The Annals of Probability
1995, Vol. 23, No. 1, 157–172

STRONG FELLER PROPERTY AND IRREDUCIBILITY FOR DIFFUSIONS ON HILBERT SPACES[1]

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It is shown that the transition semigroup $(P_t)_{t\geq 0}$ corresponding to a nonlinear stochastic evolution equation is strong Feller and irreducible, provided the nonlinearities are Lipschitz continuous and the diffusion term is nondegenerate. This result ensures the uniqueness of the invariant measure for $(P_t)_{t\geq 0}$.

0. Introduction. In the present paper we study the transition semigroup $(P_t)_{t\geq 0}$ corresponding to a nonlinear stochastic equation on a real separable Hilbert space. Our main theorems, Theorems 1.2 and 1.3, provide criteria under which $(P_t)_{t\geq 0}$ is strong Feller or is irreducible. The importance of the strong Feller property for the probabilistic potential theory in infinite dimensions has been stressed by Carmona [1] and Gross [7]. Moreover, it is well known (see [3] and [8]) that the strong Feller property and irreducibility ensure the uniqueness of the invariant measure for $(P_t)_{t\geq 0}$. Therefore our results easily imply, in particular, the uniqueness of invariant measure for stochastic heat equations (see Theorem 4.2). Under stronger conditions the uniqueness of invariant measure for stochastic heat equations has been recently obtained by Sowers [13], [14] and Mueller [12]. We believe that our approach is simpler, more natural and more general than theirs.

Strong Feller property and irreducibility for stochastic evolution equations have been studied only for equations with a constant diffusion coefficient (see [9], [10], [11], [5] and [6]). The general case has not been studied before.

The paper is organized as follows. In the first section we set up notation, terminology and we formulate the main results. Sections 2 and 3 are devoted to the proofs of the strong Feller property and irreducibility. In the final section we apply our general results to the case of stochastic heat equations.

1. Notations and formulations of the results. Let $(H, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space. By $|\cdot|$ we denote the norm on H. The spaces of bounded and Hilbert–Schmidt operators on H are denoted by $L(H)$ and $L_2(H)$, respectively. In the present paper $\|\cdot\|$ and $\|\cdot\|_2$ stand for the operator and the Hilbert–Schmidt norms. The spaces of bounded measurable and bounded continuous functions on H are denoted by $B_b(H)$ and $C_b(H)$. By $\|\cdot\|_0$ we denote the supremum norm on $B_b(H)$ or $C_b(H)$.

Received July 1993; revised December 1993.

[1]Research supported by KBN Grant 2 2003 91 02.

AMS 1991 *subject classifications*. 60J35, 60H15, 60J25.

Key words and phrases. Strong Feller property, irreducible Markov semigroups, invariant measures, stochastic evolution equations.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space with a right-continuous increasing family $\mathbf{F} = (\mathscr{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathscr{F}$ each containing $\mathbf{P}$-null sets. Let $\{e_n\}$ be an orthonormal basis in H and let $\{W_n\}$ be a sequence of independent, real-valued $\mathbf{F}$-Wiener processes. We define a *cylindrical Wiener process* on H by the series

$$W(t) = \sum_{n=1}^{\infty} W_n(t) e_n, \qquad t \geq 0,$$

which converges in a Hilbert space $\tilde{H}$ containing H with a Hilbert–Schmidt embedding.

Let $(P_t)_{t\geq 0}$ be a Markov semigroup on $B_b(H)$. Recall that $(P_t)_{t\geq 0}$ is *strong Feller* if for arbitrary $\psi \in B_b(H)$ and $t > 0$, $P_t\psi \in C_b(H)$. The semigroup $(P_t)_{t\geq 0}$ is *irreducible* if the *transition probabilities* $P_t(x, U) := P_t \chi_U(x)$ are strictly positive for all $t > 0$, $x \in H$ and nonempty open sets $U \subseteq H$. A probability measure μ on H is said to be *invariant* with respect to the semigroup $(P_t)_{t\geq 0}$ if

$$\int_H P_t\psi(x)\mu(dx) = \int_H \psi(x)\mu(dx) \quad \text{for } \psi \in B_b(H),\ t \geq 0.$$

The following classical result (see [3] and [8]) links the above concepts.

THEOREM 1.1. *Assume that a Markov semigroup $(P_t)_{t\geq 0}$ on a Polish space is strong Feller and irreducible. Then there exists at most one invariant measure for $(P_t)_{t\geq 0}$, and if μ is invariant for $(P_t)_{t\geq 0}$ then μ is ergodic and equivalent to each transition probability $P_t(x, \cdot)$. Moreover, for an arbitrary Borel set Γ, $P_t(x, \Gamma) \to \mu(\Gamma)$ as $t \to \infty$.*

In the present paper $(P_t)_{t\geq 0}$ is defined by $P_t\psi(x) = \mathbf{E}\psi(X(t, x))$, where $X(\cdot, x)$ is the solution of the stochastic Itô equation

$$dX = (AX + F(X))\,dt + B(X)dW, \qquad X(0) = x. \tag{1}$$

In (1), A is the generator of a C_0-semigroup S on H, the mappings F and B act from H into H and from H into $L(H)$, respectively. By the solution we understand the so-called *mild solution*, that is, the solution of the integral equation

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))\,ds + \int_0^t S(t-s)B(X(s))\,dW(s), \qquad t \geq 0.$$

We will say that X is strong Feller or irreducible if its transition semigroup is strong Feller or irreducible.

We will need the following assumptions on F, B and S:

(A.1) There exists a constant $L < \infty$ such that for all $z, y \in H$,

$$|F(z) - F(y)| \leq L|z - y|,$$
$$\|B(z) - B(y)\| \leq L\|z - y\|.$$

(A.2) The operators $B(z)$, $z \in H$, are invertible and

$$K = \sup_{z \in H} \|B^{-1}(z)\| < \infty.$$

(A.3) For every $T > 0$,

$$\int_0^T \|S(t)\|_2^2 \, dt < \infty.$$

(A.4) There exists a constant $\alpha > 0$ such that for every $T > 0$,

$$\int_0^T t^{-\alpha} \|S(t)\|_2^2 \, dt < \infty.$$

(A.5) Either mapping F or mapping B is bounded.

Occasionally instead of (A.1) we will assume the following weaker condition:

(B.1) B is weakly continuous and there exists a constant $L < \infty$ such that for all $z, y \in H$ and $t > 0$,

$$|F(z) - F(y)| \le L|z - y|,$$
$$\|S(t)[B(z) - B(y)]\|_2 \le L\|S(t)\|_2 |z - y|.$$

REMARK 1.1. Assume (B.1) and (A.3). Then, by Theorem A.1 of the Appendix, equation (1) has the unique solution X satisfying

$$\sup_{0 \le t \le T} \mathbf{E}\,|X(t, x)|^2 < \infty \quad \text{for every } T < \infty.$$

Under additional assumption (A.4) the process X has continuous trajectories in H (see [5], Theorem 7.6).

Since S is a C_0-semigroup there exist constants M and γ such that $\|S(t)\| \le Me^{\gamma t}$ for $t \ge 0$. Obviously we may assume that $\gamma > 0$.

We can now formulate our main results. The first theorem, which is concerned with strong Feller property, seems to be interesting even in the finite-dimensional case.

THEOREM 1.2. *Assume that* (A.1) *to* (A.3) *hold. Then for all* $\psi \in B_b(H)$ *and* $t > 0$,

$$(2) \qquad |P_t\psi(x) - P_t\psi(y)| \le c_t \|\psi\|_0 |x - y|, \qquad x, y \in H,$$

where

$$(3)\qquad \begin{cases} c_t = \dfrac{3KM}{\sqrt{t \wedge T_0}} e^{\gamma T_0} \quad \text{and } T_0 > 0 \text{ is such that} \\ M^2 L^2 T_0^2 e^{2\gamma T_0} + L^2 \displaystyle\int_0^{T_0} \|S(s)\|_2^2 \, ds \leq \dfrac{2}{9}. \end{cases}$$

COROLLARY 1.1. *If* (A.1) *to* (A.3) *hold, then the process* X *is strong Feller.*

Our second theorem deals with irreducibility.

THEOREM 1.3. *Assume that* (B.1), (A.2), (A.3) *and either* (A.4) *or* (A.5) *hold. Then the process* X *is irreducible.*

Combining Theorem 1.1 and Corollary 1.1 with Theorem 1.3 yields:

THEOREM 1.4. *Assume that* (A.1) *to* (A.3) *and either* (A.4) *or* (A.5) *hold. Then there exists at most one invariant measure for the transition semigroup of* X.

2. Proof of Theorem 1.2. Throughout this section we assume (A.1) to (A.3). We will need the following two lemmas. For the proof of the first one, we refer the reader to [5], Theorem 9.4.

LEMMA 2.1. *If* F *and* B *have bounded and continuous Fréchet derivatives* DF *and* DB, *then for arbitrary* $t \geq 0$, $x \in H$ *and* $h \in H$, *the process* $X(t,x)$ *has mean square directional derivative* $D_x X(t,x)h$ *at* x *and in the direction of* h. *Moreover, for each fixed* $h \in H$, $Y(\cdot) = D_x X(\cdot, x)h$ *is the unique solution of the equation*

$$dY = (AY + DF(X)Y)\,dt + DB(X)Y\,dW, \qquad Y(0) = h,$$

satisfying

$$(4)\qquad \sup_{0 \leq t \leq T} \mathbf{E}\,|Y(t)|^2 < \infty \quad \text{for every } T > 0.$$

LEMMA 2.2. *Let* $(P_t)_{t \geq 0}$ *be a Markov semigroup on* $B_b(H)$ *and let* $c > 0$ *and* $t > 0$ *be fixed. Then the following conditions are equivalent:*

(i) $\forall \phi \in C_b^2(H) \forall x, y \in H, |P_t\phi(x) - P_t\phi(y)| \leq c\|\phi\|_0 |x - y|$.
(ii) $\forall \phi \in B_b(H) \forall x, y \in H, |P_t\phi(x) - P_t\phi(y)| \leq c\|\phi\|_0 |x - y|$.

PROOF. We have to show that (i) $\Longrightarrow$ (ii). Let $\mathscr{K}_0 = \{\phi \in B_b(H): \|\phi\|_0 \leq 1\}$, $\mathscr{K}_1 = \{\phi \in C_b(H): \|\phi\|_0 \leq 1\}$ and $\mathscr{K}_2 = \{\phi \in C_b^2(H): \|\phi\|_0 \leq 1\}$. Since each bounded continuous function on H may be approximated pointwise by functions of the class C^2, we have

$$\sup_{\phi \in \mathscr{K}_1} |P_t\phi(x) - P_t\phi(y)| = \sup_{\phi \in \mathscr{K}_2} |P_t\phi(x) - P_t\phi(y)| \quad \text{for all } x, y \in H.$$

As a simple consequence of the Hahn decomposition theorem we have

$$\sup_{\phi \in \mathscr{K}_1} |P_t\phi(x) - P_t\phi(y)| = \operatorname{Var}(P_t(x, \cdot) - P_t(y, \cdot)).$$

Therefore for all $x, y \in H$ we have

$$\operatorname{Var}(P_t(x, \cdot) - P_t(y, \cdot)) \le c|x - y|$$

and consequently for all $\phi \in B_b(H)$,

$$\begin{aligned} |P_t\phi(x) - P_t\phi(y)| &= \left| \int_H \phi(z)(P_t(x, dz) - P_t(y, dz)) \right| \\ &\le \|\phi\|_0 \operatorname{Var}(P_t(x, \cdot) - P_t(y, \cdot)) \\ &\le c\|\phi\|_0 |x - y|, \end{aligned}$$

which is the desired conclusion. □

The proof of Theorem 1.2 consists of two parts. In the first part we prove the desired assertion under the following additional assumption:

(A.6) F, B are twice Fréchet differentiable functions with bounded and continuous derivatives up to the second order.

In the second part we show how to dispense with assumption (A.6).

Part 1. We have divided this part into a sequence of lemmas. The formulations and proofs of the two first lemmas are similar to those in Steps 1 and 2 in the proof of Theorem 2.2 from [4]. However our formulations are slightly more general. We present the proofs for the convenience of the reader. In what follows $\psi \in B_b(H)$ is regarded as fixed. For abbreviation, we write $v(t, x)$ instead of $P_t\psi(x)$.

LEMMA 2.3. *Assume that* (A.6) *holds and* $\psi \in C_b^2(H)$. *Then* $v \in C^{1,2}([0, \infty) \times H)$ *and*

$$\psi(X(t, x)) = v(t, x) + \int_0^t \langle D_z v(t - s, X(s, x)), B(X(s, x))\, dW(s) \rangle. \tag{5}$$

PROOF. For each n, let $X_n(\cdot, x)$ be the solution of the equation

$$dX_n = (A_n X_n + F(X_n))\,dt + B(X_n)Q_n\,dW, \qquad X_n(0) = x,$$

where $A_n = nA(nI - A)^{-1}$ is the Yosida approximation of A and Q_n is the finite dimensional projection of H onto $\mathrm{lin}\{e_1, \dots, e_n\}$. Applying twice Lemma 2.1, one can prove that the function $v^{(n)}(t,x) = \mathbf{E}\,\psi(X_n(t,x))$ belongs to the class $C^{1,2}([0,\infty) \times H)$. Moreover, it follows easily that $v^{(n)}$ satisfies the Kolmogorov equation

$$D_t v^{(n)}(t,x) = \tfrac{1}{2}\operatorname{Tr}[B(x)Q_n B^*(x) D^2_{xx} v^{(n)}(t,x)] + \langle A_n x + F(x), D_x v^{(n)}(t,x)\rangle,$$
$$v^{(n)}(0,x) = \psi(x).$$

Therefore, applying the Itô formula to the process $s \to v^{(n)}(t-s, X_n(s,x))$ we obtain

$$\psi(X_n(t,x)) = v^{(n)}(t,x) + \int_0^t \langle D_z v^{(n)}(t-s, X_n(s,x)), B(X_n(s,x))Q_n\,dW(s)\rangle.$$

Letting $n \to \infty$ gives the desired result (for more details see [4]). □

LEMMA 2.4. *Assume that* (A.6) *holds and* $\psi \in C_b^2(H)$. *Then the directional derivatives* $D_x v(t,x)h$ *are given by*

$$D_x v(t,x)h = \frac{1}{t}\mathbf{E}\Big\{\psi(X(t,x)) \int_0^t \langle B^{-1}(X(s,x)) D_x X(s,x)h, dW(s)\rangle\Big\}.$$

PROOF. Multiplying the both sides of (5) by the term

$$\int_0^t \langle B^{-1}(X(s,x)) D_x X(s,x)h, dW(s)\rangle$$

and taking the expectation, we get

$$\begin{aligned}
\mathbf{E}\Big\{&\psi(X(t,x)) \int_0^t \langle B^{-1}(X(s,x)) D_x X(s,x)h, dW(s)\rangle\Big\} \\
&= \mathbf{E}\int_0^t \langle B^*(X(s,x)) D_z v(t-s, X(s,x)), B^{-1}(X(s,x)) D_x X(s,x)h\rangle\,ds \\
&= \mathbf{E}\int_0^t \langle D_z v(t-s, X(s,x)), D_x X(s,x)h\rangle\,ds \\
&= \int_0^t D_x(\mathbf{E}(v(t-s, X(s,x))))h\,ds \\
&= \int_0^t D_x(P_s P_{t-s}\psi(x))h\,ds = tD_x v(t,x)h.
\end{aligned}$$

□

LEMMA 2.5. *Assume* (A.6). *Then for all* $t > 0$ *and* $\psi \in B_b(H)$ *the function* $P_t\psi(\cdot)$ *is Lipschitz continuous. Moreover, for arbitrary* $x, y \in H$,

$$|P_t\psi(x) - P_t\psi(y)| \le c_t \|\psi\|_0 |x-y|,$$

the constant c_t *being given by* (3).

PROOF. First let $\psi \in C_b^2(H)$. From Lemma 2.4, we have

$$
\begin{aligned}
|D_x v(t,x)h| &\le \frac{1}{t}\|\psi\|_0 \mathbf{E}\Big|\int_0^t \langle B^{-1}(X(s,x))D_xX(s,x)h, dW(s)\rangle\Big| \\
&\le \frac{1}{t}\|\psi\|_0\Big(\mathbf{E}\int_0^t |B^{-1}(X(s,x))D_xX(s,x)h|^2\,ds\Big)^{1/2} \qquad (6)\\
&\le \frac{1}{t}\|\psi\|_0 K\Big(\mathbf{E}\int_0^t |D_xX(s,x)h|^2\,ds\Big)^{1/2}.
\end{aligned}
$$

Our goal is to evaluate the term

$$\Big(\mathbf{E}\int_0^t |D_xX(s,x)h|^2\,ds\Big)^{1/2}.$$

To this end, write $Y(t) = D_xX(t,x)h$. Lemma 2.1 gives

$$
\begin{aligned}
|Y(t)| \le |S(t)h| &+ \int_0^t \|S(t-s)\|\ |DF(X(s,x))Y(s)|\,ds \\
&+ \Big|\int_0^t S(t-s)DB(X(s,x))Y(s)\,dW(s)\Big|.
\end{aligned}
$$

Let as denote the right-hand side of the above inequality by $I_1(t)+I_2(t)+I_3(t)$. Then we have

$$\int_0^t \mathbf{E}\,|Y(s)|^2\,ds \le 3\Big(\int_0^t I_1^2(s)\,ds + \int_0^t \mathbf{E}\,I_2^2(s)\,ds + \int_0^t \mathbf{E}\,I_3^2(s)\,ds\Big).$$

By the definition of M and γ,

$$\int_0^t I_1^2(s)\,ds \le M^2|h|^2te^{2\gamma t}.$$

Since $\|DF(z)\| \le L$ for $z \in H$, we have

$$
\begin{aligned}
\int_0^t \mathbf{E}\,I_2^2(s)\,ds &\le \int_0^t M^2L^2\mathbf{E}\Big(\int_0^s e^{\gamma(s-\tau)}|Y(\tau)|d\tau\Big)^2\,ds \\
&\le M^2L^2t^2e^{2\gamma t}\int_0^t \mathbf{E}\,|Y(s)|^2\,ds.
\end{aligned}
$$

Using standard arguments we obtain

$$\int_0^t \mathbf{E}\,I_3^2(s)\,ds \le L^2\int_0^t \|S(s)\|_2^2\,ds \int_0^t \mathbf{E}\,|Y(s)|^2\,ds.$$

Combining the above estimates gives

$$
\begin{aligned}
\int_0^t \mathbf{E}\,|Y(s)|^2\,ds &\le 3M^2|h|^2te^{2\gamma t} \\
&\quad + 3\Big(M^2L^2t^2e^{2\gamma t} + L^2\int_0^t \|S(s)\|_2^2\,ds\Big)\int_0^t \mathbf{E}\,|Y(s)|^2\,ds.
\end{aligned}
$$

From (4) we have

$$\int_0^t \mathbf{E}\,|Y(s)|^2\,ds < \infty.$$

Consequently for $t \leq T_0$ (T_0 appears in the formulation of Theorem 1.2) we have

$$\int_0^t \mathbf{E}\,|Y(s)|^2\,ds \leq 9M^2|h|^2 t e^{2\gamma t}. \tag{7}$$

Combining (6) with (7) gives

$$|D_x v(t,x)h| \leq \frac{3KM|h|}{\sqrt{t}} e^{\gamma t} \|\psi\|_0 \quad \text{for} \quad t \leq T_0. \tag{8}$$

Since

$$\begin{aligned} |P_t\psi(x) - P_t\psi(y)| &\leq \sup_{z\in H} |D_z(P_t\psi(z))(y-x)| \\ &\leq \sup_{z\in H} |D_z v(t,z)(y-x)|, \end{aligned}$$

(8) gives the desired estimate (2) for all $\psi \in C_b^2(H)$ and $t \leq T_0$. According to Lemma 2.2, (2) holds for all $\psi \in B_b(H)$ and $t \leq T_0$. Since for $t \geq T_0$ we have

$$\begin{aligned} |P_t\psi(x) - P_t\psi(y)| &\leq |P_{T_0+(t-T_0)}\psi(x) - P_{T_0+(t-T_0)}\psi(y)| \\ &\leq c_{T_0}\|P_{t-T_0}\psi\|_0|x-y| \leq c_{T_0}\|\psi\|_0|x-y|, \end{aligned}$$

and (2) holds for all t. □

Part 2. In this part we want to apply Theorem A.1 from the Appendix. Thus the task is now to find good approximations F_n of F and B_n of B. For this purpose we take a sequence of nonnegative twice differentiable functions $\{\rho_n\}$ such that

$$\operatorname{supp}(\rho_n) \subseteq \{\xi \in \mathbb{R}^n : |\xi|_{\mathbb{R}^n} \leq 1/n\} \quad \text{and} \quad \int_{\mathbb{R}^n} \rho_n(\xi)\,d\xi = 1.$$

Let Q_n be the orthonormal projection of H onto $\operatorname{lin}\{e_1,\ldots,e_n\}$. Recall that $\{e_n\}$ stands for the orthonormal basis in H. We will identify $\mathbb{R}^n$ with $\operatorname{lin}\{e_1,\ldots,e_n\}$. The mappings $F_n: H \to H$ and $B_n: H \to L(H)$ are defined by

$$\begin{aligned} F_n(x) &= \int_{\mathbb{R}^n} \rho_n(\xi - Q_n x) F\Big(\sum_{i=1}^n \xi_i e_i\Big)\,d\xi, \\ B_n(x) &= \int_{\mathbb{R}^n} \rho_n(\xi - Q_n x) B\Big(\sum_{i=1}^n \xi_i e_i\Big)\,d\xi. \end{aligned} \tag{9}$$

Observe that F_n and B_n are twice Fréchet differentiable functions with bounded and continuous derivatives. Moreover, for all x, y and n,

$$|F_n(x) - F_n(y)| = \left| \int_{\mathbb{R}^n} \rho_n(\xi)\left[F\left(\sum_{i=1}^{n} \xi_i e_i + Q_n x\right) - F\left(\sum_{i=1}^{n} \xi_i e_i + Q_n y\right)\right] d\xi \right|$$

$$\leq L|Q_n(x-y)| \int_{\mathbb{R}^n} \rho_n(\xi)\, d\xi \leq L|x-y|.$$

In the same manner we can see that $\|B_n(x) - B_n(y)\| \leq L|x-y|$.

We next prove that for n sufficiently large the operators $B_n(z)$, $z \in H$, are invertible and

$$\limsup_{n\to\infty} \sup_{z\in H} \|B_n^{-1}(z)\| \leq K,$$

the constant K being defined in (A.2). To this end, observe that the operators $B(Q_n z)$, $z \in H$, $n \in \mathbb{N}$, are invertible,

$$\sup_{n\in\mathbb{N}} \sup_{z\in H} \|B^{-1}(Q_n z)\| \leq \sup_{z\in H} \|B^{-1}(z)\| \leq K$$

and

$$\sup_{z\in H} \|B(Q_n z) - B_n(z)\|$$

$$= \sup_{z\in H} \left\| \int_{\mathbb{R}^n} \rho_n(\xi)\left[B(Q_n z) - B\left(\sum_{i=1}^{n} \xi_i e_i + Q_n z\right)\right] d\xi \right\|$$

$$\leq \int_{\mathbb{R}^n} \rho_n(\xi) L \left| \sum_{i=1}^{n} \xi_i e_i \right| d\xi \leq L n^{-1}.$$

Consequently, for n sufficiently large the operators $B_n(z)$, $z \in H$, are invertible and

$$\limsup_{n\to\infty} \sup_{z\in H} \|B_n^{-1}(z)\|$$

$$= \limsup_{n\to\infty} \sup_{z\in H} \|[B(Q_n z) - (B(Q_n z) - B_n(z))]^{-1}\|$$

$$\leq \limsup_{n\to\infty} \sup_{z\in H} \|B^{-1}(Q_n z)\| \sum_{j=0}^{\infty} \|B^{-1}(Q_n z)(B(Q_n z) - B_n(z))\|^j$$

$$\leq \limsup_{n\to\infty} K \sum_{j=0}^{\infty} K^j L^j n^{-j} = K.$$

Summing up, we have actually proved that:

1. F_n and B_n, $n \in \mathbb{N}$, are twice Fréchet differentiable functions with bounded and continuous derivatives.
2. B_n, F_n, $n \in \mathbb{N}$, satisfy the Lipschitz condition with the constant L.

3. For n sufficiently large the operators $B_n(z)$, $z \in H$, are invertible and $\forall \tilde{K} > K \exists n_0 \in \mathbb{N}$ such that

$$\sup_{z\in H} \|B_n^{-1}(z)\| \le \tilde{K} \quad \text{for } n \ge n_0.$$

Moreover, it is easy to check that:

4. For every $z \in H$,

$$\lim_{n\to\infty} |F_n(z) - F(z)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B_n(z) - B(z)\| = 0.$$

Now, let $X_n(\cdot, x)$ be the solution of the equation

$$dX_n = (AX_n + F_n(X_n)) + B_n(X_n)\, dW, \qquad X_n(0) = x,$$

and let $(P_t^{(n)})_{t\ge 0}$ be the corresponding semigroup. According to Lemma 2.2 it is enough to show that (2) holds for every $\psi \in C_b^2(H)$. To do this fix $t > 0$ and a number $\tilde{c}_t$ greater than c_t. Lemma 2.5 applied to $(P_t^{(n)})_{t\ge 0}$ shows that for all $\psi \in C_b^2(H)$, $x, y \in H$, and n sufficiently large we have

$$|P_t^{(n)}\psi(x) - P_t^{(n)}\psi(y)| \le \tilde{c}_t \|\psi\|_0 |x - y|.$$

Theorem A.1 now implies that for all t and x there exists a subsequence such that $X_{n_j}(t, x) \to X(t, x)$ a.s. as $j \to \infty$. Since ψ is a bounded continuous function, we have

$$|P_t\psi(x) - P_t\psi(y)| \le c_t \|\psi\|_0 |x - y|$$

and the proof of Theorem 1.2 is complete. □

3. Proof of Theorem 1.3. Throughout this section we assume that (B.1), (A.2) and (A.3) hold. Only in the proof of Lemma 3.1 will we need (A.4) or (A.5).

For given $a \in H$ and $r > 0$, let $B(a, r)$ stand for the ball $\{z \in H: |a - z| < r\}$. Note that X is irreducible iff for all $x \in H$, $t > 0$, $a \in H$ and $r > 0$,

$$\mathbf{P}(X(t, x) \in B(a, r)) > 0.$$

From now on, x, t, a and r are fixed. We need the following lemma:

LEMMA 3.1. *Assume that* (A.4) *or* (A.5) *is fulfilled. Let* $t_1 \in (0, t)$ *and let* $f: [t_1, t] \times H \to H$ *be a bounded and measurable mapping. If* Z *is the solution of the equation*

$$\begin{aligned} &dZ(s) = AZ(s)\, ds + B(Z(s))\, dW(s) && \text{on } [0, t_1], \\ (10) \quad &dZ(s) = (AZ(s) + f(s, Z(t_1)))\, ds + B(Z(s))\, dW(s) && \text{on } (t_1, t], \\ &Z(0) = x, \end{aligned}$$

then the laws in H *of* $X(t, x)$ *and* $Z(t)$ *are equivalent.*

PROOF. Let $\alpha(s) = B^{-1}(Z(s))[f_1(s) - F(Z(s))]$, where $f_1(s) = f(s, Z(t_1))$ for $s \in (t_1, t]$ and 0 otherwise. Finally, let

$$\Xi = \exp\Big(-\int_0^t \langle \alpha(s), dW(s)\rangle - \tfrac{1}{2}\int_0^t |\alpha(s)|^2\, ds\Big).$$

If F is bounded or if B is bounded, then by Novikov's criterion (see [5], Theorem 10.18) and by Theorem 4.1 from [2], $\mathbf{E}\,\Xi = 1$ and $\mathbf{P}(\Xi > 0) = 1$. Therefore by the Girsanov theorem (see [5]) the process

$$W^*(s) = W(s) + \int_0^s \alpha(u)\, du, \qquad s \in [0, t],$$

is a cylindrical Wiener process in H defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P}^*)$, where $d\mathbf{P}^* = \Xi\, d\mathbf{P}$. Note that $\mathbf{P}$ and $\mathbf{P}^*$ are equivalent. Now, since

$$\begin{aligned} Z(s) &= S(s)x + \int_0^s S(s-u) f_1(u)\, du + \int_0^s S(s-u) B(Z(u))\, dW(u) \\ &= S(s)x + \int_0^s S(s-u) F(Z(u))\, du + \int_0^s S(s-u) B(Z(u))\, dW^*(u), \end{aligned}$$

Z is the solution of (1) on the probability space $(\Omega, \mathscr{F}, \mathbf{P}^*)$. Since the law of the solution does not depend on the particular choice of the probability space, for every Borel set $\Gamma \subseteq H$ we have $\mathbf{P}(X(t,x) \in \Gamma) = \mathbf{P}^*(Z(t) \in \Gamma)$. Consequently, since $\mathbf{P}$ and $\mathbf{P}^*$ are equivalent the same is true for the laws of $X(t,x)$ and $Z(t)$, which is our claim.

If S satisfies (A.4) then the processes $X(\cdot, x)$ and Z have continuous trajectories (see Remark 1.1). Now, let for an arbitrary $l \in \mathbb{N}$, F_l be a bounded Lipschitz mapping identical with F on the ball $B(0, l)$, and let X_l be a continuous solution of (1) with $F = F_l$. Then, again by the Girsanov theorem, the laws of X_l and Z on the space of trajectories $C([0,t]; H)$ are equivalent. It is clear that for every l the laws of X and X_l are identical on $C([0,t]; B(0,l))$. These two facts imply the equivalence of X and Z. □

We can now proceed to the proof of Theorem 1.3. What is left is to show that there exists a function f satisfying the assumptions specified in Lemma 3.1 such that for the corresponding solution Z, $\mathbf{P}(Z(t) \in B(a, r)) > 0$.

Let us denote by $\tilde{Z}$ the solution of the equation

$$d\tilde{Z} = A\tilde{Z}\, dt + B(\tilde{Z})\, dW, \qquad \tilde{Z}(0) = x.$$

From Theorem A.1 there exists a constant $R > 0$ such that

$$\sup_{0 \le \tau \le t} \mathbf{E}\, |\tilde{Z}(\tau)|^2 \le \frac{R^2}{4}. \tag{11}$$

Take an element $\tilde{a}$ of $D(A)$ such that $|a - \tilde{a}| < r/3$. From now on R and $\tilde{a}$ are fixed. For any $\tau < t$ let us denote by f_τ a bounded and measurable extension of the function $\tilde f_\tau$ defined by

$$\tilde f_\tau(s,y) = \begin{cases} 0, & \text{if } |y| \ge 2R, \\ \dfrac{1}{t-\tau} S(s-\tau)(\tilde a - y) - A\tilde a, & \text{if } |y| \le R. \end{cases}$$

We will show that for a certain $t_1 < t$ the function $f = f_{t_1}$ has the desired properties. Obviously we may assume that there exist constants c_1, c_2 and extensions f_τ of $\tilde f_\tau$ such that

$$|f_\tau(s,y)| \le c_1(t-\tau)^{-1} + c_2 \quad \text{for all } 0 \le \tau < t, s \in [\tau, t], y \in H. \tag{12}$$

Let Z_τ be the solution of (10) with f equal to f_τ and $t_1 = \tau$. From Theorem A.1, (11) and (12) we have

$$\sup_{0\le\tau<t} \sup_{0\le s\le t} \mathbf{E}\,|Z_\tau(s)|^2 < \infty.$$

Hence from (B.1) we may find and fix $t_1 < t$ such that

$$\mathbf{E}\Big|\int_{t_1}^t S(t-s)B(Z_{t_1}(s))\,dW(s)\Big|^2 \le \frac{r^2}{18}$$

and consequently

$$\mathbf{P}\Big(\Big|\int_{t_1}^t S(t-s)B(Z_{t_1}(s))\,dW(s)\Big| \ge \frac{r}{3}\Big) \le \frac{1}{2}. \tag{13}$$

Let us observe now that for all $y \in H$ such that $|y| \le R$,

$$\begin{aligned} &S(t-t_1)y + \int_{t_1}^t S(t-s)f_{t_1}(s,y)\,ds \\ &\quad = S(t-t_1)y + \frac{1}{t-t_1}\int_{t_1}^t S(t-s)S(s-t_1)(\tilde a - y)\,ds \\ &\qquad - \int_{t_1}^t S(t-s)A\tilde a\,ds \\ &\quad = S(t-t_1)\tilde a + \int_{t_1}^t \frac{d}{ds}S(t-s)\tilde a\,ds = \tilde a. \end{aligned} \tag{14}$$

Set $Z = Z_{t_1}$ and $f = f_{t_1}$. Then Z satisfies

$$\begin{aligned} Z(t) &= \Big[S(t-t_1)Z(t_1) + \int_{t_1}^t S(t-s)f(s,Z(t_1))\,ds\Big] \\ &\quad + \int_{t_1}^t S(t-s)B(Z(s))\,dW(s) =: I_1 + I_2. \end{aligned}$$

Combining (11) with (14) gives

$$\mathbf{P}(I_1 = \tilde a) \ge \mathbf{P}(|Z(t_1)| \le R) \ge \tfrac{3}{4}.$$

From (13),

$$\mathbf{P}\left(|I_2| \geq \frac{r}{3}\right) \leq \frac{1}{2}.$$

Consequently, as $|\tilde{a} - a| < r/3$ we have

$$\begin{aligned}
\mathbf{P}(Z(t) \in B(a,r)) &= \mathbf{P}(|Z(t) - a| < r) \\
&= \mathbf{P}(|(I_1 - \tilde{a}) + I_2 + \tilde{a} - a| < r) \\
&\geq \mathbf{P}\left(I_1 = \tilde{a} \text{ and } |I_2| \leq \frac{r}{3}\right) \\
&\geq \mathbf{P}(I_1 = \tilde{a}) - \mathbf{P}\left(|I_2| \geq \frac{r}{3}\right) \\
&\geq \frac{3}{4} - \frac{1}{2} = \frac{1}{4} > 0,
\end{aligned}$$

which completes the proof of Theorem 1.3. □

4. Stochastic heat equation. Let S^1 be the unit circle and let $W(\cdot,\cdot)$ be a Brownian sheet on $[0,\infty) \times S^1$. In this section we are concerned with the equations of the form

$$\begin{aligned}
&\frac{\partial X(t)}{\partial t}(\xi) = \frac{\partial^2 X(t)}{\partial \xi^2}(\xi) - \alpha X(t)(\xi) + f(X(t)(\xi)) + b(X(t)(\xi))\frac{\partial^2 W}{\partial t \partial \xi}, \\
&X(0)(\cdot) = x(\cdot) \in L^2(S^1),
\end{aligned} \tag{15}$$

where α is a constant and f, b are real-valued functions. Note that (15) is a special case of (1), with

$$H = L^2(S^1), \qquad Ax = \left(\frac{d^2}{d\xi^2} - \alpha\right)x, \qquad D(A) = W^{1,2}(S^1),$$

and the mappings F and B given for $\xi \in S^1$ and $x, y \in L^2(S^1)$ by

$$\begin{aligned}
F(x)(\xi) &= f(x(\xi)), \\
B(x)[y](\xi) &= b(x(\xi))y(\xi).
\end{aligned} \tag{16}$$

THEOREM 4.1. *Assume that:*

(C.1) *f and b are Lipschitz continuous.*
(C.2) *b is bounded and there exists a constant $m > 0$ such that $|b(\xi)| \geq m$ for all $\xi \in \mathbb{R}$.*

Then the process X given by (15) *is strong Feller and irreducible.*

PROOF. Let us observe first that the semigroup S generated by A is given by

$$S(t)x = e^{-xt}\langle x, h_0\rangle h_0 + \sum_{n=1}^{\infty} e^{-(n^2+\alpha)t}\{\langle x, h_{2n-1}\rangle h_{2n-1} + \langle x, h_{2n}\rangle h_{2n}\},$$

where $h_0 \equiv (2\pi)^{-1/2}$, $h_{2n-1}(\xi) = \pi^{-1/2}\cos n\xi$ and $h_{2n}(\xi) = \pi^{-1/2}\sin n\xi$. Thus S satisfies (A.4) with an arbitrary $\alpha \in (0, 1/2)$. Obviously (A.2) holds. Unfortunately B given by (16) is not a Lipschitz continuous mapping acting from H into $L(H, H)$. However, it is easy to check that (B.1) holds. Thus the irreducibility of X is a direct consequence of Theorem 1.3. As far as the strong Feller property is concerned we note that if we replace (A.1) by (B.1), then the method applied in the proof of Theorem 1.2 works with only one exception. Namely, we do not know whether mappings $B_n(z)$ given by (9) are invertible. In the present particular situation their invertibility is a simple consequence of the fact that either $b(\xi) \geq m$ for all ξ or $b(\xi) \leq -m$ for all ξ. □

In [12], [13] and [14] the existence and uniqueness of the invariant measure for (15) in the space $C(S^1)$ of the continuous function is shown, provided that (C.1) and (C.2) are fulfilled and:

(C.3) f is bounded, $\alpha > 0$ and the Lipschitz constant for f is strictly smaller than α.

The last condition is needed only in the proof of existence. In [13] and [14] some specific properties of heat kernels are used. In [12] the idea of coupling is applied.

As a direct corollary of Theorems 1.4 and 4.1 we have the following stronger version of the uniqueness results from [13], [14] and [12].

THEOREM 4.2. *If* (C.1) *and* (C.2) *hold, then there exists at most one invariant measure for* (15) *in the space* $L^2(S^1)$. *Moreover, if* μ *is the invariant measure and if* $P_t(\cdot,\cdot)$ *is the transition probabilities for* (15), *then for every Borel set* Γ, $P_t(x,\Gamma) \to \mu(\Gamma)$ *as* $t \to \infty$.

APPENDIX

Throughout the Appendix we assume that S, F and B satisfy (B.1) and (A.3).

THEOREM A.1. (i) *For all* $x \in H$, (1) *has the unique solution* $X(\cdot, x)$ *satisfying*

$$\sup_{0\leq t\leq T} \mathbf{E}\,|X(t,x)|^2 < \infty \quad \textit{for } T < \infty.$$

(ii) *Let $F_n: H \to H$, $B_n: H \to L(H)$ be a sequence of mappings such that B_n are weakly continuous and for all $n \in \mathbb{N}$ and $z, y \in H$,*

$$|F_n(z) - F_n(y)| \le L|z - y|,$$
$$\big\|S(t)[B_n(z) - B_n(y)]\big\|_2 \le L\|S(t)\|_2|z - y|$$

and for all z and $t > 0$,

$$\lim_{n\to\infty} |F_n(z) - F(z)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|S(t)[B_n(z) - B(z)]\|_2 = 0.$$

Let $X_n(\cdot, x)$ be the solution of the equation

$$dX_n = (AX_n + F_n(X_n))\,dt + B_n(X_n)\,dW, \qquad X_n(0) = x. \tag{17}$$

Then for all $T > 0$ and $x \in H$,

$$\lim_{n\to\infty} \sup_{0\le t\le T} \mathbf{E}\,|X_n(t,x) - X(t,x)|^2 = 0.$$

PROOF. For given $T \ge 0$ and $\Delta > 0$ let $\mathscr{K}_{T,\Delta}$ denote the space of all predictable processes $Z: [T, T+\Delta] \times \Omega \to H$ such that

$$|||Z|||_{T,\Delta} := \Big(\sup_{T\le t\le T+\Delta} \mathbf{E}\,|Z(t)|^2\Big)^{1/2} < \infty.$$

Obviously $(\mathscr{K}_{T,\Delta}, |||\cdot|||_{T,\Delta})$ is a Banach space. For arbitrary $Z \in \mathscr{K}_{T,\Delta}$, $t \in [T, T+\Delta]$ and $\xi \in L^2(\Omega, \mathscr{F}_T, d\mathbf{P}; H)$ set

$$\mathscr{J}_{T,\Delta}(\xi, Z)(t) = S(t-T)\xi + \int_T^t S(t-s)F(Z(s))\,ds + \int_T^t S(t-s)B(Z(s))\,dW(s),$$

$$\mathscr{J}^{(n)}_{T,\Delta}(\xi, Z)(t) = S(t-T)\xi + \int_T^t S(t-s)F_n(Z(s))\,ds + \int_T^t S(t-s)B_n(Z(s))\,dW(s).$$

By the hypothesis $\mathscr{J}_{T,\Delta}$ and $\mathscr{J}^{(n)}_{T,\Delta}$ are well defined mappings with values in $\mathscr{K}_{T,\Delta}$, and for all ξ and Z, $\mathscr{J}^{(n)}_{T,\Delta}(\xi, Z)$ converges to $\mathscr{J}_{T,\Delta}(\xi, Z)$ in $\mathscr{K}_{T,\Delta}$. Moreover, it is easy to check that for every $\tilde{T} > 0$ there exist $\Delta_0 > 0$ and a constant $c < 1$ such that for all $T \le \tilde{T}$, ξ, Z, $\tilde{Z}$ and $n \in \mathbb{N}$,

$$|||\mathscr{J}_{T,\Delta_0}(\xi, Z) - \mathscr{J}_{T,\Delta_0}(\xi, \tilde{Z})|||_{T,\Delta_0} \le c|||Z - \tilde{Z}|||_{T,\Delta_0},$$
$$|||\mathscr{J}^{(n)}_{T,\Delta_0}(\xi, Z) - \mathscr{J}^{(n)}_{T,\Delta_0}(\xi, \tilde{Z})|||_{T,\Delta_0} \le c|||Z - \tilde{Z}|||_{T,\Delta_0}.$$

Thus by the local inversion theorem (see [5]), for all T and Δ, (1) and (17) have the unique solutions $X(\cdot, x)$ and $X_n(\cdot, x)$ in $\mathscr{K}_{T,\Delta}$ and

$$\lim_{n\to\infty} X_n(\cdot, x) = X(\cdot, x) \ \text{ in } \mathscr{K}_{T,\Delta},$$

which is the desired conclusion. □

Acknowledgments. The authors would like to thank Professor G. Da Prato for discussions leading to the formulation of condition (B.1).

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