theorem 3.1 will allow us to complete a substantial part of the proof without using regularity. [To see that (3.2) and (3.2$'$) are equivalent, just note that, since $b \nearrow$ and $G \searrow$, if $2^r < n \leq 2^{r+1}$, then $2^{r(k+1)} G(b(2^{r+1}))^k \leq n^{k-1} G(b(n))^k \leq 2^{(r+1)(k-1)} G(b(2^r))^k$.]

At the end of the section we discuss the regularity conditions for G and b_n that are required in Section 4.

It will be useful to rewrite conditions (3.2) and (3.3) in integral form. Let $b(t)$, $t \geq 0$, be a positive increasing function. We will write $b_n := b(n)$, $b_n^* := b(2^n)$, $n \in \mathbb{N}$, and, in general, $b^*(t) := b(2^t)$, $t > 0$. Since b is increasing, condition (3.2) is equivalent to (3.2′), hence to

$$\int_0^\infty (2^t G(b^*(t)))^k \, dt < \infty. \tag{3.2″}$$

By writing $I_{|X|<b_n^*}$ as

$$\sum_{k=0}^\infty I_{2^{-(k+1)}b_n^* \leq |X| < 2^{-k}b_n^*},$$

so that

$$(b_n^*)^{-2} EX^2 I_{|X|<b_n^*} \leq 4 \sum_{k=1}^\infty 2^{-2k} G(2^{-k} b_n^*),$$

and then expressing the sums as integrals, (3.3) turns out to be implied by

$$\int_1^\infty \left(2^t \int_0^1 uG(ub^*(t))\, du\right)^k dt < \infty. \tag{3.3′}$$

THEOREM 3.1. *Assume $2^{-\beta t} b^*(t) \nearrow$ for some $\beta > \frac{1}{2}$ and that $G(x)$ is bounded, nonincreasing and left continuous. If (3.2″) holds for some $k > 0$ (not necessarily an integer), then so does (3.3′) (for the same k).*

PROOF. Define $\bar{G}(x) = \sup_{u \leq 1} u^{2-\varepsilon} G(ux)$ for $0 < \varepsilon < 2$ to be specified below, and note that $\bar{G}$ is continuous, nonincreasing and

$$\int_0^1 uG(ux)\, du = \int_0^1 u^{2-\varepsilon} G(ux) \frac{du}{u^{1-\varepsilon}} \leq \frac{1}{\varepsilon} \bar{G}(x).$$

So it is enough to show

$$\int_1^\infty (2^t \bar{G}(b^*(t)))^k < \infty.$$

Now $S = \{z : \bar{G}(z) > G(z)\}$ is open since $\bar{G}$ is continuous and G is left continuous and nonincreasing. Thus S consists of a union of disjoint intervals. Let (x, y) be such an interval. Then for $z \in (x, y)$, $\bar{G}(z) = (x/z)^{2-\varepsilon} G(x)$. To see this note that $\bar{G}(z) = (w/z)^{2-\varepsilon} G(w)$ for some $w < z$ since G is left continuous and nonincreasing, and $z \in S$. If $\bar{G}(z) > (x/z)^{2-\varepsilon} G(x)$ and $w < x$, then $(w/x)^{2-\varepsilon} G(w) > G(x)$, implying $x \in S$, which is a contradiction. If $w > x$, then $\bar{G}(z) = (w/z)^{2-\varepsilon} G(w) \geq (t/z)^{2-\varepsilon} G(t)$ for all $t \leq z$ so that

$$G(w) \geq \sup_{t \leq w} \left(\frac{t}{w}\right)^{2-\varepsilon} G(t),$$

implying $G(w) = \bar{G}(w)$, which contradicts $w \in S$. Now, for every defining interval (x, y) of S define (u, v) by $b^*(u) = x, b^*(v) = y$ and write

$$\int_u^v (2^t \bar{G}(b^*(t)))^k \, dt = \int_u^v \left(2^t \left(\frac{b^*(u)}{b^*(t)}\right)^{2-\varepsilon} G(b^*(u))\right)^k dt. \tag{3.4}$$

Since $2^{-\beta t} b^*(t) \nearrow \infty$ for some $\beta > \frac{1}{2}$, we can choose $\varepsilon > 0$ so that $2^t(b^*(t))^{-(2-\varepsilon)} \le 2^{t_0}(b^*(t_0))^{-(2-\varepsilon)} 2^{\varepsilon(t_0 - t)}$ for $t \ge t_0$, implying that the quantity in (3.4) is bounded by a constant times

$$(2^u G(b^*(u)))^k \min(1, v-u) \le \int_u^v (2^t G(b^*(t-1)))^k \, dt \quad \text{for } u \ge 1.$$

Thus

$$\begin{aligned} \int_{S\cap\{t>1\}} (2^t \bar{G}(b^*(t)))^k \, dt &\lesssim \int_1^\infty (2^t G(b^*(t-1)))^k \, dt \\ &\lesssim 2^k \int_0^\infty (2^t G(b^*(t)))^k \, dt < \infty \end{aligned}$$

by (3.2′). The finiteness of the integral of $(2^t \bar{G}(b^*(t)))^k$ over S^c is trivial since $\bar{G} = G$ on this set. □

By way of illustration, we give a version of the statement of this theorem in the particular case $k = 2$ and $b(t) = t^{1/\alpha}$, $1 < \alpha < 2$. Note that if Y is an independent copy of X,

$$\sum 2^{2n}(P\{|X| > 2^{n/\alpha}\})^2 = \sum 2^{2n} E I_{|X|>2^{n/\alpha}, |Y|>2^{n/\alpha}} = E\left[\sum_{n:(|X|\wedge|Y|)^\alpha > 2^n} 2^{2n}\right],$$

which is equivalent to $E(|X| \wedge |Y|)^{2\alpha}$ up to fixed multiplicative and additive constants. Similarly,

$$\sum \left(\frac{2^n E X^2 I_{|X|<2^{n/\alpha}}}{2^{2n/\alpha}}\right)^2 \simeq E(|X| \wedge |Y|)^2 (|X| \vee |Y|)^{2(\alpha-1)}$$

(up to multiplicative and additive constants). Therefore, Theorem 3.1 shows that, without any assumptions on the law of X,

$$E(|X| \wedge |Y|)^{2\alpha} < \infty \quad \Rightarrow \quad E(|X| \wedge |Y|)^2 (|X| \vee |Y|)^{2(\alpha-1)} < \infty, \tag{3.5}$$

as mentioned in the Introduction.

Here is the result for truncated sums:

THEOREM 3.2. *Assume X, X_i, $i \in \mathbb{N}$, are i.i.d. and let $G(x) = P\{|X| \ge x\}$. Let $b(t)$, $t \ge 0$, be a positive function such that $t^{-\beta} b(t) \nearrow$ for some $\beta > \frac{1}{2}$, and let $b_n = b(n)$. If*

$$\lim_{n\to\infty} \frac{n}{b_n} E X I_{|X|<\varepsilon b_n} = 0 \tag{3.6}$$

for all small enough $\varepsilon > 0$, *and*

$$\sum_{n=1}^{\infty} \frac{1}{n}(nG(b_n))^k < \infty \tag{3.2}$$

for some positive integer k, then

$$\limsup_{n\to\infty} \frac{1}{b_n}\left|\sum_{i=1}^{n} X_i I_{|X_i|<b_n}\right| \le k-1 \quad a.s. \tag{3.7}$$

PROOF. Since $t^{-\beta}b(t)$ is increasing, for any $\varepsilon > 0$ there is $m(\varepsilon) < \infty$ such that $\varepsilon b_n > b_{n-m(\varepsilon)}$ and therefore $\sum(1/n)(nG(\varepsilon b_n))^k < \infty$. Thus, by Kiefer's theorem, $P\{|X_{k:n}| > \varepsilon b_n \text{ i.o.}\} = 0$. Hence,

$$\limsup_{n\to\infty} \frac{1}{b_n}\left|\sum_{i=1}^{n} X_i I_{\varepsilon b_n<|X_i|<b_n}\right| \le k-1$$

and it is enough to show

$$\limsup_{n\to\infty} \frac{1}{b_n}\left|\sum_{i=1}^{n} X_i I_{|X_i|<\varepsilon b_n}\right| \le 3k\varepsilon.$$

So, redefining b_n as εb_n, the proof of the theorem reduces to showing that the conditions

$$\lim_{n\to\infty} \frac{n}{b_n} EXI_{|X|<b_n} = 0 \tag{3.8}$$

and (3.2) imply

$$\limsup_{n\to\infty} \frac{1}{b_n}\left|\sum_{i=1}^{n} X_i I_{|X_i|<b_n}\right| \le 3k \quad \text{a.s.} \tag{3.9}$$

Letting $\tilde{b}_n = b(2^\ell)$ for $2^\ell < n \le 2^{\ell+1}$, we also have $\sum(1/n)(nG(\tilde{b}_n))^k < \infty$ so that by Kiefer's theorem, $\limsup_{n\to\infty}(1/b_n)|\sum_{i=1}^n X_i I_{\tilde{b}_n<|X_i|<b_n}| \le k-1$. Hence, proving (3.9) further reduces to showing

$$\limsup_{n\to\infty} \frac{1}{b_n}\left|\sum_{i=1}^{n} X_i I_{|X_i|<\tilde{b}_n}\right| \le 2k \quad \text{a.s.} \tag{3.9$'$}$$

Also, since $2^\ell G(b(2^\ell)) \to 0$, $(n/b_n)EXI_{\tilde{b}_n\le|X|<b_n} \le nG(\tilde{b}_n) \to 0$ or, by (3.8), $\lim_{n\to\infty}(n/b_n)EXI_{|X|<\tilde{b}_n} = 0$; hence, we can center in (3.9′). Then, by the Borel–Cantelli lemma, it suffices to prove

$$\sum_{n=1}^{\infty} P\left\{\frac{1}{b_n^*}\max_{\ell\le 2^{n+1}}\left|\sum_{j=1}^{\ell} Y_j\right| > M\right\} < \infty$$

for all $M > 2k$, where $Y_i = X_i I_{|X_i|<b_n^*} - EX_i I_{|X_i|<b_n^*}$ and $b_n^* = b(2^n)$. By the exponential inequality in Klass and Teicher [(1977), Lemma 1] it is enough to show

$$\sum_n \exp\{-Mt_n b_n^* + \tfrac{1}{2}s_n^2 t_n^2 \exp(t_n b_n^*)\} < \infty$$

for some sequence $\{t_n\}$ of positive constants, where $s_n^2 = 2^{n+1}EX^2 I_{|X|<b_n^*}$. If we set $x_n = t_n b_n^*$ and $C_n = 2M(b_n^*)^2/s_n^2$, the expression at the left side becomes

$$\sum_n \exp\left\{\frac{1}{2}\left(\frac{s_n}{b_n^*}\right)^2 [-C_n x_n + x_n^2 e^{x_n}]\right\}.$$

The x_n which minimizes the nth exponent satisfies

$$e^{x_n} = \frac{C_n}{2x_n + x_n^2}. \tag{3.10}$$

Thus, we must show that

$$\sum_n \exp\left\{\frac{1}{2}\left(\frac{s_n}{b_n^*}\right)^2 \left[-C_n x_n\left(1 - \frac{1}{2 + x_n}\right)\right]\right\} < \infty,$$

which, since $x_n > 0$, reduces to showing

$$\begin{aligned} \sum \exp\left\{-\frac{1}{4}\left(\frac{s_n}{b_n^*}\right)^2 C_n x_n\right\} &= \sum \exp\left\{-\frac{1}{2}M x_n\right\} \\ &= \sum \left(\frac{C_n}{2x_n + x_n^2}\right)^{-M/2} \\ &\le \sum \left(\frac{C_n}{(1 + x_n)^2}\right)^{-M/2} < \infty. \end{aligned} \tag{3.11}$$

By Theorem 3.1, condition (3.2) implies

$$\sum_n C_n^{-k} < \infty. \tag{3.12}$$

Since, by (3.10), $x_n \sim \log C_n$ [note that $C_n \to \infty$ by (3.12)], (3.11) follows from (3.12) if $M = 2k + \varepsilon$, $\varepsilon > 0$. □

The following example shows that the bound $k - 1$ in (3.7) is in general the best possible.

EXAMPLE 3.3. Let $X \ge 1$ be such that $P\{X \ge x\} = 1/(x^\alpha (\log x)^\sigma)$, for some $0 < \alpha < 1$ and $1/k < \sigma < 1/(k-1)$, and let $b_n = n^{1/\alpha}$, so that $G(b(2^n)) \simeq 1/2^n n^\sigma$. Then the conditions of Theorem 3.2 hold (this only requires $\sigma > 1/k$). However, for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} \sum_{i=1}^n I_{X_i \in [(1-\varepsilon)b_n, b_n)} \ge k - 1$$

by the Borel–Cantelli lemma since

$$\begin{aligned}\sum_{\ell} P\left\{\sum_{i=2^{\ell-1}+1}^{2^{\ell}} I_{X_i\in[(1-\varepsilon)b_n,b_n)} \geq k-1\right\} \\ \simeq \sum_{\ell}(2^{\ell-1}[G((1-\varepsilon)b(2^{\ell}))-G(b(2^{\ell}))])^{k-1} \\ \simeq 2^{-k}\sum_{\ell}\left(\frac{1}{\ell^{\sigma}}\right)^{k-1}=\infty.\end{aligned}$$

Thus

$$\limsup_{n\to\infty}\frac{1}{b_n}\sum X_i I_{|X_i|<b_n} \geq (1-\varepsilon)(k-1) \quad \text{for all } \varepsilon>0.$$

The sufficiency part of Mori's (1977) theorem on lightly trimmed sums follows very easily from Theorem 3.2. Here we state this theorem and give a proof, different from Mori's, of its sufficiency part in the case $c_n \to 0$.

THEOREM 3.4 [Mori (1977)]. *Let $b(t)$, b_n be as in Theorem 3.2 and let X, X_i, $i\in\mathbb{N}$, be i.i.d. with $G(x)=P\{|X|\geq x\}$. Then conditions (3.2) and (3.6) are necessary and sufficient for*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=k}^{n} X_{j:n}=0 \quad a.s. \tag{3.13}$$

PROOF OF SUFFICIENCY. Assume the limits (3.2) and (3.6) hold. Then, by the result of Kiefer (1972), mentioned above,

$$P\{|X_{k:n}|>\varepsilon b_n \text{ i.o.}\}=0.$$

Hence, given $\varepsilon>0$, there is $n(\omega)$ a.s. finite such that, for $n>n(\omega)$,

$$\sum_{j=k}^{n} X_{j:n}=\sum_{i=1}^{n} X_i I_{|X_i|<\varepsilon b_n}-\sum_{j=1}^{k-1} X_{j:n} I_{|X_{j:n}|<\varepsilon b_n}$$

and, therefore,

$$\frac{1}{b_n}\left|\sum_{j=k}^{n} X_{j:n}\right| \leq \varepsilon\left[\frac{1}{\varepsilon b_n}\sum_{i=1}^{n} X_i I_{|X_i|<\varepsilon b_n}+(k-1)\right].$$

This and Theorem 3.2 give

$$\limsup_{n\to\infty}\frac{1}{b_n}\left|\sum_{j=k}^{n} X_{j:n}\right| \leq 2\varepsilon(k-1) \quad \text{a.s.}$$

for all $\varepsilon>0$, and (3.13) follows. □

REMARK 3.5. It follows from the above proofs that the condition $t^{-\beta}b(t) \nearrow \infty$ is not required for the validity of Theorems 3.2 and 3.4 if either G is regularly varying with exponent $-\alpha$, $0 < \alpha < 2$, or $b(t)$ is regularly varying with exponent $\lambda > \frac{1}{2}$.

Actually, the centering condition (3.6) holds automatically under regularity of G and/or $b(t)$, as we show next. It is convenient to formally define the required regularity since it plays a role in the next section.

DEFINITION 3.6. In the context of this article, a random variable X, or its tail probability function $G(x) = P\{|X| \geq x\}$, $x \geq 0$, is said to be regular if either:

(a) G is regularly varying (at infinity) with exponent $-\alpha$, $0 < \alpha < 2$, and additionally $EX = 0$ for $1 < \alpha < 2$ or X is symmetric for $\alpha = 1$.
(b) $t^{\alpha}G(t) \nearrow$ for some $0 < \alpha < 2$ and X is symmetric.
(c) $t^{\alpha}G(t) \nearrow$ for some $1 < \alpha < 2$, $G(2t) \leq 2^{-1-\delta}G(t)$ for some $\delta > 0$ and all t large enough, and $EX = 0$.
(d) $t^{\alpha}G(t) \nearrow$ for some $0 < \alpha < 1$.

DEFINITION 3.7. In the context of this article, a positive continuous function $b(t)$, $t \geq 0$, such that $b(t) \nearrow \infty$ is said to be regular for X if either:

(a) b is regularly varying (at infinity) with exponent β satisfying:
  (a.1) $\beta > \frac{1}{2}$ if X is symmetric,
  (a.2) $\beta > \frac{1}{2}$, $\beta \neq 1$, if X is not symmetric, but $E|X| < \infty$ and $EX = 0$,
  (a.3) $\beta > 1$ otherwise.
(b) $t^{-\beta}b(t) \nearrow$ for some exponent β satisfying:
  (b.1) $\beta > \frac{1}{2}$ if X is symmetric,
  (b.2) $\beta > \frac{1}{2}$ if $E|X| < \infty$, $EX = 0$ and $b(2t) \leq 2^{1-\delta}b(t)$ for some $\delta > 0$ and all t large enough,
  (b.3) $\beta > 1$ otherwise.

These definitions are motivated by the following elementary propositions.

PROPOSITION 3.8. (a) *If G is regular, then there exist $C < \infty$ and $x_0 < \infty$ such that for all $x \geq x_0$,*

$$|EXI_{|X|\leq x}| \leq CxG(x) \quad \text{and} \quad EX^2I_{|X|\leq x} \leq Cx^2G(x). \tag{3.14}$$

[$|X| \leq x$ can be replaced by $|X| < x$ in (3.14).*]*
(b) *If b is regular for X and $tG(b(t)) \to 0$ as $t \to \infty$, then, with $b_n = b(n)$,*

$$\frac{n}{b_n}EXI_{|X|\leq \varepsilon b_n} \to 0 \quad \text{and} \quad \frac{n}{b_n^2}EX^2I_{|X|\leq \varepsilon b_n} \to 0 \tag{3.15}$$

for all $\varepsilon > 0$. [$|X| \leq \varepsilon b_n$ can be replaced by $|X| < \varepsilon b_n$ in (3.15).*]*

PROOF (Sketch). Statement (3.14) follows from Definition 3.6(a), by the asymptotic properties of regularly varying functions [Feller (1971), VIII.9, Theorem 1]. The second inequality in (3.14) follows immediately from $t^\alpha G(t) \nearrow$ for some $0 < \alpha < 2$, and so does the first if $\alpha < 1$. We prove only the first inequality in (3.14) under condition (c): since $E|X| < \infty$ and $EX = 0$, we have

$$|EXI_{|X|\le x}| = |EXI_{|X|>x}| \le xG(x) + \int_x^\infty G(t)\,dt$$

and, since $G(2t) \le 2^{-1-\delta}G(t)$,

$$\int_x^\infty G(t)\,dt = \sum_{k=0}^\infty \int_{2^k x}^{2^{k+1}x} G(t)\,dt \le \sum_{k=0}^\infty 2^{k+1}xG(2^k x) \le 2\biggl(\sum_{k=0}^\infty 2^{-\delta k}\biggr)xG(x).$$

For part (b) note that εb_n is regular so that $tG(\varepsilon b(t)) \to 0$ if $tG(b(t)) \to 0$, and thus it suffices to prove (3.15) for $\varepsilon = 1$. The second limit in (3.15) requires only that b be regularly varying with exponent $\beta > \frac{1}{2}$ or that $t^{-\beta}b(t) \nearrow$ for some $\beta > \frac{1}{2}$. The proofs being similar, we prove it only under the second hypothesis. Let $\beta' \in (\frac{1}{2}, \beta)$. Then $\tau_n := b_n^{1-1/2\beta} \to \infty$, so that $\varepsilon_n := \sup_{t>\tau_n} b^{-1}(t)P\{|X| > t\} \to 0$. Note also $t^{1/\beta}/b^{-1}(t) \nearrow$. So, we have

$$\begin{aligned}\frac{n}{b_n^2}EX^2 I_{|X|<b_n} &\le \frac{n}{b_n^2}\tau_n^2 + 2\varepsilon_n \frac{n}{b_n^2}\int_{\tau_n}^{b_n} \frac{t\,dt}{b^{-1}(t)}\\ &\le \frac{n}{(b_n)^{1/\beta}} + 2\varepsilon_n b_n^{1/\beta-2}\int_{\tau_n}^{b_n} t^{1-1/\beta}\,dt\\ &\le \frac{n}{(b_n)^{1/\beta}} + \frac{2\beta}{2\beta-1}\varepsilon_n \to 0.\end{aligned}$$

Suppose now $E|X| < \infty$, $EX = 0$ and $b(2t) \le 2^{1-\delta}b(t)$ and let us prove the first limit in (3.15). For simplicity, set $c = 2^{1-\delta} > 1$ and $\varepsilon_n = \sup_{t\ge b_n} b^{-1}(t)P\{|X| \ge t\}$, which tends to zero. Then

$$\begin{aligned}\frac{n}{b_n}|EXI_{|X|>b_n}| &\le nP\{|X| > b_n\} + \varepsilon_n \frac{n}{b_n}\int_{b_n}^\infty \frac{dt}{b^{-1}(t)}\\ &= o(1) + \varepsilon_n \frac{n}{b_n}\sum_{r=1}^\infty \int_{c^{r-1}b_n}^{c^r b_n} \frac{dt}{b^{-1}(t)}\end{aligned}$$

and the limit is zero because

$$\int_{c^{r-1}b_n}^{c^r b_n} \frac{dt}{b^{-1}(t)} \le \frac{c^r b_n}{b^{-1}(c^{r-1}b_n)} \le \frac{2^{(1-\delta)r}b_n}{b^{-1}(b(2^{r-1}n))} = \frac{2}{2^\delta r}\frac{b_n}{n}.$$

The rest of the cases are treated similarly, and they are even easier under regular variation. □

Theorems 3.2 and 3.4, Remark 3.5 and Proposition 3.8 give the following corollary.

COROLLARY 3.9. *If either G is regular or b is regular for X, condition* (3.2) *implies* (3.7) *and is necessary and sufficient for* (3.13) *to hold.*

PROOF. In view of the previous observations it is sufficient to check that (3.2) implies $tG(b(t)) \to 0$ as $t \to \infty$. This follows from (3.2″), monotonicity of $b(t)$ and the obvious inequality

$$[2^n G(b_n^*)]^k \le 2^k \int_n^{n+1} [2^t G(b^*(t))]^k \, dt. \qquad \square$$

Finally, combining Proposition 3.8 with the general weak law of large numbers for triangular arrays [e.g., Araujo and Giné (1980), Theorem 2.4.7, case of degenerate limits] yields the following fact that we will use below.

PROPOSITION 3.10. *If either G is regular or b is regular for X and if*

$$nG(b_n) \to 0, \tag{3.16}$$

then

$$\frac{1}{b_n} \sum_{i=1}^{n} X_i \to 0 \text{ in pr.} \tag{3.17}$$

4. Quadratic forms. Finally we consider a.s. convergence to zero of normalized sums of products of independent random variables. The first two results give necessary and sufficient conditions for symmetric variables, whereas the third shows the equivalence of the law of large numbers for sums and for maxima when the variables are regular (in the sense of Definition 3.6), but not necessarily symmetric. Only regular normalizing sequences are considered.

THEOREM 4.1. *Let Y, X, X_i, $i \in \mathbb{N}$, be i.i.d. symmetric random variables and let $\gamma(t)$, $t \ge 0$, be a continuous function increasing to ∞ such that $b(t) = (\gamma(t))^{1/2}$ is regular for X and $\gamma(2t) \le C\gamma(t)$ for some $C < \infty$ and all t large enough. Let, as usual, $\gamma_n = \gamma(n)$, $\gamma_k^* = \gamma(2^k)$ and $u_k = G^{-1}(2^{-k})$, with $G(x) = P\{|X| \ge x\}$. Then, the law of large numbers*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1 \le i < j \le n} X_i X_j = 0 \quad \text{a.s.} \tag{4.1}$$

holds if and only if the following two conditions are satisfied:

$$\sum_{k=1}^{\infty} 2^{2k} P\{|XY| > \varepsilon\gamma_k^*, |X| > u_k, |Y| > u_k\} < \infty \tag{4.2}$$

and

$$\sum_{k=1}^{\infty} 2^k P\{|X| > \varepsilon w_k\} < \infty \tag{4.3}$$

for all $\varepsilon > 0$, *where*

$$w_k = \frac{\gamma_k^*}{[2^k E(|X| \wedge u_k)^2]^{1/2}}. \tag{4.4}$$

In fact conditions (4.2) *and* (4.3) *are necessary for* (4.1) *without any regularity assumptions on the nondecreasing normalizing sequence* γ_n.

While (4.2) simply reiterates (2.2′), condition (4.3) may be stronger than (2.3′) since $w_k \le v_k := \gamma_k^*/u_k$. These conditions are equivalent when X is regular [Proposition 3.8(a)], but it is not difficult to construct examples for which $\limsup (v_k/w_k) = \infty$. Whether (4.3) is stronger than (2.3′) when (4.2) holds and $b(t)$ is regular is unclear, but the possiblity remains that replacing both sums in $\sum_{j=1}^{n}\sum_{i=1}^{j-1} X_i X_j$ by maxima may change the rate of a.s. convergence when the tail probability of X is not regular. In any case, the following result shows that if only one sum is replaced by a maximum, then the rate is unchanged, at least for symmetric random variables. Here, the order statistics $X_{k:n}$ are as defined in Section 2.

THEOREM 4.2. *Let* X_i *be i.i.d. symmetric or nonnegative random variables and let* γ *satisfy the same regularity conditions as in the previous theorem. Then the law of large numbers* (4.1) *holds if and only if*

$$\frac{1}{\gamma_n} X_{1:n} \sum_{k=2}^{n} X_{k:n} \to 0 \quad a.s. \tag{4.5}$$

The following theorem gives the equivalence between the laws of large numbers for sums and maxima in the case of regular tails.

THEOREM 4.3. *Let* X, X_i, $i \in \mathbb{N}$, *be i.i.d. and let* $\gamma(t) \nearrow \infty$, $\gamma_n = \gamma(n)$. *Consider the statements*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \max_{1\le i<j\le n} |X_i X_j| = 0 \quad a.s. \tag{4.6}$$

and

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i<j\le n} X_i X_j = 0 \quad a.s. \tag{4.1}$$

Then (4.1) *implies* (4.6). *If moreover* G *is regular,* $b(t) := \gamma(t)^{1/2}$ *is regular for* X *and* $\gamma(2t) \le C\gamma(t)$ *for some* $C < \infty$ *and all* t *large enough, then* (4.6) *implies* (4.1).

The assumption that X is symmetric when $\alpha = 1$ in Theorem 4.3 can be relaxed at the expense of extra technical detail. The problem arises in the proof of Theorem 3.2 where the centering of truncated sums must be accommodated as assumed in (3.6) (or, what is the same, in symmetrization—see Propositions 4.7 and 4.8 below). Use of the methods of Feller (1946), will improve these results, but we prefer to avoid the added complications induced.

As the following example shows, (4.6) does not imply (4.1) in general.

EXAMPLE 4.4. Consider Example 2.3 with $1 < \alpha < 2$ so that

$$\limsup \frac{1}{\gamma_n} \max_{1\le i<j\le n} X_i X_j \ge 1 \quad \text{a.s.}$$

The computations for this example also show that

$$\limsup \frac{1}{(1+\varepsilon)\gamma_n} \max_{1\le i<j\le n} X_i X_j \le 1 \quad \text{a.s.}$$

for all $\varepsilon > 0$. Thus,

$$\limsup_{n\to\infty} \frac{1}{\gamma_n} \max_{1\le i<j\le n} X_i X_j = 1 \quad \text{a.s.}$$

With the same notation as in Example 2.3 (so, b_n here has a different meaning than in Section 3 or in the rest of Section 4) we have that for $m_n = [a_{n+1/2} - 1]$ and $\ell_n < m_n$,

$$\begin{aligned} &P\{X_{1:m_n} \ge b_{n+1}, X_{\ell_n+1:m_n} = b_n\} \\ &\quad\ge \frac{m_n}{a_{n+1}} \sum_{\ell\ge\ell_n} \binom{m_n-1}{\ell}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)^{\ell}\left(1-\frac{1}{a_n}\right)^{m_n-\ell-1} \\ &\quad\ge (1-\varepsilon_n)\frac{m_n}{a_{n+1}} \sum_{\ell\ge\ell_n} \binom{m_n-1}{\ell}\left(\frac{1}{a_n}\right)^{\ell}\left(1-\frac{1}{a_n}\right)^{m_n-\ell-1}, \end{aligned}$$

where $\varepsilon_n > 0$ tends to zero as $n \to \infty$. Now $(m_n - 1)/a_n \approx a_{n+1/2}/a_n \approx n^{\alpha/2}$, so that for $\ell_n = [n^{\alpha/2}]$,

$$\inf_n P\left\{B\left(\frac{1}{a_n}, m_n - 1\right) \ge \ell_n\right\} > 0.$$

Then, since $a_{n+3/2} - a_{n+1/2} \approx a_{n+3/2}$, we can apply Borel–Cantelli to the blocks $[a_{n+1/2}, a_{n+1+1/2})$ and obtain

$$P\{X_{1:m_n} X_{\ell_n+1:m_n} \ge b_n b_{n+1} \text{ i.o.}\} = 1.$$

This implies

$$\limsup_{n\to\infty} \frac{1}{n^{\alpha/2}\gamma_{m_n}} \sum_{1\le i<j\le m_n} X_i X_j \ge \limsup_{n\to\infty} \frac{1}{\gamma_{m_n}} X_{1:m_n} X_{\ell_n+1:m_n} \ge 1.$$

Replacing γ_n by $c_n\gamma_n$ for a sequence c_n barely tending to infinity, we see that the normalized maxima tend to zero a.s. whereas the lim sup of the normalized sums tends to infinity a.s. Note also that taking $b_n = a_n^{\tau}$, $\tau > \frac{1}{2}$, gives $EX^2 = \infty$ and $(n/\gamma_n)EXI_{X\le\gamma_n} \to 0$, but the sequence $\gamma_n^{1/2}$ is not regular. With little extra effort one can extend the example to make X symmetric (with the extra factor in the norming sequence replaced by $n^{\alpha/4}$) and replace $a_{n+1/2}$ with $a_{n+\varepsilon}$, $\varepsilon > 0$, with $n^{-\tau}\gamma_n$ increasing on $[a_{n+\varepsilon}, a_{n+1}]$, but γ is still not regular, although it nearly is. It is an open question whether an example such as this one is possible for regular γ.

We are also interested in decoupled versions of the above theorems so, at the risk of becoming somewhat prolix, we will treat decoupling and randomization in some detail.

4.1. *Some randomization and the proofs of Theorems* 4.1 *and* 4.2. Adapting some arguments from Giné and Zinn (1994), we first randomize the sums by products of Rademacher variables and then we conclude that if $(1/\gamma_n)\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j X_i X_j \to 0$ a.s., then also $(1/\gamma_n^2)\sum_{1\le i<j\le n} X_i^2 X_j^2 \to 0$ a.s. giving, in particular, the law of large numbers for maxima. The corresponding decoupled statement is also obtained. Here is the randomization lemma:

LEMMA 4.5. *Let X_i, $i \in \mathbb{N}$, be i.i.d. random variables and let ε_i, $i \in \mathbb{N}$, be independent Rademacher variables independent of $\{X_i\}$. Let $\{\gamma_n\}$ be a nondecreasing sequence of positive numbers tending to infinity. Then*

$$\lim_{n\to\infty}\frac{1}{\gamma_n}\sum_{1\le i<j\le n} X_i X_j = 0 \ \ a.s. \quad \Longrightarrow \quad \lim_{n\to\infty}\frac{1}{\gamma_n}\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j X_i X_j = 0 \ \ a.s.$$

PROOF. Let A be a subset of $\mathbb{N}$ and let $A_n = A\cap\{1,\dots,n\}$, $n\in\mathbb{N}$. Let

$$S_n(A) = \frac{1}{\gamma_n}\sum_{\substack{i,j\in A_n\\ i<j}} X_i X_j$$

with $S_n = S_n(\mathbb{N})$ and, if B is another subset of $\mathbb{N}$ disjoint with A, let

$$S_n(A,B) = \frac{1}{\gamma_n}\sum_{\substack{(i,j)\in A_n\times B_n\cup B_n\times A_n\\ i<j}} X_i X_j.$$

Assume $S_n \to 0$ a.s. Then, for any $A\subset\mathbb{N}$, $S_n(A)\to 0$ a.s. Hence, $S_n(A,A^c) = S_n - S_n(A) - S_n(A^c) \to 0$ a.s. Applying this observation to $A_\varepsilon = \{i\in\mathbb{N}: \varepsilon_i = 1\}$ and noting that $(1/\gamma_n)\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j X_i X_j = S_n(A_\varepsilon)+S_n(A_\varepsilon^c)-S_n(A_\varepsilon, A_\varepsilon^c)$, it follows that

$$P_X\left\{\lim_{n\to\infty}\frac{1}{\gamma_n}\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j X_i X_j = 0\right\} = 1$$

for all fixed sequences $\{\varepsilon_i = \pm 1\}$, where P_X denotes integration with respect to the X's only. Now the result follows by Fubini's theorem. □

Let, as usual, $\{X'_i\}$ be an independent copy of $\{X_i\}$ and ε_i, ε'_i, $i \in \mathbb{N}$, i.i.d. Rademacher variables independent of $\{X_i, X'_i\}$. With this notation we have the following corollary to the proof of Lemma 4.5:

COROLLARY 4.6. *Assume $\gamma(t) \nearrow \infty$ and $\gamma(2t) \le C\gamma(t)$ for some $C < \infty$ and all $t \ge$ some finite t_0. Let $\gamma_n = \gamma(n)$. If the law of large numbers* (4.1) *holds, then we also have*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i,j\le n} X_i X'_j = 0 \quad a.s. \tag{4.7}$$

and

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i,j\le n} \varepsilon_i \varepsilon'_j X_i X'_j = 0 \quad a.s. \tag{4.8}$$

PROOF. Taking A to be the even numbers in the proof of Lemma 4.5 and noting that $\{X_{2i}\}$ and $\{X_{2i-1}\}$ are two independent sequences, we obtain

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i\ne j\le n} X_i X_j = 0 \text{ a.s.} \implies \lim_{n\to\infty} \frac{1}{\gamma_{2n}} \sum_{1\le i,j\le n} X_i X'_j = 0 \text{ a.s.} \tag{4.9}$$

So (4.7) holds. (4.1) also implies

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i\ne j\le n} \varepsilon_i \varepsilon_j X_i X_j = 0 \quad \text{a.s.}$$

by Lemma 4.5. Thus, applying (4.9) with $\varepsilon_i X_i$ instead of X_i, we obtain (4.8). □

In the next proposition we combine an inequality of Bonami (1970) with an argument of Paley and Zygmund [e.g., Kahane (1968), page 6] to obtain a.s. convergence to zero of the normalized sums of products of squares. Bonami's inequality can be by-passed at the expense of some tedious computations.

PROPOSITION 4.7. *With the notation of Lemma* 4.5, *the law of large numbers* (4.1) *implies*

$$\lim_{n\to\infty} \frac{1}{\gamma_n^2} \sum_{1\le i<j\le n} X_i^2 X_j^2 = 0 \quad a.s.$$

and, in particular, the law of large numbers (4.6) *for maxima. If, moreover, $\gamma(t)$ satisfies the conditions of Corollary* 4.6, *then* (4.1) *also implies*

$$\lim_{n\to\infty} \frac{1}{\gamma_n^2} \sum_{1\le i,j\le n} X_i^2 X_j'^2 = 0 \quad a.s.$$

and, in particular, the law of large numbers (2.11′) *for decoupled maxima.*

PROOF. Without loss of generality we can assume X_i and ε_i, $i \in \mathbb{N}$, defined on a product probability space $\Omega \times \Omega'$ with X_i depending only on ω and ε_i on ω'. E_ε (P_ε) will denote integration (probability) with respect to ω' only. Lemma 4.5 and Fubini's theorem give that ω-a.s.,

$$\frac{1}{\gamma_n} \sum_{1 \le i < j \le n} \varepsilon_i \varepsilon_j X_i(\omega) X_j(\omega) \to 0 \qquad \omega'\text{-a.s.} \tag{4.10}$$

In particular, these ω'-random variables tend to zero in probability for almost every ω. To ease notation, we fix $n \in \mathbb{N}$ and ω such that (4.10) holds, and let $a_{i,j} := (1/\gamma_n) X_i(\omega) X_j(\omega)$, $\xi := \sum_{1 \le i < j \le n} \varepsilon_i \varepsilon_j a_{i,j}$ and $K := E_\varepsilon \xi^2 = \sum_{1 \le i < j \le n} a_{i,j}^2$. By developing the power in $(\sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j a_{i,j})^4$ and using the Cauchy–Schwarz inequality it can be easily (but tediously) seen that

$$E_\varepsilon \xi^4 \le C K^2$$

for some finite, positive constant C independent of n and $a_{i,j}$. [For the best constant and a much more general result, see Bonami (1970).] Hölder's inequality gives that, for any $t > 0$,

$$E_\varepsilon \xi^2 \le t^2 + E_\varepsilon \xi^2 I_{|\xi| > t} \le t^2 + (E_\varepsilon \xi^4)^{1/2} (P_\varepsilon\{|\xi| > t\})^{1/2}.$$

Combining the preceding two inequalities, we obtain

$$P_\varepsilon\{|\xi| > t\} \ge \left(\frac{(K - t^2)_+}{(E_\varepsilon \xi^4)^{1/2}} \right)^2 \ge \frac{(K - t^2)_+^2}{CK^2} \ge \frac{1}{4C} I_{K \ge 2t^2}.$$

This implies that $(1/\gamma_n^2) \sum_{1 \le i < j \le n} X_i^2 X_j^2 < 2t^2$ as soon as

$$P_\varepsilon \left\{ \left| \frac{1}{\gamma_n} \sum_{1 \le i < j \le n} \varepsilon_i \varepsilon_j X_i(\omega) X_j(\omega) \right| > t \right\} < \frac{1}{4C},$$

which eventually happens for almost every ω by (4.10). Hence,

$$\frac{1}{\gamma_n^2} \sum_{1 \le i < j \le n} X_i^2 X_j^2 \to 0 \quad \text{a.s.}$$

To prove the second limit, we just apply the previous arguments starting with (4.8) (which holds by Corollary 4.6) instead of (4.10). □

Let us recall from Section 2 that $u_k = G^{-1}(2^{-k})$, $\gamma_k^* = \gamma(2^k)$, $b_k^* = (\gamma_k^*)^{1/2}$ and $v_k = \gamma_k^* / u_k$, $k \in \mathbb{N}$.

PROOF OF THEOREM 4.1. (a) *Sufficiency of conditions* (4.2) *and* (4.3). As observed above, $w_k \le v_k$ so that (4.2) and (4.3) imply the law of large numbers for maxima [that is, (4.6)]. Since $\gamma_{2n} \le C\gamma_n$ eventually, it follows that $\max_{1 \le i < j \le 2^k} |X_i X_j| < \gamma_{k-1}^*$ eventually a.s. Also, condition (4.3) implies

$\max_{i\le 2^k}|X_i| < w_{k-1}$ eventually a.s. So, we can ignore large values in (4.1), that is, (4.1) will follow if

$$\frac{1}{\gamma_n}\sum_{1\le i<j\le n} X_iX_jI_{|X_iX_j|<\gamma^*_{k(n)},|X_i|<w_{k(n)},|X_j|<w_{k(n)}} \to 0 \quad \text{a.s.},$$

where $k(n) = \max\{k : 2^k < n\}$. Borel–Cantelli reduces this to proving

$$\sum_{k=1}^{\infty} P\left\{\max_{2^{k-1}<n\le 2^k}\frac{1}{\gamma^*_{k-1}}\left|\sum_{1\le i<j\le n} X_iX_jI_{|X_iX_j|<\gamma^*_{k-1},|X_i|<w_{k-1},|X_j|<w_{k-1}}\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$. Thus, decomposing the event $\{|X_iX_j| < \gamma^*_k, |X_i| < w_k, |X_j| < w_k\}$ into the union of the five disjoint events

$$\{|X_i| < b^*_k, |X_j| < b^*_k\}, \qquad \{|X_i| \le u_k, b^*_k \le |X_j| < w_k\},$$
$$\{|X_j| \le u_k, b^*_k \le |X_i| < w_k\},$$
$$\{u_k < |X_i| < b^*_k, b^*_k \le |X_j| < w_k, |X_iX_j| < \gamma^*_k\},$$
$$\{u_k < |X_j| < b^*_k, b^*_k \le |X_i| < w_k, |X_iX_j| < \gamma^*_k\},$$

the proof of (4.2) reduces to showing that the following three inequalities hold:

$$\sum_{k=1}^{\infty} P\left\{\max_{1\le n\le 2^k}\left|\sum_{1\le i<j\le n} X_iI_{|X_i|<b^*_{k-1}}X_jI_{|X_j|<b^*_{k-1}}\right| > \varepsilon\gamma^*_{k-1}\right\} < \infty, \tag{4.11}$$

$$\sum_{k=1}^{\infty} P\left\{\max_{1\le n\le 2^k}\left|\sum_{1\le i<j\le n} X_iI_{u_{k-1}<|X_i|<b^*_{k-1}}X_jI_{b^*_{k-1}\le|X_j|<w_{k-1}} \times I_{|X_iX_j|<\gamma^*_{k-1}}\right| > \varepsilon\gamma^*_{k-1}\right\} < \infty, \tag{4.12}$$

$$\sum_{k=1}^{\infty} P\left\{\max_{1\le n\le 2^k}\left|\sum_{1\le i<j\le n} X_iI_{|X_i|\le u_{k-1}}X_jI_{b^*_{k-1}\le|X_j|<w_{k-1}}\right| > \varepsilon\gamma^*_{k-1}\right\} < \infty. \tag{4.13}$$

By symmetry, the sums inside these expressions are martingales relative to the σ-fields $\mathscr{F} = \sigma(X_1,\dots,X_n)$ so that we can apply Doob's maximal inequality and further reduce our problem to showing

$$\sum_{k=1}^{\infty}\frac{2^{2k}}{(\gamma^*_k)^2}E[X^2I_{|X|<b^*_k}Y^2I_{|Y|<b^*_k}] < \infty, \tag{4.14}$$

$$\sum_{k=1}^{\infty}\frac{2^{2k}}{(\gamma^*_k)^2}E[X^2I_{u_k<|X|<b^*_k}Y^2I_{b^*_k\le|Y|<w_k}I_{|XY|<\gamma^*_k}] < \infty, \tag{4.15}$$

$$\sum_{k=1}^{\infty}\frac{2^{2k}}{(\gamma^*_k)^2}E[X^2I_{|X|\le u_k}Y^2I_{b^*_k\le|Y|<w_k}] < \infty, \tag{4.16}$$

where Y is an independent copy of X. Condition (4.3) implies (2.3′), which, in turn, implies that $\varepsilon\gamma_k^* \geq u_k^2$ eventually (as observed in the proof of Theorem 2.1); this, together with (4.2), gives

$$\sum_{k=1}^{\infty}(2^k P\{|X| > b_k^*\})^2 < \infty. \tag{4.17}$$

Hence, condition (3.2′) in Theorem 3.2 holds with $k = 2$ so that, by this theorem, (3.3) holds too, giving (4.14). To prove (4.15) first we observe that if $t^{-\beta}b(t) \nearrow$ for some $\beta > \frac{1}{2}$ or if $b(t)$ is regularly varying with exponent $\beta > \frac{1}{2}$, then there is $C < \infty$ such that for all $x > 0$,

$$\sum_{k:\ 2^k \geq \gamma^{-1}(x)} \frac{2^{2k}}{(\gamma_k^*)^2} \leq C\left(\frac{\gamma^{-1}(x)}{x}\right)^2. \tag{4.18}$$

[We omit the straightforward proof of (4.18).] So, (4.18) holds by regularity of $b(t)$ and gives

$$\sum_{k=1}^{\infty} \frac{2^{2k}}{(\gamma_k^*)^2} X^2 Y^2 I_{|XY|<\gamma_k^*} \leq X^2 Y^2 \sum_{k:\ 2^k \geq \gamma^{-1}(|XY|)} \frac{2^{2k}}{(\gamma_k^*)^2} \leq C[\gamma^{-1}(|XY|)]^2. \tag{4.19}$$

Now, if $u_k < |X| < b_k^*$, $|Y| \geq |X|$ and $|XY| < \gamma_k^*$ (note $u_k w_k \leq u_k v_k = \gamma_k^*$), then $\gamma^{-1}(|XY|) \leq 2^k \leq 1/(G(|X|)) \leq 1/(G(|Y|))$ so that (4.19) yields

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{2^{2k}}{(\gamma_k^*)^2} X^2 I_{u_k<|X|<b_k^*} Y^2 I_{b_k^* \leq |Y|} I_{|XY|<\gamma_k^*} \\ &\qquad \leq C[\gamma^{-1}(|XY|)]^2 I_{\gamma^{-1}(|XY|) \leq 1/(G(|X|)) \wedge 1/(G(|Y|))} \end{aligned}$$

and the expected value of this random variable is bounded by (2.4) [i.e., by (4.2)]. Inequality (4.15) is thus proved.

To prove (4.16) we first perform an integration by parts. Let

$$S_k := EX^2 I_{|X|\leq u_k}, \qquad T_k := EY^2 I_{b_k^* < |Y| < w_k}, \qquad Q_k := \sum_{j\geq k} \frac{2^{2j}}{(\gamma_j^*)^2} \simeq \frac{2^{2k}}{(\gamma_k^*)^2}$$

[where $\simeq$ denotes a two-sided inequality up to finite positive multiplicative constants; see (4.18) for the sum of the series defining Q_k]. Then the nth partial sum of the series in (4.16) equals

$$S_n T_n Q_{n+1} - S_1 T_1 Q_1 + \sum_{k=2}^{n}(S_k - S_{k-1}) T_k Q_k + \sum_{k=2}^{n}(T_k - T_{k-1}) S_{k-1} Q_k. \tag{4.20}$$

The definition of w_k and the regularity of $\{\gamma_n\}$ give

$$S_n T_n Q_{n+1} = \frac{2^{2(n+1)}}{(\gamma_{n+1}^*)^2} EX^2 I_{|X|\leq u_n} EY^2 I_{b_n^* < |Y| < w_n} \lesssim 2^n P\{|Y| > b_n^*\} \tag{4.21}$$

and this last quantity tends to zero by (4.17). So the proof of (4.16) reduces to showing that

$$\sum(S_k - S_{k-1})T_k Q_k = \sum \frac{2^{2k}}{(\gamma_k^*)^2} EX^2 I_{u_{k-1}<|X|\le u_k} EY^2 I_{b_k^*<|Y|<w_k} < \infty \tag{4.22}$$

and

$$\sum \frac{2^{2k}}{(\gamma_k^*)^2} EX^2 I_{|X|\le u_{k-1}} EY^2 I_{w_{k-1}<|Y|<w_k} < \infty. \tag{4.23}$$

Note that the series in (4.23) dominates the positive terms in the series $\sum_{k=2}^{\infty}(T_k - T_{k-1})S_{k-1}Q_k$; thus, since all the terms in the series (4.16) are nonnegative, (4.22), (4.23) and the convergence to zero of the expression in (4.21) imply (4.16). In order to bound the series in (4.22), let us define $k^* = \min\{k : |XY| \le \gamma_k^*,\ G(|X|) \le 2^{1-k}\}$ and $k^* = \infty$ if this set is empty. Then, since $u_k w_k \le u_k v_k = \gamma_k^*$, we have, by (4.18),

$$\begin{aligned}\sum \frac{2^{2k}}{(\gamma_k^*)^2} & EX^2 I_{u_{k-1}<|X|\le u_k} EY^2 I_{b_k^*<|Y|<w_k} \\ &\le E\Bigg[X^2Y^2 \sum_{|X|\le|Y|,|XY|\le\gamma_k^*,G(|X|)\le 2^{1-k}} \frac{2^{2k}}{(\gamma_k^*)^2}\Bigg] \\ &\lesssim E\Bigg[\frac{2^{2k^*}}{(\gamma_{k^*}^*)^2} X^2Y^2 I_{k^*<\infty} I_{|X|\le|Y|}\Bigg].\end{aligned}$$

Now, on the set $\{k^* < \infty\}$,

$$[\gamma^{-1}(|XY|)]^2 \le 2^{2k^*} \le \frac{4}{G(|X|)^2}$$

and, therefore, on this set,

$$\gamma^{-1}(|XY|) \wedge \frac{1}{G(|X|)} \ge \frac{1}{2}\gamma^{-1}(|XY|)$$

and

$$\frac{2^{k^*}}{\gamma_{k^*}^*}|XY| = \gamma^{-1}(|XY|)\frac{2^{k^*}}{\gamma_{k^*}^*}\frac{|XY|}{\gamma^{-1}(|XY|)} \le K_1 + K_2\gamma^{-1}(|XY|).$$

[The second inequality follows from the regularity of $b(t)$: it holds with $K_1 = 0$ and $K_2 = 1$ if $t^{-1}\gamma(t) \nearrow$, and is a simple consequence of the representation theorem for slowly varying functions if γ is regularly varying with exponent larger than 1.] Therefore,

$$\begin{aligned}
E&\left[\frac{2^{2k^*}}{(\gamma^*_{k^*})^2}X^2Y^2 I_{k^*<\infty} I_{|X|\le|Y|}\right]\\
&\le 2K_1^2+2K_2^2E[\gamma^{-1}(|XY|)]^2 I_{k^*<\infty} I_{|X|\le|Y|}\\
&\le 2K_1^2+8K_2^2E\left[\left(\gamma^{-1}(|XY|)^2\wedge\frac{1}{G(|X|)^2}\right)I_{|X|\le|Y|}\right].
\end{aligned}$$

Now (4.22) follows since this last integral is finite by hypothesis (4.2). To prove finiteness of the second series, let A be the set of $k\in\mathbb{N}$ such that $w_k > w_{k-1}$ [the sum in the series (4.23) extends only over $k\in A$], and let us observe that, by the definition of w_k and hypothesis (4.3),

$$\begin{aligned}
\sum&\frac{2^{2k}}{(\gamma^*_k)^2}EX^2 I_{|X|\le u_{k-1}} EY^2 I_{w_{k-1}<|Y|<w_k}\\
&=\sum_{k\in A}\frac{2^{2k}}{(\gamma^*_k)^2}EX^2 I_{|X|\le u_{k-1}} EY^2 I_{w_{k-1}<|Y|<w_k}\\
&\le\sum_{k\in A}\left(\frac{2^{2k}}{(\gamma^*_k)^2}EX^2 I_{|X|\le u_{k-1}}\right)w_k^2 P\{|Y|>w_{k-1}\}\\
&\le 2\sum 2^k P\{|Y|>w_k\}<\infty,
\end{aligned}$$

proving (4.23), hence (4.16) and the direct part of the theorem.

(b) *Necesity of conditions* (4.2) *and* (4.3). By Proposition 4.7, the law of large numbers (4.1) for sums implies the law of large number (4.6) for maxima. Therefore, Theorem 2.1′ gives convergence of the series in (4.2) and also of the series in (2.3′). To prove (4.3) we note first that we also have, by Proposition 4.7,

$$\frac{1}{(\gamma_n)^2}\sum_{1\le i,j\le n} X_i^2Y_j^2\to 0\quad\text{a.s.},\tag{4.24}$$

where we write Y_j instead of X'_j. Hence, in particular,

$$\frac{1}{(\gamma^*_k)^2}\max_{2^{k-1}<j\le 2^k} Y_j^2 I_{|Y_j|\le v_k}\sum_{2^{k-1}<i\le 2^k}(X_i^2\wedge u_k^2)\to 0\quad\text{a.s.}$$

Conditionally on the Y_j's, this is a normalized sum of independent nonnegative random variables. Since $u_kv_k=\gamma^*_k$ by definition, the normalized summands are bounded by 1 so that, by bounded convergence,

$$\frac{1}{(\gamma^*_k)^2}E_X\left[\max_{2^{k-1}<j\le 2^k} Y_j^2 I_{|Y_j|\le v_k}\max_{2^{k-1}<i\le 2^k} X_i^2\wedge u_k^2\right]\to 0\quad\text{a.s.},$$

where E_X denotes expectation with respect to the X_i's only. It is also easy to see that the conditional α-quantiles of the normalized sums tend to zero for almost every sequence $\{Y_j\}$, for every $\alpha>0$. Therefore, Hoffmann-Jørgensen's inequality [Hoffmann-Jørgensen (1974); e.g., reproduced in Araujo and Giné (1980), page 107], which works also for nonnegative random variables, yields

$$\frac{1}{(\gamma_k^*)^2} E_X\Big[\max_{2^{k-1}<j\le 2^k} Y_j^2 I_{|Y_j|\le v_k} \sum_{2^{k-1}<i\le 2^k} (X_i^2 \wedge u_k^2)\Big] \to 0 \quad \text{a.s.};$$

that is,

$$\frac{1}{w_k^2} \max_{2^{k-1}<j\le 2^k} Y_j^2 I_{|Y_j|\le v_k} \to 0 \quad \text{a.s.}$$

This now yields by Borel–Cantelli that

$$\sum 2^k P\{|Y| I_{|Y|\le v_k} > \varepsilon w_k\} < \infty$$

for all $\varepsilon > 0$. This, $v_k \ge w_k$ and Theorem 2.1′ (2.3′) imply condition (4.3). □

PROOF OF THEOREM 4.2. $\sum_{1\le i<j\le n} X_i X_j$ can be decomposed in terms of order statistics, as follows:

$$\sum_{1\le i<j\le n} X_i X_j = X_{1:n} \sum_{k=2}^{n} X_{k:n} + \frac{1}{2}\Big(\sum_{k=2}^{n} X_{k:n}\Big)^2 - \frac{1}{2}\sum_{k=2}^{n} X_{k:n}^2. \tag{4.25}$$

If the law of large numbers (4.1) holds then, by Proposition 4.7, so does the law of large numbers (4.6) for maxima. Therefore, the conditions (2.2′) and (2.3′) in Theorem 2.1′ are satisfied, implying

$$\sum_{k=1}^{\infty} (2^k P\{|X| > b_k^*\})^2 < \infty, \tag{4.17}$$

as indicated in the previous proof. Hence, the last two summands at the right-hand side of (4.25) tend to zero a.s. when divided by γ_n by Mori's theorem (Corollary 3.9), and therefore so does the first summand that is, the limit (4.5) holds. Note that this part of the proof does not require symmetry or positivity of X.

Conversely, if (4.5) holds and X is symmetric, then replacing X_i by $\varepsilon_i X_i$, with X_i depending only on ω and ε_i on ω' (as in the proof of Proposition 4.7), we have that ω-a.s.,

$$\frac{1}{\gamma_n} X_{1:k}(\omega) \sum_{k=2}^{n} \varepsilon_{k(n,\omega)}(\omega') X_{k:n}(\omega) \to 0, \qquad \omega'\text{-a.s.}$$

for suitable indices $k(n, \omega)$. Hence, as in the proof of Proposition 4.7, we also have

$$\frac{1}{\gamma_n^2} X_{1:k}^2 \sum_{k=2}^{n} X_{k:n}^2 \to 0 \quad \text{a.s.}$$

and, in particular, $(1/\gamma_n)\max_{1\le i<j\le n} |X_i X_j| \to 0$ a.s. (this is obvious if the variables X_i are nonnegative). So (4.17) also holds and, by Mori's theorem,

the last two terms at the right of (4.25) tend to zero a.s. when divided by γ_n. Therefore, (4.1) holds. □

4.2. *More on symmetrization and decoupling and the proof of Theorem* 4.3. Whereas in the case of sums of i.i.d. random variables neither the lack of symmetry nor the lack of regularity poses any problems for the equivalence between converge to zero a.s. of normalized sums and maxima, in the case of products these factors seem to play a role. So we do not know how to desymmetrize in Theorems 4.1 and 4.2, in general, or how to prove equivalence between convergence to zero a.s. of normalized sums of products and maxima of products. In the last subsection we dealt with the regularity problem under symmetry, whereas here we deal with the symmetry problem under regularity (of course, as mentioned in the introduction to this section, this leaves some open questions).

PROPOSITION 4.8. *Let γ satisfy the same regularity condition as in Corollary* 4.6 *and let $\gamma_n = \gamma(n)$.*

(a) *If*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i\ne j\le n} \varepsilon_i \varepsilon_j X_i X_j = 0 \quad a.s. \tag{4.26}$$

[*which follows from* (4.1) *by Lemma* 4.5], *then we also have*

$$\frac{1}{\gamma_n} \sum_{1\le i\ne j\le n} (X_i - X_i')(X_j - X_j') \to 0 \quad a.s. \tag{4.27}$$

and

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \max_{1\le i,j\le n} |X_i X_j'| = 0 \quad a.s. \tag{4.28}$$

(b) *Assume in addition that $\gamma(t)^{1/2}$ is regular for X. Then if* (4.28) *holds, we have*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{i=1}^{n} X_i X_i' = 0 \quad a.s. \tag{4.29}$$

and therefore the limits (4.7), (4.8) *and* (4.28) *also hold with the diagonal terms excluded.*

PROOF. To prove part (a) we show first that (4.26) is equivalent to

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \sum_{1\le i\ne j\le n} \varepsilon_i \varepsilon_j' X_i X_j = 0 \quad \text{a.s.} \tag{4.30}$$

By Fubini's theorem it suffices to prove this conditionally on the variables X_i. Set $a_{ij} = X_i X_j$ and, for each $n \in \mathbb{N}$, define the following sequences:

$$A_{ij}^{(n)} = \left(0, \overset{n)}{\ldots}, 0, \frac{a_{ij}}{\gamma_n}, \frac{a_{ij}}{\gamma_{n+1}}, \ldots\right) \quad \text{for } i \vee j \le n,$$

$$A_{ij}^{(n)} = \left(0, \overset{\ell)}{\ldots}, 0, \frac{a_{ij}}{\gamma_\ell}, \frac{a_{ij}}{\gamma_{\ell+1}}, \ldots\right) \quad \text{for } i \vee j = \ell > n,$$

where $1 \le i \ne j < \infty$. Note that each of these sequences tends to zero for almost every choice of $\{X_i\}_{i=1}^\infty$, that is, for every such choice they are in the Banach space c_0. Also,

$$\sum_{1 \le i \ne j < \infty} \varepsilon_i \varepsilon_j' A_{ij}^{(n)} = \left(0, \overset{n)}{\ldots}, 0, \frac{1}{\gamma_n} \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' a_{ij}, \frac{1}{\gamma_{n+1}} \sum_{1 \le i \ne j \le n+1} \varepsilon_i \varepsilon_j' a_{ij}, \ldots\right).$$

So, letting $\|A\|$ denote the sup of the terms of any sequence A, we have

$$\sup_{k \ge n} \frac{1}{\gamma_k} \left| \sum_{1 \le i \ne j \le k} \varepsilon_i \varepsilon_j a_{ij} \right| = \left\| \sum_{1 \le i \ne j < \infty} \varepsilon_i \varepsilon_j A_{ij}^{(n)} \right\| := Z_n$$

and

$$\sup_{k \ge n} \frac{1}{\gamma_k} \left| \sum_{1 \le i \ne j \le k} \varepsilon_i \varepsilon_j' a_{ij} \right| = \left\| \sum_{1 \le i \ne j < \infty} \varepsilon_i \varepsilon_j' A_{ij}^{(n)} \right\| := Z_n'.$$

By hypercontractivity of Banach valued Rademacher chaos [Borell (1979), Theorem 1.1, and (1984), Lemma 2.1], the sequence $\{Z_n\}$ converges to zero in probability (we are now assuming the a_{ij}'s fixed) if and only if it converges to zero in L_2, and likewise for $\{Z_n'\}$. However, by Kwapień (1987), Theorem 2, $Z_n \to 0$ in L_2 if and only if $Z_n' \to 0$ in L_2. Since (4.26) holds if and only if $Z_n \to 0$ in pr and (4.30) holds if and only if $Z_n' \to 0$ in pr, we conclude that the statements (4.30) and (4.26) are equivalent. Suppose now that (4.26) holds and consider the "usual" symmetrization, $(1/\gamma_n) \sum_{1 \le i \ne j \le n} (X_i - X_i')(X_j - X_j')$. Since the sequences $\{\varepsilon_i(X_i - X_i')\}$ and $\{X_i - X_i'\}$ have the same joint distribution, $(1/\gamma_n) \sum_{1 \le i \ne j \le n} (X_i - X_i')(X_j - X_j') \to 0$ a.s. if and only if $(1/\gamma_n) \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j (X_i - X_i')(X_j - X_j') \to 0$ a.s. and, by the previous argument, this holds if and only if $(1/\gamma_n) \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' (X_i - X_i')(X_j - X_j') \to 0$ a.s. Now

$$\begin{aligned} &\frac{1}{\gamma_n} \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' (X_i - X_i')(X_j - X_j') \\ &\quad = \frac{1}{\gamma_n} \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' X_i X_j + \frac{1}{\gamma_n} \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' X_i' X_j' \\ &\qquad - \frac{1}{\gamma_n} \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' X_i X_j' - \frac{1}{\gamma_n} \sum_{1 \le i \ne j \le n} \varepsilon_i \varepsilon_j' X_i' X_j. \end{aligned}$$

The first two terms on the right tend to zero a.s. by (4.30), and so do the third and fourth by Corollary 4.6 applied, respectively, to εX and to $\varepsilon X'$ instead of X. So (4.27) holds. (4.28) follows from Proposition 4.7 applied to εX and from Corollary 2.4. Part (a) is proved.

In the decoupled case the diagonals are easily treated because the limit (4.28) gives $\lim_{n\to\infty}(1/\gamma_n)\max_{i\le n}|X_iX_i'| = 0$ a.s. so that, by regularity and Feller's theorem [Feller (1946); e.g., Stout (1974), page 132]

$$\lim_{n\to\infty}\frac{1}{\gamma_n}\sum_{i=1}^{n} X_iX_i' = 0 \quad \text{a.s.},$$

and part (b) follows. [The hypotheses of Feller's theorem are satisfied, even with room to spare, if $\gamma(t)/t^{\lambda} \nearrow$ for some $\lambda > 1$, but Feller's theorem also follows easily if γ is regularly varying with exponent larger than one; so, Feller's theorem holds with normalizers γ_n if $\gamma^{1/2}$ is regular for (any) X.] □

To prove strong laws of large numbers for sums of independent random variables it suffices to consider the symmetric case since, as Kuelbs and Zinn (1979) observe, $(1/\gamma_n)\sum_{i=1}^{n} X_i \to 0$ a.s. if and only if both $(1/\gamma_n)\sum_{i=1}^{n}(X_i - X_i') \to 0$ a.s. and $(1/\gamma_n)\sum_{i=1}^{n} X_i \to 0$ in probability. We do not know if an exact analog of this statement is true for quadratic forms, but we can prove the following proposition, based on a similar idea.

PROPOSITION 4.9. *Let X be a random variable such that $G(x) = P\{|X| \ge x\}$ is regular and let $\gamma(t)$ be a positive function increasing to infinity such that $\gamma^{1/2}$ is regular for X and $\gamma(2t) \le C\gamma(t)$ for some $C < \infty$ and all $t \ge$ some finite t_0. Let ε be a Rademacher variable independent of X. Then, if εX satisfies the law of large numbers* (4.1), *so does X.*

PROOF. Since X satisfies (4.26) by hypothesis, then it also satisfies (4.6) by Proposition 4.7, and (4.27), (4.28) and (4.29) by Proposition 4.8. (4.27) and (4.29) give

$$\frac{1}{\gamma_n}\sum_{1\le i\ne j\le n} X_iX_j + \frac{1}{\gamma_n}\sum_{1\le i\ne j\le n} X_i'X_j' - 2\left(\frac{\sum_{i=1}^{n} X_i'}{\gamma_n}\right)\sum_{i=1}^{n} X_i \to 0 \quad \text{a.s.} \tag{4.31}$$

Since $(1/\gamma_n)\max_{i,j\le n}|X_iX_j'| \to 0$ in probability [a.s. by (4.28)], $(P\{\max_{i\le n}|X_i| > (\gamma_n)^{1/2}\})^2 \to 0$, hence $nP\{|X| > (\gamma_n)^{1/2}\} \to 0$. By Proposition 3.10, this implies that $(1/\gamma_n)\sum_{i=1}^{n} X_i^2 \to 0$ in pr and that $(1/(\gamma_n)^{1/2})\sum_{i=1}^{n} X_i \to 0$ in pr and therefore that $(1/\gamma_n)\sum_{i\ne j\le n} X_iX_j \to 0$ in pr. It follows from this and (4.31) that

$$\frac{1}{\gamma_n}\sum_{1\le i\ne j\le n} X_i'X_j' - 2\left(\frac{\sum_{i=1}^{n} X_i'}{\gamma_n}\right)\sum_{i=1}^{n} X_i \to 0 \tag{4.32}$$

in $\{X_i\}$-probability, $\{X_i'\}$-a.s.

Define $A_n = (2/\gamma_n)\sum_{i=1}^n X_i'$ and $B_n = (\operatorname{sign} A_n)(1/\gamma_n)\sum_{i\neq j\leq n} X_i'X_j'$, and note that $A_n \to 0$ a.s. [since $\max_{i\leq n}|X_i| \to \infty$ a.s. and, by (4.28),

$$\frac{1}{\gamma_n}\max_{1\leq i,j\leq n}|X_iX_j'| = \frac{1}{\gamma_n}\max_{i\leq n}|X_i'|\max_{i\leq n}|X_i| \to 0 \quad \text{a.s.},$$

we have that $(1/\gamma_n)\max_{i\leq n}|X_i| \to 0$ a.s., which implies $A_n \to 0$ a.s. by Feller's theorem]. Fix now a sequence $\{X_i'\}$ so that $A_n \to 0$. Then the system $\{|A_n|X_i: i \leq n\}_{n=1}^{\infty}$ is infinitesimal and, by (4.32), its row sums are shift convergent (weakly) to zero, with shifts $-B_n$. Then the converse weak law of large numbers [e.g., Araujo and Giné (1980)] implies $nP\{|X| > |A_n|^{-1}\} \to 0$ and $B_n \approx n|A_n|EXI(|X| \leq |A_n|^{-1})$ as $n \to \infty$. The first limit and regularity of G implies [by (3.17)] that the second quantity tends to zero. That is, $(1/\gamma_n)\sum_{i\neq j\leq n} X_i'X_j' \to 0$ a.s. □

PROOF OF THEOREM 4.3. (4.1) ⇒ (4.6) by Proposition 4.7. To prove the converse, X being regular, we can assume X is also symmetric by Proposition 4.9. Proceeding as in the proof of sufficiency in Theorem 4.1, but replacing w_k by v_k, the proof reduces to showing that the analogues in the series (4.14)–(4.16) converge. The first two can be dealt with exactly as in the proof of Theorem 4.1, and we are only left with showing that the third of these series converges, that is, that

$$\sum_{k=1}^{\infty}\frac{2^{2k}}{(\gamma_k^*)^2}E[X^2 I_{|X|\leq u_k}Y^2 I_{b_k^*\leq |Y|<v_k}] < \infty. \tag{4.33}$$

For this we use the regularity hypothesis on X. Note that the series in (4.33) is dominated by

$$\sum_{i=1}^{\infty}\frac{2^{2k}}{(\gamma_k^*)^2}E[X^2Y^2 I_{b_k^*<|Y|<v_k, |XY|<\gamma_k^*}]. \tag{4.34}$$

Since X is regular we can apply Proposition 3.8 [the second inequality in (3.14)] to X and obtain that the series (4.34) is in turn dominated by a constant times

$$\begin{aligned}\sum 2^{2k}E\left[G\left(\frac{\gamma_k^*}{|Y|}\right)I_{b_k^*<|Y|<v_k}\right] &= \sum 2^{2k}P\{|XY|\geq\gamma_k^*,\ b_k^*<|Y|<v_k\}\\ &\leq \sum 2^{2k}P\{|XY|\geq\gamma_k^*, |X|>u_k, |Y|>b_k^*\}\\ &\lesssim \sum 2^{2k}P\{|XY|\geq\gamma_k^*, |X|, |Y|>u_k\},\end{aligned}$$

where in the last inequality we use (2.6). This last series is finite by Theorem 2.1. Therefore the series in (4.34) converges, proving (4.33) and the theorem. □

REMARK 4.10. *Another proof of the sufficiency part of Theorem* 4.3. The sufficiency part of Mori's theorem and Theorem 3.2 provide another proof of

(4.6) $\Rightarrow$ (4.1) for X regular. However, the above proof, which uses only Theorem 3.1 from Section 3 (in fact not even this if we are willing to use regularity of G throughout), is more elementary since Mori's theorem not only requires Theorem 3.1, but also an exponential inequality (see Section 3, proofs of Theorems 3.2 and 3.4). This second proof is interesting for its use of order statistics, and we indicate it now. With $X_{j:n}$ as defined in Section 2, we have, by (4.25),

$$\left|\sum_{i<j\le n} X_i X_j\right| \le \left|X_{1:n}\sum_{k=2}^{n} X_{k:n}\right| + \frac{1}{2}\left|\sum_{k=2}^{n} X_{k:n}\right|^2 + \frac{1}{2}\sum_{k=2}^{n} X_{k:n}^2. \tag{4.35}$$

As observed above, Theorem 2.1′ implies condition (4.17). Hence,

$$\frac{1}{\gamma_n}\left[\sum_{k=2}^{n} X_{k:n}\right]^2 \to 0 \quad \text{a.s.}$$

by Theorem 3.4. If $b(t)$ is regular for X, then $\gamma(t)$ is regular for X^2 and therefore (4.17) also implies

$$\frac{1}{\gamma_n}\sum_{k=2}^{n} X_{k:n}^2 \to 0 \quad \text{a.s.}$$

by Corollary 3.9. So, we only need to show that γ_n^{-1} times the first term at the right of (4.35) tends to zero a.s. By Theorem 3.4, it suffices to consider this term over the set $|X_{1:n}| > b_n$ for each n and, as observed at the beginning of the previous proof, large values of the variables X_i can be ignored. Thus, the proof of (4.1) is reduced to showing that

$$\frac{1}{\gamma_{k-1}^*} X_{1:k} I_{b_{k-1}^* < |X_{1,k}| < v_{k-1}} \max_{2^{k-1}<n\le 2^k}\left|\sum_{i=1}^{n} X_i I_{|X_{1:k}X_i| < \gamma_{k-1}^*}\right| \to 0 \quad \text{a.s.},$$

where, for simplicity of notation, we set $X_{1:2^k} = X_{1,k}$. By Borel–Cantelli, this will follow from

$$\sum_{k=1}^{\infty} P\left\{|X_{1:k}| I_{b_{k-1}^* < |X_{1:k}| < v_{k-1}} \max_{m\le 2^k}\left|\sum_{i=1}^{m} X_i I_{|X_{1:k}X_i| < \gamma_{k-1}^*}\right| > \varepsilon\gamma_{k-1}^*\right\} < \infty$$

for all $\varepsilon > 0$. Now, the left side is bounded from above by

$$\sum_{k=1}^{\infty} P\left\{\max_{j\le 2^k}\left[|X_j| I_{b_{k-1}^* < |X_j| < v_{k-1}} \max_{m\le 2^k}\left|\sum_{i=1}^{m} X_i I_{|X_i X_j| < \gamma_{k-1}^*}\right|\right] > \varepsilon\gamma_{k-1}^*\right\}$$

$$\le \sum_{k=1}^{\infty} 2^k P\left\{\max_{2\le m\le 2^k}\left|X_1 I_{b_{k-1}^* < |X_1| < v_{k-1}} \sum_{i=2}^{m} X_i I_{|X_1 X_i| < \gamma_{k-1}^*}\right| > \varepsilon\gamma_{k-1}^*\right\}.$$

Applying Kolmogorov's maximal inequality conditionally on X_1, it follows that the last series is dominated by a constant times

$$\sum \frac{2^{2k}}{(\gamma_k^*)^2} E[X^2 Y^2 I_{b_k^* < |X| < v_k, |XY| < \gamma_k^*}], \tag{4.34}$$

which has been shown to be finite at the end of the previous proof.

The regularity hypothesis on G has been invoked twice in the proof of Theorem 4.3: first for symmetrization and then to prove convergence of the series in (4.33). It would be interesting to decide whether it is superfluous.

4.3. *Decoupled and/or symmetrized versions of the previous theorems.* Collecting Corollary 2.4, Theorem 4.3 and Propositions 4.7 and 4.8, we obtain the following theorem.

THEOREM 4.11. *Assuming $\gamma(t)$ nondecreasing, $\gamma(2t) \le C\gamma(t)$ for some $C < \infty$ and all $t \ge$ some finite t_0, and $\gamma^{1/2}(t)$ regular for X, then any of the conditions* (4.1) *or its symmetrized* (4.26) *or its decoupled* (4.7) *or its decoupled and symmetrized* (4.8) *implies both* (4.6) *and* (4.28). *If in addition X is regular, then conversely any of the conditions* (4.6) *or* (4.28) *implies* (4.1), (4.7), (4.8) *and* (4.26).

Corollary 2.4, Proposition 4.7 and minor formal changes in the proof of Theorem 4.1 also give the next theorem.

THEOREM 4.12. *If $\gamma(t)$ satisfies the usual regularity conditions (as in the previous theorem) and X is symmetric, then the law of large numbers* (4.1) *is equivalent to its decoupled version* (4.7).

Using Theorem 4.12 and Lévy's inequality for necessity, and a slight modification of the corresponding part of the proof of Theorem 4.2 for sufficiency, we obtain the decoupled version of Theorem 4.2:

THEOREM 4.13. *The law of large numbers* (4.1) *for X symmetric holds if and only if*

$$\frac{1}{\gamma_n} X_{1:n} \sum_{i=1}^{n} X_i' \to 0 \quad a.s.$$

Theorem 4.12 also follows from recent general results on decoupling by de la Peña and Montgomery-Smith (1994). (The present manuscript was already completed when we received their preprint.)

EXAMPLE 4.14. For Example 2.9, assuming X symmetric if $\alpha = 1$ and $EX = 0$ if $\alpha > 1$, the conditions (4.1), (4.6), (4.7), (4.8), (4.26) and (4.28) are all equivalent, and equivalent to $\beta > \frac{1}{2}$.

APPENDIX

We give the proof of Corollary 2.8 and of Corollaries 2.5′, 2.7′ and 2.8′, and complete the proof of Theorem 2.10.

PROOF OF COROLLARY 2.8. We assume first that G is continuous [and satisfies (a) in Corollary 2.8]. We assume further that:

(c) $\gamma_k^*/((\log\gamma_k^*)^{2p}) \le u_k^2 \le \gamma_k^*$ eventually.

Let $b_k^* = (\gamma_k^*)^{1/2}$. By (a) and (c), for $x \in [u_k, b_k^*]$,

$$G\left(\frac{\gamma_k^*}{x}\right) = G\left(\frac{b_k^*}{x}b_k^*\right)G\left(\frac{x}{b_k^*}b_k^*\right)/G(x) \approx \frac{G^2(b_k^*)}{G(x)}. \tag{A.1}$$

Hence, letting $F(x) = 1 - G(x)$,

$$\begin{aligned}
&\sum 2^{2k}P\{XY > \gamma_k^*;\ X, Y > u_k\} \\
&\quad = 2\sum 2^{2k}P\{XY > \gamma_k^*;\ X \ge Y > u_k\} \\
&\quad = 2\sum 2^{2k}\int_{u_k}^{\infty} G\left(\frac{\gamma_k^*}{x} \vee x\right)dF(x) \\
&\quad = 2\sum 2^{2k}\int_{u_k}^{b_k^*} G\left(\frac{\gamma_k^*}{x}\right)dF(x) + 2\sum 2^{2k}\int_{b_k^*}^{\infty} G(x)\,dF(x) \qquad \text{(A.2)} \\
&\quad \approx 2\sum 2^{2k}G^2(b_k^*)\int_{u_k}^{b_k^*}\frac{dF(x)}{G(x)} + \sum 2^{2k}G^2(b_k^*) \\
&\quad = 2\sum 2^{2k}G^2(b_k^*)|\log(2^kG(b_k^*))| + \sum 2^{2k}G^2(b_k^*).
\end{aligned}$$

Therefore, (2.2) [hence (2.1) by Corollary 2.5] is equivalent to both $2^kG(b_k^*) \to 0$ and $\sum 2^{2k}G^2(b_k^*)|\log(2^kG(b_k^*))| < \infty$. However, these two conditions are equivalent to (2.17) by regular variation of G and regularity of $\{\gamma_n\}$.

Now, if we let

$$\delta(t) = \left[G^{-1}\left(\frac{1}{t(\log t)^r}\right)\right]^2, \qquad t > 1,$$

with $\frac{1}{2} < r < \alpha p$, it is easy to check that $\gamma(t) := \delta(t)$ satisfies (b), (c) and (2.17); therefore, by the previous paragraph, also (2.1). As a consequence, if γ is any function (increasing to infinity and) satisfying (b), γ satisfies (2.17) if and only if $\gamma \wedge \delta$ does and, likewise, γ satisfies (2.1) if and only if $\gamma \wedge \delta$ does. So it suffices to prove the corollary for $\gamma \wedge \delta$. However, if γ satisfies (b) and either (2.1) or (2.17), then $\gamma \wedge \delta$ satisfies (c) and therefore (A.2) gives the result for $\gamma \wedge \delta$, hence for γ.

The following argument reduces the general case to the case of continuous G. Let U be a nonnegative bounded (e.g., by 1) random variable with continuous distribution, independent of X, and let $Z = X + U$. Then, $G_Z(x) = P\{Z \ge x\}$ is continuous and satisfies $G(x) \ge G_Z(x) \ge G(x+1)$. So, by regular variation of G we get $G_Z(x)/G(x) \to 1$ as $x \to \infty$. It follows that G_Z is regularly varying, satisfies (A.1) (with the original b_k^*'s) and (2.17) holds for $a_n = G_Z(\gamma_n^{1/2})$ if and only if it holds for the original $a_n = G(\gamma_n^{1/2})$. The same comments apply to $Z' = (X \vee 1) - U$. Moreover, if Z satisfies (2.1), so does X, and if X satisfies (2.1), so does Z'. □

SKETCH OF THE PROOF OF COROLLARIES 2.5′, 2.7′ AND 2.8′. Parts (b) and (c) follow from part (a). To prove (a) note that (2.11) implies (2.1) and that, by Corollary 2.5, (2.1), (2.2) and (2.10) are equivalent. If (2.1) holds then, as noted above, by monotonicity of the sequence $\{u_k\}$,

$$\sum 2^{2k} P\{XY > \gamma^*_{k-\ell};\ X, Y \geq u_k\} < \infty$$

for all $\ell \in (-\infty, \infty)$. Given $\varepsilon > 0$ there is $\ell < \infty$ such that $\gamma^*_{k-\ell} \leq \varepsilon\gamma^*_k$ for all $k > \ell$. Hence, $\sum 2^{2k} P\{XY > \varepsilon\gamma_k;\ X,\ Y \geq u_k\} < \infty$. So, by Corollary 2.5, $P\{\max_{1\leq i<j\leq n} X_i X_j > \varepsilon\gamma_n \text{ i.o.}\} = 0$ and (2.11) follows. □

COMPLETION OF THE PROOF OF THEOREM 2.10. Only the last statement of the theorem is left to prove. To establish (2.29), note that, as in the proof of Corollary 2.8, we can take G to be continuous. Then the lower limit of integration in (2.19) can be replaced by u_k and (2.19) implies (2.20). The equivalence of (2.19) and (2.21) follows from (2.16) since by (A.1), (2.19) is equivalent to

$$\sum_k 2^{k\ell} G^2(b^*_k) \int_{u_k}^{\infty} G^{\ell-3}(x)\, dF(x)$$
$$= (\ell-2)^{-1} \sum 2^{2k} G^2(b^*_k) \simeq \sum n^{-1}(na_n)^2. \qquad \square$$

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