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A LIMIT THEOREM FOR A CLASS OF INTERACTING PARTICLE SYSTEMS

BY ITALO SIMONELLI

Temple University

Let S be a countable set and Λ the collection of all subsets of S. We consider interacting particle systems (IPS) $\{\eta_t\}$ on Λ, with duals $\{\tilde{\eta}_t\}$ and duality equation $P[|\eta_t^{\zeta} \cap A|\ odd] = \tilde{P}[|\tilde{\eta}_t^{A} \cap \zeta|\ odd]$, $\zeta, A \subset S$, A finite. Under certain conditions we find all the extreme invariant distributions that arise as limits of translation invariant initial configurations. Specific systems will be considered. A new property of the annihilating particle model is then used to prove a limiting relation between the annihilating and coalescing particle models.

1. Introduction. Let S be a countable set and Λ the collection of all subsets of S. In this paper we are going to study the limiting behavior of a certain class of particle systems $\{\eta_t\}$ on Λ, characterized by the existence of different processes $\{\tilde{\eta}_t\}$ that satisfy the equation

$$(1) \qquad P\big[|\eta_t^{\zeta} \cap A|\text{odd}\big] = \tilde{P}\big[|\tilde{\eta}_t^{A} \cap \zeta\,|\text{odd}\big],$$

where $\tilde{\eta}_0^A = A \subset S$, $|A| < +\infty$, $\eta_0^{\zeta} = \zeta \subset S$,

$$|\eta_t^{\zeta} \cap A| = \big|\{x_i \in S : x_i \in \eta_t^{\zeta} \cap A\}\big| \quad \text{and} \quad |\tilde{\eta}_t^{A} \cap \zeta\,| = \big|\{x_i \in S : x_i \in \tilde{\eta}_t^{A} \cap \zeta\,\}\big|.$$

The processes $\{\tilde{\eta}_t\}$ are called dual processes, and (1) is their corresponding duality equation.

Among the systems that can be defined via a percolation structure, with $S = Z^d$, the d-dimensional integer lattice, a well known class that satisfies (1) is that of cancellative systems. In 1979, David Griffeath proved that if η_t is any cancellative system without spontaneous birth, and $\mu_{1/2}$ is the product measure with density 1/2, then there is a distribution ν_* such that

$$\mu_{1/2}S(t) \to \nu_* \quad \text{as } t \to +\infty,$$

where $S(t)$ denotes the Feller semigroup associated with η_t, $\mu_{1/2}S(t)$ denotes a version of the process with initial distribution $\mu_{1/2}$ and $\to$ denotes weak convergence. Furthermore, the distribution ν_* is such that $\forall A \subset Z^d$ finite,

$$(2) \qquad \nu_*[\eta : |\eta \cap A|\text{odd}] = \tfrac{1}{2}\tilde{P}\big[\tilde{\eta}_s^A \neq \varnothing\ \forall s \geq 0\big].$$

The generality of this raises the question of whether this is an isolated instance or if, instead, there is a unifying theory for the asymptotic behavior of those systems. We have a partial answer to this question. Throughout this paper ν_* will denote the distribution defined by (2), and for arbitrary $\eta \subset S$,

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δ_η will denote the point-mass measure concentrated at η. Our main result is the following.

THEOREM 1. *Assume η_t and $\tilde{\eta}_t$ satisfy the duality equation* (1) *and the following conditions*:

(a) *For each $t > 0$, $x \in S$, $\zeta \subset S$, $\zeta \neq \varnothing$, we have $0 < P[\eta_t^\zeta(x) = 1] < 1$.*

(b) *Let $\tilde{\mathscr{A}}_\infty = \{\tilde{\eta}: \tilde{\eta}_t \neq \varnothing, \forall t \geq 0\}$. Then on $\tilde{\mathscr{A}}_\infty$, $|\tilde{\eta}_t| \to +\infty$ as $t \to +\infty$ a.s.*

Then we have:

(i) *For all translation invariant distribution ν,*

$$\lim_{t \to +\infty} \nu S(t) = \nu(\{\varnothing\})\delta_\varnothing + (1 - \nu(\{\varnothing\}))\nu_*.$$

(ii) *If $\nu_* = \delta_\varnothing$, then $\delta_\varnothing$ is the unique invariant distribution.*

(iii) *If $\nu_* \neq \delta_\varnothing$, then $\nu_*(\{\varnothing\}) = 0$.*

The above result was motivated by a theorem proved by Harris (1976). Even though the statements of these two theorems are quite similar, there are two important differences between them. One is that Harris' result applies to particle systems with duality equation

$$P\big[\eta_t^\zeta \cap A \neq \varnothing\big] = \tilde{P}\big[\tilde{\eta}_t^A \cap \zeta \neq \varnothing\big].$$

This equation implies that the function

$$f(\zeta) = P\big[\eta_t^\zeta \cap A \neq \varnothing\big]$$

is monotone, and this property is an important characterization of the systems to which his result applies as well as an essential ingredient in his proof. The other difference is that Harris did not include part (b) of our theorem as part of his assumptions. Part (b) is equally important in both theorems, but he proved it to be a consequence of part (a).

This paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we use Theorem 1 to study the asymptotic behavior of specific particle systems: the branching annihilating processes and the branching annihilating random walk are the systems that we consider. In Section 4 we consider the annihilating and coalescing particle models, with common irreducible probability density $p(x, y)$, $x, y \in Z^d$. We denote these two processes by η_t and ξ_t, respectively. For all finite $A \subset Z^d$ we prove that

$$\lim_{t \to +\infty} \frac{P\big[|\eta_t^{Z^d} \cap A| \text{odd}\big]}{P\big[\xi_t^{Z^d} \cap A \neq \varnothing\big]} = \frac{1}{2},$$

extending a previous result of Arratia (1981). An important ingredient in the proof of this result is a property of the annihilating particle model proved in Section 2.

2. Proof of Theorem 1. Before proving Theorem 1 we need to prove some related results. Let $C_0(\Lambda)$ be the collection of all continuous functions on Λ that depend only on a finite number of coordinates. Since in Λ the class of cylindrical sets is a class of convergence-determining sets, the weak limit of a distribution μ is uniquely determined by limits of the form

$$\lim_{t\to+\infty}\int_X f\,d\mu\,S(t) \quad \text{with } f\in C_0(\Lambda).$$

The next result further restricts the collection of convergence-determining functions that we need to consider. Since this result is already known [Griffeath (1979), Chapter 3], we will only give a sketch of its proof.

Lemma 1. *Let $f\in C_0(\Lambda)$. Then f can be written as a linear combination of functions of the type*

$$1_{|\eta\cap A|odd} \text{ or } 1_{|\eta\cap A|even} \quad \text{with } A\subset S, |A|<+\infty.$$

Proof. It suffices to consider functions f of the type

$$f(\eta)=H(\eta,A)=\prod_{x\in A}\eta(x),$$

where A is any fixed finite subset of S, $\eta(x)=1$ if $x\in\eta$ and $\eta(x)=0$ otherwise. Furthermore, for such functions the result will follow once we show that Lemma 1 holds for functions g, $g(\eta)=\eta(z)\cdot 1_{|\eta\cap A|\text{odd}}$, where $z\in S$ and A is as before. If $z\notin A$, it is easy to show that

$$\eta(z)\cdot 1_{|\eta\cap A|\text{odd}}=\tfrac{1}{2}\left[1_{|\eta\cap A|\text{odd}}+1_{|\eta\cap A\cup\{z\}|\text{even}}-1_{|\eta\cap\{z\}|\text{even}}\right].$$

If $z\in A$,

$$\eta(z)\cdot 1_{|\eta\cap A|\text{odd}}=\eta(z)-\eta(z)\cdot 1_{|\eta\cap A\setminus\{z\}|\text{odd}}$$

and we are back to the previous case. The result now follows. □

Definition. *For a any distribution μ, $A\subset S$ finite and $t\geq 0$, let*

$$\hat{\mu}_t(A)=\mu S(t)[\eta:|\eta\cap A|odd]. \tag{3}$$

From the above definition we can derive the following relations:

$$\begin{aligned}\hat{\mu}_{t+s}(A)&=\int_\Lambda 1_{|\zeta\cap A|\text{odd}}\,d\mu\,S(t+s)(\zeta)=\int_\Lambda S(s)1_{|\zeta\cap A|\text{odd}}\,d\mu\,S(t)(\zeta)\\&=\int_\Lambda P\left[|\eta_s^\zeta\cap A|\text{odd}\right]d\mu\,S(t)(\zeta)=\int_\Lambda \tilde{P}\left[|\tilde{\eta}_s^A\cap\zeta|\text{odd}\right]d\mu\,S(t)(\zeta)\\&=\int_\Lambda\sum_{\substack{C\subset S\\|C|<+\infty}}\tilde{P}\left[\tilde{\eta}_s^A=C\right]1_{|C\cap\zeta|\text{odd}}\,d\mu\,S(t)(\zeta).\end{aligned}$$

Thus we have

$$
\begin{aligned}
\hat{\mu}_{t+s}(A) &= \sum_{\substack{C\subset S\\ |C|<+\infty}} \tilde{P}\left[\tilde{\eta}_s^A = C\right]\int_{\Lambda} 1_{|C\cap\zeta|odd}\, d\mu S(t)(\zeta)\\
&= \sum_{\substack{C\subset S\\ |C|<+\infty}} \tilde{P}\left[\tilde{\eta}_s^A = C\right] P\left[|\eta_t^{\mu}\cap C| odd\right] \qquad (4)\\
&= \sum_{\substack{C\subset S\\ |C|<+\infty}} \tilde{P}\left[\tilde{\eta}_s^A = C\right]\hat{\mu}_t(C).
\end{aligned}
$$

Furthermore, if μ is invariant, that is, $\mu S(t) = \mu\ \forall t \geq 0$, from (4) we have

$$\hat{\mu}(A) = \sum_{\substack{C\subset S\\ |C|<+\infty}} \tilde{P}\left[\tilde{\eta}_s^A = C\right]\hat{\mu}(C).$$

The next two results are very important for the proofs of Theorem 1 and Theorem 6 in Section 4. The motivation is the following. Let η_t be a system that satisfies Theorem 1 and view η_t as a collection of particles. Then, if μ is any translation invariant distribution with $\mu(\{\varnothing\}) = 0$ and t is fixed, η_t^{μ} will contain an infinite number of particles, with probability 1, and they will uniformly cover the space. Moreover, if $A \subset S$ is large enough, A will also contain a very large number of particles. If the coordinates of η_t^{μ} were independent, it would immediately follow that for large A the probability of having an odd number of particles in A will be close to $1/2$, with the difference going to zero as the size of A goes to infinity. The next two results will show that the conditions of Theorem 1 are sufficient for this to occur.

LEMMA 2. *Let $Y_1, Y_2, \ldots, Y_k$ be random variables with $0 \leq Y_i \leq 1$ and*

$$P[1-\delta \leq Y_i \leq \delta] \geq 1-\varepsilon, \qquad 1 \leq i \leq k \quad \textit{for some } \varepsilon > 0,\ \delta \in \left(\tfrac{1}{2}, 1\right).$$

Then $E[|1-2Y_1|\cdots|1-2Y_k|] \leq \varepsilon + (2\delta-1)^k$.

PROOF. Let $X_i = |1-2Y_i|$. Then $0 \leq X_i \leq 1$, $P[X_i \geq (2\delta-1)] \leq \varepsilon$ and we need to show

$$E[X_1 \cdots X_k] \leq \varepsilon + (2\delta-1)^k.$$

The proof is based on applying Hölder inequality $k-1$ times, each time with different conjugates p_i and q_i $(1/p_i + 1/q_i = 1)$. Precisely, $p_1 = k$, $p_2 = k -$

$1, \dots, p_{k-1} = 2$. Using these conjugates we obtain

$$\begin{aligned} E[X_1 \cdots X_k] &\le \left(EX_1^k\right)^{1/k}\left(E(X_2 \cdots X_k)^{k/(k-1)}\right)^{(k-1)/k} \\ &\le \left(EX_1^k\right)^{1/k}\left(EX_2^k\right)^{1/k}\left(E(X_3 \cdots X_k)^{k/(k-2)}\right)^{(k-2)/k} \\ &\ \ \vdots \\ &\le \left(EX_1^k\right)^{1/k}\left(EX_2^k\right)^{1/k} \cdots \left(EX_k^k\right)^{1/k} \\ &\le \max_{1 \le i \le k} E\left(X_i^k\right). \end{aligned}$$

Since for arbitrary i, $1 \le i \le k$, we have

$$E\left(X_i^k\right) = \int_{\{X_i > (2\delta - 1)\}} X_i^k \, dP + \int_{\{X_i \le (2\delta - 1)\}} X_i^k \, dP \le \varepsilon + (2\delta - 1)^k,$$

we obtain $E[X_1 \cdots X_k] \le \varepsilon + (2\delta - 1)^k$ as desired, and the lemma is proved. □

THEOREM 2. *Let μ be a translation invariant measure with $\mu(\{\varnothing\}) = 0$. Then $\forall \varepsilon > 0$, $\forall t > 0$, $\exists k = k(\varepsilon, t)$ such that if $A \subset S$ with $|A| \ge k$,*

$$|P[|\eta_t^\mu \cap A| odd] - P[|\eta_t^\mu \cap A| even]| \le \varepsilon.$$

PROOF. Let $\varepsilon > 0$, $t > 0$ be given. For every $s \in [0, \varepsilon]$, assumption (a) of Theorem 1 implies that

$$\mu S(t)[\eta : P^\eta[\eta_s(0) = 1] = 1] = 0.$$

Therefore, $\exists \delta_1 \in (\frac{1}{2}, 1)$ such that $\forall s \in [0, \varepsilon]$,

$$\mu S(t)[\eta : P^\eta[\eta_s(0) = 1] > \delta_1] < \frac{\varepsilon}{6}.$$

A similar argument, combined with the assumption $\mu(\{\varnothing\}) = 0$, gives the existence of $\delta_2 \in (\frac{1}{2}, 1)$ such that $\forall s \in [0, \varepsilon]$,

$$\mu S(t)[\eta : P^\eta[\eta_s(0) = 0] > \delta_2] < \frac{\varepsilon}{6}.$$

Let $\delta = \max\{\delta_1, \delta_2\}$. Then $\forall s \in [0, \varepsilon]$ we have

$$\mu S(t)[\eta : 1 - \delta < P^\eta[\eta_s(0) = 1] < \delta] > 1 - \frac{\varepsilon}{3}. \tag{5}$$

Let A be an arbitrary subset of S, that is, $A = \{x_1, x_2, \dots, x_n\}$, and let

$$f_{\mu_t}(A) = P[|\eta_t^\mu \cap A| \text{odd}] - P[|\eta_t^\mu \cap A| \text{even}] = \int_\Lambda \left(1_{|\eta \cap A| \sigma} - 1_{|\eta \cap A| \rho}\right) d\mu S(t),$$

where $\sigma = $ odd and $\rho = $ even. For $1 \le i \le n$ we define $A_0 = A$ and $A_i =$

$A_{i-1} \setminus \{x_i\}$. Writing

$$\begin{aligned}
&1_{|\eta \cap A_i|\sigma} - 1_{|\eta \cap A_i|\rho} \\
&\quad = (1 - \eta(x_i))1_{|\eta \cap A_{i-1}|\sigma} + \eta(x_i)1_{|\eta \cap A_{i-1}|\rho} - (1 - \eta(x_i))1_{|\eta \cap A_{i-1}|\rho} \\
&\qquad - \eta(x_i)1_{|\eta \cap A_{i-1}|\sigma} \\
&\quad = (1 - 2\eta(x_i))(1_{|\eta \cap A_{i-1}|\sigma} - 1_{|\eta \cap A_{i-1}|\rho}),
\end{aligned}$$

we have

$$f_{\mu_t}(A) = \int_\Lambda -(1 - 2\eta(x_1)) \cdots (1 - 2\eta(x_n))\, d\mu S(t).$$

Since $g_i(\eta) = (1 - 2\eta(x_i)) \in C_0(\Lambda)$, and the map $s \mapsto S(s)f$ is continuous for every fixed $f \in C_0(\Lambda)$ [Liggett (1985), Chapter 1], we can find an r_i, $0 < r_i < \varepsilon$, such that if $0 \le h_i \le r_i$,

$$\|S(h_i)g_i - g_i\| \le \frac{\varepsilon}{3|A|} = \frac{\varepsilon}{3 \cdot n}.$$

Moreover, if we let $r = \min\{r_1, \cdots, r_n\}$ we have

$$\begin{aligned}
g_1(\eta) \cdots g_n(\eta) &\le (1 - 2S(r)\eta(x_1))g_2(\eta) \cdots g_n(\eta) + \frac{3}{3 \cdot n} \\
&\le (1 - 2S(r)\eta(x_1))(1 - 2S(r)\eta(x_2)) \cdots g_n(\eta) + \frac{2\varepsilon}{3 \cdot n} \\
&\ \ \vdots \\
&\le (1 - 2S(r)\eta(x_1))(1 - 2S(r)\eta(x_2)) \cdots (1 - 2S(r)\eta(x_n)) \\
&\qquad + \frac{n\varepsilon}{3 \cdot n}.
\end{aligned}$$

Hence,

$$g_1(\eta) \cdots g_n(\eta) \le |1 - 2S(r)\eta(x_1)| \cdots |1 - 2S(r)\eta(x_n)| + \frac{\varepsilon}{3}.$$

Since $-(g_1(\eta)g_2(\eta) \cdots g_n(\eta))$ also satisfies the above inequality, we have

$$|f_{\mu_t}(A)| \le \int_\Lambda |1 - 2S(r)\eta(x_1)| \cdots |1 - 2S(r)\eta(x_n)|\, d\mu S(t) + \frac{\varepsilon}{3}.$$

Next let $Y_i(\eta) = S(r)\eta(x_i) = P[\eta_r^\eta(x_i) = 1]$. Then Y_i is a random variable on Λ, $0 \le Y_i \le 1$, and by (5) we have that $\mu S(t)[\eta : (1 - \delta) \le Y_i \le \delta] \ge 1 - \varepsilon/3$. Therefore, the Y_i's satisfy the conditions of Lemma 2, and we have

$$|f_{\mu_t}(A)| \le \frac{\varepsilon}{3} + (2\delta - 1)^n + \frac{\varepsilon}{3}.$$

Hence, if we let $k(\varepsilon, t) = k_0$ be any integer such that $(2\delta - 1)^{k_0} < \varepsilon/3$, then $|f_{\mu_t}(A)| \le \varepsilon$ whenever $|A| \ge k_0$ and the proof of Theorem 2 is complete. □

REMARK. The requirement $t > 0$ in Theorem 2 can be relaxed if we exclude the case $\mu = \delta_S$.

Finally we are ready to prove Theorem 1.

PROOF OF THEOREM 1. First we note that the existence of (1) implies that $\delta_\varnothing$ is an invariant measure, that is $\delta_\varnothing S(t) = \delta_\varnothing$ $\forall t \geq 0$. Next let ν be an arbitrary translation invariant distribution. Then ν can be written as

$$\begin{aligned}\nu &= \nu(\{\varnothing\})\,\delta_\varnothing + (1 - \nu(\{\varnothing\}))\left[\frac{\nu - \nu(\{\varnothing\})\,\delta_\varnothing}{1 - \nu(\{\varnothing\})}\right] \\ &= \nu(\{\varnothing\})\,\delta_\varnothing + (1 - \nu(\{\varnothing\}))\,\mu,\end{aligned}$$

where μ is translation invariant and $\mu(\{\varnothing\}) = 0$. Hence

$$\begin{aligned}\lim_{t \to +\infty} \nu S(t) &= \lim_{t \to +\infty}\left[\nu(\{\varnothing\})\,\delta_\varnothing + (1 - \nu(\{\varnothing\}))\left[\frac{\nu - \nu(\{\varnothing\})\,\delta_\varnothing}{1 - \nu(\{\varnothing\})}\right]\right] S(t) \\ &= \nu(\{\varnothing\})\,\delta_\varnothing + (1 - \nu(\{\varnothing\}))\lim_{t \to +\infty} \mu S(t).\end{aligned}$$

Therefore, we need to show that $\lim_{t \to +\infty} \mu S(t) = \nu_*$, and by Lemma 1, it suffices to show that for all finite $A \subset S$ $\lim_{t \to +\infty} \hat{\mu}_t(A) = \hat{\nu}_*(A)$, where $\hat{\mu}_t(A)$ and $\hat{\nu}_*(A)$ are defined by (3). First let us assume that $\tilde{P}^A[\tilde{\mathscr{A}}_\infty] = 0$ and note that (1) also implies that $\{\varnothing\}$ is an absorbing state for $\tilde{\eta}_t$. Then

$$\begin{aligned}\lim_{t \to +\infty} \hat{\mu}_t(A) &= \lim_{t \to +\infty} P[|\eta_t^\mu \cap A|\text{odd}] = \lim_{t \to +\infty} \int_\Lambda \tilde{P}[|\tilde{\eta}_t^A \cap \zeta|\text{odd}]\, d\mu(\zeta) \\ &\leq \lim_{t \to +\infty} \tilde{P}[\tilde{\eta}_t^A \neq \varnothing] = \tilde{P}^A[\tilde{\mathscr{A}}_\infty] = 0.\end{aligned}$$

Next we assume that $\tilde{P}^A[\tilde{\mathscr{A}}_\infty] \neq 0$. Then

$$\begin{aligned}\lim_{t \to +\infty} \hat{\mu}_t(A) &= \lim_{t \to +\infty}\left[\sum_{\substack{C \subset S \\ |C| < +\infty}} \tilde{P}[\tilde{\eta}_{t-s}^A = C] \cdot \hat{\mu}_s(C)\right] \qquad \text{[by (4)]} \\ &= \lim_{t \to +\infty}\left[\sum_{\substack{C \subset S \\ |C| < +\infty}} \tilde{P}[\tilde{\eta}_{t-s}^A = C|\tilde{\mathscr{A}}_\infty] \cdot \tilde{P}^A[\tilde{\mathscr{A}}_\infty] \cdot \hat{\mu}_s(C)\right],\end{aligned}$$

where the above the equality holds because

$$\lim_{t \to +\infty}\left[\sum_{\substack{C \subset S \\ |C| < +\infty}} \tilde{P}[\tilde{\eta}_{t-s}^A = C|\tilde{\mathscr{A}}_\infty^C] \cdot \tilde{P}^A[\tilde{\mathscr{A}}_\infty^C] \cdot \hat{\mu}_s(C)\right] = 0.$$

Let $\varepsilon > 0$ be given. By assumption (b) of Theorem 1, on $\tilde{\mathscr{A}}_\infty$, $|\tilde{\eta}_t| \to +\infty$ as $t \to +\infty$ a.s., that implies

$$\exists t_1(\varepsilon) \text{ such that on } \tilde{\mathscr{A}}_\infty, |\tilde{\eta}_t| \geq k(\varepsilon, s) \qquad \forall t \geq t_1(\varepsilon),$$

where $k(\varepsilon, s)$ is chosen as in Theorem 2. Hence, by Theorem 2, $\forall t \geq t_1(\varepsilon)$ we have

$$\left| \sum_{\substack{C \subset S \\ |C| < +\infty}} \tilde{P}\left[\tilde{\eta}^A_{t-s} = C | \tilde{\mathscr{A}_\infty}\right] \cdot \tilde{P}^A\left[\tilde{\mathscr{A}_\infty}\right] \cdot \hat{\mu}_s(C) - \frac{\tilde{P}^A\left[\tilde{\mathscr{A}_\infty}\right]}{2} \right| \leq \varepsilon.$$

Therefore,

$$\lim_{t \to +\infty} \hat{\mu}_t(A) = \tfrac{1}{2}\tilde{P}^A[\tilde{\eta}_t \neq \varnothing \, \forall t \geq 0].$$

Since the above limit also covers the case $\tilde{P}^A[\tilde{\mathscr{A}_\infty}] = 0$, part (i) of Theorem 1 is proved. Next let us assume $\nu_* = \delta_\varnothing$ and let μ be an arbitrary distribution. Then $\forall A \subset S$ finite,

$$\begin{aligned} P\left[|\eta_t^\mu \cap A| \text{odd}\right] &= \int_\Lambda P\left[|\eta_t^\zeta \cap A| \text{odd}\right] d\mu(\zeta) \\ &= \int_\Lambda \tilde{P}\left[|\tilde{\eta}_t^A \cap \zeta| \text{odd}\right] d\mu(\zeta) \leq \tilde{P}\left[\tilde{\eta}_t^A \neq \varnothing\right]. \end{aligned}$$

Since, $\nu_* = \delta_\varnothing$ if and only if $\tilde{P}^A[\tilde{\mathscr{A}_\infty}] = 0$ for all A finite, then

$$\lim_{t \to +\infty} P\left[|\eta_t^\mu \cap A| \text{odd}\right] \leq \lim_{t \to +\infty} \tilde{P}\left[\tilde{\eta}_t^A \neq \varnothing\right] = 0.$$

Therefore, $\lim_{t \to +\infty} \mu S(t) = \delta_\varnothing$ and part (ii) of Theorem 1 is also proved. For part (iii) we assume $\nu_* \neq \delta_\varnothing$ and want to show $\nu_*(\{\varnothing\}) = 0$. Let $\varepsilon > 0$ be given. We are going to show $\nu_*(\{\varnothing\}) < \varepsilon$. By definition,

$$\nu_*(\{\varnothing\}) = \lim_{t \to +\infty} \int_\Lambda \prod_{x \in S} (1 - \eta(x)) \, d\mu S(t) = \lim_{t \to +\infty} h(t),$$

where μ is any translation invariant distribution with $\mu(\{\varnothing\}) = 0$. Let $\varepsilon' > 0$ be arbitrary but fixed. Then $\exists t_1$ such that if $t \geq t_1$,

$$|\nu_*(\{\varnothing\}) - h(t)| \leq \frac{\varepsilon'}{3}.$$

Moreover,

$$\begin{aligned} \nu_*(\{\varnothing\}) &\leq \int_\Lambda \prod_{x \in S} (1 - \eta(x)) \, d\mu S(t) + \frac{\varepsilon'}{3} \\ &\leq \int_\Lambda \prod_{i=1}^{n} (1 - \eta(x_i)) \, d\mu S(t) + \frac{\varepsilon'}{3} \end{aligned}$$

for all integers $n > 0$ and for all $\{x_1, x_2, \ldots, x_n\} \subset S$. Let $t \geq t_1$. If we proceed similarly as in the proof of Theorem 2, we can find a δ, $\frac{1}{2} < \delta < 1$, such that

$$\int_\Lambda \prod_{i-1}^{n} (1 - \eta(x_i)) \, d\mu S(t) \leq \frac{2\varepsilon'}{3} + \delta^n.$$

Hence, if we let $\varepsilon' < \varepsilon/3$ and choose n large enough, we have

$$\nu_*(\{\varnothing\}) \leq \frac{2\varepsilon'}{3} + \delta^n + \frac{\varepsilon'}{3} \leq 2\varepsilon' < \varepsilon.$$

Since ε was arbitrary, this implies that $\nu_*(\{\emptyset\}) = 0$, and the proof of Theorem 1 is now complete. □

It is interesting to note that the invariant measure ν_1 obtained by Harris is defined by

$$\nu_1(A) = \lim_{t \to +\infty} P[\eta_t^\mu \cap A \neq \emptyset] = \tilde{P}[\tilde{\eta}_s^A \neq \emptyset \, \forall s \geq 0],$$

suggesting some similarities between the limiting behavior of these two classes of systems.

3. Applications (Part 1). In this section we are going to use Theorem 1 to study the asymptotic behavior of specific systems. That is, we are going to show that they satisfy the conditions of Theorem 1. Throughout this section $S = Z^d$.

Branching annihilating processes (BAP) with death rate $\delta \geq 0$. The BAP ($\delta \geq 0$) is an interacting particle system η_t which evolves according to the following dynamics:

1. Each particle gives birth to a new particle on a neighboring site at rate 1.
2. If there is a birth on a site that is already occupied, annihilation occurs.
3. Each particle dies at rate $\delta \geq 0$.

THEOREM 3. *The BAP ($\delta \geq 0$) satisfies the conditions of Theorem* 1.

REMARK. The result that, for the BAP ($\delta \geq 0$), ν_* is the unique measure that arises as the weak limit of translation invariant distributions μ's, with $\mu(\{\emptyset\}) = 0$, has been already proved by Bramson, Din and Durrett (1991), and for the case $\delta = 0$ independently by Sudbury (1990). For the case $\delta = 0$, Bramson, Din and Durrett and Sudbury also proved that indeed $\delta_\emptyset$ and ν_* are the only invariant measures. Moreover, their proofs contain the proof of Theorem 3.

PROOF OF THEOREM 3 (Sketch). Let η_t be a BAP ($\delta \geq 0$). Using a graphical representation for the construction of the η_t, Bramson, Din and Durrett have shown that indeed there is a dual $\tilde{\eta}_t$ that satisfies the duality equation

$$P[|\eta_t^\zeta \cap A| \text{odd}] = \tilde{P}[|\tilde{\eta}_t^A \cap \zeta| \text{odd}] \qquad \forall \zeta, A \subset Z^d, |A| < \infty,$$

and that $\tilde{\eta}_t$ is an independent copy of the original η_t. Condition (a) of Theorem 1 can easily be verified by looking at the dynamics of the process. It is left to show that on $\tilde{\mathscr{A}}_\infty$, $|\tilde{\eta}_t| \to +\infty$ a.s. For the case $\delta > 0$ this follows from the result that

$$(6)\qquad \inf_{0<|B|\le n} P\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0\right] \ge \left(\frac{\delta}{2d+\delta}\right)^n > 0,$$

where the above bound follows easily from the dynamics of the process (see Bramson Din and Durret for the details, or proof of the next theorem for similar arguments). For the case $\delta = 0$ the above argument cannot be used because the LHS of (6) equals 0. In this case the desired result follows from showing that the $\tilde{\eta}_t$ dominates an oriented percolation process with parameter $1 - \varepsilon$, with the property that if ε is small enough,

$$\text{on } \Omega_\infty = \left\{W_n^0 \neq \varnothing \text{ for all } n\right\}, \qquad |W_n^0| \to +\infty \text{ a.s. as } n \to +\infty,$$

where $W_n^0 = \{y \in Z^d\colon$ there is an open path from $(0,0)$ to $(y,n)\}$ [Bramson, Din and Durrett (1991)]. □

Branching annihilating random walk (*BARW*). The BARW is an interacting particle system η_t with the following rate:

1. A particle at site x jumps to a site y with probability $p(x,y)$ at rate $\alpha > 0$.
2. A particle at site x gives birth to a particle on site y with probability $p(x,y)$ at rate 1.
3. If two particles occupy the same site, annihilation occurs.

We will assume throughout this section that the Markov chain associated with the $p(x,y)$'s is *irreducible*, that is, $\forall x,\ y \in Z^d$, $\exists n > 0$ such that $p^n(x,y) > 0$.

The BARW, with $p(x,y) = 1/(2d)$ was studied by Bramson and Gray (1985) in the context of extinction and survival of the process. They showed that in Z if $\alpha < 1/100$, the process starting from any finite and fixed configuration will survive with positive probability, and if α is "large enough," then extinction is certain. In 1991, Bramson, Din and Durrett gave a much simpler proof of the survival result of the BARW with small α, and they extended it to all $d \ge 1$, and $p(x,y)$ any arbitrary random walk.

We are going to prove the following theorem.

THEOREM 4. *The BARW satisfies the condition of Theorem* 1.

PROOF. It is clear from the dynamics of the process that condition (a) of Theorem 1 is satisfied. Next we want to show the existence of a dual $\tilde{\eta}_t$ such that

$$P\left[|\eta^\zeta \cap A| \text{odd}\right] = \tilde{P}\left[|\tilde{\eta}_t^A \cap \zeta| \text{odd}\right] \qquad \forall \zeta,\ A \subset Z^d, |A| < +\infty.$$

To accomplish this we will construct the BARW using a graphical representation. The following approach is based on defining a percolation structure invented by Harris (1978) and developed by Griffeath (1979).

Graphical representation of BARW. Let us consider $Z^d \times [0, \infty)$, and think of this as assigning a time line to each $x \in Z^d$. For each x, $y \in Z^d$ with $p(x, y) > 0$, let $\{T_i^{(x,y)}(t), t \ge 0\}$, $i = 1, 2$, be two independent Poisson processes with rate 1 and α, respectively, and $\{T_{i,n}^{(x,y)}, n \ge 1\}$ their respective arrival times. At each $T_{1,n}^{(x,y)}$ we draw an arrow from $(x, T_{1,n}^{(x,y)})$ to $(y, T_{1,n}^{(x,y)})$, and at each $T_{2,n}^{(x,y)}$ we draw an arrow from $(x, T_{2,n}^{(x,y)})$ to $(y, T_{2,n}^{(x,y)})$ and put a δ at $(x, T_{2,n}^{(x,y)})$. We say that there is a path from (x, s) to (y, t), $s < t$, if there is a chain of upward vertical and directed horizontal segments leading from (x, s) to (y, t) that does not go through any δ's.

Let $N_t^x(y)$ be the number of paths from $(x, 0)$ to (y, t) and

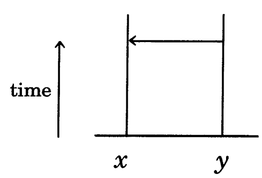
$$N_t^A(y) = \sum_{x \in A} N_t^x(y).$$

Define

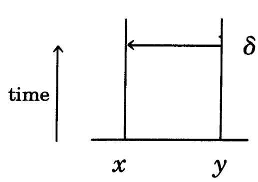
$$\eta_t^A = \{y \in Z^d : N_t^A(y) \equiv 1 \bmod 2\}. \tag{7}$$

Lemma 3. *η_t^A is a BARW.*

Proof. Since the dynamics of the BARW guarantees its uniqueness as an interacting particle system [Liggett (1985), Chapter 1], we need only to check that the effect of the arrows is consistent with the dynamics of the BARW. Let $\eta(x) = 1$ if $x \in \eta$, $\eta(x) = 0$ otherwise.



before arrow $(\eta(x), \eta(y))$	after arrow $(\eta(x), \eta(y))$
(0, 0)	(0, 0)
(0, 1)	(1, 1)
(1, 0)	(1, 0)
(1, 1)	(0, 1)



before arrow $(\eta(x), \eta(y))$	after arrow $(\eta(x), \eta(y))$
(0, 0)	(0, 0)
(0, 1)	(1, 0)
(1, 0)	(1, 0)
(1, 1)	(0, 0)

□

Next, by reversing time and the direction of the arrows in the percolation structure, we can define in the same probability space a new process $\tilde{\eta}_t$, with the property that $\forall \zeta$, $A \subset Z^d$, with A finite,

$$P\big[|\eta_t^\zeta \cap A| \text{odd}\big] = \tilde{P}\big[|\tilde{\eta}_t^A \cap \zeta| \text{odd}\big],$$

where $|\tilde{\eta}_t^A \cap \zeta|$ denotes the number of paths in the new percolation structure from (A, t) down to $(\zeta, 0)$ [Griffeath (1979), Chapter 3]. By analyzing the effect of the reversed arrows, it is easy to see that $\tilde{\eta}_t$ is a spin system. That is, change in the configuration only occurs at one site at a time, and $\tilde{\eta}(x)$ changes to $1 - \tilde{\eta}(x)$ at rate given by

$$(8) \quad c(x, \tilde{\eta}) = \begin{cases} \alpha \sum_y p(x, y)\tilde{\eta}(y) + \sum_y p(x, y)\tilde{\eta}(y), & \text{if } \tilde{\eta}(x) = 0, \\ \alpha \sum_y p(x, y)(1 - \tilde{\eta}(y)) + \sum_y p(x, y)\tilde{\eta}(y), & \text{if } \tilde{\eta}(x) = 1. \end{cases}$$

Next we are going to show that

$$(9) \qquad \inf_{0 < |B| \le n} P\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0\right] > 0,$$

and this will imply that on $\tilde{\mathscr{A}}_\infty$, $|\tilde{\eta}_t| \to +\infty$ a.s. [Bramson, Din and Durrett (1991)]. We will prove (9) only for the case $p(x, y)$ a symmetric random walk on Z^d; the same type of calculation applies to the general case. Let B, $0 < |B| \le n$, be arbitrary, and suppose first that $\alpha \le 1$. Hence from (8) we have

$$\begin{aligned} &P\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0\right] \\ &\quad \ge \tilde{P}\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0, |\tilde{\eta}_s^B| \ge |\tilde{\eta}_{s+u}^B| \,\forall s, u \ge 0\right] \\ &\quad \ge \left(\frac{\alpha}{2\alpha + 1}\right)^n > 0, \end{aligned}$$

which implies (9) whenever $\alpha \le 1$. Next suppose $\alpha > 1$. Let

$$c_1(x, \eta) = \begin{cases} c(x, \eta), & \text{if } \eta(x) = 0, \\ 1, & \text{if } \eta(x) = 1, \end{cases}$$

and denote by ζ_t the process defined by c_1. Hence we have

$$\begin{aligned} &P\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0\right] \\ &\quad \ge \tilde{P}\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0, |\tilde{\eta}_s^B| \ge |\tilde{\eta}_{s+u}^B| \,\forall s, u \ge 0\right] \\ &\quad \ge \tilde{P}\left[\zeta_t^B = \varnothing \text{ for some } t \ge 0, |\zeta_s^B| \ge |\zeta_{s+u}^B| \,\forall s, u \ge 0\right] \\ &\quad \ge \left(\frac{1}{\alpha + 2}\right)^n > 0, \end{aligned}$$

which implies (9) whenever $\alpha > 1$. Therefore, $\forall \alpha$'s we have

$$\inf_{0 < |B| \le n} P\left[\tilde{\eta}_t^B = \varnothing \text{ for some } t \ge 0\right] \ge \min\left\{\left(\frac{\alpha}{2\alpha + 1}\right)^n, \left(\frac{1}{\alpha + 2}\right)^n\right\} > 0,$$

and the proof of Theorem 4 is now complete. □

4. Applications (Part 2). In this section we consider the evolution of an arbitrary collection of independent particles that perform a continuous time random walk on Z^d, that is, each particle at a site x waits an exponential time with mean 1 and then jumps into a site y with probability $p(x, y)$, where $p(x, y)$'s are the transition probabilities associated with an irreducible Markov chain. The evolution is further characterized by introducing one of the following collision rules: annihilation or coalescence. In the annihilation case whenever one particle jumps into a site that is already occupied, both particles disappear; in the coalescence case the two particles merge into one. These two processes are known as the annihilating particle model (a.p.m) η_t, and the coalescing particle models (c.p.m) ξ_t, respectively.

In 1978, Griffeath proved that both models are ergodic, with $\delta_{\varnothing}$ the only invariant distribution. Furthermore, he observed that for arbitrary initial configurations $B \subset Z^d$, it is possible to construct a coupling such that for all $t \geq 0$,

$$\eta_t^B \subset \xi_t^B.$$

This relation motivates the study of the limiting behavior of ratios of the type

$$\frac{P\left[\eta_t^B(0) = 1\right]}{P\left[\xi_t^B(0) = 1\right]}.$$

The most general result in this context was proved by Arratia (1981). To avoid introducing new quantities, we will state a somewhat simpler version of his Theorem 3.

THEOREM 5. *Let η_t be an a.p.m. and ξ_t a c.p.m. with common probability density $p(x, y)$. If p is any genuinely multidimensional random walk, or a random walk on the integers with $\sum_i p(0, x_i)|x_i| = +\infty$, then*

$$\lim_{t \to +\infty} \frac{P\left[\eta_t^{Z^d}(0) = 1\right]}{P\left[\xi_t^{Z^d}(0) = 1\right]} = \frac{1}{2}. \tag{10}$$

REMARK. Even in its general form, Arratia's theorem does not cover the case when "p is a non-nearest neighbor walk on the integers with finite expectation."

In this section we are going to extend Theorem 5. In fact, we are going to prove the following theorem.

THEOREM 6. *Let η_t be an a.p.m. and ξ_t a c.p.m. with common-irreducible random walk $p(x, y)$. Then for all finite $A \subset Z^d$,*

$$\lim_{t \to +\infty} \frac{P\left[|\eta_t^{Z^d} \cap A| odd\right]}{P\left[\xi_t^{Z^d} \cap A \neq \varnothing\right]} = \frac{1}{2}. \tag{11}$$

REMARK. If we let $A = \{0\}$ in Theorem 6, then (10) and (11) are equivalent.

To prove Theorem 6 we need the following results. It is well known [Griffeath (1978)] that there is an interacting particle system ζ_t, the voter model, that $\forall BA \subset Z^d$ with A finite, satisfies the duality equations

$$(12)\qquad P\big[|\eta_t^B \cap A|\text{odd}\big] = P\big[|\zeta_t^A \cap B|\text{odd}\big]$$

and

$$(13)\qquad P\big[\xi_t^B \cap A \neq \varnothing\big] = P\big[\zeta_t^A \cap B \neq \varnothing\big].$$

Furthermore, the voter model satisfies the following property [Arratia (1981)], which we state as a lemma.

LEMMA 4. *For the voter model ζ_t based on an arbitrary nontrivial random walk $p(x, y)$ and for every positive integer m,*

$$P\big[|\zeta_t^0| \geq m|\zeta_t^0 \neq \varnothing\big] \to 1 \quad \textit{as } t \to +\infty.$$

REMARK. The result of Lemma 4 easily generalizes to any finite initial distribution.

REMARK. The a.p.m. satisfies the conditions of Theorem 1. The only condition that has not been verified yet is condition (a), that as usual can be verified by analyzing the rate of the system.

We are now ready to prove Theorem 6.

PROOF OF THEOREM 6. Let $A \subset \mathrm{Z}^d$ be finite. We want to show that

$$\lim_{t\to+\infty} \frac{P\big[|\eta_t^{Z^d} \cap A|\text{odd}\big]}{P\big[\xi_t^{Z^d} \cap A \neq \varnothing\big]} = \frac{1}{2}.$$

Let $s > 0$ be arbitrary and define

$$g(s,t) = \frac{P\big[|\eta_{t+s}^{Z^d} \cap A|\text{odd}\big]}{P\big[\xi_t^{Z^d} \cap A \neq \varnothing\big]}.$$

Then by (12) and (13),

$$\begin{aligned} g(s,t) &= \frac{P\big[|\eta_s^{Z^d} \cap \zeta_t^A|\text{odd}\big]}{P\big[\zeta_t^A \neq \varnothing\big]} \\ &= \frac{\sum_{C\subset Z^d, |C|<+\infty} P\big[\zeta_t^A = C\big] P\big[|\eta_s^{Z^d} \cap C|\text{odd}\big]}{P\big[\zeta_t^A \neq \varnothing\big]} \\ &= \sum_{\substack{C\subset Z^d\\ |C|<+\infty}} P\big[\zeta_t^A = C|\zeta_t^A \neq \varnothing\big] P\big[|\eta_s^{Z^d} \cap C|\text{odd}\big]. \end{aligned}$$

Hence,

$$\lim_{t\to+\infty} g(s,t) = \lim_{t\to+\infty} \sum_{\substack{C\subset Z^d \\ |C|<+\infty}} P\left[\zeta_t^A = C \mid \zeta_t^A \neq \varnothing\right] P\left[|\eta_s^{Z^d} \cap C| \text{odd}\right] = \tfrac{1}{2},$$

where the last equality follows from the remarks after Lemma 4, and Theorem 2. Next, let $\{t_k\}_{k=1}^{+\infty}$ be a sequence such that $t_k \to +\infty$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \frac{P\left[|\eta_{t_k}^{Z^d} \cap A| \text{odd}\right]}{P\left[\xi_{t_k}^{Z^d} \cap A \neq \varnothing\right]} = \alpha,$$

for some $\alpha \geq 0$. Choose an arbitrary $k > 0$, and let $0 < s < t_k$. Then

$$\begin{aligned}
\frac{P\left[|\eta_{t_k}^{Z^d} \cap A| \text{odd}\right]}{P\left[\xi_{t_k}^{Z^d} \cap A \neq \varnothing\right]} &= \frac{P\left[|\zeta_{t_k-s}^A \cap \eta_s^{Z^d}| \text{odd}\right]}{P\left[\zeta_{t_k}^A \neq \varnothing\right]} \\
&= \frac{P\left[|\zeta_{t_k-s}^A \cap \eta_s^{Z^d}| \text{odd}\right]}{P\left[\zeta_{t_k}^A \neq \varnothing \mid \zeta_{t_k-s}^A \neq \varnothing\right] P\left[\zeta_{t_k-s}^A \neq \varnothing\right]} \\
&= \frac{g(s, t_k - s)}{P\left[\zeta_{t_k}^A \neq \varnothing \mid \zeta_{t_k-s}^A \neq \varnothing\right]},
\end{aligned}$$

where the second equality holds because $\{\varnothing\}$ is an absorbing state for ζ_t. Hence,

$$\alpha = \lim_{k\to+\infty} \frac{g(s, t_k - s)}{P\left[\zeta_{t_k}^A \neq \varnothing \mid \zeta_{t_k-s}^A \neq \varnothing\right]}.$$

Since $\lim_{k\to+\infty} g(s, t_k - s) = \frac{1}{2}$ for all $s > 0$, then

$$\lim_{k\to+\infty} P\left[\zeta_{t_k}^A \neq \varnothing \mid \zeta_{t_k-s}^A \neq \varnothing\right]$$

also exists and is independent of s. By Lemma 5 this limit is equal to one. Therefore, all convergent subsequences will converge to the same limit and

$$\lim_{t\to+\infty} \frac{P\left[|\eta_t^{Z^d} \cap A| \text{odd}\right]}{P\left[\xi_t^{Z^d} \cap A \neq \varnothing\right]} = \frac{1}{2}. \qquad \square$$

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DEPARTMENT OF MATHEMATICS
TEMPLE UNIVERSITY
PHILADELPHIA, PENNSYLVANIA 19122