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LAPLACE APPROXIMATIONS FOR LARGE DEVIATIONS OF NONREVERSIBLE MARKOV PROCESSES. THE NONDEGENERATE CASE

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We are investigating Markov process expectations for large time of the form $\exp(TF(L_T))$, where L_T is the empirical measure of a uniformly ergodic Markov process and F is a smooth functional. Such expressions are evaluated up to a factor which converges to 1. In contrast to earlier work on the subject, it is not assumed that the process is reversible.

1. Statement of the result. Let E be a compact metric space and $\mathscr{E}$ its Borel field. $C(E)$ is the set of continuous mappings $E \to \mathbb{R}$, and $\| \ \|_\infty$ is the supremum metric on $C(E)$. $C^+(E)$ denotes the set of strictly positive functions on E. $\mathscr{M}(E)$ is the set of signed measures on $(E, \mathscr{E})$ and $\mathscr{M}_1^+(E)$ is the set of probability measures. The path space $\Omega = D([0, \infty), E)$ is the set of right continuous functions $\omega: [0, \infty) \to E$ with left-hand limits. The well-known Skorohod metric gives Ω the structure of a Polish space whose Borel field $\mathscr{F}$ is generated by the evaluation mappings $X_t(\omega) = \omega(t)$, $t \geq 0$. We also write $\mathscr{M}_1^+$, C, C^+ instead of $\mathscr{M}_1^+(E), \dots,$ if there is no danger of confusion.

We consider an $\mathscr{E}$-measurable family $(\mathbb{P}_x)_{x \in E}$ of time-homogeneous Markovian probability measures on $(\Omega, \mathscr{F})$ with $\mathbb{P}_x(X_0 = x) = 1$, $x \in E$. We assume that the corresponding semigroup $(P_t)_{t \geq 0}$ is a semigroup of contractions on $(C(E), \| \ \|_\infty)$. Furthermore, we make a strong uniform ergodicity assumption:

ASSUMPTION 1.1. There exists a (P_t)-invariant probability measure π, and for each $t > 0$, there exist transition densities $(p_t(x, y))_{x,y \in E}$ of P_t w.r.t. π which satisfy $p_t \in C^+(E \times E)$.

Let $L_T: \Omega \to \mathscr{M}_1^+(E)$ be the empirical measure

$$L_T(\omega) = \frac{1}{T} \int_0^T \delta_{X_s(\omega)} \, ds.$$

Under our assumptions, L_T satisfies a strong uniform large deviation principle with rate functions $J: \mathscr{M}_1^+ \to [0, \infty]$:

$$J(\mu) = \sup\left\{ -\int \frac{Lu}{u} d\mu : u \in C^+ \cap \mathscr{D}_L \right\},$$

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where L is the infinitesimal generator of (P_t) on $C(E)$, and $\mathscr{D}_L$ is its domain (see [9], Theorem 4.2.4). As a consequence, if $F\colon M_1^+ \to \mathbb{R}$ is bounded and continuous, then

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{E}_x(\exp(TF(L_T))) = \sup\{F(\mu) - J(\mu)\colon \mu \in \mathscr{M}_1^+\} \equiv b_F.$$

We define

$$K_F \equiv \{\mu \in \mathscr{M}_1^+\colon F(\mu) - J(\mu) = b_F\}.$$

K_F is not empty and compact in $\mathscr{M}_1^+$.

It is the aim of this paper to give a more precise evaluation of $\mathbb{E}_x(\exp(TF(L_T)))$ in the case where F is smooth. Such evaluations have recently been obtained in the reversible case, that is, when the P_t are self-adjoint on $L_2(\pi)$, by Brydges and Maya [7] for processes with finite state space, with the help of Berezin integration, and Kusuoka and Tamura [10] for more general cases, but still only for reversible processes. For sums of i.i.d. random variables, see [4] and [5].

To formulate the appropriate smoothness, we imbed $\mathscr{M}(E)$ in a suitable Hilbert space which is the same as that used in [10]:

Let $(\psi_n)_{n\in\mathbb{N}}$, $\psi_n \in C(E)$, be a complete orthonormal system in $L_2(E, \pi)$ and let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers, satisfying

$$\lim_{n\to\infty} a_n = 0 \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} a_n \|\psi_n\|_\infty^2 = 1. \tag{1.3}$$

If $\nu, \mu \in \mathscr{M}(E)$, let

$$\langle \nu, \mu\rangle_a = \sum_{n=1}^{\infty} a_n \int \psi_n \, d\nu \int \psi_n \, d\mu,$$

and $\|\mu\|_a \equiv \sqrt{\langle \mu, \mu\rangle_a}$. We will give concrete examples for the choices of (a_n) and (ψ_n) in the case of diffusions on compact manifolds in Section 5.

From (1.3), we immediately get

$$\|\mu\|_a \le \|\mu\|_{\mathrm{var}}, \tag{1.4}$$

where $\|\ \|_{\mathrm{var}}$ is the total variation norm. Therefore $\|\ \|_a$ is finite on $\mathscr{M}(E)$. We denote by H_a the completion of $\mathscr{M}(E)$ w.r.t. $\|\ \|_a$. The imbedding $\mathscr{M}(E) \to H_a$ is continuous when $\mathscr{M}(E)$ is equipped with the weak topology, that is, the topology induced by $\mu \to \int f\,d\mu$, $f \in C$. As $\mathscr{M}_1^+(E)$ is compact in the weak topology, it is compact in H_a as well. Therefore, any continuous function $F\colon \mathscr{M}_1^+ \to \mathbb{R}$ can be extended to a bounded continuous function $\tilde{F}\colon H_a \to \mathbb{R}$. We will use the following smoothness and uniqueness conditions:

ASSUMPTION 1.5. *F has an extension to H_a which is bounded, continuous and has two bounded and continuous Frechet derivatives. We will denote this extension by F, too.*

(We will give conrete examples of (a_n) and (ψ_n) and differentiable F in the case of diffusions on compact manifolds in Section 5).

We denote by $DF(\mu) \in H_a$ and $D^2F(\mu) \in H_a \otimes H_a$ the first and second derivative at $\mu \in \mathscr{M}_1^+(E)$. We also need a nondegeneracy condition:

ASSUMPTION 1.6. $F - J$ has nonvanishing curvature at κ, for any $\kappa \in K_F$.

As it stands, Assumption 1.6 is not a mathematically precise statement because J is not smooth at all. To state it in a precise form needs some preparation. The formal definition is given in Section 2 [cf. Assumption 2.26].

THEOREM 1.7. *Under the Assumptions* 1.1, 1.5 *and* 1.6, K_F *contains at most finitely many points* $\{\kappa_1, \kappa_2, \dots, \kappa_n\}$ *and there exists* $h_{\kappa_i} \in C^+(E)$, $d_{F,\kappa_i} \in \mathbb{R}^+$, $i = 1, \dots, n$, *such that as* $T \to \infty$,

$$\mathbb{E}_x(\exp(TF(L_T))) = \sum_{i=1}^{n} a_{x_i}(x) \exp(Tb_F)(1 + o(1)),$$

where

$$a_{x_i}(x) = d_{F,x_i} h_{x_i}(x) \int_E \frac{1}{h_{x_i}(y)} x_i(dy), \qquad i = 1, \dots, n, x \in E,$$

For $\kappa \in K_F$, the function h_κ is the unique $L^2(\kappa)$-normalized eigenfunction associated with b_F, the principal eigenvalue of the operator $L + \phi^\kappa$, where $\phi^\kappa \in C^+(E)$ is given by

$$E \ni x \to \phi^x(x) \equiv \langle DF(x), \delta_x \rangle_a;$$

compare (2.4) below. $d_{F,\kappa} \in (0, \infty)$ can be described in terms of a determinant:

$$d_{F,\kappa} \equiv \left[\det\left(I - D^2F(\kappa) \circ S_\kappa\right)\right]^{-1},$$

where $D^2F(\kappa)$ is the second derivative of F at κ, interpreted as a bounded linear operator $H_a \to H_a$, and S_κ is a trace class operator, essentially the second derivative of J, which will be described in the next section [cf. (2.29)].

Associated with each $\kappa \in K_F$ and $x \in E$ we construct a Markovian law $\mathbb{Q}_x^\kappa$ on $(\Omega, \mathscr{F})$, the h_κ-transform of $\mathbb{P}_x$:

$$\left.\frac{\mathbb{Q}_x^\kappa(d\omega)}{\mathbb{P}_x(d\omega)}\right|_{\mathscr{F}_t} = \exp(-b_F t) \frac{h_\kappa(X_t(\omega))}{h_\kappa(x)} \exp\left[\int_0^t h_\kappa(X_s(\omega))\, ds\right];$$

compare Section 2, which has the property that L_T converges to κ under $\mathbb{Q}_x^\kappa$.

Define the family of measures $\{\hat{\mathbb{P}}_x^T : T > 0\}$ on $(\Omega, \mathscr{F})$:

$$\hat{\mathbb{P}}_x^T(\Gamma) = \frac{\mathbb{E}_x(\exp(TF(L_T)); L_T \in \Gamma)}{\mathbb{E}_x(\exp(TF(L_T)))}, \qquad \Gamma \in \mathscr{F},$$

where we use the notation $\mathbb{E}(X; B)$ for $\mathbb{E}(X1_B)$.

In [6] we show that $\{\hat{\mathbb{P}}_x^T, T > 0\}$ is tight, and that any limit point can be expressed as a mixture of $\mathbb{Q}_x^\kappa$, $\kappa \in K_F$. In the nondegenerate case, we can identify the mixture coefficients explicitly in the following convergence theorem which is a direct consequence of Theorem 1.7; compare [6] and [10].

THEOREM 1.8. *With respect to the weak convergence on Ω we have*

$$\lim_{T\to\infty} \hat{\mathbb{P}}_x^T = \sum_{i=1}^n \alpha_{\kappa_i}(x)\mathbb{Q}_x^{\kappa_i},$$

where $\alpha_{\kappa_i}(x) = a_{\kappa_i}(x)/(\sum_{j=1}^n a_{\kappa_j}(x))$.

We fix some notations: If f_1, f_2 are measurable real-valued functions defined on E, and $\mu \in \mathscr{M}(E)$, we write $\langle f_1, f_2\rangle_\mu$ for $\int f_1 f_2\, d\mu$ and $\langle f_1\rangle_\mu$ for $\int f_1\, d\mu$, if they are defined. This should not be confounded with the notation $\langle \mu_1, \mu_2\rangle_a$ for $\mu_1, \mu_2 \in \mathscr{M}(E)$ introduced above.

We will use $k, k_1, k_2, \dots$ for generic positive constants, not necessarily the same along different computations.

The rest of the paper is divided into four sections. In Section 2 we give a precise form of the nondegeneracy Assumption 1.6. The argument is based on a perturbation of the rate function J at the equilibrium points κ_i. In Section 3 we derive the Gaussian behavior of L_T near κ_i. In particular, we prove a uniform moderate deviation result; compare Proposition 3.2. Section 4 gives the proof of Theorem 1.7, following the argument of Bolthausen in [4]. Finally in Section 5 we present a few examples focussing on the computation of the rate function J and the nondegeneracy condition 1.6.

2. Perturbations. We recall some facts discussed in [6] and give a precise form of Assumption 1.6 and the trace class operator S_κ associated with $\kappa \in K_F$. If $\varphi \in C(E)$, let

$$P_t^\varphi(x, A) = \mathbb{E}_x\left(\exp\left(\int_0^t \varphi(X_s)\, ds\right); X_t \in A\right),$$

$A \in \mathscr{E}$;

$$\Lambda(\varphi) = \sup\left\{\int \varphi\, d\mu - J(\mu) : \mu \in \mathscr{M}_1^+(E)\right\} \tag{2.1}$$

is the logarithmic spectral radius of (P_t^φ):

$$\Lambda(\varphi) = \lim_{t\to\infty} \frac{1}{T}\log\|P_t^\varphi\|_{\mathrm{op}}, \tag{2.2}$$

where $\|\ \|_{\mathrm{op}}$ is the operator norm on $(C(E), \|\ \|_\infty)$. Furthermore, one has the following duality relation:

$$J(\mu) = \sup\left\{\int \varphi\, d\mu - \Lambda(\varphi) : \varphi \in C(E)\right\}. \tag{2.3}$$

There exist right- and left-hand principal eigenfunctions h^φ, $l^\varphi \in C^+(E)$ of

P_t^φ, that is,

$$P_t^\varphi h^\varphi = \exp(\Lambda(\varphi)t)h^\varphi, \qquad t \geq 0, \tag{2.4}$$

$$\int \pi(dy) l^\varphi(y) P_t^\varphi(y, dz) = \exp(\Lambda(\varphi)t) l^\varphi(z)\pi(dz); \tag{2.5}$$

h^φ, l^φ are unique if they are appropriately normed. We require

$$\int (h^\varphi)^2 \, d\pi = 1, \tag{2.6}$$

$$d\pi^\varphi \equiv l^\varphi h^\varphi \, d\pi \in \mathscr{M}_1^+(E). \tag{2.7}$$

$\Lambda(\varphi)$, h^φ, l^φ depend continuous on φ. For proofs of these facts, see [6], Section 2. From (2.5) and (2.7) it follows that π^φ is the stationary measure of the transition kernels

$$Q_t^\varphi(x, dy) \equiv \exp(-\Lambda(\varphi)t) \frac{1}{h^\varphi(x)} P_t^\varphi(x, dy) h^\varphi(y).$$

We write $\mathbb{Q}_x^\varphi$ for the corresponding Markov measure on $(\Omega, \mathscr{F})$. Q_t^φ has continuous strictly positive transition densities $q_t^\varphi(x, y)$ w.r.t. π, and therefore

$$\|q_t^\varphi(x, \cdot) - l^\varphi h^\varphi\|_\infty \to 0$$

exponentially fast, as $t \to \infty$, uniformly in x.

We set

$$g^\varphi(x, y) = l^\varphi(x) h^\varphi(x) \int_0^\infty (q_t^\varphi(x, y) - l^\varphi(y) h^\varphi(y)) \, dt$$

and

$$\bar{g}^\varphi(x, y) = g^\varphi(x, y) + g^\varphi(y, x).$$

If $\varphi = 0$, then we write just g and $\bar{g}$. G is the operator on $C(E)$ defined by $Gf(x) \equiv \int g(x, y) f(y) \pi(dy)$, and $G^* f(x) \equiv \int g(y, x) f(y) \pi(dy)$, $\overline{G} \equiv G + G^*$. Note that they are bounded operators on $C(E)$.

We will need some information about how these quantities behave for $\varphi \sim 0$. Let $f \in C(E)$ satisfy $\int f \, d\pi = 0$. We set $\Lambda(\varepsilon) \equiv \Lambda(\varepsilon f)$, $h^\varepsilon \equiv h^{\varepsilon f}$ and so forth. We also set

$$\Lambda_{T,x}(\varepsilon) \equiv \frac{1}{T} \log \mathbb{E}_x \exp\left(\varepsilon \int_0^T f(X_s) \, ds\right).$$

LEMMA 2.8. *There exists $K > 0$ such that for $0 < \varepsilon \leq 1$ and $T \geq 1$,*

$$\left|\Lambda_{T,x}(\varepsilon) - \frac{\varepsilon^2}{2}\langle f, \overline{G} f\rangle_\pi\right| \leq K\left(\frac{\varepsilon\|f\|_\infty}{T} + \frac{\varepsilon^2}{T}\|f\|_\infty^2 + \varepsilon^3\|f\|_\infty^3(1 + \exp(K\varepsilon\|f\|_\infty))\right).$$

PROOF. For $y \in E$ set

$$\Lambda_{T,x,y}(\varepsilon) \equiv \frac{1}{T} \log \mathbb{E}_x\left[\exp\left[\varepsilon \int_0^T f(X_s) \, ds\right] \middle| X_T = y\right].$$

It is enough to show that

$$(2.9)\qquad \left|\Lambda_{T,x,y}(\varepsilon) - \frac{\varepsilon^2}{2}\langle f, \overline{G}f\rangle_\pi\right| \le r_T(\varepsilon),$$

where

$$r_T(\varepsilon) \le K\left(\frac{\varepsilon}{T}\|f\|_\infty + \frac{\varepsilon^2}{T}\|f\|_\infty^2 + \varepsilon^3\|f\|_\infty^3(1 + \exp(K\varepsilon\|f\|_\infty))\right).$$

By Assumption 1.1, $\{P_t: t > 0\}$ is uniformly mixing in the sense that

$$P_t(x, dy) = p_t(x, y)\pi(dy), \qquad \|\log p_t\|_\infty < \infty \quad \text{for } t > 0.$$

By Lemma 2.5 of [6] we have

$$\|\log h^\varepsilon\|_\infty \le 2\varepsilon\|f\|_\infty + \log c_1,$$

where $c_t = \exp(\|\log p_t\|_\infty)$. (Note that c_t in [6] should be replaced by e^{c_t}.) The corresponding semigroup $\{Q_t^\varepsilon: t > 0\}$ is uniformly mixing with invariant distribution $\pi^\varepsilon(dx) = \rho^\varepsilon(x)\pi(dx)$, where $\rho^\varepsilon = h^\varepsilon l^\varepsilon$, and $Q_t^\varepsilon(x, dy) = q_t^\varepsilon(x, y)\pi(dy)$ with

$$\|\log q_1^\varepsilon(x, \cdot)\|_\infty \le 6\varepsilon\|f\|_\infty + 3\log c_1.$$

In particular we have the uniform exponential convergence estimate

$$\|q_t^\varepsilon(x, \cdot) - \rho^\varepsilon\|_\infty \le K_\varepsilon \exp(-\alpha_\varepsilon t), \qquad t \ge 1,$$

where

$$(2.10)\qquad K_\varepsilon = 2\exp(3c_1)\exp(6\varepsilon\|f\|_\infty), \quad \alpha_\varepsilon = \exp(-8\varepsilon\|f\|_\infty - 4c_1).$$

For these estimates, see the proof of Lemma 2.5(h) in [6].

We will prove Lemma 2.8 in several steps. Let

$$A_T = \int_0^T f(X_s)\, ds$$

and

$$\mathbb{E}_{x,y}^{\varepsilon,T}[\cdot] = \mathbb{E}_{\mathbb{Q}_x^\varepsilon}[\cdot \mid X_T = y].$$

Note that if F is bounded $\mathscr{F}_T$ measurable, then

$$\mathbb{E}_{x,y}^{\varepsilon,T}[F] = \frac{\mathbb{E}_{x,y}^{0,T}[Fe^{\varepsilon A_T}]}{\mathbb{E}_{x,y}^{0,T}[e^{\varepsilon A_T}]}.$$

A simple computation yields the next lemma.

LEMMA 2.11. *Set* $\phi_T(\varepsilon) = \mathbb{E}_{x,y}^{0,T}[\exp(\varepsilon A_T)]$. *Then*

$$\Lambda_{T,x,y}(\varepsilon) = \frac{1}{T}\log \phi_T(\varepsilon)$$

and

$$\Lambda'_{T,x,y}(\varepsilon) = \frac{d}{d\varepsilon}\Lambda_{T,x,y}(\varepsilon) = \frac{1}{T}\frac{\phi'_T(\varepsilon)}{\phi_T(\varepsilon)} = \frac{1}{T}\mathbb{E}^{\varepsilon,T}_{x,y}[A_T]$$

$$\Lambda''_{T,x,y}(\varepsilon) = \frac{1}{T}\left\{\frac{\phi''_T(\varepsilon)}{\phi_T(\varepsilon)} - \left(\frac{\phi'_T(\varepsilon)}{\phi_T(\varepsilon)}\right)^2\right\} = \frac{1}{T}\mathbb{E}^{\varepsilon,T}_{x,y}\Big[\big(A_T - \mathbb{E}^{\varepsilon,T}_{x,y}[A_T]\big)^2\Big]$$

$$\Lambda'''_{T,x,y}(\varepsilon) = \frac{1}{T}\left\{\frac{\phi'''_T(\varepsilon)}{\phi_T(\varepsilon)} - 3\frac{\phi''_T(\varepsilon)}{\phi_T(\varepsilon)}\frac{\phi'_T(\varepsilon)}{\phi_T(\varepsilon)} + 2\left(\frac{\phi'_T(\varepsilon)}{\phi_T(\varepsilon)}\right)^3\right\}$$
$$= \frac{1}{T}\mathbb{E}^{\varepsilon,T}_{x,y}\Big[\big(A_T - \mathbb{E}^{\varepsilon,T}_{x,y}[A_T]\big)^3\Big].$$

LEMMA 2.12. *There exists a constant $K > 0$ such that for $T \geq 1$,*

$$|\Lambda'_{T,x,y}(0)| \leq \frac{K}{T}\|f\|_\infty, \tag{2.13}$$

$$|\Lambda''_{T,x,y}(0) - \langle f, \overline{G}f\rangle_\pi| \leq \frac{K}{T}\|f\|^2_\infty. \tag{2.14}$$

PROOF. By Lemma 2.11,

$$|\Lambda'_{T,x,y}(0)| = \frac{1}{T}|\mathbb{E}^{0,T}_{x,y}[A_T]| \leq \frac{\|f\|_\infty}{T} + \frac{1}{T}|\mathbb{E}^{0,T}_{x,y}[A_{T-1}]|.$$

Next, let $\{P^*_t : t > 0\}$ be the π-adjoint of $\{P_t : t > 0\}$. Then

$$\begin{aligned}|p_T(x,y)\mathbb{E}^{0,T}_{x,y}[A_{T-1}]| &\leq \int_0^{T-1}\|P_s(fp_{T-s}(\cdot,y))\|_\infty\,ds\\ &\leq \int_0^{T-1}|\langle f, p_{T-s}(\cdot,y)\rangle_\pi|\,ds + k_1\int_0^{T-1}e^{-\alpha_0 s}\,ds\|f\|_\infty\\ &\leq \int_0^{T-1}|P^*_{T-s}f(y)|\,ds + k_2\|f\|_\infty \leq k_3\|f\|_\infty,\end{aligned}$$

where we have used (2.10) and the fact that (2.10) holds for $\{P^*_t : t > 0\}$ with $\varepsilon = 0$. Finally we get (2.13) since $p_T(x,y) \geq 1/c_1$ for $T \geq 1$.

The proof of (2.14) goes along the same lines:

$$\Lambda''_{T,x,y}(0) = \frac{1}{T}\mathbb{E}^{0,T}_{x,y}[A_T^2] - \frac{1}{T}\mathbb{E}^{0,T}_{x,y}[A_T]^2.$$

By (2.13) we know

$$\frac{1}{T}\mathbb{E}^{0,T}_{x,y}[A_T]^2 \leq \frac{K^2}{T}\|f\|^2_\infty$$

and

$$\Big|\mathbb{E}^{0,T-1}_{x,X_{T-1}}[A_{T-1}]\Big| \leq k_3\|f\|_\infty.$$

This together with the Markov property imply

$$\left|\mathbb{E}_{x,y}^{0,T}[A_T^2] - \mathbb{E}_{x,y}^{0,T}[A_{T-1}^2]\right| \le k_4\|f\|_\infty^2.$$

Next we have

$$p_T(x, y)\mathbb{E}_{x,y}^{0,T}[A_{T-1}^2] = 2\int_0^{T-1} ds \int_s^{T-1} dt\, P_s(fP_{t-s}(fp_{T-t}(\cdot, y)))(x).$$

By the first part of the proof,

$$\sup_{0\le s\le T-1} \int_s^{T-1} \|P_{t-s}(fp_{T-t}(\cdot, y))\|_\infty\, dt \le k_5\|f\|_\infty,$$

this shows that

$$\left|p_T(x, y)\mathbb{E}_{x,y}^{0,T}[A_{T-1}^2] - 2\int_0^{T-1} ds \int_s^{T-1} dt\langle f, P_{T-s}(fp_{T-t}(\cdot, y))\rangle_\pi\right|$$
$$\le k_6\int_0^{T-1} e^{-\alpha_0 s}\, ds\|f\|_\infty^2 \le k_7\|f\|_\infty^2.$$

Also we have

$$\langle f, P_{T-s}(fp_{T-t}(\cdot, y))\rangle_\pi = P_{T-t}^*(fP_{t-s}^*(f))(y)$$

and

$$\left|\int_0^{T-1} ds \int_s^{T-1} dt\{P_{T-t}^*(fP_{t-s}^*(f))(y) - \langle f, P_{t-s}^* f\rangle_\pi\}\right|$$
$$\le k_8\|f\|_\infty^2 \int_0^{T-1} ds \int_s^{T-1} \exp(-\alpha_0(T-t))\exp(-\alpha_0(t-s))\, dt \le k_9\|f\|_\infty^2.$$

Finally we have

$$\left|\int_0^{T-1} ds \int_s^{T-1} dt\langle f, P_{t-s}^* f\rangle_\pi - \int_0^{T-1} ds \int_0^\infty dt\langle f, P_t^* f\rangle_\pi\right| \le k_{10}\|f\|_\infty^2$$

with

$$\int_0^{T-1} ds \int_0^\infty dt\langle f, P_t^* f\rangle_\pi = (T-1)\langle f, G^*f\rangle_\pi = (T-1)\langle f, Gf\rangle_\pi.$$

Putting things together we get

$$|p_T(x, y)\Lambda''_{T,x,y}(0) - 2\langle f, Gf\rangle_\pi| \le \frac{k_{11}}{T}\|f\|_\infty^2,$$

which implies (2.14) since $|p_T(x, y) - 1| \le K_0 e^{-\alpha_0 T}$. □

LEMMA 2.15. *There is a constant $K > 0$ such that*

$$|\Lambda'''_{T,x,y}(\varepsilon)| \le K(1 + \exp(\varepsilon\|f\|_\infty K))\|f\|_\infty^3.$$

PROOF. Basically we can use the same ideas as above with K_ε and α_ε

instead of K_0 and α_0. In particular one can first show that

$$|\Lambda'_{T,x,y}(\varepsilon)| \le k_1(1+\exp(\varepsilon\|f\|_\infty k_1))\|f\|_\infty,$$
$$|\Lambda''_{T,x,y}(\varepsilon)| \le k_2(1+\exp(\varepsilon\|f\|_\infty k_2))\|f\|_\infty^2$$

and it is enough to prove that

$$\frac{1}{T}\left|\mathbb{E}_{x,y}^{\varepsilon,T}\left[(A_{T-1}^\varepsilon)^3\right]\right| \le k_3(1+\exp(\varepsilon\|f\|_\infty k_3))\|f\|_\infty^3,$$

where $A_{T-1}^\varepsilon = \int_0^{T-1} f^\varepsilon(X_s)\,ds$ with $f^\varepsilon = f - \langle f\rangle_{\pi^\varepsilon}$. Note first that

$$q_T^\varepsilon(x,y)\mathbb{E}_{x,y}^{\varepsilon,T}\left[(A_{T-1}^\varepsilon)^3\right]$$
$$= 6\int_0^{T-1}ds\int_s^{T-1}dt\int_t^{T-1}du\, Q_s^\varepsilon(f^\varepsilon Q_{t-s}^\varepsilon(f^\varepsilon Q_{u-t}^\varepsilon(f^\varepsilon q_{T-u}^\varepsilon(\cdot,y))))(x).$$

As above we may replace Q_s^ε by π^ε, since by induction

$$\frac{1}{T}\int_s^{T-1}dt\int_t^{T-1}du\|Q_{t-s}^\varepsilon(f^\varepsilon Q_{u-t}^\varepsilon(f^\varepsilon q_{T-u}^\varepsilon(\cdot,y)))\|_\infty \le k_4(1+\exp(\varepsilon\|f\|_\infty k_4))\|f\|_\infty^2.$$

Next we have

$$\langle f^\varepsilon, Q_{t-s}^\varepsilon(f^\varepsilon Q_{u-t}^\varepsilon(f^\varepsilon q^\varepsilon(\cdot,y)))\rangle_{\pi^\varepsilon} = Q_{T-u}^{\varepsilon,*}(f^\varepsilon Q_{u-t}^{\varepsilon,*}(f^\varepsilon Q_{t-s}^{\varepsilon,*}(f^\varepsilon)))(y),$$

where $\{Q_t^{\varepsilon,*}: t>0\}$, the π^ε adjoint of $\{Q_t^\varepsilon: t>0\}$, is also uniformly mixing and satisfies Assumption 1.1. Moreover,

$$\|Q_{T-u}^{\varepsilon,*}(f^\varepsilon Q_{u-t}^{\varepsilon,*}(f^\varepsilon Q_{t-s}^{\varepsilon,*}(f^\varepsilon)))\|_\infty$$
$$\le \|Q_{T-u}^{\varepsilon,*}(f^\varepsilon Q_{u-t}^{\varepsilon,*}(f^\varepsilon Q_{t-s}^{\varepsilon,*}(f^\varepsilon) - \langle f^\varepsilon, Q_{t-s}^{\varepsilon,*}(f^\varepsilon)\rangle_{\pi^\varepsilon}))\|_\infty$$
$$+ \|Q_{T-u}^{\varepsilon,*}(f^\varepsilon)\langle f^\varepsilon, Q_{t-s}^{\varepsilon,*}(f^\varepsilon)\rangle_{\pi^\varepsilon}\|_\infty$$
$$\le k_6K_\varepsilon\{\exp(-\alpha_\varepsilon(u-t))\exp(-\alpha_\varepsilon(t-s))$$
$$+\exp(-\alpha_\varepsilon(T-u))\exp(-\alpha_\varepsilon(t-s))\}\|f^\varepsilon\|_\infty^3.$$

However, this implies

$$\frac{1}{T}\int_0^{T-1}ds\int_s^{T-1}dt\int_t^{T-1}du|\langle f^\varepsilon, Q_{t-s}^\varepsilon(f^\varepsilon Q_{u-t}^\varepsilon(f^\varepsilon q_{T-u}^\varepsilon(\cdot,y)))\rangle_{\pi^\varepsilon}|$$
$$\le k_7(1+\exp(\varepsilon\|f\|_\infty k_7))\|f^\varepsilon\|_\infty^3$$

and the lemma is proved. □

PROOF OF (2.9). By the mean value theorem we know that there exist $\tilde\varepsilon \in [-\varepsilon,\varepsilon]$ such that

$$\Lambda_{T,x,y}(\varepsilon) = \varepsilon\Lambda'_{T,x,y}(0) + \frac{\varepsilon^2}{2}\Lambda''_{T,x,y}(0) + \frac{\varepsilon^3}{3!}\Lambda'''_{T,x,y}(\tilde\varepsilon)$$

and the result follows from the above lemmas. □

COROLLARY 2.16. *We have that*

$$\Lambda(\varepsilon) = \frac{\varepsilon^2}{2}\langle f, \overline{G}f\rangle_\pi + O(\varepsilon^3).$$

LEMMA 2.17. (a) $h^\varepsilon = 1 + \varepsilon Gf + r_1(\varepsilon)$.
(b) $l^\varepsilon = 1 + \varepsilon G^* f + r_2(\varepsilon)$.
(c) $\pi^\varepsilon = (1 + \varepsilon \overline{G}f + r_3(\varepsilon))\pi$.
(d) $J(\pi^\varepsilon) = \varepsilon^2/2\langle f, \overline{G}f\rangle_\pi + o(\varepsilon^2)$, *where* $\|r_i(\varepsilon)\|_\infty = o(\varepsilon)\|f_i\|_\infty$.

PROOF. First note that by (2.4) and (2.5), h^ε and $l^\varepsilon \in C_b(E; \mathbb{R}^+)$ are $L^2(\mu)$-normalized positive eigenfunctions:

$$(L + \varepsilon f)h^\varepsilon = \Lambda(\varepsilon)h^\varepsilon, \quad (L^* + \varepsilon f)l^\varepsilon = \Lambda(\varepsilon)l^\varepsilon. \tag{2.18}$$

Next, by continuity,

$$h^\varepsilon = 1 + o(1), \quad l^\varepsilon = 1 + o(1). \tag{2.19}$$

Also since $\varepsilon\langle f, h^\varepsilon\rangle_\pi - \Lambda(\varepsilon)\langle h^\varepsilon\rangle_\pi = 0$, we have from (2.18),

$$L(h^\varepsilon - \langle h^\varepsilon\rangle_\pi - \varepsilon Gf) = \varepsilon(f(1 - h^\varepsilon) - \langle f, 1 - h^\varepsilon\rangle_\pi) + \Lambda(\varepsilon)(h^\varepsilon - \langle h^\varepsilon\rangle_\pi).$$

This yields

$$h^\varepsilon - \langle h^\varepsilon\rangle_\pi - \varepsilon Gf = \varepsilon G(f(1 - h^\varepsilon) - \langle f, 1 - h^\varepsilon\rangle_\pi) + \Lambda(\varepsilon)G(h^\varepsilon - \langle h^\varepsilon\rangle_\pi) \equiv q(\varepsilon).$$

From Corollary 2.16 we know that

$$\Lambda(\varepsilon) = \frac{\varepsilon^2}{2}\langle f, (G + G^*)f\rangle_\pi + o(\varepsilon^2). \tag{2.20}$$

Also, since $\{P_t : t > 0\}$ and $\{P_t^* : t > 0\}$ are uniformly mixing, G and G^* are bounded:

$$\|Gf\|_\infty \le K\|f\|_\infty, \qquad \|G^*f\|_\infty \le K\|f\|_\infty. \tag{2.21}$$

Now by (2.18), (2.19) and (2.21) we see that $\|q(\varepsilon)\|_\infty = o(\varepsilon)\|f\|_\infty$. From this and $\langle q(\varepsilon)\rangle_\pi = 0$ we also get

$$1 = \langle (h^\varepsilon)^2\rangle_\pi = \langle h^\varepsilon\rangle_\pi^2 + \varepsilon^2\langle (Gf)^2\rangle_\pi + o(\varepsilon^2);$$

thus,

$$\langle h^\varepsilon\rangle_\pi = 1 + o(\varepsilon)$$

and

$$h^\varepsilon = \langle h^\varepsilon\rangle_\pi + \varepsilon Gf + q(\varepsilon) = 1 + \varepsilon Gf + r_1(\varepsilon),$$

where $r_1(\varepsilon)$ has the required property. Using a similar argument one shows

$$l^\varepsilon = 1 + \varepsilon G^* f + r_2(\varepsilon).$$

Finally we have $d\pi^\varepsilon = l^\varepsilon h^\varepsilon$ and $d\pi = (1 + \varepsilon(Gf + G^*f) + r_3(\varepsilon))\,d\pi$ with

$$J(\mu^\varepsilon) = \langle -\frac{Lh^\varepsilon}{h^\varepsilon}\rangle_{\mu^\varepsilon} = \langle l^\varepsilon, (-L)h^\varepsilon\rangle_{\mu^\varepsilon} = \varepsilon\langle f\rangle_{\mu^\varepsilon} - \Lambda(\varepsilon);$$

compare [6]. This yields

$$\begin{aligned}J(\mu^{\varepsilon}) &= \varepsilon\langle f\rangle_{\mu^{\varepsilon}} - \Lambda(\varepsilon)\\ &= \varepsilon\langle f, 1+\varepsilon(G+G^*)f + r_3(\varepsilon)\rangle_{\pi} - \frac{\varepsilon^2}{2}\langle f, (G+G^*)f\rangle_{\pi} + o(\varepsilon^2)\\ &= \frac{\varepsilon^2}{2}\langle f, (G+G^*)f\rangle_{\pi} + o(\varepsilon^2). \qquad \square\end{aligned}$$

Let $F: \mathscr{M}_1^+(E) \to \mathbb{R}$ be smooth in the sense of Assumption 1.5. If $\mu \in \mathscr{M}_1^+$, the first derivative of F at μ is denoted by $DF(\mu) \in H_a$. We define

$$\varphi^{\mu}(x) \equiv \langle DF(\mu), \delta_x\rangle_a.$$

As $E \ni x \to \delta_x \in H_a$ is continuous, we have $\varphi^{\mu} \in C(E)$. By a slight abuse of notation, we write $\Lambda(\mu)$, h^{μ}, π^{μ}, and so forth, instead of $\Lambda(\varphi^{\mu})$, $h^{\varphi^{\mu}}$, $\pi^{\varphi^{\mu}}, \dots$.

LEMMA 2.22. *If $\mu \in K_F$, then $\pi^{\mu} = \mu$.*

PROOF. If $\varphi \in C(E)$, let J^{φ} be the rate function corresponding to $(Q_t^{\varphi})_{t\geq 0}$. Using (2.3), one sees

$$J^{\varphi}(\mu) = J(\mu) - \int \varphi\, d\mu + \Lambda(\varphi).$$

It is well known that $J^{\varphi}(\mu) = 0$ if and only if $\mu = \pi^{\varphi}$.

If $F(\mu) - J(\mu) = b_F$, then by the convexity of J, we have

$$\int \varphi^{\mu}\, d\mu - J(\mu) = \sup_{\nu \in \mathscr{M}_1^+} \left(\int \varphi^{\mu}\, d\nu - J(\nu)\right).$$

Therefore, the function $\mathscr{M}_1^+ \ni \nu \to J^{\mu}(\nu)$ is minimal at μ and so $\mu = \pi^{\mu}$. $\square$

Another important property of the elements in K_F is that $F - J$ has nonpositive curvature at points in K_F. Because J is not differentiable, the formulation needs some care. An appropriate formulation is given in the following proposition.

PROPOSITION 2.23. *Let $\kappa \in K_F$. Then for any $f \in C(E)$,*

$$\langle f, \overline{G}^{\kappa} f\rangle_{\pi} \geq D^2F(\kappa)\big[(\overline{G}^{\kappa} f)\pi, (\overline{G}^{\kappa} f)\pi\big]. \tag{2.24}$$

Here, if $g \in C(E)$, then $g\pi$ is the measure $g(x)\pi(dx)$. $D^2F(\kappa)$ is interpreted as a bilinear form on $\mathscr{M}(E) \subset H_a$.

PROOF. It is convenient to write everything in terms of the densities w.r.t. $\kappa(=\pi^{\kappa})$: Let $\hat{q}_t^{\kappa}(x, y)$ be the densities of Q_t^{κ} w.r.t. κ; that is, $\hat{q}_t^{\kappa}(x, y) = q_t^{\kappa}(x, y)/l^{\kappa}(y)h^{\kappa}(y)$, and let

$$\begin{aligned}\hat{g}(x, y) &\equiv \int_0^{\infty}(q_t^{\kappa}(x, y) - 1)\,dt + \int_0^{\infty}(q_t^{\kappa}(y, x) - 1)\,dt\\ &= \bar{g}^{\kappa}(x, y)/l^{\kappa}(x)l^{\kappa}(y)h^{\kappa}(x)h^{\kappa}(y).\end{aligned}$$

Then

$$\langle f, \overline{G}^{\kappa} f\rangle_{\pi} = \langle f, \hat{G}^{\kappa} f\rangle_{\kappa},$$

where, by an abuse of notation,

$$\hat{G}^{\kappa} f = \int \hat{g}^{\kappa}(x, y) f(y) \kappa(dy).$$

Furthermore

$$(\overline{G}^{\kappa} f)\pi = (\hat{G}^{\kappa} f)\kappa.$$

Replacing F by the function

$$F^{\kappa}(x) = F(x) - F(\kappa) - \langle DF(\kappa), x - \kappa\rangle_a, \tag{2.25}$$

we see that, for the sake of proving the proposition, we may assume that

$$\kappa = \pi \in K_F.$$

From Corollary 2.16 and Lemma 2.17, Proposition 2.23 follows. □

We can now give a precise formulation of our nondegeneracy assumption:

ASSUMPTION 2.26. For any $f \in C(E)$ with $\overline{G}^{\kappa} f \not\equiv 0$,

$$\langle f, \overline{G}^{\kappa} f\rangle_{\pi} > D^2F(\kappa)\left[(\overline{G}^{\kappa} f)\pi, (\overline{G}^{\kappa} f)\pi\right]. \tag{2.27}$$

We will interpret the quadratic form $f \to \langle f, \overline{G}^{\kappa} f\rangle_{\pi}$ as one coming from a symmetric positive trace class operator on H_a. Note that

$$\langle f, \overline{G}^{\kappa} f\rangle_{\pi} = \lim_{T\to\infty} \operatorname{var}_{\mathbb{Q}^{\kappa}}\left(\int f\, dl_T\right), \tag{2.28}$$

where $l_T \equiv \sqrt{T}(L_T - \kappa)$. If $f \in C(E)$, we would like to write the mapping $u \to \int f\, d\mu$ in the form $\langle \hat{f}, \mu\rangle_a$ for some $\hat{f} \in H_a$. This is not always possible, as $\int f\, d\mu$ may not be continuous in μ on H_a. However, $\hat{\psi}_n$ is certainly well defined and just

$$\hat{\psi}_n = \frac{1}{a_n} \psi_n \pi.$$

We write $C_0(E)$ for the set of finite linear combinations of the ψ_n. Then, if $f \in C_0(E)$, $\hat{f}$ is well defined. We put

$$l_n = \frac{1}{\sqrt{a_n}} \psi_n \pi, \qquad n \in \mathbb{N},$$

which obviously is a complete orthonormal system in H_a and we define the bounded linear operator S_{κ} on H_a by

$$S_{\kappa} l_n = \sum_m s_{\kappa}(n, m) l_m, \tag{2.29}$$

where $s_{\kappa}(n, m) = \sqrt{a_n a_m}\, \langle \psi_n, \overline{G}^{\kappa} \psi_m\rangle_{\pi}$.

LEMMA 2.30. *Let* $\kappa \in K_F$. *Then*:

(a) S_κ *is a symmetric, positive semidefinite trace class operator on* H_a.
(b) *If* $f \in C_0(E)$, *then* $S_\kappa \hat{f} = (\overline{G}^\kappa f)\pi$ *and therefore* $\langle f, \overline{G}^\kappa f\rangle_\pi = \langle \hat{f}, S_\kappa \hat{f}\rangle_a$.
(c) $\langle x, S_\kappa x\rangle_a \geq \langle S_\kappa x, D^2F(\kappa)S_\kappa x\rangle_a$ *for all* $x \in H_a$.

PROOF. The lemma is obvious from the definition and Proposition 2.23. □

If $x \in H_a$, let $\Gamma(x) = \inf\{|y|_a^2 : x = \sqrt{S}y\}$, where we drop κ in the notation and where $\inf\varnothing = \infty$.

$$H_\Gamma \equiv \{x \in H : \Gamma(x) < \infty\}$$

is a linear subspace of H_a and Γ is a Hilbert norm on H_Γ. $\Gamma: H_a \to [0, \infty]$ is convex, lower semicontinuous and has compact level sets, that is, $\{x: \Gamma(x) \leq c\}$ is compact for $c \in (0, \infty)$. Obviously, SH is a dense subspace in (H_Γ, Γ). From Lemma 2.30(c), we therefore obtain

$$\Gamma(x) \geq \langle x, D^2F(\kappa)x\rangle_a \tag{2.31}$$

for all $x \in H_\Gamma$, and therefore also for $x \in H_a$.

LEMMA 2.32. *Assume* (2.27). *Then* $\Gamma(x) > \langle x, D^2F(\kappa)x\rangle_a$ *for all* $x \in H_\Gamma$ *with* $\Gamma(x) \neq 0$.

PROOF. Assume that for some $x \in H_\Gamma$,

$$\Gamma(x) = \langle x, D^2F(\kappa)x\rangle_a = 1.$$

We claim that for some $\alpha \in \mathbb{R} \setminus \{0\}$,

$$x = \alpha SpD^2F(\kappa)x, \tag{2.33}$$

where p is the projection of H_a on the closure $\overline{H}_\Gamma$ of H_Γ in H_a.

To prove this, let $y \in H_\Gamma$ satisfy

$$\langle y, D^2F(\kappa)x\rangle_a = 0.$$

We put $x_t \equiv (x + ty)/[1 + t^2\langle D^2F(\kappa)y, y\rangle_a]^{1/2}$, t in a neighborhood of 0. Then

$$\langle x_t, D^2F(\kappa)x_t\rangle_a = 1$$

and therefore $\Gamma(x_t) \geq 1$. It follows that $\Gamma(x, y) = 0$, where $\Gamma(x, y)$ is the inner product in H_Γ. Using this, (2.33) follows. We now put

$$g(\xi) \equiv \alpha\langle pD^2F(\kappa)x, \delta_\xi\rangle_a,$$

which is in $C(E)$ and satisfies $\hat{g} = \alpha pD^2F(\kappa)x$. Therefore,

$$x = S\hat{g}$$

and using Lemma 2.30, we have

$$\langle g, \overline{G}^\kappa g\rangle_\pi = \langle \overline{G}^\kappa g\pi, D^2F(\kappa)\overline{G}^\kappa g\pi\rangle_a = D^2F(\kappa)\left[\overline{G}^\kappa g\pi, \overline{G}^\kappa g\pi\right],$$

which contradicts (2.27). □

3. Gaussian behavior near π. It will suffice to discuss the limiting behavior of the law of L_T near π. As S is a trace class operator, there exists a unique centered Gaussian measure γ on H_a satisfying

$$\int \langle x, \xi \rangle_a \langle y, \xi \rangle_a \gamma(d\xi) = \langle x, Sy \rangle_a.$$

PROPOSITION 3.1. *(l_T, X_T) converges weakly to $\gamma \otimes \pi$ for $T \to \infty$ on $H_a \times E$.*

PROOF. $l_T = 1/\sqrt{T}(\int_0^T \delta_{X_s}\, ds - T\pi)$, and $E \ni \xi \to \delta_\xi \in H_a$ is a bounded continuous function. The proposition then follows by standard central limit theorems for Markov processes. □

PROPOSITION 3.2. *If $A \subset H_a$ is closed, then*

$$\limsup_{c \to \infty} \sup_{t,T} \left\{ \frac{1}{t^2} \log \mathbb{P}_x(l_T \in tA) : c \le t \le \sqrt{T}/c \right\} \le -\Gamma(A),$$

where $\Gamma(A) = \inf_{x \in A} \Gamma(x)$.

PROOF FOR A COMPACT. Let A be compact, and satisfy $\Gamma(A) < \infty$. We may assume $\Gamma(A) > 0$. If $0 < \varepsilon < \Gamma(A)$, then

$$\{x \in H_a : \Gamma(x) > \Gamma(A) - \varepsilon\}$$

is open and contains A. As Γ is lower semicontinuous, we may cover A with finitely many balls

$$U_i = B_{r_i}(x_i) \equiv \{y \in H_a : \|y - x_i\|_a < r_i\}, \qquad 1 \le i \le m,$$

with

$$\Gamma(U_i) > \Gamma(A) - \frac{\varepsilon}{2}.$$

Let

$$C_i \equiv \left\{x : \Gamma(x) \le \Gamma(U_i) - \frac{\varepsilon}{2}\right\}, \qquad 1 \le i \le m,$$

which is compact and convex. Therefore, there exists $y_i \in H_a$ with

$$U_i \subset \{x : \langle x, y_i \rangle_a > 1\} \subset \left\{x : \Gamma(x) > \Gamma(U_i) - \frac{\varepsilon}{2}\right\}.$$

By continuity, we may assume that $y_i = \hat{f}_i$ with $f_i \in C_0(E)$, and therefore

$$\{\mu \in \mathscr{M} : \langle \mu, y_i \rangle_a > 1\} = \left\{\mu \in \mathscr{M} : \int f_i\, d\mu > 1\right\}.$$

From this we get

$$\mathbb{P}_x(l_T \in tA) \le \sum_{i=1}^m \mathbb{P}_x\left(\int f_i\, dl_T > t\right).$$

By Lemma 2.8 and the standard exponential estimates, this yields

$$\limsup_{c\to\infty} \sup\left\{\frac{1}{t^2}\log\sup_x \mathbb{P}_x(l_T \in tA): c \le t \le \frac{\sqrt{T}}{c}\right\} \le -\frac{1}{2}\min_{1\le i\le m}\langle f_i, \overline{G}f_i\rangle_\pi^{-1}.$$

Using

$$\{x: \langle \hat{f}_i, x\rangle_a > 1\} \subset \{x: \Gamma(x) > \Gamma(U_i) - \varepsilon\} \subset \{x: \Gamma(x) - \varepsilon\}$$

and Lemma 2.30, this proves the claim in the case where A is compact and $\Gamma(A) < \infty$. The case $\Gamma(A) = \infty$ follows by an obvious modification (replacing the condition $\Gamma(U_i) > \Gamma(A) - \varepsilon/2$ by $\Gamma(U_i) > 1/\varepsilon$, etc.).

It remains to consider the case where A is only closed. This needs some preparation. Let $b = (b_n)$ be a sequence of strictly positive real numbers, satisfying $b_n \to 0$,

$$\lim_{n\to\infty} b_n/a_n = \infty \tag{3.3}$$

and

$$\sum_n b_n \|\psi_n\|_\infty^2 = 1.$$

The Hilbert space H_b is a subspace of H_a, and by (3.3), the imbedding $H_b \subset H_a$ is compact.

LEMMA 3.4.

$$\varrho(b) \equiv -\limsup_{c\to\infty}\sup_{t,T}\left\{\frac{1}{t^2}\log\sup_x \mathbb{P}_x(\|l_T\|_b > t): c \le t \le \frac{\sqrt{T}}{c}\right\} > 0.$$

We prove the lemma in several steps.

LEMMA 3.5. *Let* $\{f_n, n \in \mathbb{Z}^+\} \subseteq C(E)$ *satisfy*

$$\sup_n \|f_n - \langle f_n\rangle_\pi\|_\infty \equiv M < \infty \tag{3.6}$$

and set

$$\Lambda_\pi(f) \equiv \log\int_E \exp(f - \langle f\rangle_\pi)\, d\pi.$$

Then

$$\sup\{\Lambda_\pi(\varepsilon f_n) \vee \Lambda_\pi(-\varepsilon f_n): n \in \mathbb{Z}^+\} \le \frac{\varepsilon^2}{2}L + \varepsilon^3 K(\varepsilon),$$

where $L = \sup_n \|f_n - \langle f_n\rangle_\pi\|_{L^2(\pi)}^2 \le M^2$ *and* $K(\varepsilon) = 8M^3 e^{2\varepsilon M}/6$.

PROOF. Write $\phi_n(\varepsilon) = \Lambda_\pi(\varepsilon f_n)$ and $\bar{f}_n = f_n - \langle f_n\rangle_\pi$. Then $\phi_n'(0) = 0$,

$$\phi_n''(0) = \int_E (\bar{f}_n)^2\, d\pi \le L$$

and

$$|\phi_n'''(\varepsilon)| = \left|\frac{\int_E (\bar{f}_n^\varepsilon)^3 e^{\varepsilon\bar{f}_n}\, d\pi}{\int_E e^{\varepsilon\bar{f}_n}\, d\pi}\right| \le 8M^3 e^{2\varepsilon M} = 3!K(\varepsilon),$$

where

$$\bar{f}_n^{\varepsilon} = \bar{f}_n - \frac{\langle \bar{f}_n, e^{\varepsilon \bar{f}_n} \rangle_\pi}{\langle e^{\varepsilon \bar{f}_n} \rangle_\pi}.$$

Now the result follows from the mean value theorem. □

LEMMA 3.7. *There exists $2 < \beta < \infty$ such that*

$$\Lambda_T(f) \equiv \frac{1}{T} \log \mathbb{E}_\pi \left[\exp\left(\int_0^T f(X_s)\, ds \right) \right] \le \frac{1}{\beta} \Lambda_\pi(\beta f), \qquad f \in C(E).$$

PROOF. By Hölder's inequality we have

$$T\Lambda_T(f) \le \frac{[T]\Lambda_{[T]}(2f)}{2} + \frac{(T - [T])\Lambda_{T-[T]}(2f)}{2}.$$

Note that $\{P_t : t > 0\}$ is π-hypercontractive (cf. [9]). Thus by Jensen's inequality,

$$\begin{aligned}
[T]\Lambda_{[T]}(2f) &= \log \mathbb{E}_\pi \left[\exp\left(\int_0^1 \left(\sum_{k=0}^{[T]-1} 2f(X_{s+k}) \right) ds \right) \right] \\
&\le \log \int_0^1 \mathbb{E}_\pi \left[\exp\left(\sum_{k=0}^{[T]-1} 2f(X_{s+k}) \right) \right] ds \\
&= \log \mathbb{E}_\pi \left[\prod_{k=0}^{[T]-1} \exp(2f(X_k)) \right] \\
&\le \frac{[T]}{\beta'} \Lambda_\pi(2\beta' f)
\end{aligned}$$

for some $1 < \beta' < \infty$ (cf. [9]). On the other hand, again by Jensen's inequality we have

$$\begin{aligned}
(T - [T])\Lambda_{T-[T]}(2f) &= \log \mathbb{E}_\pi \left[\exp\left(\frac{1}{T - [T]} \int_0^{T-[T]} (T - [T]) 2f(X_s)\, ds \right) \right] \\
&\le \Lambda_\pi(2(T - [T])f) \le \frac{T - [T]}{\beta'} \Lambda_\pi(2\beta' f)
\end{aligned}$$

since $\beta'/(T - [T]) > 1$. This proves the lemma. □

Set $b_n' = b_n \|\psi_n\|_\infty^2$, $\psi_n' = \psi_n / \|\psi_n\|_\infty$ and $\bar{\psi}_n' = \psi_n' - \langle \psi_n' \rangle_\pi$. Then

$$\|m\|_b \equiv \left(\sum_n b_n' \langle \psi_n' \rangle_m^2 \right)^{1/2}.$$

LEMMA 3.8. *Under the above assumptions,*

$$\mathbb{E}_\pi[\exp(\varepsilon T \|L_T - \pi\|_b)] \le \exp\left[T\left(\frac{\varepsilon^2}{2} L' + \varepsilon^3 K'(\varepsilon) \right) + 1 \right],$$

where $L' = 4\beta L$ *and* $K'(\varepsilon) = 8\beta^2 K(\varepsilon)$. *In particular, for each* $t > 0$ *we have*

$$\text{(3.9)} \qquad \frac{1}{Tt^2}\log \mathbb{P}_\pi(\|L_T - \pi\|_b \geq t) \leq -\frac{1}{2L'} + \frac{tK'(t/L')}{(L')^3} + \frac{1}{t^2 T}$$

and

$$\text{(3.10)} \qquad \begin{aligned}&\limsup_{c\to\infty} \sup_{T,t} \left\{ \frac{1}{t^2}\log \mathbb{P}_\pi\left(T^{1/2}\|L_T - \pi\|_b > t\right): c \leq t \leq \frac{T^{1/2}}{c} \right\} \\ &\quad \leq -\frac{1}{2L'} < 0.\end{aligned}$$

PROOF. Note that $t \to \Psi(t) \equiv e^{t^{1/2}}$ is convex on $[1, \infty)$. Thus, by Jensen's inequality we have

$$\begin{aligned}\mathbb{E}_\pi\left[\exp(\varepsilon T\|L_T - \pi\|_b)\right] &= \mathbb{E}_\pi\left[\exp(\|\varepsilon T(L_T - \pi)\|_b)\right] \\ &= \mathbb{E}_\pi\left[\Psi\left(\sum_n b'_n \varepsilon^2 T^2 \langle \bar{\psi}'_n\rangle_{L_T}^2\right)\right] \\ &\leq \mathbb{E}_\pi\left[\Psi\left(\sum_n b'_n \varepsilon^2 T^2 \langle \bar{\psi}'_n\rangle_{L_T}^2 \vee 1\right)\right] \\ &\leq \sum_n b'_n \mathbb{E}_\pi\left[\Psi\left(\varepsilon^2 T^2 \langle \bar{\psi}'_n\rangle_{L_T}^2 \vee 1\right)\right] \\ &\leq \sum_n b'_n \mathbb{E}_\pi\left[\exp\left(\varepsilon T|\langle \bar{\psi}'_n\rangle_{L_T}| + 1\right)\right] \\ &= e^1 \sum_n b'_n \mathbb{E}_\pi\left[\exp\left(\varepsilon T|\langle \bar{\psi}'_n\rangle_{L_T}|\right)\right].\end{aligned}$$

Using the above lemmas and Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}_\pi\left[\exp\left(\varepsilon T|\langle \bar{\psi}'_n\rangle_{L_T}|\right)\right] &\leq \mathbb{E}_\pi\left[\exp\left(2\varepsilon T\langle \bar{\psi}'_n\rangle_{L_T}^+\right)\right]^{1/2} \mathbb{E}_\pi\left[\exp\left(2\varepsilon T\langle \bar{\psi}'_n\rangle_{L_T}^-\right)\right]^{1/2} \\ &\leq \mathbb{E}_\pi\left[\exp\left(2\varepsilon T\langle \bar{\psi}'_n\rangle_{L_T}\right)\right]^{1/2} \mathbb{E}_\pi\left[\exp\left(-2\varepsilon T\langle \bar{\psi}'_n\rangle_{L_T}\right)\right]^{1/2} \\ &= \exp\left[T\left(\Lambda_T(2\varepsilon\bar{\psi}'_n) + \Lambda_T(-2\varepsilon\bar{\psi}'_n)\right)/2\right] \\ &\leq \exp\left[T\left(\Lambda_\pi(2\beta\varepsilon\bar{\psi}'_n) + \Lambda_\pi(-2\beta\varepsilon\bar{\psi}'_n)\right)/(2\beta)\right] \\ &\leq \exp\left[T\left(\tfrac{1}{2}\varepsilon^2 4\beta L + \varepsilon^3 8\beta^2 K(\varepsilon)\right)\right] \\ &= \exp\left[T\left(\tfrac{1}{2}\varepsilon^2 L' + \varepsilon^3 K'(\varepsilon)\right)\right].\end{aligned}$$

Finally note that for each $\varepsilon > 0$ we have

$$\begin{aligned}\mathbb{P}_\pi(\|L_T - \pi\|_b \geq t) &\leq \exp(-Tt\varepsilon)\mathbb{E}_\pi\left[\exp(\varepsilon T\|L_T - \pi\|_b)\right] \\ &\leq \exp\left[-T\left(t\varepsilon - \frac{\varepsilon^2}{2}L' - \varepsilon^3 K'(\varepsilon)\right) + 1\right].\end{aligned}$$

Choosing $\varepsilon = t/L'$ yields (3.9), and (3.10) follows from (3.9). □

PROOF OF LEMMA 3.4. By assumption there exists $R < \infty$ such that $P_1(x, dy) \le R\pi(dy)$, $x \in E$. Note that $\|m\|_b \le \|m\|_{\mathrm{var}}$ and since $\|L_T - L_T \circ \theta_1\|_{\mathrm{var}} \le 2/T$, where $L_T \circ \theta_1(\omega) = (1/T)\int_1^{T+1} \delta_{X_s(\omega)}\, ds$, we have

$$\begin{aligned}\mathbb{P}_x(\|L_T - \pi\|_b > t) &\le \mathbb{P}_x\left(\|L_T \circ \theta_1 - \pi\|_b > \left(t - \frac{2}{T}\right)\right)\\ &= \int_E P_1(x, dy)\mathbb{P}_y\left(\|L_T - \pi\|_b > \left(t - \frac{2}{T}\right)\right)\\ &\le R\mathbb{P}_\pi\left(\|L_T - \pi\|_b > \left(t - \frac{2}{T}\right)\right).\end{aligned}$$

From (3.10) we get

$$\limsup_{c\to\infty} \sup_{T,t} \left\{\frac{1}{t^2}\log\sup_{x\in E} \mathbb{P}_x\left(T^{1/2}\|L_T - \pi\|_b > t\right): c \le t \le \frac{T^{1/2}}{c}\right\} < 0. \quad \square \tag{3.11}$$

PROOF OF PROPOSITION 3.2 FOR CLOSED A. Let $D_t \equiv \{x \in H_a: \|x\|_b > t\}$. D_t^c is compact in H_a:

$$\begin{aligned}&\sup\left\{\frac{1}{t^2}\log\sup_x \mathbb{P}_x(l_T \in tA): c \le t \le \frac{\sqrt{T}}{c}\right\}\\ &\quad\le \frac{\log 2}{c^2} + \sup\left\{\frac{1}{t^2}\log\sup_x \mathbb{P}_x\left(l_T \in t(A \cap D_r^c)\right): c \le t \le \frac{\sqrt{T}}{c}\right\}\\ &\qquad \vee \sup\left\{\frac{r^2}{t^2}\log\sup_x \mathbb{P}_x(l_T \in D_t): rc \le t \le \frac{r\sqrt{T}}{c}\right\},\end{aligned}$$

for an arbitrary $r > 0$. Therefore,

$$\begin{aligned}\limsup_{c\to\infty}\sup_{t,T}\left\{\frac{1}{t^2}\log\sup_x \mathbb{P}_x(l_T \in A): c \le t \le \frac{\sqrt{T}}{c}\right\} &\le -\Gamma(A \cap D_r^c) \wedge r^2\varrho(b)\\ &\le -\Gamma(A) \wedge r^2\varrho(b).\end{aligned}$$

Finally, letting $r \to \infty$ gives the desired result. $\square$

4. Proof of the theorem. Besides Assumption 1.5, we assume that all elements $\kappa \in K_F$ satisfy (2.27). As remarked before, this implies that K_F is finite. In fact, if K_F is infinite, then there exists $\kappa \in K_F$ which is an accumulation point of other elements in K_F, and this κ clearly would not satisfy (2.27). By splitting $\mathbb{E}(\exp(TF(L_T)))$ into the contribution coming from small neighborhoods near the elements in K_F, we easily see that we may assume that K_F contains just one element, $K_F = \{\kappa\}$. If we introduce

$$\tilde{F}(\mu) = F(\mu) - F(\kappa) - \langle DF(\kappa), \mu - \kappa\rangle_a$$

and the Markovian measure $\tilde{\mathbb{P}}_x \equiv \mathbb{Q}_x^\kappa$ on $(\Omega, \mathscr{F})$, we have

$$\mathbb{E}_x(\exp(TF(L_T))) = \exp(Tb_F)h(x)\tilde{\mathbb{E}}_x\left(\exp\left(T\tilde{F}(L_T)\right)\frac{1}{h(X_T)}\right),$$

where $h = h^{\kappa}$. $\tilde{\mathbb{P}}$ has the same properties as $\mathbb{P}$ and the stationary measure is κ. Therefore, w.l.o.g., we may assume that $K_F = \{\pi\}$ and

$$F(\pi) = 0, \qquad DF(\pi) = 0,$$

but we have to investigate slightly more general expressions

$$\mathbb{E}_x\big(\exp(TF(L_T))\varphi(X_T)\big),$$

with $\varphi \in C(E)$.

If c_1, $c_2 > 0$, let

$$I_1(c_1, T) \equiv \mathbb{E}_x\big(\exp(TF(L_T))\varphi(X_T); \|l_T\|_a \le c_1\big),$$
$$I_2(c_1, c_2, T) \equiv \mathbb{E}_x\big(\exp(TF(L_T))\varphi(X_T); c_1 < \|l_T\|_a \le c_2\sqrt{T}\big),$$
$$I_3(c_2, T) \equiv \mathbb{E}_x\big(\exp(TF(L_T))\varphi(X_T); c_2\sqrt{T} \le \|l_T\|_a\big),$$

where $l_T = \sqrt{T}(L_T - \pi)$.

LEMMA 4.1. *$\lim_{T\to\infty} I_3(c_2, T) = 0$ for all $c_2 > 0$.*

PROOF. By the large deviation principle for L_T, we have

$$\limsup_{T\to\infty} \frac{1}{T}\log I_3(c_2, T) \le \sup\big(F(\mu) - J(\mu) : \|\mu\|_a \ge c_2\big) < 0. \qquad \square$$

LEMMA 4.2. *We have that*

$$\hat{I}_1(c_1) = \lim_{T\to\infty} I_1(c_1, T)$$

exists for all but countably many $c_1 > 0$, and

$$\lim_{c_1\to\infty} \hat{I}_1(c_1) = \big[\det(I - D^2F(\pi)\circ S)\big]^{-1/2}\langle\varphi\rangle_\pi.$$

PROOF. On $\|l_T\| \le c_1$, we have

$$TF(L_T) = \tfrac{1}{2}\langle D^2F(\pi)l_T, l_T\rangle_a + o(1).$$

Therefore, by Proposition 3.1, we have for all but countably many $c_1 > 0$,

$$\lim_{T\to\infty} I_1(c_1, T) = \int_{\{x:\|x\|_a\le c_1\}} \exp(\langle D^2F(\pi)x, x\rangle_a)\gamma(dx)\langle\varphi\rangle_\pi,$$

and therefore

$$\lim_{c_1\to\infty}\hat{I}_1(c_1) = \int\exp\big(\tfrac{1}{2}\langle D^2F(\pi)x, x\rangle_a\big)\gamma(dx)\langle\varphi\rangle_\pi$$
$$= \big[\det(I - D^2F(\pi)\circ S)\big]^{-1/2}\langle\varphi\rangle_\pi. \qquad \square$$

The most delicate part is the treatment of I_2.

LEMMA 4.3. *If $c_2 > 0$ is small enough, then*

$$\lim_{c_1\to\infty}\sup_T I_2(c_1, c_2, T) = 0.$$

PROOF. We first claim that if $\varepsilon > 0$ is small enough, then

$$\Gamma\Big(\big\{x \in H_a : \langle D^2F(\pi)x, x\rangle_a + \varepsilon\|x\|_a^2 \geq 1\big\}\Big) > 1. \tag{4.4}$$

Let $A_\delta = \{x \in H_a : \langle D^2F(\pi)x, x\rangle_a \geq 1 - \delta\}$. For any $r > 0$,

$$\Gamma\Big(\big\{x : \langle D^2F(\pi)x, x\rangle_a + \varepsilon\|x\|_a^2 \geq 1\big\}\Big) \geq \Gamma(\{x : \|x\|_a \geq r\}) \wedge \Gamma(A_{r\delta}).$$

Obviously, $\Gamma(A_{r\delta}) = (1 - r\delta)\Gamma(A_0)$ and $\Gamma(\{x : \|x\|_a \geq r\}) \to \infty$ for $r \to \infty$. It therefore suffices to prove that $\Gamma(A_0) > 1$. This, however, is immediate from Lemma 4.3.

Assume now that $\varepsilon > 0$ is such that (4.4) holds true. Then, if c_2 is small enough and $\|l_T\|_a \leq c_2\sqrt{T}$, we have

$$TF(L_T) \leq \tfrac{1}{2}\langle D^2F(\pi)l_T, l_T\rangle_a + \tfrac{1}{2}\varepsilon\|l_T\|_a^2.$$

Therefore,

$$\begin{aligned} I_2(c_1, c_2, T) &\leq \mathbb{E}_x\Big(\exp\big(\tfrac{1}{2}\langle D^2F(\pi)l_T, l_T\rangle_a + \tfrac{1}{2}\varepsilon\|l_T\|_a^2\big); c_1 \leq \|l_T\|_a \leq c_2\sqrt{T}\Big) \\ &= \int_{-\infty}^{\infty} dt\, e^t \mathbb{P}_x\big(\tfrac{1}{2}\langle D^2F(\pi)l_T, l_T\rangle_a + \tfrac{1}{2}\varepsilon\|l_T\|_a^2 \geq t\big) \\ &= \int_{-\infty}^{\infty} dt\, e^t \mathbb{P}_x\big(l_T \in \sqrt{t}\,C_\varepsilon; c_1 \leq \|l_T\|_a \leq c_2\sqrt{T}\big), \end{aligned}$$

where $C_\varepsilon = \{x : \frac{1}{2}\langle D^2F(\pi)x, x\rangle_a + \frac{1}{2}\varepsilon\|x\|_a^2 \geq 1\}$. Note that if $k = \sup\{\langle D^2F(\pi)x, x\rangle_a : \|x\|_a = 1\}$, then $\inf\{\|x\|_a : x \in C_\varepsilon\} \geq \sqrt{2/(k+\varepsilon)}$.

According to Proposition 3.2, there exists $c > 0$ and $q > 1$ such that

$$\mathbb{P}_x\big(l_T \in \sqrt{t}\,C_\varepsilon\big) \leq \exp(-qt)$$

for $c \leq \sqrt{t} \leq \sqrt{T}/c$. If $c_2 < (1/c)\sqrt{2/(k+\varepsilon)}$, then

$$\big(\sqrt{t}\,C_\varepsilon\big) \cap \big\{x : \|x\|_a \leq c_2\sqrt{T}\big\} = \varnothing$$

if $\sqrt{t} \geq \sqrt{T}/c$. Therefore, if $d > c$, we have

$$I_2(c_1, c_2, T) \leq e^d \mathbb{P}_x(\|l_T\|_a \geq c_1) + \int_d^{\infty} e^{t(1-q)}\, dt.$$

Letting first $c_1 \to \infty$ and then $d \to \infty$ yields

$$\lim_{c_1 \to \infty} \sup_T I_2(c_1, c_2, T) = 0$$

if $c_2 > 0$ is small enough. □

PROOF OF THE THEOREM.

$$\lim_{T \to \infty} \mathbb{E}_x(\exp(TF(L_T))\varphi(X_T)) = \langle \varphi \rangle_\pi \big(\det(I - D^2F(\pi) \circ S)\big)^{-1/2}$$

is an immediate consequence of Lemma 4.1–4.3, in the case $F(\pi) = 0$, $DF(\pi) = 0$, $K_F = \{\pi\}$ and π is nondegenerate.

As remarked at the beginning of this section, this suffices for the proof. □

5. Examples. In this section we present a few examples. We focus on the computation of the rate function J and on the nondegeneracy condition Assumption 1.6. [more precisely (2.27)].

EXAMPLE 5.1. *The finite-dimensional situation.* Let E be a finite set with $|E| = n$ and let $\mathbb{P}_x$ be the law of the (time continuous) Markov chain on E starting at $x \in E$ with infinitesimal generator given by the matrix Q:

$$Q(x, y) \geq 0, \qquad x \neq y, \quad \text{and} \quad Q(x, x) = -\sum_{y \neq x} Q(x, y).$$

We write $Qf(x) = \sum_{y \neq x} Q(x, y)(f(y) - f(x))$. We will assume that the chain is irreducible and denote by $\pi \in \mathscr{M}_1^+(E)$ the invariant distribution. Let $J: \mathscr{M}_1^+(E) \to [0, \infty)$ be the rate function

$$\begin{aligned} J(\kappa) &= \sup\left\{\int_E -\frac{Qu}{u}\, d\kappa : u > 0\right\} \\ &= \sup\left\{\sum_x \sum_{y \neq x} \kappa(x) Q(x, y)\left(1 - \frac{u(y)}{u(x)}\right) : u > 0\right\} \\ &= \sum_x \sum_{y \neq x} \kappa(x) Q(x, y) - \inf\left\{\sum_x \sum_{y \neq x} \kappa(x) Q(x, y) \frac{u(y)}{u(x)} : u > 0\right\}. \end{aligned}$$

Assume that $\kappa > 0$, then the supremum is obtained at a regular point $u > 0$ and a simple computation of the gradient shows that u is unique up to multiplication by a positive constant and satisfies

$$(5.2) \quad \sum_{y \neq z}\left(\kappa(y) Q(y, x) \frac{u(x)}{u(y)} - \kappa(x) Q(x, y) \frac{u(y)}{u(x)}\right) = 0 \quad \text{for all } x \in E.$$

If Q^κ denotes the transformed transition matrix

$$Q^\kappa(x, y) = Q(x, y) \frac{u(y)}{u(x)}, \qquad x \neq y,$$

then (5.2) is equivalent with the κ-invariance of Q^κ. $(Q^\kappa)'$, the κ-adjoint of Q^κ, is of the form

$$(Q^\kappa)'(x, y) = Q(y, x) \frac{l(y)}{l(x)}, \qquad x \neq y \quad \text{where } l(x) = \frac{\kappa(x)}{u(x)}.$$

If Q is π-symmetric we simply have $u(x) = l(x) = \sqrt{\kappa(x)/\pi(x)}$ and

$$J(\kappa) = \tfrac{1}{2} \sum_x \sum_{x \neq y} \pi(x) Q(x, y)(u(x) - u(y))^2.$$

Let Π_κ be the projection from the $\mathbb{R}^n$ to $\mathfrak{W}_\kappa = \{f : \langle 1, f \rangle_\kappa = 0\}$ and define on $\mathfrak{W}_\kappa$, $G^\kappa = (-Q^\kappa)^{-1}$, $(G^\kappa)'$ the κ-adjoint of G^κ and $\overline{G}^\kappa = G^\kappa + (G^\kappa)'$ the symmetrized. Next take $F \in C^2(E)$ with second derivative D^2F viewed as a symmetric matrix. The nondegeneracy condition is of the form

$$\langle h, (\overline{G}^\kappa)^{-1} h \rangle_\kappa > \langle h, D^2 F h \kappa \rangle_\kappa, \qquad h \in \mathfrak{W}_\kappa, h \neq 0,$$

or equivalently for $f \neq 0$ with $\Sigma_x f(x) = 0$,

$$\sum_{x,y} f(x)(\overline{G}^{\kappa})^{-1}(x,y)\frac{1}{\kappa(y)}f(y) > \sum_{x,y} f(x)D^2(x,y)f(y).$$

EXAMPLE 5.3. *Random walk on the torus.* In general it is quite difficult to compute J explicitly. For example, let $E = \{1, 2, \dots, n\}$ be a discrete one-dimensional torus and let Q be the generator of a Markov chain with jumps to the nearest neighbor only, that is,

$$Q(x,y) = \begin{cases} -(r(x)+l(x)), & x = y, \\ r(x), & y = x+1, \\ l(x), & y = x-1, \\ 0, & |x-y| \geq 2. \end{cases} \tag{5.4}$$

The chain is irreducible if and only if $\Pi_x r(x) \neq 0$ or $\Pi_x l(x) \neq 0$. We will assume $\Pi_x r(x) \neq 0$. The invariant distribution π is the solution to the equation

$$\pi(x+1)l(x+1) + \pi(x-1)r(x-1) - \pi(x)(r(x)+l(x)) = 0, \quad x \in E.$$

Also the chain is π reversible if and only if

$$\pi(x)r(x) = \pi(x+1)l(x+1), \qquad x \in E,$$

and this is the case if and only if

$$\prod_x r(x) = \prod_x l(x).$$

Next take $\kappa \in \mathscr{M}_1^+(E)$ with $\kappa > 0$. Then in the computation of $J(\kappa)$, (5.2) is equivalent with

$$\kappa(x+1)l(x+1)\frac{1}{\alpha(x)} - \kappa(x)r(x)\alpha(x)$$
$$= \left(\kappa(x)l(x)\frac{1}{\alpha(x-1)} - \kappa(x-1)r(x-1)\alpha(x-1)\right), \qquad x \in E,$$

where we set $\alpha(x) = (u(x+1))/(u(x))$. Thus

$$-\kappa(x+1)l(x+1)\frac{1}{\alpha(x)} + \kappa(x)r(x)\alpha(x) = 2\Delta, \qquad x \in E,$$

for some constant $\Delta = \Delta(\kappa) \in \mathbb{R}$. Since $r > 0$, we have

$$\alpha(x) = \frac{\Delta + \sqrt{\Delta^2 + \kappa(x)\kappa(x+1)r(x)l(x+1)}}{\kappa(x)r(x)},$$

where Δ is chosen such that $\Pi_x \alpha(x) = 1$, which is equivalent with

$$\Phi(\Delta) = \sum_x \log\left(\Delta + \sqrt{\Delta^2 + \kappa(x)\kappa(x+1)r(x)l(x+1)}\right)$$
$$- \sum_x \log \kappa(x) - \sum_x \log r(x) = 0.$$

Note that Φ is a continuous strictly monotone increasing function with $\lim_{\Delta\to-\infty}\Phi(\Delta) = -\infty$ and $\lim_{\Delta\to\infty}\Phi(\Delta) = +\infty$. Thus the above equation has a unique solution. Also

$$\prod_x r(x) = \prod_x l(x) \quad \text{if and only if } \Delta = 0,$$

and this is equivalent with the π-symmetry of Q. In general we have

$$\prod_x r(x) < \prod_x l(x) \quad \text{if and only if } \Delta < 0,$$
$$\prod_x r(x) > \prod_x l(x) \quad \text{if and only if } \Delta > 0.$$

Once Δ is identified we have

$$\begin{aligned} J(\kappa) &= \sum_x \Big(\kappa(x)(r(x)+l(x)) - 2\sqrt{\Delta^2 + \kappa(x)\kappa(x+1)r(x)l(x+1)}\Big) \\ &\geq \sum_x \Big(\sqrt{\kappa(x)r(x)} - \sqrt{\kappa(x+1)l(x+1)}\Big)^2, \end{aligned}$$

where equality holds if and only if $\Delta = 0$ which corresponds to the symmetric case. The corresponding Q^κ is given in the form (5.4) with

$$r^\kappa(x) = \frac{\Delta + \sqrt{\Delta^2 + \kappa(x)\kappa(x+1)r(x)l(x+1)}}{\kappa(x)},$$
$$l^\kappa(x) = \frac{-\Delta + \sqrt{\Delta^2 + \kappa(x-1)\kappa(x)r(x-1)l(x)}}{\kappa(x)}.$$

In the degenerate case where $\kappa(x_0) = 0$ for some $x_0 \in E$, we can compute J explicitly: One can take $\alpha(x_0) = \infty$, $\alpha(x_0 - 1) = 0$, $\Delta = 0$ and get

$$J(\kappa) = \sum_x \Big(\sqrt{\kappa(x)r(x)} - \sqrt{\kappa(x+1)l(x+1)}\Big)^2.$$

Finally consider the explicit example of a random walk on E with jump to the right only: $r > 0$ and l identically 0. Then $\pi(x) = c/(r(x))$, where $c = (\sum_x 1/(r(x)))^{-1}$ and

$$J(\kappa) = \sum_x \kappa(x)r(x) - 2n\Delta(\kappa) \quad \text{with } \Delta(\kappa) = \tfrac{1}{2}\Big(\prod_x \kappa(x)r(x)\Big)^{1/n}.$$

Also we have the following expressions for the transition matrix Q^κ and $G^\kappa = (-Q^\kappa)^{-1}$ and $(G^\kappa)'$ on $\mathfrak{W}_\kappa$:

$$r^\kappa(x) = \frac{2\Delta(\kappa)}{\kappa(x)}, \quad l_\kappa(x) = 0, \quad G^\kappa = \Pi_\kappa \cdot A_\kappa \cdot \Pi_\kappa, \quad (G^\kappa)' = \Pi_\kappa \cdot A'_\kappa \cdot \Pi_\kappa,$$

where

$$A_\kappa(x,y) = \frac{\kappa(y)}{2\Delta(\kappa)} 1_{\{x \geq y\}}, \quad A'_\kappa(x,y) = \frac{\kappa(y)}{2\Delta(\kappa)} 1_{\{x \leq y\}}.$$

This yields

$$\overline{G}^{\kappa} = G_{\kappa} + (G^{\kappa})' = \Pi_{\kappa} \cdot (A_{\kappa} + A'_{\kappa}) \cdot \Pi_{\kappa} = \Pi_{\kappa} \cdot \tilde{A}_{\kappa} \cdot \Pi_{\kappa}$$

with

$$\tilde{A}_{\kappa}(x, y) = \frac{\kappa(x)}{2\Delta(\kappa)} 1_{\{x=y\}}.$$

The rate function J_{κ} associated with Q^{κ} is given by

$$J_{\kappa}(\mu) = 2\Delta(\kappa)\left\{\sum_x \frac{\mu(x)}{\kappa(x)} - n\left(\prod_x \frac{\mu(x)}{\kappa(x)}\right)^{1/n}\right\}.$$

If $\kappa_{\varepsilon}(x) = (1 + \varepsilon h(x))\kappa(x)$ with $\langle h, 1\rangle_{\kappa} = 0$, then a simple computation shows $(d/d\varepsilon)J_{\kappa}(\kappa_{\varepsilon})\mid_{\varepsilon=0} = 0$,

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}J_{\kappa}(\kappa_{\varepsilon})\mid_{\varepsilon=0} &= 2\Delta(\kappa)\left\{\sum_x h^2(x) - \frac{1}{n}\left(\sum_x h(x)\right)^2\right\}\\ &= \frac{\Delta(\kappa)}{n}\sum_{x,y}(h(x) - h(y))^2.\end{aligned}$$

If we compute the inverse of $\overline{G}_{\kappa}$ on $\mathfrak{W}_{\kappa}$ we see that

$$\frac{d^2}{d\varepsilon^2}J_{\kappa}(\kappa_{\varepsilon})\mid_{\varepsilon=0} = \langle h, \overline{G}_{\kappa}^{-1}h\rangle_{\kappa}$$

as it should. Take $F \in C^2(\mathbb{R}^n)$ with second derivative D^2F. Then the nondegeneracy condition is of the form

$$\begin{aligned}\langle h, D^2F(\kappa)\kappa h\rangle_{\kappa} &= \sum_{x,y}\kappa(x)h(x)D^2F(\kappa)(x, y)\kappa(y)h(y)\\ &< \frac{\Delta(\kappa)}{n}\sum_{x,y}(h(x) - h(y))^2,\end{aligned}$$

for $h \in \mathfrak{W}_{\kappa}$ with $h \neq 0$.

EXAMPLE 5.5. *Diffusion on a compact manifold.* Another situation of interest is when L is the generator of a diffusion on a compact N-dimensional manifold M. Let $X_0, X_1, \ldots, X_d$ be a collection of smooth vector fields and consider the operator $L: C^{\infty}(M) \to C^{\infty}(M)$ in Hörmander form

$$L = \sum_{i=1}^{d} X_i \circ X_i + X_0.$$

We will assume the strong Hörmander hypothesis:

$$\mathrm{Lie}(X_1, \ldots, X_d)(x) = T(M)(x), \qquad x \in M,$$

that is, the Lie algebra generated by $X_1, \ldots, X_d$ is the full tangent bundle over M at each $x \in M$. Let $\mathbb{P}_x$ be the law of the diffusion generated by L.

Then the rate function is given by

$$J(\kappa) = \sup\left\{-\int_M \frac{Lu}{u}\,d\kappa : u \in C_+^\infty(M)\right\}$$
$$= \sup\left\{\int_M \left(L\psi - \sum_{i=1}^d |X_i\psi|^2\right) d\kappa : \psi \in C^\infty(M)\right\},$$

where $C_+^\infty(M) \equiv C^\infty(M) \cap C^+(M)$. For $u \in C_+^\infty(M)$ define

$$L^u(\psi) \equiv \frac{L(u\cdot\psi)}{u} - \frac{Lu}{u}\cdot\psi = L\psi + 2\sum_{i=1}^d X_i(\log u)X_i\psi, \qquad \psi \in C^\infty(M).$$

Then

$$J(\kappa) = -\int_M \frac{Lu^*}{u^*}\,d\kappa$$

with $u^* \in C_+^\infty(M)$ if and only if L^{u^*} is κ-invariant.

Let $\pi \in \mathscr{M}_1^+(M)$ be a fixed smooth reference measure. For a vector field X, let $X^* = -X + g_X$, $g_X \in C^\infty(M)$, denote the π-adjoint of X, that is,

$$\langle \psi, X\phi\rangle_\pi = \langle X^*\psi, \phi\rangle_\pi, \qquad \phi, \psi \in C^\infty(M).$$

We can rewrite L as the sum of a π-symmetric part $\tilde{L} = -\sum_{i=1}^d X_i^* \circ X_i$ and a drift $Y = X_0 - \sum_{i=1}^d g_{X_i} X_i$:

$$L = \tilde{L} + Y.$$

We will assume that π is the (unique) invariant distribution for the process. This is equivalent with $g_Y = 0$ or $Y^* = -Y$, and L^*, the π-adjoint of L, is of the form

$$L^* = \tilde{L} - Y.$$

The process is π-symmetric if and only if $Y = 0$.

In the more general situation we had in Section 2, it is not clear if $J((1 + \varepsilon h)\pi)$ is smooth in ε near 0, for a sufficiently rich class of functions h satisfying $\langle h\rangle_\pi = 0$. For this reason, we had to resort to a slightly more delicate perturbation argument. In our more concrete situation here, the above expression is in fact smooth if $h \in C^\infty(M)$. It may be instructive to calculate the derivatives. Let $h \in C^\infty(M)$ with $\langle h\rangle_\pi = 0$ and set

$$\pi_\varepsilon(dx) \equiv (1 + \varepsilon h(x))\pi(dx) \equiv f_\varepsilon(x)\pi(dx).$$

Take ε small enough such that $f_\varepsilon \in C_+^\infty(M)$. Next for a given smooth vector field X, let X_ε^* be the π_ε-adjoint of X:

$$X_\varepsilon^*(\psi) \equiv X^*(\psi) - X(\log f_\varepsilon)\psi, \qquad \psi \in C^\infty(M),$$

and set

$$\tilde{L}_\varepsilon \equiv -\sum_{i=1}^d X_{i,\varepsilon}^* \circ X_i.$$

$\tilde{L}_\varepsilon$ is π_ε symmetric and we can define the corresponding Green operator

$\tilde{G}_\varepsilon = (-\tilde{L}_\varepsilon)^{-1}$ on $\mathfrak{W}_{\pi_\varepsilon} = \{\psi \in L^2(\pi_\varepsilon): \langle \psi \rangle_{\pi_\varepsilon} = 0\}$. With this notation we have

$$\begin{aligned} J(\pi_\varepsilon) &= \sup\left\{\int_M \left(L\psi - \sum_{i=1}^{d} |X_i\psi|^2\right) d\pi_\varepsilon : \psi \in C^\infty(M)\right\} \\ &= \sup\left\{\left\langle \frac{L^*(f_\varepsilon)}{f_\varepsilon}, \psi \right\rangle_{\pi_\varepsilon} - \sum_{i=1}^{d} \langle |X_i\psi|^2 \rangle_{\pi_\varepsilon} : \psi \in C^\infty(M)\right\} \\ &= \frac{1}{4}\left\langle \frac{L^*(f_\varepsilon)}{f_\varepsilon}, \left(-\tilde{L}_\varepsilon\right)^{-1}\left(\frac{L^*(f_\varepsilon)}{f_\varepsilon}\right)\right\rangle_{\pi_\varepsilon} \\ &= \frac{\varepsilon^2}{4}\left\langle L^*h, \tilde{G}_\varepsilon\left(\frac{L^*h}{f_\varepsilon}\right)\right\rangle_\pi . \end{aligned}$$

Note that

$$\lim_{\varepsilon \to 0}\left\langle L^*h, \tilde{G}_\varepsilon\left(\frac{L^*h}{f_\varepsilon}\right)\right\rangle_\pi = \langle L^*h, \tilde{G}(L^*h)\rangle_\pi = 2\langle h, \overline{G}^{-1}h\rangle_\pi ,$$

where $\overline{G} = G + G^* = (-L)^{-1} + (-L^*)^{-1}$ on $\mathfrak{W}_\pi$. This shows that $(d/d\varepsilon)J(\pi_\varepsilon)\,|_{\varepsilon=0} = 0$ and

$$\frac{d^2}{d\varepsilon^2}J(\pi_\varepsilon)\,|_{\varepsilon=0} = \langle h, \overline{G}^{-1}h\rangle_\pi .$$

We describe now a possible choice for the sequences (a_n) and (ψ_n) and the Hilbert space H_a of Section 1. We take the operator $(-\tilde{L})$ as reference: let $\{\psi_n: n \in \mathbb{N}\}$ and $\{\lambda_n: n \in \mathbb{N}\} \subseteq \mathbb{R}^+ \cup \{0\}$ be the eigenfunctions and eigenvalues of $-\tilde{L}$. Denote by $\{\tilde{P}_t: t > 0\}$ the corresponding symmetric semigroup. Then it is well known that there exist $\nu \in (0,\infty)$ and $c \in (0,\infty)$ such that

$$\tilde{p}_t(x,x) \le \frac{c}{t^{\nu/2}}, \qquad t \in (0,1], \tag{5.6}$$

see [1]. In the elliptic case where $d = N$, $\nu = N$ is the dimension of the manifold. In the hypoelliptic case, $\nu > N$, is the maximal graded dimension of the operator; cf. [1].

The estimate (5.6) is equivalent to the Sobolev type inequality

$$\|f\|_{L^2(\pi)}^{2+4/\nu} \le A\tilde{\mathscr{E}}(f,f)\|f\|_{L^1(\pi)}^{4/\nu}, \qquad f \in C^\infty(M),$$

for some constant $A \in (0,\infty)$; where $\tilde{\mathscr{E}}(f,f) = \sum_{i=1}^{d}\langle |X_i f|^2\rangle_\pi$ is the Dirichlet form associated with $(-\tilde{L})$ (cf. [8]). In particular, if $\nu > 2$, (5.6) is equivalent with the usual Sobolev inequality

$$\|f\|_{L^p(\pi)}^2 \le A'\tilde{\mathscr{E}}(f,f), \qquad f \in C^\infty(M), \langle f, 1\rangle_\pi = 0,$$

for some constants A', $\delta \in (0,\infty)$ with $p = 2\nu/(\nu-2)$. From (5.6) we have

$$\operatorname{trace}\left(\tilde{P}_t\right) = \int_E \tilde{p}_t(x,x)\pi(dx) \le \frac{c}{t^{\nu/2}}, \qquad t \in (0,1],$$

and using Weyl's formula one gets

$$(\lambda_k)^{\nu/2} \geq c_1 k, \qquad k \to \infty,$$

for some constants $c_1 \in (0, \infty)$. Also (5.6) implies the following estimate of the supremum norm of the eigenfunctions:

$$\|\psi_k\|_\infty^2 \leq c_3(\lambda_k)^{\nu/2}, \qquad k \to \infty,$$

for some $c_3 \in (0, \infty)$; cf. (48), page 155 of [11]. Now take $\{a_k: k \in \mathbb{Z}^+\}$ of the form $a_k = \text{const.}(\lambda_k)^{-\theta}$, where $\theta > 0$ is chosen such that $\sum_k a_k \|\psi_k\|_\infty^2 = 1$. In view of the above, we may take any $\theta > \nu$.

Let F be a C^2 functional on $\mathscr{M}$ with second derivative D^2F of the form

$$D^2F(\pi)(\nu, \nu) = \iint_{E\times E} V_\pi(x, y)\nu(dx)\nu(dy),$$

where $V_\pi \in C(E \times E; \mathbb{R})$ is a symmetric function of the form

$$V_\pi(x, y) = \int_\Sigma v(x, \tau)v(y, \tau)\sigma(d\tau).$$

where σ is a finite signed measure on a compact space Σ, and $v \in C_b(M \times \Sigma; \mathbb{R})$; cf. [2].

Now differentiability in H_a requires that

$$|D^2F(\pi)(\nu, \nu)| \leq K\|\nu\|_a^2$$

for some $K \in (0, \infty)$. Expressing ν in terms of the basis $\{l_n: n \in \mathbb{Z}^+\}$ and using Schwarz's inequality we see that

$$K \leq \int_\Sigma \sum_{n\in\mathbb{Z}^+} \frac{1}{a_n}\langle v(\cdot, \tau), \phi_n\rangle_\pi^2 |\sigma|(d\tau) = \int_\Sigma \langle (-\tilde{L})^\theta v(\cdot, \tau), v(\cdot, \tau)\rangle_\pi^2 |\sigma|(d\tau).$$

In particular the r.h.s. is finite if $v(\cdot, \tau) \in W_2^{(r)}(\mathbf{X}; \pi)$ for $r = [\theta/2] + 1$, where $W_2^{(r)}(\mathbf{X}; \pi)$ is the closure of $C^\infty(M)$ with respect to the Sobolev norm

$$\|g\|_{r,2} \equiv \sum_{0\leq|\alpha|\leq r} \|X_\alpha g\|_{L^2(\pi)}$$

where, for $\alpha = (\alpha_1, \dots, \alpha_k)$, $X_\alpha g = X_{\alpha_1} \circ \cdots \circ X_{\alpha_k} g$ (cf. [3]). The situation is especially simple when the function V is diagonizable with respect to $\{\psi_n: n \in \mathbb{Z}^+\}$, that is, if

$$V(x, y) = \sum_{n\in\mathbb{Z}^+} \beta_n \psi_n(x)\psi_n(y)$$

for some $\{\beta_n\} \subseteq \mathbb{R}$ with $\sum_n |\beta_n| \|\psi_n\|_\infty^2 < \infty$. Then $K < \infty$ if

$$\sum_{n\in\mathbb{Z}^+} |\beta_n|\lambda_n^\theta < \infty.$$

Things simplify considerably if L and L^* *commute* or equivalently if Y and $\tilde{L}$ commute. Then L is a normal operator, and we will denote by $\{f_n: n \in \mathbb{N}\}$ and $\{\mu_n: n \in \mathbb{N}\} \subseteq \mathbb{C}$ the eigenfunctions and eigenvalues of $-L$. Due to the normality of $-L$, we know that $\psi_{2n} = \Re(f_n)/\|\Re(f_n)\|_{L^2(\kappa)}$ and $\psi_{2n+1} =$

$\mathfrak{I}(f_n)/\|\mathfrak{I}(f_n)\|_{L^2(x)}$ are (real) eigenfunctions of $-\tilde{L}$ with positive eigenvalues $\lambda_{2n} = \lambda_{2n+1} = \mathfrak{R}(\mu_n)$. A direct computation shows

$$\begin{aligned}\langle h, \overline{G}h\rangle_\pi &= \sum_{n\in\mathbb{Z}^+}\left(\frac{\mu_n+\overline{\mu}_n}{\mu_n\overline{\mu}_n}\right)\left(\langle h, \psi_{2n}\rangle_\pi^2 + \langle h, \psi_{2n+1}\rangle_\pi^2\right)\\ &= \sum_{n\in\mathbb{Z}^+}\frac{2\lambda_n}{\lambda_n^2+\rho_n^2}\langle h, \psi_n\rangle_\pi^2,\end{aligned}$$

where $\rho_{2n} = \rho_{2n+1} = \mathfrak{I}(\mu_n)$. Thus

$$\begin{aligned}\langle h, \overline{G}^{-1}h\rangle_\pi &= \frac{1}{2}\sum_{n\in\mathbb{Z}^+}\left(\lambda_n + \frac{\rho_n^2}{\lambda_n}\right)\langle h, \psi_n\rangle_\pi^2\\ &= \frac{1}{2}\langle h, (-\tilde{L})h\rangle_\pi + \frac{1}{2}\langle h, (-\overline{Q})h\rangle_\pi.\end{aligned}$$

Here $-\tilde{Q}$ is the operator with eigenfunctions $\{\psi_n: n\in\mathbb{Z}^+\}$ and eigenvalues $\{\rho_n^2/\lambda_n: n\in\mathbb{Z}^+\}$. For a diagonizable functional F of the nondegeneracy condition reads

$$\beta_n < \frac{\lambda_n^2+\rho_n^2}{2\lambda_n}, \qquad n\in\mathbb{Z}^+,$$

and we get the explicit expression

$$d_{F,\pi} = \det\left(I - D^2F(\kappa)\circ S\right) = \prod_{n\in\mathbb{Z}^+}\left(1 - \frac{2\lambda_n\beta_n}{\lambda_n^2+\rho_n^2}\right).$$

Consider the concrete example where $M = T_N = (-\pi, \pi]^N$ is the N-dimensional torus:

$$X_i = \frac{\partial}{\partial x_i}, \qquad i = 1,\dots,N \quad\text{and}\quad X_0 = Y = \sum_{i=1}^N b_iX_i = b\cdot\nabla,$$

for some constant vector $b = (b_1,\dots,b_N)\in\mathbb{R}^N\setminus\{0\}$. In this case we simply have $\pi(dx) = (2\pi)^{-N}dx$, the normalized Lebesgue measure on T_N, $X_i^* = -X_i$, $\overline{L} = \Delta$, the Laplacian operator, $L = \Delta + b\cdot\nabla$ and $L^* = \Delta - b\cdot\nabla$. L and L^* commute and

$$\left\{f_k(x) = \exp(ix\cdot k): k = (k_1,\dots,k_N)\in\mathbb{N}^N\right\},\quad \left\{u_k = \left(|k|^2 - ik\cdot b\right): k\in\mathbb{N}^N\right\}$$

are the eigenfunctions and eigenvalues of $-L$. The corresponding eigenvalues and eigenfunctions of $-\tilde{L} = -\Delta$ are $\{|k|^2: k\in\mathbb{N}^N\}$ and $\{\cos(k\cdot), \sin(k\cdot): k\in\mathbb{N}^N\}$. Note that the eigenfunctions are uniformly bounded; thus $\theta > N/2$ would be sufficient. Further we get

$$\langle h, \overline{G}h\rangle_\pi = 2\sum_{k\neq 0}\frac{|k|^2}{|k|^4 + (b\cdot k)^2}\left(\langle h, \cos(k\cdot)\rangle_\pi^2 + \langle h, \sin(k\cdot)\rangle_\pi^2\right)$$

and

$$\langle h, \overline{G}^{-1}h\rangle_\pi = \frac{1}{2}\sum_{k\neq 0}\left(|k|^2 + \frac{(k\cdot b)^2}{|k|^2}\right)\left(\langle h, \cos(k\,\cdot)\rangle_\pi^2 + \langle h, \sin(k\,\cdot)\rangle_\pi^2\right)$$

$$= \frac{1}{2}\langle(-\Delta)h, h\rangle_\pi + \frac{1}{2}\langle h, (-\tilde{Q})h\rangle_\pi,$$

where $\tilde{Q}h = b\cdot\nabla(b\cdot\nabla\tilde{G}h)$. Note that this is a nonlocal operator. Next take the quadratic functional

$$F(\nu) = -\beta\int_{T_N}\int_{T_N}\|\Xi(x) - \Xi(y)\|^2\nu(dx)\nu(dy),$$

where $\beta > 0$ and, in polar coordinates, $\Xi_i(x) = (\cos(x_i), \sin(x_i))$, $i = 1,\dots, N$. Thus F is diagonizable,

$$F(\nu) = -\beta N + \beta\sum_{i=1}^N\left\{\left(\int_{T_N}\cos(x_i)\nu(dx)\right)^2 + \left(\int_{T_N}\sin(x_i)\nu(dx)\right)^2\right\}$$

with

$$V_\pi(x, y) = 2\beta\sum_{i=1}^N\{\cos(x_i)\cos(y_i) + \sin(x_i)\sin(y_i)\}.$$

For sufficiently small $\beta > 0$, π is the unique solution of the variational problem. In view of the above we get

$$d_{F,\pi} = \det(I - D^2F(\kappa)\circ S) = \prod_{i=1}^N\left(1 - \frac{4\beta}{1 + b_i^2}\right)^2.$$

EXAMPLE 5.7. *Diffusion on the circle*. Let $E = \mathbf{S}^1$ be the unit circle and dx be the Lebesgue measure on $\mathbf{S}^1$. In this special case we can compute J explicitly. Let a, $b \in C^\infty(\mathbf{S}^1)$ with $a > 0$ and $\int_{\mathbf{S}^1} a^{-1}(y)\,dy = 1$. Define the vector field $X = a(\partial/\partial x)$ and the measure $\lambda \in \mathbf{S}^1$, $\lambda(dx) = a^{-1}(x)\,dx$. Consider the diffusion operator L on $C^\infty(\mathbf{S}^1)$:

$$Lf = (X\circ X + bX)f = a^2 f'' + a(a' + b)f'.$$

Next let $\{\mathbb{P}_x: x \in \mathbf{S}^1\}$ be the Markovian family associated with the diffusion process generated by L. The density of the invariant measure $\rho = d\pi/d\lambda \in C^\infty(\mathbf{S}^1)$ is the solution of the divergence equation

$$(X\circ X)(\rho) - X(\rho b) = 0,$$

that is,

$$X(\rho) = \rho b - c_\pi \quad \text{with } c_\pi = \langle b\rangle_\lambda\langle\rho^{-1}\rangle_\lambda^{-1}.$$

This gives the following explicit solution: set $B(x) = \int_0^x b(y)a^{-1}(y)\,dy$. Then

$$\rho(x) = \rho(0)e^{B(x)}\left\{1 - \frac{1 - e^{-\langle b\rangle\lambda}}{\langle e^{-B}\rangle_\lambda}\int_0^x e^{-B(y)}a^{-1}(y)\,dy\right\},$$

where $\rho(0)$ is chosen such that $\langle\rho\rangle_\lambda = 1$.

Let X^* be the π-adjoint of X. Then $X^* = -X - X(\log \rho)$. If L^* denote the π-adjoint of L and $\tilde{L} = (L + L^*)/2$ denote the symmetrized, then we have

$$L = -X^* \circ X + c_\pi \rho^{-1} X, \qquad L^* = -X^* \circ X - c_\pi \rho^{-1} X, \qquad \tilde{L} = -X^* \circ X,$$

that is, refering to the previous example, $Y = c_\pi \rho^{-1} X$ and the process is π-symmetric if and only if $c_\pi = \langle b \rangle_\lambda = 0$.

We get the following expression for the rate function $J\colon \mathscr{M}_1^+(M) \to [0, \infty]$: if $d\kappa/d\pi = f$ with $f^{1/2} \in H^1(\mathbf{S}^1)$ (the usual Sobolev space), then

$$J(\kappa) = \langle |X(f^{1/2})|^2 \rangle_\pi + \frac{\langle b \rangle_\lambda^2}{4} \{ \langle f, \rho^{-1} \rangle_\lambda \langle \rho^{-1} \rangle_\lambda^{-2} \langle f^{-1}, \rho^{-1} \rangle_\lambda^{-1} \};$$

otherwise $J(\kappa) = \infty$. This follows from [3] since $J(\kappa) = \langle |X(f^{1/2})|^2 \rangle_\pi + I^Y(\kappa)$ with

$$\begin{aligned} I^Y(\kappa) &= \sup\{ \langle Y\psi \rangle_\kappa - \langle |X\psi|^2 \rangle_\kappa : \psi \in C^\infty(\mathbf{S}^1) \} \\ &= \sup\{ c_\pi \langle \rho^{-1}, X\psi \rangle_\kappa - \langle |X\psi|^2 \rangle_\kappa : \psi \in C^\infty(\mathbf{S}^1) \} \\ &= \langle |X\psi^*|^2 \rangle_\kappa = \frac{c_\pi^2}{4} \{ \langle \rho^{-2} \rangle_\kappa - \langle \rho^{-1} \rangle_\lambda^2 \langle f^{-1}, \rho^{-1} \rangle_\lambda^{-1} \} \\ &= \frac{\langle b \rangle_\lambda^2}{4} \{ \langle f, \rho^{-1} \rangle_\lambda \langle \rho^{-1} \rangle_\lambda^{-2} - \langle f^{-1}, \rho^{-1} \rangle_\lambda \}, \end{aligned}$$

with

$$X\psi^* = \frac{c_\pi}{2} \rho^{-1} \left(1 - \frac{\langle \rho^{-1} \rangle_\lambda}{\langle f^{-1}, \rho^{-1} \rangle_\lambda} f^{-1} \right).$$

Next suppose that $f = d\kappa/d\lambda \in C^\infty(\mathbf{S}^1)$ with $f > 0$. Then the corresponding L_κ is of the form

$$L_\kappa = -X' \circ X + c_\kappa f^{-1} X, \qquad L'_\kappa = -X' \circ X - c_\kappa f^{-1} X \quad \text{with } c_\kappa = \langle b \rangle_\lambda \langle f^{-1} \rangle_\lambda^{-1},$$

where $X' = -X - X(\log f)$ and L'_κ denote the κ-adjoint of X and L_κ. Let J_κ be the rate function associated with L_κ. Then we have

$$J_\kappa(\mu) = \langle |X(g^{1/2})|^2 \rangle_\kappa + \frac{\langle b \rangle_\lambda^2}{4} \{ \langle g, f^{-1} \rangle_\lambda \langle f^{-1} \rangle_\lambda^{-2} - \langle g^{-1}, f^{-1} \rangle_\lambda^{-1} \}$$

if $d\mu/d\kappa = g$ with $g^{1/2} \in H^1(\mathbf{S}^1)$; $J_\kappa(\mu) = \infty$ otherwise.

Next let $d\kappa_\varepsilon = (1 + \varepsilon h) d\kappa = g_\varepsilon d\kappa$, with $\langle h \rangle_\kappa = 0$ and $g_\varepsilon = (1 + \varepsilon h)$. Then

$$\begin{aligned} \frac{d}{d\varepsilon} J_\kappa(\kappa_\varepsilon) &= \langle X(g_\varepsilon^{1/2}), X(g_\varepsilon^{-1/2} h) \rangle_k \\ &\quad + \frac{\langle b \rangle_\lambda^2}{4} \{ \langle h, f^{-1} \rangle_\lambda \langle f^{-1} \rangle_\lambda^{-2} - \langle g_\varepsilon^{-1}, f^{-1} \rangle_\lambda^{-2} \langle g_\varepsilon^{-2} h, f^{-1} \rangle_\lambda \} \end{aligned}$$

and

$$\frac{d^2}{d\varepsilon^2}J_\kappa(\kappa_\varepsilon) = \frac{1}{2}\langle X(g_\varepsilon^{-1/2}h), X(g_\varepsilon^{-1/2}h)\rangle_\kappa - \frac{1}{2}\langle X(g_\varepsilon^{1/2}), X(g_\varepsilon^{-3/2}h^2)\rangle_\kappa$$
$$+ \frac{\langle b\rangle_\lambda^2}{2}\{\langle g_\varepsilon^{-1}, f^{-1}\rangle_\lambda^{-2}\langle g_\varepsilon^{-3}h^2, f^{-1}\rangle_\lambda$$
$$-\langle g_\varepsilon^{-1}, f^{-1}\rangle_\lambda^{-3}\langle g_\varepsilon^{-2}h, f^{-1}\rangle_\lambda^2\}.$$

Thus we get at $\varepsilon = 0$, $(d/d\varepsilon)J_\kappa(\kappa_\varepsilon)\,|_{\varepsilon=0} = 0$ and

$$\frac{d^2}{d\varepsilon^2}J(\kappa_\varepsilon)\,|_{\varepsilon=0} = \frac{1}{2}\langle |X(h)|^2\rangle_\kappa$$
$$+ \frac{\langle b\rangle_\lambda^2}{2}\{\langle f^{-1}\rangle_\lambda^{-2}\langle h^2, f^{-1}\rangle_\lambda - \langle f^{-1}\rangle_\lambda^{-3}\langle h, f^{-1}\rangle_\lambda^2\}$$
$$= \frac{1}{2}\langle |X(h)|^2\rangle_\kappa$$
$$+ \frac{\langle b\rangle_\lambda^2}{4}\langle f^{-1}\rangle_\lambda^{-1}\int_{\mathbf{S}^1}\int_{\mathbf{S}^1}(h(x) - h(y))^2\kappa^{-1}(dx)\kappa^{-1}(dy)$$
$$\equiv \langle h, (\overline{G}^\kappa)^{-1}h\rangle_\kappa,$$

with $\kappa^{-1}(dx) = \langle f^{-1}\rangle_\lambda^{-1}f^{-1}(x)\lambda(dx)$ and

$$\overline{G}_\kappa^{-1} = \tfrac{1}{2}\left(-\tilde{L}_\kappa - Q_\kappa\right).$$

Here L_κ is the generator of the symmetrized diffusion $\tilde{L}_\kappa = (-X' \circ X)$ and Q_κ is the generator of the jump process

$$Q_\kappa h(x) = \langle b\rangle_\lambda^2\langle f^{-1}\rangle_\lambda^{-2}f^{-2}(x)\int_{\mathbf{S}^1}(h(y) - h(x))\kappa^{-1}(dy).$$

Note the similarity with (5.2). Also it is interesting to see that although both L_κ and $(L_\kappa)'$ are local operators, $(\overline{G}^\kappa)^{-1}$ is nonlocal.

Take a $C^2(\mathbf{S}^1)$ functional F with second derivative D^2F. Then the nondegeneracy condition is of the form

$$D^2F(\kappa)[h\kappa, h\kappa] < \frac{1}{2}\langle |X(h)|^2\rangle_\kappa + \frac{\langle b\rangle_\lambda^2}{4}\langle f^{-1}\rangle_\lambda^{-1} \tag{5.8}$$
$$\times \int_{\mathbf{S}^1}\int_{\mathbf{S}^1}(h(x) - h(y))^2\kappa^{-1}(dx)\kappa^{-1}(dy),$$

for $h \in C^\infty(\mathbf{S}^1)$ with $h \not\equiv 0$.

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