The Annals of Probability
1995, Vol. 23, No. 2, 938–947

A NOTE ON THE ASYMPTOTIC INDEPENDENCE OF THE SUM AND MAXIMUM OF STRONGLY MIXING STATIONARY RANDOM VARIABLES[1]

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It is shown that $\sum_{i=1}^{n} X_n$ and $\max_{i=1}^{n} X_i$ are asymptotically independent if $\{X_i\}$ is strongly mixing and $\sum_{i=1}^{n} X_i$ is asymptotically Gaussian. This generalizes a result of Anderson and Turkman.

1. Introduction. Let $\{X_i\}$ be a weakly dependent strictly stationary sequence of random variables, where weak dependence is, for the time being, loosely interpreted to mean that the dependence between the members of the sequence weakens as time separation increases. It is well known that in the infinite variance case, the asymptotic behavior of $S_n := \sum_{i=1}^{n} X_i$ is dominated by that of the extreme order statistics, so that S_n and $M_n := \max_{i=1}^{n} X_i$ are asymptotically dependent. See Lévy (1953), Chow and Teugels (1978), LePage, Woodroofe and Zinn (1981) and Davis and Hsing (1995). Chow and Teugels (1978) also showed that if $\{X_i\}$ is iid and $P[X_1 \le x]$ is in both the sum-domain of attraction of the normal distribution and the max-domain of attraction of an extreme value distribution, then S_n and M_n are asymptotically independent. Anderson and Turkman (1991) extended Chow and Teugels' result by showing that the asymptotic independence of S_n and M_n still holds true if (i) $\{X_i\}$ is a zero mean strictly stationary sequence which has a nonzero extremal index and satisfies the strong mixing condition, (ii) for constants a_n, c_n, d_n with $a_n \to \infty$, S_n/a_n converges in distribution to Normal(0, 1) and $(M_n - d_n)/c_n$ converges in distribution to a Gumbel distribution or a Pareto distribution with tail index greater than 2 and (iii) $\{X_i\}$ satisfies

$$\lim_{k\to\infty} \limsup_{n\to\infty} k \sum_{j=1}^{[n/k]} E\Big[\big|\exp\big(it(S_{[n/k]} - X_j)/a_n\big) - 1\big| I_{(X_j > c_n x + d_n)}\Big] = 0, \quad \text{all } x.$$

The purpose of the present paper is to show that for a stationary sequence, strong mixing and the asymptotic normality of S_n are basically enough to guarantee the asymptotic independence of the sum and maximum. In doing so, we also clarify that this is a special case of a more general result which follows as a consequence of asymptotic normality but has little to do with order statistics.

Received April 1993; revised October 1993.
[1] Research supported by Navy ONR Grant N00014-92-J-1007.
AMS 1991 *subject classification*. 60F05.
Key words and phrases. Central limit theorem, extreme values, strong mixing.

The main result of this paper is stated as follows.

THEOREM 1.1. *Let $\{X_i\}$ be a strictly stationary, strongly mixing sequence of random variables with zero mean and finite variance. Assume that $\sigma_n^2 := ES_n^2 \to \infty$ and $\{S_n/\sigma_n\}$ satisfies the central limit theorem (i.e., S_n/σ_n converges in distribution to standard normal). Then for any sequence B_n of Borel sets in $(-\infty, \infty)$ such that*

$$\liminf_{n\to\infty} P\left[\bigcap_{i=1}^{n} (X_i \in B_n)\right] > 0, \tag{1.1}$$

the conditional distribution $F_n(x) := P[S_n/\sigma_n \le x | \bigcap_{i=1}^{n}(X_i \in B_n)]$ converges weakly to the standard normal distribution.

Clearly, if for some monotone increasing function $u_n(y)$, $y \in (-\infty, \infty)$, for which

$$P[M_n \le u_n(y)] \to_w G(y),$$

where G is a probability distribution, then letting $B_n = (-\infty, u_n(y)]$ in Theorem 1.1 gives

$$P[S_n/\sigma_n \le x, M_n \le u_n(y)] \to_w \Phi(x)G(y),$$

generalizing Anderson and Turkman's result.

The proof of Theorem 1.1 is given in Section 2. The intuitive explanation of the asymptotic independence of the sum and maximum comes from the following observation. If $\{X_i\}$ is weakly dependent and S_n is asymptotically normal, then the individual summands must be asymptotically negligible and therefore the extreme terms should play no role in the limiting distribution. In our proof, the asymptotic negligibility of the summands is described by a Lindeberg-type condition, based on a central limit theorem for strongly mixing random variables in Ibragimov and Linnik (1971).

2. Technical details. To be precise, we define the strong mixing condition as follows. For $l \ge 1$, the mixing coefficient $\alpha(l)$ is defined by

$$\alpha(l) = \sup\{|P[A \cap B] - P[A]P[B]|: A \in \mathscr{F}_{1,m}, B \in \mathscr{F}_{m+l+1,\infty}, \text{ for all } m \ge 1\},$$

where $\mathscr{F}_{i,j}$ is the σ-field generated by X_k, $i \le k \le j$. The strong mixing condition is said to hold for $\{X_i\}$ if $\alpha(l) \to 0$ as $l \to \infty$. The strong mixing condition is one of a number of standard weak dependence conditions under which limit theorems concerning S_n are derived; see Peligrad (1986) and Philipp (1986).

Before proving Theorem 1.1, we first cite the following fundamental theorem, which is crucial for the proof.

THEOREM 2.1 [Ibragimov and Linnik (1971), Theorem 18.4.1]. *Assume that $\{X_j\}$ is a zero mean strictly stationary strongly mixing sequence with* *mixing coefficient α and finite second moments. In order for the central limit theorem to hold for $\{S_n/\sigma_n\}$, where $\sigma_n^2 = E(S_n^2)$ and $\lim_{n\to\infty}\sigma_n = \infty$, it is necessary that*:

(i) *$\sigma_n^2 = nh(n)$, where h is a slowly varying function defined on the positive axis.*

(ii) *For any pair of sequences of integers $p = p(n)$, $q = q(n)$, such that* (a) *$p \to \infty$, $q \to \infty$ and $q = o(p)$, $p = o(n)$ as $n \to \infty$,* (b) *$\lim_{n\to\infty} n^{1-\beta}q^{1+\beta}/p^2 = 0$ for all $\beta > 0$ and* (c) *$\lim_{n\to\infty} n\alpha(q)/p = 0$, we have*

$$\lim_{n\to\infty}\frac{n}{p\sigma_n^2}E\left[S_p^2 I_{(|S_p|>\varepsilon\sigma_n)}\right] = 0 \quad \text{for all } \varepsilon > 0. \tag{2.1}$$

Conversely, if condition (i) *is fulfilled and* (2.1) *holds for some p, q satisfying* (a), (b) *and* (c), *then the central limit theorem holds for $\{S_n/\sigma_n\}$.*

For $1 \le i \le j \le n$ and $n \ge 1$, define the event

$$H_{i,j}^{(n)} = \bigcap_{m=i}^{j}(X_m \in B_n),$$

where $B_1, B_2, \ldots$ are as in Theorem 1.1.

Proof of Theorem 1.1. Let p, q be chosen as in (a), (b) and (c) of Theorem 2.1, which is always possible under strong mixing. Since the central limit theorem holds for $\{S_n/\sigma_n\}$, (2.1) holds. Let $k = k_n$ be the integer part of $n/(p+q)$. Clearly, $k \sim n/p$.

The notation P is reserved for the probability measure of the probability space on which $\{X_i\}$ is defined and, as usual, the expectation with respect to P is denoted by E. Consider the sequence of random variables $\{\tilde{X}_i\}$ defined on another probability space with probability measure $\tilde{P}$ and with the distribution of $\{\tilde{X}_i\}$ determined by

$$\begin{aligned}\tilde{P}\left[\tilde{X}_1 \le x_1,\ldots,\tilde{X}_j \le x_j\right] &= P\left[X_1 \le x_1,\ldots,X_j \le x_j \,\middle|\, \bigcap_{i=1}^{n}(X_i \in B_n)\right]\\ &= \frac{P\left[(X_1 \le x_1,\ldots,X_j \le x_j)\cap H_{1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right]}.\end{aligned}$$

Also the expectation with respect to $\tilde{P}$ is denoted by $\tilde{E}$. Define

$$\tilde{\xi}_i = \sum_{j=i(p+q)+1}^{(i+1)p+iq}\tilde{X}_j, \qquad 0 \le i \le k-1,$$

$$\tilde{\eta}_i = \sum_{j=(i+1)p+iq+1}^{(i+1)(p+q)}\tilde{X}_j, \qquad 0 \le i \le k-1,$$

$$\tilde{\eta}_k = \sum_{j=k(p+q)+1}^{n} \tilde{X}_j,$$

$$\sigma_n^{-1}\tilde{S}_n = \sigma_n^{-1}\tilde{S}'_n + \sigma_n^{-1}\tilde{S}''_n := \sigma_n^{-1}\sum_{i=1}^{k-1}\tilde{\xi}_i + \sigma_n^{-1}\sum_{i=0}^{k}\tilde{\eta}_i.$$

Clearly, we are to show that $\{\tilde{S}_n/\sigma_n\}$ satisfies the central limit theorem. By the Schwarz inequality,

$$\begin{aligned}\tilde{E}\Big[\big(\sigma_n^{-1}\tilde{S}''_n\big)^2\Big] &= \sigma_n^{-2}\Bigg(\sum_{0\le i,\,j\le k-1}\tilde{E}\big(\tilde{\eta}_i\tilde{\eta}_j\big)\\ &\qquad +2\sum_{0\le i\le k-1}\tilde{E}\big(\tilde{\eta}_i\tilde{\eta}_k\big)+\tilde{E}\big(\tilde{\eta}_k^2\big)\Bigg)\\ &\le \sigma_n^{-2}\Bigg(\sum_{0\le i,\,j\le k-1}\tilde{E}^{1/2}\big(\tilde{\eta}_i^2\big)\tilde{E}^{1/2}\big(\tilde{\eta}_j^2\big)\\ &\qquad +2\sum_{0\le i\le k-1}\tilde{E}^{1/2}\big(\tilde{\eta}_i^2\big)\tilde{E}^{1/2}\big(\tilde{\eta}_k^2\big)+\tilde{E}\big(\tilde{\eta}_k^2\big)\Bigg).\end{aligned}\tag{2.2}$$

By the definition of $\tilde{E}$,

$$\tilde{E}\big(\tilde{\eta}_i^2\big) = \frac{E\big(\eta_i^2 I_{H_{1,n}^{(n)}}\big)}{P\big[H_{1,n}^{(n)}\big]} \le \frac{E\big(\eta_i^2\big)}{P\big[H_{1,n}^{(n)}\big]},$$

where

$$\eta_i = \sum_{j=(i+1)p+iq+1}^{(i+1)(p+q)} X_j, \qquad 0\le i\le k-1,$$

$$\eta_k = \sum_{j=k(p+q)+1}^{n} X_j.$$

Thus the right-hand side of (2.2) is bounded by

$$\frac{1}{\sigma_n^2 P\big[H_{1,n}^{(n)}\big]}\Big(k^2E\big(\eta_0^2\big) + 2kE^{1/2}\big(\eta_0^2\big)E^{1/2}\big(\eta_k^2\big) + E\big(\eta_k^2\big)\Big).$$

In view of (1.1), this tends to zero by (18.4.8) of Ibragimov and Linnik (1971). Thus $\sigma_n^{-1}\tilde{S}_n$ has the same limiting distribution, if any, as $\sigma_n^{-1}\tilde{S}'_n$. Using arguments similar to those on page 338 of Ibragimov and Linnik (1971), and by Lemma 2.2 below and the notation there, we obtain

$$\begin{aligned}\Big|\tilde{E}\big[\exp\big(it\sigma_n^{-1}\tilde{S}'_n\big)\big] - \tilde{E}^k\big[\exp\big(it\sigma_n^{-1}\tilde{\xi}_0\big)\big]\Big| &\le 16k\,\tilde{\alpha}_n(q)\\ &\le 64k\frac{P\Big[\big(H_{1,q}^{(n)}\big)^c\Big]+\alpha(q)}{P^3\big[H_{1,n}^{(n)}\big]}.\end{aligned}\tag{2.3}$$

By Lemma 2.3, the choice of p, q and (1.1), the right-hand side of (2.3) tends to zero. Thus, $\sigma_n^{-1}\tilde{S}_n'$ and hence $\sigma_n^{-1}\tilde{S}_n$ have the same limiting distribution, if any, as

$$\sigma_n^{-1}\sum_{i=0}^{k-1}\tilde{\xi}_i',$$

where $\tilde{\xi}_i'$, $0 \le i \le k-1$, are independent random variables and $\tilde{\xi}_i'$ has the same distribution as $\tilde{\xi}_i$. By Lemmas 2.4 and 2.5 below,

$$\tilde{E}\left(\sigma_n^{-1}\sum_{i=0}^{k-1}\tilde{\xi}_i'\right) \to 0 \quad \text{and} \quad \tilde{E}\left(\sigma_n^{-1}\sum_{i=0}^{k-1}\tilde{\xi}_i'\right)^2 \to 1.$$

Thus, to show that $\{\sigma_n^{-1}\tilde{S}_n\}$ satisfies the central limit theorem, it remains to verify the Lindeberg condition for $\sigma_n^{-1}\tilde{\xi}_0', \dots, \sigma_n^{-1}\tilde{\xi}_{k-1}'$. However,

$$\sigma_n^{-2}\sum_{i=0}^{k-1}\tilde{E}\left[\left(\tilde{\xi}_i'\right)^2 I_{(|\tilde{\xi}_i'|>\varepsilon\sigma_n)}\right] \le \frac{n}{p\sigma_n^2 P\left[H_{1,n}^{(n)}\right]} E\left[S_p^2 I_{(|S_p|>\varepsilon\sigma_n)}\right],$$

which tends to zero for all $\varepsilon > 0$ by (1.1) and (2.1). This concludes the proof. □

The following lemmas, which assume the conditions and notation of the preceding proof, serve to fill the gaps therein.

Lemma 2.2. *Let $\tilde{\mathscr{F}}_{i,j}$ denote the σ-field generated by $\tilde{X}_i, \dots, \tilde{X}_j$. Then for* $0 \le l \le n-1$,

$$\begin{aligned}\tilde{\alpha}_n(l) &:= \sup\Big\{|\tilde{P}[\tilde{A} \cap \tilde{B}] - \tilde{P}[\tilde{A}]\tilde{P}[\tilde{B}]|: \tilde{A} \in \tilde{\mathscr{F}}_{1,m}, \tilde{B} \in \tilde{\mathscr{F}}_{m+l+1,n}, \\ &\qquad \textit{for all } m \textit{ satisfying } 1 \le m \le m+l+1 \le n\Big\} \\ &\le 4\frac{P\left[\left(H_{1,l}^{(n)}\right)^c\right] + \alpha(l)}{P^3\left[H_{1,n}^{(n)}\right]},\end{aligned} \tag{2.4}$$

where $(H_{1,l}^{(n)})^c$ is the complement of $H_{1,l}^{(n)}$ and α is the strong mixing coefficient of $\{X_i\}$.

Proof. Denote by $\mathscr{F}_{i,j}$, $i \le j$, the σ-field generated by $X_i, \dots, X_j$. For convenience, if A is some event in $\mathscr{F}_{i,j}$, the corresponding event in $\tilde{\mathscr{F}}_{i,j}$ will be denoted by $\tilde{A}$ and vice versa. For $1 \le m \le m+l+1 \le n$, let $\tilde{A} \in \tilde{\mathscr{F}}_{1,m}$ and $\tilde{B} \in \tilde{\mathscr{F}}_{m+l+1,n}$. Then

$$\begin{aligned}\tilde{P}[\tilde{A} \cap \tilde{B}] - \tilde{P}[\tilde{A}]\tilde{P}[\tilde{B}] &= \frac{P\left[A \cap B \cap H_{1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right]} - \frac{P\left[A \cap H_{1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right]}\frac{P\left[B \cap H_{1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right]} \\ &= \sum_{i=1}^{6} C_{n,i},\end{aligned}$$

where

$$C_{n,1} = \frac{P[A \cap B \cap H_{1,n}^{(n)}]}{P[H_{1,n}^{(n)}]} - \frac{P[A \cap B \cap H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]}{P[H_{1,n}^{(n)}]},$$

$$C_{n,2} = \frac{P[A \cap B \cap H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]}{P[H_{1,n}^{(n)}]} - \frac{P[A \cap H_{1,m}^{(n)}]P[B \cap H_{m+l+1,n}^{(n)}]}{P[H_{1,n}^{(n)}]},$$

$$C_{n,3} = \frac{P[A \cap H_{1,m}^{(n)}]P[B \cap H_{m+l+1,n}^{(n)}]}{P[H_{1,n}^{(n)}]} - \frac{P[A \cap H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]P[B \cap H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]}{P[H_{1,m}^{(n)}]P[H_{m+l+1,n}^{(n)}]P[H_{1,n}^{(n)}]},$$

$$C_{n,4} = \frac{P[A \cap H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]P[B \cap H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]}{P[H_{1,m}^{(n)}]P[H_{m+l+1,n}^{(n)}]P[H_{1,n}^{(n)}]} - \frac{P[A \cap H_{1,n}^{(n)}]P[B \cap H_{1,n}^{(n)}]}{P[H_{1,m}^{(n)}]P[H_{m+l+1,n}^{(n)}]P[H_{1,n}^{(n)}]},$$

$$C_{n,5} = \frac{P[A \cap H_{1,n}^{(n)}]P[B \cap H_{1,n}^{(n)}]}{P[H_{1,m}^{(n)}]P[H_{m+l+1,n}^{(n)}]P[H_{1,n}^{(n)}]} - \frac{P[A \cap H_{1,n}^{(n)}]P[B \cap H_{1,n}^{(n)}]}{P[H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]P[H_{1,n}^{(n)}]},$$

$$C_{n,6} = \frac{P[A \cap H_{1,n}^{(n)}]P[B \cap H_{1,n}^{(n)}]}{P[H_{1,m}^{(n)} \cap H_{m+l+1,n}^{(n)}]P[H_{1,n}^{(n)}]} - \frac{P[A \cap H_{1,n}^{(n)}]P[B \cap H_{1,n}^{(n)}]}{P^2[H_{1,n}^{(n)}]}.$$

Clearly,

$$|C_{n,1}| \le \frac{P[(H_{1,l}^{(n)})^c]}{P[H_{1,n}^{(n)}]}, \qquad |C_{n,4}| \le 2\frac{P[(H_{1,l}^{(n)})^c]}{P^3[H_{1,n}^{(n)}]} \quad \text{and} \quad |C_{n,6}| \le \frac{P[(H_{1,l}^{(n)})^c]}{P^3[H_{1,n}^{(n)}]}.$$

By the strong mixing condition,

$$|C_{n,2}| \le \frac{\alpha(l)}{P[H_{1,n}^{(n)}]}, \qquad |C_{n,3}| \le 2\frac{\alpha(l)}{P^3[H_{1,n}^{(n)}]} \quad \text{and} \quad |C_{n,5}| \le \frac{\alpha(l)}{P^3[H_{1,n}^{(n)}]}.$$

Thus,

$$|\tilde{P}[\tilde{A} \cap \tilde{B}] - \tilde{P}[\tilde{A}]\tilde{P}[\tilde{B}]| \le 4\frac{P[(H_{1,l}^{(n)})^c] + \alpha(l)}{P^3[H_{1,n}^{(n)}]}.$$

Since the right-hand side is independent of m, we have shown (2.4). □

LEMMA 2.3. *For k, q defined in the proof,*

(2.5) $$\lim_{n\to\infty} kP[(H_{1,q}^{(n)})^c] = 0.$$

PROOF. First fix r to be a positive integer. For large n, let i_s and j_s, $1 \le s \le kr$, be integers in $\{1,\dots,n\}$ such that each of the intervals $[i_s, j_s]$ contains exactly q integers, and the intervals are separated from one another by at least q integers. This is possible if n is large enough since $kq = o(n)$. Now consider the quantity $P[\bigcap_{s=1}^{kr} H_{i_s,j_s}^{(n)}]$. By the definition of the mixing coefficient α and stationarity,

$$\left| P\left[\bigcap_{s=1}^{kr} H_{i_s,j_s}^{(n)}\right] - P^{kr}\left[H_{1,q}^{(n)}\right]\right| \le kr\alpha(q),$$

which tends to 0 by (c) of Theorem 2.1. Thus

$$\liminf_{n\to\infty} P^{kr}\left[H_{1,q}^{(n)}\right] = \liminf_{n\to\infty} P\left[\bigcap_{s=1}^{kr} H_{i_s,j_s}^{(n)}\right] \ge \liminf_{n\to\infty} P\left[\bigcap_{i=1}^{n} (X_i \in B_n)\right] > 0$$

by (1.1). Since this holds for all positive integers r, the only possibility is that $P^k[H_{1,q}^{(n)}] \to 1$, which is equivalent to (2.5). □

LEMMA 2.4. *Uniformly for all* $0 \le i \le k-1$,

$$\frac{k}{\sigma_n^2}\tilde{E}\left(\tilde{\xi}_i^2\right) \to 1 \quad \text{as } n \to \infty.$$

PROOF. An ingredient of the proof of Theorem 2.1 is

$$E\left(S_p^2\right) \sim \sigma_n^2/k. \tag{2.6}$$

See the first equation on page 273 of Denker (1986). Thus it suffices to show that uniformly for all $0 \le i \le k-1$,

$$\tilde{E}\left(\tilde{\xi}_i^2\right) \sim E\left(S_p^2\right) \quad \text{as } n \to \infty.$$

For convenience of notation we illustrate the proof for $i = 0$. The general proof is basically the same. First fix $\varepsilon > 0$ and write

$$\frac{\tilde{E}\left(\tilde{\xi}_0^2\right)}{E\left(S_p^2\right)} - 1 = \frac{E\left[\sigma_n^{-2}S_p^2 I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)} - 1 = \sum_{i=1}^{5} C_{n,i},$$

where

$$C_{n,1} = \frac{E\left[\sigma_n^{-2}S_p^2 I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)} - \frac{E\left[\sigma_n^{-2}S_p^2 I_{(\sigma_n^{-2}S_p^2 \le \varepsilon)} I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)},$$

$$C_{n,2} = \frac{E\left[\sigma_n^{-2}S_p^2 I_{(\sigma_n^{-2}S_p^2 \le \varepsilon)} I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)} - \frac{E\left[\sigma_n^{-2}S_p^2 I_{(\sigma_n^{-2}S_p^2 \le \varepsilon)} I_{H_{p+q+1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)},$$

$$C_{n,3} = \frac{E\left[\sigma_n^{-2}S_p^2 I_{(\sigma_n^{-2}S_p^2 \le \varepsilon)} I_{H_{p+q+1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)} - \frac{E\left[\sigma_n^{-2}S_p^2 I_{(\sigma_n^{-2}S_p^2 \le \varepsilon)}\right]P\left[H_{p+q+1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right]E\left(\sigma_n^{-2}S_p^2\right)},$$

$$C_{n,4} = \frac{E\left[\sigma_n^{-2} S_p^2 I_{(\sigma_n^{-2} S_p^2 \le \varepsilon)}\right] P\left[H_{p+q+1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)} - \frac{E\left[\sigma_n^{-2} S_p^2 I_{(\sigma_n^{-2} S_p^2 \le \varepsilon)}\right] P\left[H_{1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)},$$

$$C_{n,5} = \frac{E\left[\sigma_n^{-2} S_p^2 I_{(\sigma_n^{-2} S_p^2 \le \varepsilon)}\right] P\left[H_{1,n}^{(n)}\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)} - 1.$$

Clearly,

$$0 \le C_{n,1} = \frac{E\left[\sigma_n^{-2} S_p^2 I_{(\sigma_n^{-2} S_p^2 > \varepsilon)} I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)} \le \frac{E\left[\sigma_n^{-2} S_p^2 I_{(\sigma_n^{-2} S_p^2 > \varepsilon)}\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)},$$

which, together with (1.1), (2.1) and (2.6), implies

$$\lim_{n\to\infty} C_{n,1} = 0.$$

Next,

$$\begin{aligned} 0 \le -C_{n,2} &= \frac{E\left[\sigma_n^{-2} S_p^2 I_{(\sigma_n^{-2} S_p^2 \le \varepsilon)} I_{(H_{1,p+q}^{(n)})^c}\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)} \\ &\le \frac{\varepsilon P\left[\left(H_{1,p+q}^{(n)}\right)^c\right]}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)}. \end{aligned} \tag{2.7}$$

By Lemma 2.3 of Hsing, Hüsler and Leadbetter (1988) and (1.1),

$$P^k\left[H_{1,p+q}^{(n)}\right] - P\left[H_{1,n}^{(n)}\right] \to 0. \tag{2.8}$$

Hence it follows from (1.1), (2.6) and (2.7) that

$$\limsup_{n\to\infty} |C_{n,2}| \le b\varepsilon,$$

where

$$b = -\frac{\liminf_{n\to\infty} \log\left(P\left[H_{1,n}^{(n)}\right]\right)}{\liminf_{n\to\infty} P\left[H_{1,n}^{(n)}\right]} < \infty. \tag{2.9}$$

By the strong mixing condition and Lemma 1.1 of Peligrad (1986),

$$|C_{n,3}| \le \frac{2\pi\varepsilon\alpha(q)}{P\left[H_{1,n}^{(n)}\right] E\left(\sigma_n^{-2} S_p^2\right)},$$

which tends to 0 by (1.1), (2.6) and (c) of Theorem 2.1. The quantities $C_{n,4}$ and $C_{n,5}$ are taken care of in the same way as $C_{n,1}$ and $C_{n,2}$ are, respectively, so that

$$\limsup_{n\to\infty} \sum_{i=1}^{5} |C_{n,i}| \le 2b\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

LEMMA 2.5. *Uniformly for all* $0 \le i \le k - 1$,

$$\frac{k}{\sigma_n}\tilde{E}\left(\tilde{\xi}_i\right) \to 0 \quad as\, n \to \infty.$$

PROOF. Again we illustrate the proof by taking $i = 0$. For fixed $\varepsilon > 0$, write

$$\frac{k}{\sigma_n}\tilde{E}\left(\tilde{\xi}_0\right) = k\frac{E\left[\sigma_n^{-1}S_p I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]} = C_{n,1} + C_{n,2},$$

where

$$C_{n,1} = k\frac{E\left[\sigma_n^{-1}S_p I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]} - k\frac{E\left[\sigma_n^{-1}S_p I_{(|S_p| \le \varepsilon\sigma_n)} I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]},$$

$$C_{n,2} = k\frac{E\left[\sigma_n^{-1}S_p I_{(|S_p| \le \varepsilon\sigma_n)} I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]}.$$

By (1.1) and (2.1),

$$|C_{n,1}| = k\left|\frac{E\left[\sigma_n^{-1}S_p I_{(|S_p| > \varepsilon\sigma_n)} I_{H_{1,n}^{(n)}}\right]}{P\left[H_{1,n}^{(n)}\right]}\right| \le k\varepsilon^{-1}\frac{E\left[\sigma_n^{-2}S_p^2 I_{(|S_p| > \varepsilon\sigma_n)}\right]}{P\left[H_{1,n}^{(n)}\right]} \to 0.$$

Now, using arguments similar to those in Lemma 2.4,

$$\begin{aligned}|C_{n,2}| \le k\Biggl(2\varepsilon\frac{P\left[\left(H_{1,p+q}^{(n)}\right)^c\right]}{P\left[H_{1,n}^{(n)}\right]} \\ + 2\pi\varepsilon\frac{\alpha(q)}{P\left[H_{1,n}^{(n)}\right]} + \left|E\left[\sigma_n^{-1}S_p I_{(|S_p| \le \varepsilon\sigma_n)}\right]\right|\Biggr).\end{aligned} \tag{2.10}$$

Since S_p has mean zero,

$$E\left[\sigma_n^{-1}S_p I_{(|S_p| \le \varepsilon\sigma_n)}\right] = -E\left[\sigma_n^{-1}S_p I_{(|S_p| > \varepsilon\sigma_n)}\right].$$

Using (2.10), it follows from (1.1), (2.8), the choice of p, q and (2.1) that

$$\limsup_{n\to\infty}|C_{n,2}| \le 2b\varepsilon,$$

where b is defined in (2.9). Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Acknowledgment. The author thanks the referee for providing constructive comments.

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