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DIMENSION RESULTS FOR GAUSSIAN VECTOR FIELDS AND INDEX-α STABLE FIELDS[1]

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The Hausdorff dimension and packing dimension of the image and graph of Gaussian vector fields and index-α stable fields are obtained under general conditions which allow for different local behavior of the components and for dependence among them. These results correct some errors in print. The condition for an index-α stable field to have k-multiple points are considered and in the case of $Nk > (k-1)\sum_{j=1}^{d}\alpha_j$, the Hausdorff dimension of the set of k-multiple times is given.

1. Introduction. In this paper, we study the Hausdorff dimension and packing dimension of various random sets arising from Gaussian vector fields and their natural generalization, index-α stable fields. There has been considerable interest in the Hausdorff dimension and packing dimension of the image, graph, level sets and multiple points of Gaussian fields, including fractional Brownian motion (see [1]–[3], [7], [9], [15], [19] and references therein). Recently, Nolan [12] considered the sample path properties of stable fields and obtained the Hausdorff dimension of the image, graph and level sets for classes of index-α stable fields. His result generalized Cuzick's Theorem 1 [2]. However, there are flaws in their results and in Theorem 8.4.1 in [1].

In Section 2, we find the Hausdorff dimension of the image $X(E) = \{X(t), t \in E\}$ and graph $\operatorname{Gr} X(E) = \{(t, X(t)), t \in E\}$ of index-α Gaussian fields, where $E \subset R^N$ is an arbitrary compact set. This result corrects Theorem 1 in [2], Theorem 8.4.1 in [1] and generalizes a theorem of Kahane [9]. We also obtain the packing dimension of $X([0,1]^N)$ and $\operatorname{Gr} X([0,1]^N)$.

In Section 3, we prove an analogous result for index-α stable fields which corrects Theorem 4.1 of [12].

Kono [10], Goldman [7] and Testard [16] studied the existence of k-multiple points for fractional Brownian motion. Weber [18] obtained the Hausdorff dimension of the k-multiple times. Xiao [21], [22] found the Hausdorff dimension and packing dimension of k-multiple points and extended Weber's result to the restriction of fractional Brownian motion on disjoint compact sets in R^N. Cuzick [3] proved analogous results for more general (N, d) Gaussian fields. In Section 4, we generalize these results to certain locally nondeterministic (LND) index-α stable fields.

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2. Index-α Gaussian vector fields. Let $X(t) = (X_1(t), \dots, X_d(t))$ be a R^d-valued mean zero Gaussian vector field on R^N. We assume that the coordinate fields $X_1, \dots, X_d$ have stationary increments. Denote

$$\sigma_j^2(t) = E\big(X_j(t) - X_j(0)\big)^2.$$

If for each $j = 1, 2, \dots, d$, there exists $0 < \alpha_j \le 1$ such that

$$\alpha_j = \sup\Big\{\alpha > 0, \lim_{|t|\to 0} |t|^{-\alpha}\sigma_j(t) = 0\Big\}$$
$$= \inf\Big\{\alpha > 0, \lim_{|t|\to 0} |t|^{-\alpha}\sigma_j(t) = +\infty\Big\},$$

we call $X(t)$ an index-α Gaussian field for $\alpha = (\alpha_1, \dots, \alpha_d)$. For simplicity, we shall assume that all $\sigma_j(t)$ are bounded away from zero on $I_N^* = [-1, 1]^N$ for t bounded away from the origin. To avoid degeneracies, we make the following restriction on the type of dependence allowed between the coordinate fields $X_1, \dots, X_d$: there exists a constant $\varepsilon > 0$ such that

$$\det \operatorname{cov}(X(t) - X(s)) \ge \varepsilon \prod_{j=1}^{d} \sigma_j^2(t-s), \tag{2.1}$$

where $\operatorname{cov}(Y)$ denotes the covariance matrix of the random vector Y. This condition will be satisfied if the coordinate fields are independent. A specific example of index-α Gaussian fields that we consider is fractional Brownian motion.

Recall briefly the definition of Hausdorff dimension and packing dimension. For each $\alpha > 0$, $E \subset R^N$, the α-dimensional Hausdorff measure of E is defined by

$$s^\alpha - m(E) = \lim_{\delta\to 0} \inf\Big\{\sum_i (2r_i)^\alpha, E \subset \bigcup_i B(x_i, r_i), r_i \le \delta\Big\}, \tag{2.2}$$

where $B(x_i, r_i)$ denotes the open ball of radius r_i centered at x_i, $s^\alpha - m$ is a metric outer measure and all Borel sets are measurable. The Hausdorff dimension of E is defined by

$$\dim E = \inf\{\alpha > 0, s^\alpha - m(E) = 0\}$$
$$= \sup\{\alpha > 0, s^\alpha - m(E) = +\infty\}.$$

In [15], Taylor and Tricot defined another set function $s^\alpha - P(E)$ in which economical coverings are replaced by disjoint packings,

$$s^\alpha - P(E) = \lim_{\delta\to 0} \sup\Big\{\sum_i (2r_i)^\alpha, B(x_i, r_i) \text{ are disjoint}, x \in E, r_i \le \delta\Big\}.$$

$s^\alpha - P$ is not an outer measure because it fails to be countably subadditive. However, $s^\alpha - P$ is a premeasure, so we can obtain a metric outer measure on R^N by

$$s^\alpha - p(E) = \inf\Big\{\sum_i s^\alpha - P(E_i), E \subset \bigcup_i E_i\Big\}.$$

$s^\alpha - p(E)$ is called the α-dimensional packing measure of E. The packing dimension of E is

$$\begin{aligned}\operatorname{Dim} E &= \inf\{\alpha > 0,\ s^\alpha - p(E) = 0\}\\ &= \sup\{\alpha > 0,\ s^\alpha - p(E) = +\infty\}.\end{aligned}$$

It is known [15] that $s^\alpha - m(E) \le s^\alpha - p(E)$ for any $E \subset R^N$, so

$$0 \le \dim E \le \operatorname{Dim} E \le N \tag{2.3}$$

For each $\varepsilon > 0$ and bounded set $E \subset R^N$, let

$$\begin{aligned} M(\varepsilon, E) &= \text{smallest number of balls of radius } \varepsilon \text{ need to cover } E,\\ \delta(E) &= \liminf_{\varepsilon\to 0} \frac{\log M(\varepsilon, E)}{-\log \varepsilon},\\ \Delta(E) &= \limsup_{\varepsilon\to 0} \frac{\log M(\varepsilon, E)}{-\log \varepsilon}.\end{aligned}$$

δ and Δ are called the upper and lower entropy indices of Kolmogorov. Tricot [17] proved that

$$\operatorname{Dim}(E) = \hat{\Delta}(E) = \inf\Big\{\sup \Delta(E_i),\ E \subset \bigcup_i E_i\Big\}. \tag{2.4}$$

An excellent general reference on Hausdorff dimension and packing dimension is [4].

We use $c_1, c_2, \dots,$ or c to denote unimportant constants; they may be different from line to line.

We first prove some lemmas. Lemma 2.1 is a generalization of Lemma 8.2.1 in [1] and Theorem 6 in [9], Chapter 10. Lemma 2.1 and Lemma 2.3 show that Theorem 1 in [2] and Theorem 4.1 in [12] are incorrect.

LEMMA 2.1. *Let $E \subset R^N$ be a compact set, $f = (f_1, \dots, f_d)$: $E \to R^d$ satisfy a uniform Hölder condition of order $\alpha = (\alpha_1, \dots, \alpha_d)$ on E, that is, $\forall\ x, y \in E$,*

$$|f_j(x) - f_j(y)| \le c|x - y|^{\alpha_j}, \qquad j = 1, 2, \dots, d, \tag{2.5}$$

where $c > 0$, $0 < \alpha_j \le 1$ $(j = 1, \dots, d)$ are constants. If

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_d,$$

then

$$\dim f(E) \le \min\left\{d; \frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\ 1 \le j \le d\right\}, \tag{2.6}$$

$$\dim \operatorname{Gr} f(E) \le \min\left\{\frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\ 1 \le j \le d;\ \dim E + \sum_{i=1}^{d}(1 - \alpha_i)\right\}. \tag{2.7}$$

PROOF. Clearly $\dim f(E) \le d$, $\dim f(E) \le \dim \operatorname{Gr} f(E)$, we only prove (2.7). Take any $\gamma > \dim E$. Then by (2.2), for each $\delta > 0$, there exists a sequence of balls $\{B_l\}$ with $\operatorname{diam} B_l \le \delta$, such that

$$E \subset \bigcup_l B_l, \qquad \sum_l (\operatorname{diam} B_l)^\gamma < +\infty. \tag{2.8}$$

By (2.5), each $f(B_l)$ can be covered by a rectangle C_l of sides $c(\operatorname{diam} B_l)^{\alpha_i}(i = 1, \dots, d)$. For each fixed $1 \le j \le d$, C_l can be covered by $O((\operatorname{diam} B_l)^{\sum_{i=1}^{j}(\alpha_i - \alpha_j)})$ cubes C_{lk} of edge $(\operatorname{diam} B_l)^{\alpha_j}$. Since

$$\operatorname{Gr} f(E) \subset \bigcup_l \bigcup_k B_l \times C_{lk}, \qquad \operatorname{diam}(B_l \times C_{lk}) \le c_1 (\operatorname{diam} B_l)^{\alpha_j}$$

by (2.8), we have

$$\begin{aligned}
&\sum_{l,k} (\operatorname{diam}(B_l \times C_{lk}))^{(\gamma + \sum_{i=1}^{j}(\alpha_j - \alpha_i))/\alpha_j} \\
&\quad \le c_2 \sum_l (\operatorname{diam} B_l)^{\sum_{i=1}^{j}(\alpha_i - \alpha_j)} (\operatorname{diam} B_l)^{\gamma + \sum_{i=1}^{j}(\alpha_j - \alpha_i)} \\
&\quad = c_2 \sum_l (\operatorname{diam} B_l)^\gamma \\
&\quad < +\infty.
\end{aligned}$$

This proves that

$$\dim \operatorname{Gr} f(E) \le \min\left(\frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, j = 1, \dots, d \right). \tag{2.9}$$

On the other hand, each rectangle C_l can be covered by $O((\operatorname{diam} B_l)^{\sum_{i=1}^{d}(\alpha_i - 1)})$ cubes C'_{lk} of edge $\operatorname{diam} B_l$, and $\operatorname{Gr} f(E) \subset \bigcup_l \bigcup_k B_l \times C'_{lk}$,

$$\begin{aligned}
&\sum_{l,k} (\operatorname{diam}(B_l \times C'_{lk}))^{\gamma + \sum_{i=1}^{d}(1 - \alpha_i)} \\
&\quad \le c_3 \sum_l (\operatorname{diam} B_l)^\gamma \\
&\quad < +\infty.
\end{aligned}$$

Hence,

$$\dim \operatorname{Gr} f(E) \le \dim E + \sum_{i=1}^{d} (1 - \alpha_i). \tag{2.10}$$

From (2.9) and (2.10) we prove (2.6) and (2.7). □

By using (2.4) and an argument in [19], we can prove the following packing dimension analog.

LEMMA 2.2. *Under the conditions of Lemma* 2.1, *we have*

$$\operatorname{Dim} f(E) \le \min\left\{d; \frac{\operatorname{Dim} E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le d\right\}, \tag{2.11}$$

$$\operatorname{Dim}\operatorname{Gr} f(E) \le \min\left\{\frac{\operatorname{Dim} E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, \; 1 \le j \le d; \operatorname{Dim} E + \sum_{i=1}^{d}(1 - \alpha_i)\right\}. \tag{2.12}$$

REMARK. From the proof of Lemma 2.1, it is easy to show that if f satisfies a uniform Hölder condition of every order $\beta < \alpha$, that is, for every $\beta = (\beta_1, \dots, \beta_d)$ with $0 < \beta_j < \alpha_j$, $j = 1, \dots, d$, (2.5) holds, then (2.6), (2.7), (2.11), and (2.12) are still valid.

By a simple calculation, we can prove the following lemma.

LEMMA 2.3. *If* $0 = \alpha_0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d$ *and* $\sum_{i=0}^{k-1}\alpha_i < \dim E \le \sum_{i=1}^{k}\alpha_i$, *then*

$$\lambda = \min\left\{d; \frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, j = 1, \dots, d\right\} = \frac{\dim E + \sum_{i=1}^{k}(\alpha_k - \alpha_i)}{\alpha_k}$$

and $k - 1 < \lambda \le k$. *If* $\dim E \ge \sum_{i=1}^{d}\alpha_i$, *then* $\lambda = d$.

Now we prove the main result of this section.

THEOREM 2.1. *Let* $X(t)$ *be an* (N, d) *Gaussian field of index* α *with coordinates so arranged that the* α *satisfy*

$$0 = \alpha_0 < \alpha_1 \le \cdots \le \alpha_d \le 1$$

and let $E \subset R^N$ *be a compact set. If for any* $(s, t) \in E \times E$ (2.1) *holds, then with probability* 1,

$$\dim X(E) = \min\left\{d; \frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le d\right\} = \begin{cases} \dfrac{\dim E + \sum_{i=1}^{k}(\alpha_k - \alpha_i)}{\alpha_k}, & \text{if } \sum_{i=0}^{k-1}\alpha_i \le \dim E \le \sum_{i=1}^{k}\alpha_i, \\ d, & \text{if } \dim E > \sum_{i=1}^{d}\alpha_i, \end{cases} \tag{2.13}$$

$$\dim \operatorname{Gr} X(E) = \min\left\{\frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\right.$$

$$\left. 1 \le j \le d; \dim E + \sum_{i=1}^{d}(1-\alpha_i)\right\} \tag{2.14}$$

$$= \begin{cases} \dim X(E), & \textit{if } \dim X(E) < d, \\ \dim E + \sum_{i=1}^{d}(1-\alpha_i), & \textit{if } \dim X(E) = d. \end{cases}$$

PROOF. By Theorem 8.3.2 in [1], $X(t)$ satisfies almost surely a uniform Hölder condition of order β for every β with $0 < \beta_i < \alpha_i$ $(i = 1, \dots, d)$ on E. The right-hand sides of (2.13) and (2.14) serve as a.s. upper bounds to $\dim X(E)$ and $\dim \operatorname{Gr} X(E)$, respectively, is an immediate consequence of Lemma 2.1 and its remark. We need only to show that they also serve as lower bounds almost surely.

Consider $X(E)$. If $\dim E = 0$, there is nothing to prove. We assume $\sum_{i=1}^{k-1}\alpha_i < \dim E \le \sum_{i=1}^{k}\alpha_i$. Then by Lemma 2.3, $k - 1 < \lambda \le k$. By standard capacity arguments, it is sufficient to show that for any $k - 1 < \gamma < \lambda$, there exists a positive measure σ on E, such that

$$\int_E \int_E E(|X(t) - X(s)|^{-\gamma})\sigma(dt)\sigma(ds) < +\infty. \tag{2.15}$$

Let

$$Y_j(t,s) = \frac{X_j(t) - X_j(s)}{\sigma_j(t-s)}, \qquad j = 1, \dots, d.$$

By (2.1), we have $\det \operatorname{cov}(Y(t,s)) \ge \varepsilon$:

$$E(|X(t) - X(s)|^{-\gamma})$$

$$= \int_{R^d}\left[\sum_{i=1}^{d}(x_i\sigma_i(t-s))^2\right]^{-\gamma/2} \tag{2.16}$$

$$\times \frac{1}{(2\pi)^{d/2}\sqrt{\det \operatorname{cov}(Y)}}\exp\left[-\frac{1}{2}X \operatorname{cov}(Y)^{-1}X'\right] dx_1 \cdots dx_d,$$

where X' is the transpose of $X = (x_1, \dots, x_d)$. Take β_j $(j = 1, \dots, d)$ with $\beta_j > \alpha_j$ for $\alpha_j < 1$, $\beta_j = 1$ when $\alpha_j = 1$ and

$$\beta_1 \le \beta_2 \le \cdots \le \beta_d, \qquad \gamma\beta_k < \dim E + \sum_{i=1}^{k}(\beta_k - \beta_i). \tag{2.17}$$

Then there exist $\delta > 0$, $c_3 > 0$ such that for $|t| < \delta$,

$$\sigma_j(t) \ge c_3|t|^{\beta_j} \qquad (j = 1, \dots, d). \tag{2.18}$$

It is known that for any $d \times d$ real symmetric matrix B, there is a constant $c_4 > 0$, such that for every $X \in R^d$,

$$|XBX'| \le c_4 XX'. \tag{2.19}$$

Since $\operatorname{cov}(Y)$ is positive definite with rank d, there is a $d \times d$ invertible matrix A such that $\operatorname{cov}(Y)^{-1} = AA'$. Then by (2.19) with $B = A^{-1}(A')^{-1}$, we have

$$X \operatorname{cov}(Y)^{-1} X' \geq \frac{1}{c_4} XX'. \tag{2.20}$$

By (2.16)–(2.20), we have for $|t - s| < \delta$,

$$\begin{aligned} &E(|X(t) - X(s)|^{-\gamma}) \\ &\leq c_5 \int_{R^d} \left[\sum_{j=1}^{d} \left(x_j |t - s|^{\beta_j} \right)^2 \right]^{-\gamma/2} \\ &\qquad \times \exp\left(-\left(x_1^2 + \cdots + x_d^2 \right)/2c_4 \right) dx_1 \cdots dx_d \\ &= c_5 |t - s|^{-\gamma\beta_1} \int_{R^d} \left[x_1^2 + \left(x_2 |t - s|^{\beta_2 - \beta_1} \right)^2 + \cdots \right. \\ &\qquad \left. + \left(x_d |t - s|^{\beta_d - \beta_1} \right)^2 \right]^{-\gamma/2} \\ &\qquad \times \exp\left(-\left(x_1^2 + \cdots + x_d^2 \right)/2c_4 \right) dx_1 \cdots dx_d \end{aligned} \tag{2.21}$$

The integral in (2.21) is convergent since $\gamma < d$. Using the fact that

$$\int_0^{+\infty} \left(y^2 + a^2 \right)^{-\gamma/2} dy = c_1(\gamma) a^{-\gamma+1}, \quad \text{for } \gamma > 1, \tag{2.22}$$

$$\begin{aligned} &\int_0^{+\infty} \left(y^2 + a^2 \right)^{-\gamma/2} \exp(-y^p)\, dy \\ &\qquad \asymp c_2(\gamma) a^{-\gamma+1} + c_3(\gamma), \quad \text{for } 0 < \gamma < 1,\ p > 0, \end{aligned} \tag{2.23}$$

where $c_1(\gamma)$, $c_2(\gamma)$ and $c_3(\gamma)$ are positive constants depending only on γ, $A \asymp B$ means that there is a constant $c > 0$ such that $c^{-1} \leq A/B \leq c$. We first integrate out x_1 to obtain that (2.21) is less than a constant times

$$\begin{aligned} &|t - s|^{-\gamma\beta_1} \int_{R^{d-1}} \left[\left(x_2 |t - s|^{\beta_2 - \beta_1} \right)^2 + \cdots + \left(x_d |t - s|^{\beta_d - \beta_1} \right)^2 \right]^{-(\gamma-1)/2} \\ &\qquad \times \exp\left(-\frac{x_2^2 + \cdots + x_d^2}{2c_4} \right) dx_2 \cdots dx_d. \end{aligned} \tag{2.24}$$

Then iterate this argument for $dx_2, \ldots, dx_{k-1}$. We find that (2.24) is less than a constant times

$$\begin{aligned} &|t - s|^{-\gamma\beta_1 - (\gamma-1)(\beta_2 - \beta_1) - \cdots - (\gamma-k+1)(\beta_k - \beta_{k-1})} \\ &\quad \times \int_{R^{d-k+1}} \left[x_k^2 + \sum_{i=k+1}^{d} \left(x_i |t - s|^{\beta_i - \beta_k} \right)^2 \right]^{-(\gamma-k+1)/2} \\ &\quad \times \exp\left[-\frac{\sum_k^d x_i^2}{2c_4} \right] dx_k \cdots dx_d \end{aligned}$$

$$\le c_6|t-s|^{-\gamma\beta_1-(\gamma-1)(\beta_2-\beta_1)-\cdots-(\gamma-k+1)(\beta_k-\beta_{k-1})}$$

$$\times\int_{R^{d-k}}\left(\left[\sum_{i=k+1}^{d}\left(x_i|t-s|^{\beta_i-\beta_k}\right)^2\right]^{-(\gamma-k)/2}+c(\gamma)\right)$$

$$\times\exp\left[-\frac{\sum_{k+1}^{d}x_i^2}{2c_4}\right]dx_{k+1}\cdots dx_d$$

$$\le c_7|t-s|^{-\gamma\beta_k+\sum_{i=1}^{k}(\beta_k-\beta_i)}.$$

Since $\gamma\beta_k-\sum_{i=1}^{k}(\beta_k-\beta_i)<\dim E$, there is a positive measure σ on E with

$$\int_E\int_E\frac{\sigma(dt)\sigma(ds)}{|t-s|^{\gamma\beta_k-\sum_{i=1}^{k}(\beta_k-\beta_i)}}<+\infty.$$

Thus we have (2.15). If $\dim E>\sum_{i=1}^{d}\alpha_i$, the same computation shows that (2.15) holds for any $\gamma<d$. Therefore, (2.13) holds.

To find $\dim\mathrm{Gr}\,X(E)$, we first consider the case $\dim X(E)<d$. Since $\dim\mathrm{Gr}\,X(E)\ge\dim X(E)$, it follows from Lemma 2.1 and (2.13) that $\dim\mathrm{Gr}\,X(E)=\dim X(E)$ a.s. In the case of $\dim X(E)=d$, which implies that $\dim E\ge\sum_{i=1}^{d}\alpha_i$, the dimension of the graph can be larger. Similar manipulation as above shows that for any $d<\gamma<\dim E+\sum_{i=1}^{d}(1-\alpha_i)$, there exists a positive measure μ on E such that

$$\int_E\int_E E\left(\left[|t-s|^2+|X(t)-X(s)|^2\right]^{-\gamma/2}\right)\mu(dt)\mu(ds)<+\infty.$$

Therefore,

$$\dim\mathrm{Gr}\,X(E)\ge\dim E+\sum_{i=1}^{d}(1-\alpha_i)\quad\text{a.s.}$$

This completes the proof of (2.14). □

If $\dim E>\sum_{i=1}^{d}\alpha_i$, then $\dim X(E)=d$. In [20], we proved that $X(E)$ a.s. has positive Lebesgue measure and moreover if X is locally nondeterministic on E, then as in [6] and [14], we can show that $X(t)$ almost surely has continuous local time $\alpha(x,E)$. This implies that $X(E)$ a.s. has interior points.

If $E\doteq[0,1]^N$, then by Lemma 2.2 and (2.3) we have following theorem, in which the statements for Hausdorff dimension correct Theorem 1 in [2] and Theorem 8.4.1 in [1].

THEOREM 2.2. *Let $X(t)$ be an (N,d) Gaussian field as in Theorem* 2.1. *If* (2.1) *holds for all $s,t\in[0,1]^N$, then with probability* 1,

$$\dim X([0,1]^N)=\mathrm{Dim}\,X([0,1]^N)$$
$$=\min\left\{d;\frac{N+\sum_{i=1}^{j}(\alpha_j-\alpha_i)}{\alpha_j},1\le j\le d\right\},$$

$$\dim \operatorname{Gr} X([0,1]^N) = \operatorname{Dim} \operatorname{Gr} X([0,1]^N)$$
$$= \min\left\{ \frac{N + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le d; N + \sum_{i=1}^{d}(1-\alpha_i) \right\}.$$

3. Index-α stable fields. A real-valued random variable X is called symmetric p-stable ($0 < p \le 2$) of parameter σ if for any $\lambda \in R$,

$$E \exp(i\lambda X) = \exp(-\sigma^p |\lambda|^p). \tag{3.1}$$

Denote $\|X\|_p = \sigma$. Then $\|X\|_p = [-\log E\exp(iX)]^{1/p}$. This is a norm ($p$ quasinorm if $0 < p < 1$) on the space of symmetric p-stable random variables. If $p = 2$, X is a Gaussian variable and $\|V\|_2^2 = \frac{1}{2}\operatorname{Var}(X)$.

Let $0 < p \le 2$ and $T \subset R^N$. A real-valued random field $X = (X(t), t \in T)$ is called an $(N, 1, p)$ stable field if every finite linear combination $\sum_{j=1}^{n} \alpha_j X(t_j)$ is a symmetric p-stable random variable, that is,

$$E \exp\left(i \sum_{j=1}^{n} a_j X(t_j) \right) = \exp\left(- \left\| \sum_{j=1}^{n} a_j X(t_j) \right\|_p^p \right). \tag{3.2}$$

It is known that for any such process there is a measure space $(\mathbb{M}, \mathscr{M}, m)$ and a collection $\{k(t, \cdot), t \in T\} \subset L^p(\mathbb{M}, \mathscr{M}, m)$ such that

$$X(t) = \int k(t,u) W(du), \tag{3.3}$$

where W is the p-stable noise generated by m and

$$\left\| \sum_{j=1}^{n} a_j X(t_j) \right\|_p = \left\| \sum_{j=1}^{n} a_j k(t, \cdot) \right\|_{L^p}.$$

Conversely, given any measure space $(\mathbb{M}, \mathscr{M}, m)$, $0 < p \le 2$ and $\{k(t, \cdot), t \in T\} \subset L^p(\mathbb{M}, \mathscr{M}, m)$, one can define a p-stable process $X(t)$ by (3.3) (see [8] for details).

Nolan [12], [13] generalized the concept of local nondeterminism (LND) to stable processes and fields. We alter slightly the definition of LND to study the existence of multiple points for index-α fields.

We say that $t_1, \dots, t_m \in T$ are ordered if $t_1 < \cdots < t_m$, when $T \subset R$, or $|t_j - t_{j-1}| \le |t_j - t_i|$ for $1 \le i < j \le m$, when $T \subset R^N$, $N > 1$, and we write

$$t_1 \preccurlyeq t_2 \preccurlyeq \cdots \preccurlyeq t_m.$$

An $(N, 1, p)$ stable field X is locally nondeterministic (LND) on T if:

1. $\|X(t)\|_p > 0$, for all $t \in T$.
2. $\|X(t) - X(s)\|_p > 0$, for all $t, s \in T$ sufficiently close.
3. For any $m > 1$, there is a $c_m > 0$ such that

$$
\begin{aligned}
& c_m^{-1}\left(\|a_1 X(t_1)\|_p^p + \sum_{l=2}^{m} \|a_l(X(t_l) - X(t_{l-1}))\|_p^p\right) \\
(3.4) \qquad & \le \left\| a_1 X(t_1) + \sum_{l=2}^{m} a_l(X(t_l) - X(t_{l-1}))\right\|_p^p \\
& \le c_m\left(\|a_1 X(t_1)\|_p^p + \sum_{l=2}^{m} \|a_l(X(t_l) - X(t_{l-1}))\|_p^p\right)
\end{aligned}
$$

for all $a_1, \dots, a_m \in R$ and all ordered $t_1, \dots, t_m \in T$.

Let $X(t) = (X_1(t), \dots, X_d(t))$ $(t \in R^N)$ be a stable field with values in R^d. Each component $X_j(t)$ is an index-$\alpha_j(N, 1, p_j)$ symmetric stable field [12]. $X(t)$ is called an index-α $(N, d, \bar{p})$ stable field, where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\bar{p} = (p_1, \dots, p_d)$. We assume $X(t)$ has stationary increments throughout the rest of the paper and we shall need the following conditions:

(H_1) $X(t)$ satisfies a uniform Hölder condition of every order $\beta < \alpha = (\alpha_1, \dots, \alpha_d)$ on any cube $T \subset R^N$.

(H_2) For each $\beta > \alpha$, simultaneously for all components $j = 1, \dots, d$,

$$|h|^{\beta_j} = o\left(\|X_j(t+h) - X_j(t)\|_{p_j}\right) \quad \text{as } h \to 0.$$

(H_3) $X(t)$ has characteristic function locally approximately independently components, that is, for all $m \ge 1$, there is a $c = c(d, m) > 0$, such that for all $u_1, \dots, u_m \in R^d$ and all $t_1, \dots, t_m \in T$,

$$
\begin{aligned}
& \prod_{j=1}^{d} E \exp\left(ic^{-1} \sum_{l=1}^{m} u_{lj} X_j(t_l)\right) \le E \exp\left(i \sum_{l=1}^{m} \langle u_l, X(t_l)\rangle\right) \\
(3.5) \qquad & \qquad \le \prod_{j=1}^{d} E \exp\left(ic \sum_{l=1}^{m} u_{lj} X_j(t_l)\right).
\end{aligned}
$$

An $(N, d, \bar{p})$ stable field is LND if each component is LND and (3.5) holds.

Remark. We refer to [11] and references therein for examples of stable processes satisfying (H_1)

Theorem 3.1. *Let $X(t)$ be an $(N, d, \bar{p})$ stable field with stationary increments that satisfies* (H_1), (H_2) *and* (H_3) *for some $\alpha = (\alpha_1, \dots, \alpha_d)$ with*

$$0 = \alpha_0 < \alpha_1 \le \cdots \le \alpha_d.$$

Then for any compact set $E \subset R^N$, almost surely,

$$(3.6) \quad \dim X(E) = \min\left\{d; \frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j}, 1 \le j \le d\right\},$$

$$\dim \operatorname{Gr} X(E) = \min\left\{\dim E + \sum_{i=1}^{d}(1-\alpha_i);\ \frac{\dim E + \sum_{i=1}^{j}(\alpha_j - \alpha_i)}{\alpha_j},\ 1 \le j \le d\right\}. \tag{3.7}$$

PROOF. By (H_1) and Lemma 2.1, the right-hand sides of (3.6) and (3.7) are upper bounds for dim $X(E)$ and dim Gr $X(E)$, respectively. The lower bound for dim $X(E)$ follows from standard capacity arguments as in the proof of Theorem 2.1. We just make minor adjustments. First we note that if F is any d-dimensional distribution function with characteristic function ϕ, then for each $\gamma > 0$,

$$2^{\gamma/2-1}\Gamma\left(\frac{\gamma}{2}\right)\int_{R^d}|x|^{-\gamma}F(dx) = (2\pi)^{-d/2}\int_0^{+\infty}u^{\gamma-1}\,du\int_{R^d}\exp(-|x|^2/2)\phi(ux)\,dx. \tag{3.8}$$

This equality is obtained by replacing ϕ in the right side of (3.8) by its expression as a Fourier integral and then performing a routine calculation. Assume $\sum_{i=0}^{k-1}\alpha_i < \dim E \le \sum_{i=1}^{k}\alpha_i$, take $\beta_1, \dots, \beta_d$ to satisfy (2.17) and, as $|t-s|$ sufficiently small,

$$\|X_j(t) - X_j(s)\|_{p_j} \ge c|t-s|^{\beta_j}. \tag{3.9}$$

By (3.5),

$$\begin{aligned} &E(\exp i\langle x, X(t) - X(s)\rangle) \\ &\le \prod_{j=1}^{d} E\exp\left[icx_j(X_j(t) - X_j(s))\right] \\ &= \prod_{j=1}^{d}\exp\left(-\|cx_j(X_j(t)-X_j(s))\|_{p_j}^{p_j}\right) \\ &\le \exp\left(-\sum_{j=1}^{d}|cx_j|t-s|^{\beta_j}|^{p_j}\right). \end{aligned} \tag{3.10}$$

Thus for $0 < \gamma < \min\{d, (\dim E + \sum_{i=1}^{k}(\alpha_k - \alpha_i))/\alpha_k\}$,

$$\begin{aligned} &E(|X(t) - X(s)|^{-\gamma}) \\ &\le c_9\int_0^{+\infty}u^{\gamma-1}\,du\int_{R^d}\exp\left(-\frac{|x|^2}{2} - \sum_{j=1}^{d}|cx_j|t-s|^{\beta_j}u|^{p_j}\right)dx_1\cdots dx_d \\ &= c_9\int_{R^d}\exp\left(-\sum_{j=1}^{d}|cy_j|t-s|^{\beta_j}|^{p_j}\right)dy_1\cdots dy_d\int_0^{+\infty}u^{\gamma-d-1}\exp\left(-\frac{|y|^2}{2u^2}\right)du \\ &= c_{10}\int_{R^d}|y|^{\gamma-d}\exp\left(-\sum_{j=1}^{d}|cy_j|t-s|^{\beta_j}|^{p_j}\right)dy \end{aligned}$$

$$= c_{10}|t - s|^{-\sum_{j=1}^{d}\beta_j}\int_{R^d}\left[\sum_{j=1}^{d}\left(x_j|t - s|^{-\beta_j}\right)^2\right]^{-(d-\gamma)/2}$$

$$\times \exp\left(-\sum_{j=1}^{d}|cx_j|^{p_j}\right)dx_1\cdots dx_d$$

$$= c_{10}|t - s|^{-\sum_{j=1}^{d}\beta_j + (d-\gamma)\beta_d}\int_{R^d}\left[x_d^2 + \sum_{i=1}^{d-1}\left(x_i|t - s|^{\beta_d - \beta_i}\right)^2\right]^{-(d-\gamma)/2}$$

$$\times \exp\left(-\sum_{j=1}^{d}|cx_j|^{p_j}\right)dx_d\cdots dx_1.$$

By using (2.22) and (2.23), we integrate out $dx_d, \ldots, dx_{d-k}$ iteratively and the above integral is less than a constant times

$$|t - s|^{-\sum_{j=1}^{d}\beta_j + (d-\gamma)\beta_d}\int_{R^{d-1}}\left[\sum_{i=1}^{d-1}\left(x_i|t - s|^{\beta_d - \beta_i}\right)^2\right]^{-(d-1-\gamma)/2}$$

$$\times \exp\left(-\sum_{j=1}^{d-1}|cx_j|^{p_j}\right)dx_{d-1}\cdots dx_1.$$

The rest of the proof follows Theorem 2.1. □

REMARK. If $E = [0,1]^N$, Theorem 3.1 and Lemma 2.2 give the Hausdorff and packing dimensions for $X([0,1]^N)$ and Gr $X([0,1]^N)$, which correct Theorem 4.1 in [12].

In the case of $\dim E > \sum_{j=1}^{d}\alpha_j$, $\dim X(E) = d$ a.s., we prove a stronger result.

THEOREM 3.2. *Under the hypothesis of Theorem* 3.1, *if* $\dim E > \sum_{j=1}^{d}\alpha_j$, *then* $X(E)$ *a.s. has positive Lebesgue measure.*

PROOF. Take $\beta_j > \alpha_j$ $(j = 1, \ldots, d)$ satisfying

$$\beta_1 \le \cdots \le \beta_d, \qquad \dim E > \sum_{j=1}^{d}\beta_j.$$

Then there is a positive Borel measure σ on E such that

$$\int_E\int_E \frac{\sigma(dt)\sigma(ds)}{|t - s|^{\sum_{j=1}^{d}\beta_j}} < +\infty. \tag{3.11}$$

Let μ be the image of σ under the mapping $t \to X(t)$. Then the Fourier transform of μ is

$$\hat{\mu}(u) = \int_E \exp(i\langle u, X(t)\rangle)\sigma(dt).$$

By (3.5) and (3.11), we have

$$\begin{aligned} E\int_{R^d} |\hat{\mu}(u)|^2\,du &= \int_{R^d}\int_E\int_E E\exp(i\langle u, X(t)-X(s)\rangle)\,\sigma(dt)\sigma(ds)\,du \\ &\le \int_E\int_E\int_{R^d}\exp\left(-\sum_{j=1}^{d}\|t-s|^{\beta_j}u|^{p_j}\right)du\,\sigma(dt)\sigma(ds) \\ &= c_{12}\int_E\int_E\frac{\sigma(dt)\sigma(ds)}{|t-s|^{\sum_{j=1}^d\beta_j}} \\ &< +\infty, \end{aligned}$$

that is, $\hat{\mu}\in L^2(R^d)$ a.s. This implies that $X(t)$ a.s. has a square integrable local time and $X(E)$ a.s. has positive Lebesgue measure. □

As in [20], under the conditions of Theorem 3.1, with the added assumption that X is LND on E, if $\dim E < \sum_{j=1}^d\alpha_j$, then $\mathrm{Leb}_d(X(E)) = 0$ a.s. and for every $u\in R^d$,

$$P(X^{-1}(u)\cap E\neq\varnothing) = 0.$$

If $\dim E > \sum_{j=1}^d\alpha_j$, then for every $u\in R^d$,

$$P(X^{-1}(u)\cap E\neq\varnothing) > 0.$$

4. Multiple points of index-α stable fields. The existence of k-multiple points was shown by Kono [10] and Goldman [7] for fractional Brownian motion and by Cuzick [3] for general LND index-α Gaussian fields. In [18], Weber obtained the Hausdorff dimension of k-multiple times for fractional Brownian motion. In this section, we generalize these results to stable fields.

Let $X(t) = (X_1(t),\dots,X_d(t))$ be an $(N, d, \bar{p})$ stable field on R^N. $x\in R^d$ is called a k-multiple point of $X(t)$ if there are distinct $t_1,\dots,t_k\in R^N$ such that

$$X(t_1) = \cdots = X(t_k) = x.$$

Denote by L_k the set of k-multiple points and M_k the set of k-multiple times, that is,

$$M_k = \{(t_1,\dots,t_k)\in R^{Nk},\ t_1,\dots,t_k \text{ distinct},\ X(t_1) = \cdots = X(t_k)\}.$$

Let $P_1,\dots,P_k$ be disjoint closed cubes in R^N, $P = \prod_{l=1}^k P_l$,

$$M_k(P) = M_k\cap P.$$

We define a $(kN, (k-1)d)$ stable field $Y(\bar{t})$ by

$$Y(\bar{t}) = (X(t_2)-X(t_1), X(t_3)-X(t_2),\dots,X(t_k)-X(t_{k-1})) \qquad (\bar{t} = (t_1,\dots,t_k)).$$

Then $M_k(P) = Y^{-1}(0)\cap P$.

LEMMA 4.1. *Let $X(t)$ be a LND $(N, d, \bar{p})$ stable field on compact $T \subset R^N$ and $p(\bar{t}, \bar{x})$ be the density of $Y(\bar{t})$. Then there is a constant $c(k, \bar{p}) > 0$ depending only on k and $\bar{p}$, such that for any distinct $t_1, \dots, t_k \in T$,*

$$p(\bar{t}, \bar{x}) \le c(k, \bar{p}) J(\bar{t}_\pi), \tag{4.1}$$

where

$$J(\bar{t}_\pi) = \left[\prod_{j=1}^{d} \prod_{m=2}^{k} \|X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)})\|_{p_j} \right]^{-1}$$

and π is a permutation of $\{1, 2, \dots, k\}$ such that $t_{\pi(1)}, \dots, t_{\pi(k)}$ are ordered.

PROOF. By making a change of variables in the Fourier inverse formula, we have

$$\begin{aligned} p(\bar{t}, \bar{x}) &\le (2\pi)^{-(k-1)d} \\ &\quad \times \int_{R^{(k-1)d}} E \exp\left(i \sum_{m=2}^{k} \langle u_m, X(t_m) - X(t_{m-1}) \rangle \right) d\bar{u} \\ &= (2\pi)^{-(k-1)d} \\ &\quad \times \int_{R^{(k-1)d}} E \exp\left(i \sum_{m=2}^{k} \langle w_m, X(t_{\pi(m)}) - X(t_{\pi(m-1)}) \rangle \right) d\bar{u}, \end{aligned} \tag{4.2}$$

where

$$w_m = \sum_{l=m}^{k} (u_{\pi(l)} - u_{\pi(l)+1}), \qquad m = 2, \dots, k, \, u_{k+1} = 0.$$

and the mapping $(u_2, \dots, u_k) \to (w_2, \dots, w_k)$ is nonsingular. By LND of $X(t)$ on T, the integrand in (4.2) is dominated by

$$\exp\left\{ -c \sum_{j=1}^{d} \left(\sum_{m=2}^{k} \|w_{mj}(X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)}))\|_{p_j} \right)^{p_j} \right\}.$$

Make a change of variables: $v_{mj} = w_{mj} \|X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)})\|_{p_j}$ for $j = 1, \dots, d$, $m = 2, \dots, k$. Noting that $J(\bar{t}_\pi)$ is precisely the Jacobian of the transformation $\bar{w} \to \bar{v}$, we get (4.1). □

THEOREM 4.1. *Let $X(t)$ be an $(N, d, \bar{p})$ stable field on $\bigcup_{l=1}^{k} P_l$ that is* LND *and satisfies* (H_1). *If $Nk < (k-1)\sum_{j=1}^{d} \alpha_j$, then with probability* 1,

$$M_k(P) = \varnothing.$$

PROOF. By using Lemma 4.1, Theorem 4.1 can be proven in the same way as Theorem 2 in [2]. □

THEOREM 4.2. *Let $X(t)$ be an $(N, d, \bar{p})$ stable field on $\bigcup_{l=1}^{k} P_l$ that is* LND *and satisfies* (H_1) *and* (H_2). *If $Nk > (k-1)\sum_{j=1}^{d} \alpha_j$, then with positive* *probability, $X(t)$ has k-multiple points and*

$$P\left(\dim M_k(P) = Nk - (k-1)\sum_{j=1}^{d}\alpha_j\right) > 0. \tag{4.3}$$

PROOF. By Lemma 4.1 and (H_1), similar to the proof of Theorem 3 in [21], we can prove

$$\dim M_k(P) \le Nk - (k-1)\sum_{j=1}^{d}\alpha_j \quad \text{a.s.}$$

To complete the proof, we need only to show that with positive probability,

$$\dim M_k(P) \ge N_k - (k-1)\sum_{j=1}^{k}\alpha_j. \tag{4.4}$$

For this purpose, it is sufficient to show that for any $0 < \gamma < Nk - (k-1)\sum_{j=1}^{k}\alpha_j$, we can construct a positive measure μ on $M_k(P)$ such that

$$\int_P\int_P \frac{\mu(d\bar{t})\,\mu(d\bar{s})}{|\bar{t}-\bar{s}|^{\gamma}} < +\infty. \tag{4.5}$$

The similar methods were used by Kahane [9], Adler [1], Weber [18], Testard [16] and Xiao [19].

Let $\mathscr{M}_\gamma^+$ be the space of all nonnegative measures on R^{Nk} with finite γ energy. It is known [1] that $\mathscr{M}_\gamma^+$ is a complete metric space under the metric

$$\|\mu\|_\gamma = \int_{R^{Nk}}\int_{R^{Nk}} \frac{\mu(d\bar{t})\,\mu(d\bar{s})}{|\bar{t}-\bar{s}|^{\gamma}}.$$

We define a sequence of random positive measures μ_n on the Borel sets of R^{Nk} by

$$\mu_n(B,\omega) = \int_{P\cap B}\phi_n(\bar{t},\omega)\,d\bar{t},$$

where for each $n \ge 1$,

$$\phi_n(\bar{t},\omega) = \int_{R^{(k-1)d}} \exp\left(-\frac{1}{n}\sum_{m=2}^{k}\sum_{j=1}^{d}|u_{mj}|^{p_j} + i\langle\bar{u}, Y(\bar{t})\rangle\right) d\bar{u},$$

$\phi_n(\bar{t}) \ge 0$ is a continuous function ([5], Chapter 15) and $\phi_n(\bar{t})$ tend to zero outside any neighborhood of $Y^{-1}(0)$ as $n \to +\infty$, that is,

$$\lim_{n\to\infty}\int_P \phi_n(\bar{t})g(\bar{t})\,d\bar{t} = 0 \tag{4.6}$$

for any continuous function g that vanishes in a neighborhood of $Y^{-1}(0)$.

By a lemma of Testard [16], which simplifies the arguments in [9] and [1], if there are constants $c_{13} > 0$, $c_{14} > 0$ such that

$$E(\|\mu_n\|) \ge c_{13}, \qquad E(\|\mu_n\|^2) \le c_{14}, \tag{4.7}$$

$$E(\|\mu_n\|_\gamma) < +\infty, \tag{4.8}$$

where $\| \mu_n \| = \mu_n(P)$, then there is a subsequence of $\{\mu_n\}$, say $\{\mu_{n_k}\}$, such that $\mu_{n_k} \to \mu$ in $\mathscr{M}_\gamma^+$ and μ is strictly positive with probability greater than or equal to $c_{13}^2/2c_{14}$. Moreover by (4.6), μ has its support in $M_k(P)$ almost surely. This will give (4.5).

The last points to prove are (4.7) and (4.8). Let $S(k)$ be the set of all permutations of $\{1, 2, \dots, k\}$, for $\pi \in S(k)$,

$$\Gamma_\pi = \left\{\bar{t} \in P,\ t_{\pi(1)}, \dots, t_{\pi(k)} \text{ are ordered}\right\}.$$

Then

$$\begin{aligned} E(\| \mu_n \|) &= E\int_P \phi_n(\bar{t})\, d\bar{t} \\ &= \sum_{\pi \in S(k)} \int_{\Gamma_\pi} \int_{R^{(k-1)d}} \exp\left(-\frac{1}{n} \sum_{m=2}^{k} \sum_{j=1}^{d} |u_{mj}|^{p_j} \right) \\ &\qquad \times E \exp(i\langle \bar{u}, Y(\bar{t}) \rangle)\, d\bar{u}\, d\bar{t}. \end{aligned} \tag{4.9}$$

By LND and the proof of Lemma 4.1,

$$\begin{aligned} &E \exp(i\langle \bar{u}, Y(\bar{t}) \rangle) \\ &\quad = E \exp\left(i \sum_{m=2}^{k} \langle w_m, X(t_{\pi(m)}) - X(t_{\pi(m-1)}) \rangle \right) \\ &\quad \geq \exp\left\{ -c \sum_{j=1}^{d} \left(\sum_{m=2}^{d} \| w_{mj} (X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)})) \|_{p_j} \right)^{p_j} \right\}, \end{aligned}$$

where the mapping $(u_2, \dots, u_k) \to (w_2, \dots, w_k)$ is nonsingular. Hence the integral in (4.9) is more than a constant times

$$\begin{aligned} &\int_{\Gamma_\pi} \int_{R^{(k-1)d}} \exp\left\{ -\frac{c}{n} \sum_{m=2}^{k} \sum_{j=1}^{d} |w_{mj}|^{p_j} \right\} \\ &\qquad \times \left\{ -c \sum_{j=1}^{d} \left(\sum_{m=2}^{d} \| w_{mj} (X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)})) \|_{p_j} \right)^{p_j} \right\} d\bar{u}\, d\bar{t} \\ &\quad \geq c_{15} \int_{\Gamma_\pi} \frac{d\bar{t}}{\prod_{m=2}^{k} \prod_{j=1}^{d} \left(1/n + \| X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)}) \|_{p_j} \right)} \\ &\quad \geq c_{15} \int_{\Gamma_\pi} \frac{d\bar{t}}{\prod_{m=2}^{k} \prod_{j=1}^{d} \left(1 + \| X_j(t_{\pi(m)}) - X_j(t_{\pi(m-1)}) \|_{p_j} \right)}. \end{aligned}$$

This is the required positive constant:

$$\begin{aligned} E(\| \mu_n \|^2) &= \int_P \int_P \int_{R^{(k-1)d}} \int_{R^{(k-1)d}} \\ &\qquad \times \exp\left\{ -\frac{1}{n} \sum_{m=2}^{k} \sum_{j=1}^{d} \left(|u_{mj}^1|^{p_j} + |u_{mj}^2|^{p_j} \right) \right\} \\ &\qquad \times E \exp i(\langle \bar{u}^1, Y(\bar{t}^1) \rangle + \langle \bar{u}^2, Y(\bar{t}^2) \rangle)\, d\bar{u}^1\, d\bar{u}^2\, d\bar{t}^1\, d\bar{t}^2. \end{aligned} \tag{4.10}$$

Since $P_1, \dots, P_k$ are disjoint compact cubes,

$$\min_{i \neq j} \left(\operatorname{dist}(P_i, P_j)\right) \geq \varepsilon > 0,$$

where $\operatorname{dist}(P_i, P_j)$ is the distance between P_i and P_j. We may assume that the diameters of P_i $(i = 1, \dots, k)$ are so small that for any $t_i^1, t_i^2 \in P_i$ $(i = 1, \dots, k)$, there is a permutation $\pi \in S(k)$ such that

$$t^l_{\pi(i)} \preccurlyeq t^{\bar{l}}_{\pi(i+1)}, \qquad i = 1, \dots, k-1,\ l, \bar{l} = 1, 2.$$

Furthermore we can find k permutations $p_1, \dots, p_k$ of $\{1, 2\}$ such that

$$(4.11) \qquad t^{p_1(1)}_{\pi(1)} \preccurlyeq t^{p_1(2)}_{\pi(1)} \preccurlyeq t^{p_2(1)}_{\pi(2)} \preccurlyeq \cdots \preccurlyeq t^{p_k(2)}_{\pi(k)}$$

and the integral in (4.10) is a finite sum of the integrals over $\Gamma(\pi, p_1, \dots, p_k)$, where

$$\Gamma(\pi, p_1, \dots, p_k) = \left\{(\bar{t}^1, \bar{t}^2) \in P \times P,\ (\bar{t}^1, \bar{t}^2) \text{ satisfies } (4.11)\right\}.$$

For simplicity, we assume that $\pi, p_1, \dots, p_k$ are all identities.

We write

$$(4.12) \qquad \begin{aligned} &\sum_{l=1}^{2} \sum_{m=2}^{k} \left\langle u^l_m, X(t^l_m) - X(t^l_{m-1})\right\rangle \\ &\quad = \sum_{m=1}^{k} \sum_{l=1}^{2} \left\langle v^l_m, X(t^l_m) - X(t^{l-1}_m)\right\rangle, \end{aligned}$$

where

$$v^l_m = u^l_m + u^{l+1}_m, \qquad u^2_m \triangleq u^1_{m+1}, \qquad u^l_1 = u^l_{k+1} = 0 \qquad (l = 1, 2)$$
$$t^0_m \triangleq t^2_{m-1}, \qquad t^0_1 = 0.$$

For each $q = 1, 2, \dots, k$, the transformation

$$(\bar{u}^1, \bar{u}^2) \to V_q = \left(v^l_m, (m, l) \in C_q\right)$$

is nonsingular with Jacobian $|J| = 1$, where

$$C_q = \{(m, l) | 1 \leq m \leq k,\ l = 1, 2,\ (m, l) \neq (q, 2),\ (m, l) \neq (1, 1)\}.$$

By (4.11), (4.12) and LND, we have

$$\begin{aligned} &E \exp\left(i \sum_{l=1}^{2} \sum_{m=2}^{k} \langle u^l_m, X(t^l_m) - X(t^l_{m-1})\rangle\right) \\ &\quad = E \exp\left(i \sum_{m=1}^{k} \sum_{l=1}^{2} \langle v^l_m, X(t^l_m) - X(t^{l-1}_m)\rangle\right) \end{aligned}$$

$$\text{(4.13)}\quad \begin{aligned} &\le \prod_{j=1}^{d} E\exp\left(ic\sum_{m=1}^{k}\sum_{l=1}^{2} v_{mj}^{l}\left(X_j(t_m^l)-X_j(t_m^{l-1})\right)\right)\\ &\le \prod_{j=1}^{d}\exp\left\{-\left(\sum_{m=1}^{k}\sum_{l=1}^{2}\|cv_{mj}^{l}\left(X_j(t_m^l)-X_j(t_m^{l-1})\right)\|_{p_j}\right)^{p_j}\right\}\\ &= \prod_{q=1}^{k}\prod_{j=1}^{d}\exp\left\{-\frac{1}{k}\left(\sum_{(m,l)\in C_q}\|cv_{mj}^{l}\left(X_j(t_m^l)-X_j(t_m^{l-1})\right)\|_{p_j}\right)^{p_j}\right\}. \end{aligned}$$

Then by the generalized Hölder's inequality,

$$\text{(4.14)}\quad \begin{aligned} &\int\!\!\int_{R^{2(k-1)d}} E\exp\left(i\sum_{l=1}^{2}\sum_{m=2}^{k}\langle u_m^l, X(t_m^l)-X(t_m^{l-1})\rangle\right)d\bar{u}^1\,d\bar{u}^2\\ &\le \prod_{q=1}^{k}\Bigg[\int\!\!\int_{R^{2(k-1)d}}\prod_{j=1}^{d}\\ &\qquad\times\exp\left\{-\left(\sum_{(m,l)\in C_q}\|cv_{mj}^{l}\left(X_j(t_m^l)-X_j(t_m^{l-1})\right)\|_{p_j}\right)^{p_j}\right\}d\bar{u}^1\,d\bar{u}^2\Bigg]^{1/k}\\ &= c_{16}\prod_{q=1}^{k}\prod_{j=1}^{d}\prod_{(m,l)\in C_q}\frac{1}{\|X_j(t_m^l)-X_j(t_m^{l-1})\|_{p_j}^{1/k}}. \end{aligned}$$

Now take β_j $(j=1,\dots,d)$ satisfying

$$\beta_j > \alpha_j,\qquad Nk > (k-1)\sum_{j=1}^{d}\beta_j.$$

Then by (H_2), as $|t-s|$ sufficiently small,

$$\text{(4.15)}\qquad \|X_j(t)-X_j(s)\|_{p_j}\ge |t-s|^{\beta_j}\qquad (j=1,2,\dots,d).$$

By (4.14) and (4.15), the integral in (4.10) can be bounded by a constant times

$$\int_P\!\int_P\frac{d\bar{t}^1\,d\bar{t}^2}{\prod_{m=1}^{k}|t_m^2-t_m^1|^{((k-1)\sum_{j=1}^{d}\beta_j)/k}}<+\infty.$$

This proves (4.7).

For any $0\le\gamma\le Nk-(k-1)\sum_{j=1}^{d}\alpha_j$, we can choose $\beta_j>\alpha_j$ satisfying $\gamma<Nk-(k-1)\sum_{j=1}^{d}\beta_j$ and (4.15). Then we have

$$\begin{aligned} E(\|\mu_n\|_\gamma) &= \int_P\!\int_P E(\phi(\bar{t}^1)\phi(\bar{t}^2))\frac{d\bar{t}^1\,d\bar{t}^2}{|\bar{t}^1-\bar{t}^2|^\gamma}\\ &\le c_{17}\int_P\!\int_P\frac{d\bar{t}^1\,d\bar{t}^2}{\prod_{m=1}^{k}|t_m^2-t_m^1|^{(\gamma+(k-1)\sum_{j=1}^{d}\beta_j)/k}}\\ &< +\infty. \end{aligned}$$

Hence, with positive probability [independent of β_j $(j=1,\dots,d)$],

$$\dim M_k(P)\ge Nk-(k-1)\sum_{j=1}^{d}\beta_j.$$

This completes the proof of the theorem. □

REMARK. In [18] and [19] we obtained the Hausdorff dimension and packing dimension of L_k for fractional Brownian motion. For index-α stable fields, the problem is open.

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REFERENCES

[1] ADLER, R. J. (1981). *The Geometry of Random Fields.* Wiley, New York.
[2] CUZICK, J. (1978). Some local properties of Gaussian vector fields. *Ann. Probab.* **6** 984–994.
[3] CUZICK, J. (1982). Multiple points of a Gaussian vector field. *Z. Wahrsch. Verw. Gebiete* **61** 431–436.
[4] FALCONER, K. (1990). *Fractal Geometry—Mathematical Foundations and Applications.* Wiley, New York.
[5] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications.* Wiley, New York.
[6] GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67.
[7] GOLDMAN, A. (1981). Points multiples des trajectories de processus Gaussiens. *Z. Wahrsch. Verw. Gebiete* **57** 481–494.
[8] HARDIN, D. C. (1982). On the spectral representation of symmetric stable processes. *J. Multivariate Anal.* **12** 385–401.
[9] KAHANE, J-P. (1985). *Some Random Series of Functions,* 2nd. ed. Cambridge Univ. Press.
[10] KONO, N. (1978). Double points of a Gaussian sample path. *Z. Wahrsch. Verw. Gebiete* **45** 175–180.
[11] KONO, N. and MAEJIMA, M. (1991). Self-similar stable processes with stationary increments. In *Stable Processes and Related Topics* (S. Cambanis, G. Samorodnitsky and M. Taqqu, eds.) *Progress in Probability* **25** 275–296. Birkhäuser, Boston.
[12] NOLAN, J. P. (1988). Path properties of index-β stable fields. *Ann. Probab.* **16** 1596–1607.
[13] NOLAN, J. P. (1989). Local nondeterminism and local times for stable processes. *Probab. Theory Related Fields* **82** 387–410.
[14] PITT, L. D. (1978). Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** 309–330.
[15] TAYLOR, S. J. and TRICOT, C. (1985). Packing measure and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.* **288** 679–699.
[16] TESTARD, F. (1987). Polarité, points multiples et géomètrie de certains processus gaussiens. Thèse doctorat, Orsay.
[17] TRICOT, C. (1982). Two definitions of fractional dimension. *Math. Proc. Cambridge Phil. Soc.* **91** 57–74.
[18] WEBER, M. (1983). Dimension de Hausdorff et points multiples du mouvement brownien fractionnaire dans R^n. *C. R. Acad. Sci. Paris* **297** 357–360.
[19] XIAO, Y. (1990a). Geometric properties of certain Gaussian random fields. Ph.D. thesis, Wuhan Univ.
[20] XIAO, Y. (1990b). Hausdorff dimension of the image, graph and level sets of Gaussian random fields. *J. Wuhan Univ. Nat. Sci.* **4** 15–24.
[21] XIAO, Y. (1991). Multiple points of fractional Brownian motion and Hausdorff dimension. *Chinese Ann. Math.* **12A** 612–618.
[22] XIAO, Y. (1994). Uniform packing dimension results for fractional Brownian motion. Unpublished manuscript.

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