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POINT PROCESS AND PARTIAL SUM CONVERGENCE FOR WEAKLY DEPENDENT RANDOM VARIABLES WITH INFINITE VARIANCE

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Let $\{\xi_j\}$ be a strictly stationary sequence of random variables with regularly varying tail probabilities. We consider, via point process methods, weak convergence of the partial sums, $S_n = \xi_1 + \cdots + \xi_n$, suitably normalized, when $\{\xi_j\}$ satisfies a mild mixing condition. We first give a characterization of the limit point processes for the sequence of point processes N_n with mass at the points $\{\xi_j/a_n,\ j = 1,\dots,n\}$, where a_n is the $1 - n^{-1}$ quantile of the distribution of $|\xi_1|$. Then for $0 < \alpha < 1$ ($-\alpha$ is the exponent of regular variation), S_n is asymptotically stable if N_n converges weakly, and for $1 \le \alpha < 2$, the same is true under a condition that is slightly stronger than the weak convergence of N_n. We also consider large deviation results for S_n. In particular, we show that for any sequence of constants $\{t_n\}$ satisfying $nP[\xi_1 > t_n] \to 0$, $P[S_n > t_n]/(nP[\xi_1 > t_n])$ tends to a constant which can in general be different from 1. Applications of our main results to self-norming sums, m-dependent sequences and linear processes are also given.

1. Introduction. Let $\{\xi_j\}$ be a strictly stationary sequence of random variables with regularly varying tail probabilities, that is,

$$P[|\xi_1| > x] = x^{-\alpha}L(x), \tag{1.1}$$

where $\alpha > 0$, $L(\cdot)$ is a slowly varying function at ∞ and

$$\frac{P[\xi_1 > x]}{P[|\xi_1| > x]} \to p \quad \text{and} \quad \frac{P[\xi_1 < -x]}{P[|\xi_1| > x]} \to q, \tag{1.2}$$

as $x \to \infty$ with $0 \le p \le 1$ and $q = 1 - p$. One of the objectives of this paper is to investigate the asymptotic distributional behavior of $S_n := \sum_{j=1}^n \xi_j$ in the case where $\alpha \in (0,2)$ and the dependence between ξ_j becomes weaker as the time separation increases.

In the iid case, it is well known that (1.1) with some $\alpha \in (0,2)$ together with (1.2) are necessary and sufficient for the existence of normalizing

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constants a_n, b_n for which $(S_n - b_n)/a_n$ converges weakly to some stable law with index α [cf. Feller (1971)]. Also, if $\{\xi_j\}$ is iid, (1.1) with (1.2) imply that

$$\lim_{n\to\infty} \frac{P[S_n > t_n]}{nP[\xi_1 > t_n]} = 1 \tag{1.3}$$

for any constants t_n satisfying $nP[\xi_1 > t_n] \to 0$ [cf. Heyde (1967a, b), (1968), A. V. Nagaev (1969), S. V. Nagaev (1979) and Cline and Hsing (1994)]. Early proofs of these results are based on analytic arguments. However, the probabilistic reasoning behind them is that for heavy-tailed distributions, the distributional behavior of S_n is dominated by that of the extreme order statistics of the summands. In the context of deriving the asymptotic distribution of S_n, this was made clear for the first time in LePage, Woodroofe and Zinn (1981) for the iid case, and subsequently in Davis (1983) for the dependent case when the extremes of the process can be approximated by the extremes of some iid sequence. This approach is particularly appealing in the dependent case, since, although deriving the distribution of S_n directly (e.g., by finding the characteristic function) is generally very difficult, handling the behavior of the extreme order statistics is a typically less formidable problem.

A useful way to capture the information contained in the extreme order statistics in a sample in this context is through a point process N_n, defined as follows. Let a_n be such that

$$nP[|\xi_1| > a_n] \to 1 \tag{1.4}$$

(one can choose a_n to be the $1 - 1/n$ quantile of the distribution function of $|\xi_1|$). Define the point process

$$N_n = \sum_{j=1}^{n} \delta_{\xi_j/a_n}, \tag{1.5}$$

where δ_x represents unit point measure at the point x. In order to achieve distributional stability of N_n as $n \to \infty$, it is necessary to allow for a buildup of infinite mass at 0. This is handled, for our problem, by defining the state space for N_n to be $\mathbb{R} - \{0\}$ so that compact sets in this space are closed sets which are bounded away from 0 and $\pm\infty$. With that, weak convergence of N_n is equivalent to (joint) weak convergence of the extreme order statistics. It is easy to see that (1.1) together with (1.2) are equivalent to

$$nP[\xi_1/a_n \in \cdot\,] \to_v \mu(\cdot), \tag{1.6}$$

where μ is the measure

$$\mu(dx) = \left(p\alpha x^{-\alpha-1} I_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1} I_{(-\infty,0)}(x)\right) dx$$

and $\to_v$ denotes vague convergence on $\mathbb{R} - \{0\}$. Moreover, in the case that $\{\xi_j\}$ is iid, (1.6) is equivalent to convergence of N_n to a Poisson point process with intensity measure μ [see Resnick (1987)], from which the asymptotic distribution of S_n is readily obtained. This was the essential idea of LePage, Woodroofe and Zinn (1981) and Davis (1983). Our goal in this paper is then to consider the extensions to where the sequence is weakly dependent and yet has nonnegligible local dependence.

A very general model is one where the point process N_n converges weakly to a point process which has the representation

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{P_i Q_{ij}},$$

where $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process on $(0, \infty)$ and $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$, $i \geq 1$, are iid point processes on $[-1, 1] - \{0\}$ also independent of the Poisson process. We show in Theorem 2.3 that any weak limit of N_n must have this form if $\{\xi_j\}$ satisfies (1.1) and a mixing condition similar in spirit to the condition D used in extreme value theory. Mori (1977) characterized all weak limits of a point process related to N_n under the strong-mixing condition. However, he took the state space to be $(0, \infty)$ so that N_n only effectively describes the joint distributions of the upper extreme order statistics. Here, we are interested in the joint distributions of the upper and lower extreme order statistics in a dependent setting substantially weaker than strong mixing. Thus our characterization can be considered an extension of the corresponding result in Mori (1977). We also establish a new characterization in Theorem 2.5 for convergence of N_n which is useful for deriving the limit of N_n and for statistical applications. As an illustration, we consider in Theorem 2.7 the important example where all finite-dimensional distributions of $\{\xi_j\}$ are jointly regularly varying.

In Section 3, we establish the weak convergence of the normalized partial sums $S_n := \sum_{j=1}^{n} \xi_j$ from the weak convergence of N_n for the case $0 < \alpha < 2$. Some connected results can be found in Samur (1984), Leadbetter and Rootzén (1988), Denker and Jakubowski (1989) and Jakubowski and Kobus (1989). See also Durrett and Resnick (1978). However, our results in Section 3 are the most general up to now in this context. Theorem 3.1 shows that for $0 < \alpha < 1$, S_n is asymptotically stable if N_n converges weakly and for $1 \leq \alpha < 2$, the same is true under a condition that is slightly more than the weak convergence of N_n. The parameters of the limiting stable distribution are given in terms of the limiting point process in Theorem 3.2.

In Section 4, we consider large deviation results for S_n under the general model as in Section 3 in the $0 < \alpha < 2$ case. The novelty of our approach is using probability arguments to link the probabilities of large deviations of S_n and the asymptotic behavior of extremes. As far as we know this provides the first nonanalytic proof of (1.3). The benefit of this approach becomes obvious when $\{\xi_j\}$ is weakly dependent. We show in Theorem 4.4 that for t_n satisfying $nP[\xi_1 > t_n] \to 0$, $P[S_n > t_n]/(nP[\xi_1 > t_n])$ tends to a constant which can, in general, be different from 1. This is an extension of (1.3).

Finally, in Section 5, we consider some examples of our main results. These include self-norming sums, m-dependent sequences and linear processes.

2. Point process convergence. In this section, we first characterize in Theorem 2.3 the weak convergence of the point process N_n defined in Section

1 under a weak dependence condition. We will also provide a useful device, contained in Theorem 2.5, for the purpose of identifying the limit point process. Point process convergence for the case where the finite-dimensional distributions of $\{\xi_j\}$ are regularly varying will be presented as an application in Theorem 2.7.

We will follow the point process theory in Kallenberg (1983). As mentioned in Section 1, N_n is a point process on $\mathbb{R} - \{0\}$. Write $\mathscr{B}$ for the collection of bounded Borel sets in $\mathbb{R} - \{0\}$. (Bounded here means bounded away from 0 and $\pm\infty$.) Let $\mathscr{F}$ be the collection of bounded nonnegative continuous functions on $\mathbb{R} - \{0\}$ with bounded support and let $\mathscr{F}_s$ be the collection of step functions on $\mathbb{R} - \{0\}$ with bounded support. Write M for the collection of Radon counting measures on $\mathbb{R} - \{0\}$. The null measure is denoted by o. M is assumed to be equipped with the vague topology and the corresponding Borel σ-field. When speaking of a subclass of M, we assume that it has the relative topological and measure theoretic structures.

Suppose that $\{\xi_j\}$ is a strictly stationary sequence satisfying (1.1) and that $\{a_n\}$ satisfies (1.4). The mixing condition appropriate for this section is defined as follows. Say that the condition $\mathscr{A}(\{a_n\})$ holds for $\{\xi_j\}$ if there exists a set of positive integers $\{r_n\}$ such that $r_n \to \infty$, $r_n/n \to 0$ as $n \to \infty$ and

$$E\exp\left(-\sum_{j=1}^{n} f(\xi_j/a_n)\right) - \left(E\exp\left(-\sum_{j=1}^{r_n} f(\xi_j/a_n)\right)\right)^{[n/r_n]} \to 0 \quad \text{as } n \to \infty \text{ for all } f \in \mathscr{F}_s. \tag{2.1}$$

The convergence in (2.1) is not required to be uniform in f. This is indeed a very weak condition and is implied by many known mixing conditions, in particular the strong-mixing condition. Note that the condition is independent of the specific a_n sequence; if both a_n and a'_n satisfy (1.4), then the conditions $\mathscr{A}(\{a_n\})$ and $\mathscr{A}(\{a'_n\})$ are equivalent. A corresponding statement can be made for the weak convergence of N_n. Define $k_n = [n/r_n]$ and

$$\tilde{N}_n = \sum_{i=1}^{k_n} \tilde{N}_{r_n,i},$$

with $\tilde{N}_{r_n,i}$, $1 \le i \le k_n$, iid distributed as $\tilde{N}_{r_n} := \sum_{j=1}^{r_n} \delta_{\xi_j/a_n}$. In view of (2.1), the condition $\mathscr{A}(\{a_n\})$ implies that N_n converges weakly if and only if $\tilde{N}_n$ does and they have the same limit. This is an important ingredient in the proof of Lemma 2.1 below. Also, for any $y \ge 0$, define

$$M_y = \left\{\mu \in M \colon \mu([-y, y]^c) > 0 \text{ and } \mu([-x, x]^c) = 0 \text{ for some } 0 < x(= x_\mu) < \infty\right\}.$$

LEMMA 2.1. *Suppose that the condition $\mathscr{A}(\{a_n\})$ holds for $\{\xi_j\}$. Then $N_n \to_d$ some $N \neq o$ if and only if there exists some nonzero measure λ on $M - \{o\}$*

with $\int(1 - e^{-\mu(B)})\lambda(d\mu) < \infty$, $B \in \mathscr{B}$, *such that*

$$\int(1 - e^{-\mu f})\lambda_n(d\mu) \to \int(1 - e^{-\mu f})\lambda(d\mu), \qquad f \in \mathscr{F}, \tag{2.2}$$

where $\lambda_n = k_n P \circ \tilde{N}_{r_n}^{-1}$. *In this case,* N *is an infinitely divisible point process with Laplace transform* $\exp\{-\int(1 - e^{-\mu f})\lambda(d\mu)\}$, N *has no fixed atoms and the support of* $P \circ N^{-1}$ *and that of* λ *are both contained in* M_0.

PROOF. As mentioned above, N_n converges in distribution if and only if $\tilde{N}_n$ converges to the same limit under $\mathscr{A}(\{a_n\})$. Also it is clear that $\tilde{N}_{r_n,i}$, $1 \le i \le k_n$, form a "null array." It follows from Theorem 6.1 of Kallenberg (1983), that $\tilde{N}_n \to_d N$ is equivalent to (2.2) with λ satisfying the described property. By Theorem 7.1 of Kallenberg (1983), N can have at most countably many fixed atoms whose collection is denoted here by D. Thus for any $x \ne 0$ and $\varepsilon_o > 0$, there exists an $\varepsilon \in (0, \varepsilon_o)$ with $x - \varepsilon$ and $x + \varepsilon$ both in D^c and having the same sign. It follows by Lemma 4.4 of Kallenberg (1983) that

$$P[N(\{x\}) > 0] \le P[N(x - \varepsilon, x + \varepsilon) > 0] = \lim_{n\to\infty} P[N_n(x - \varepsilon, x + \varepsilon) > 0]$$
$$\le \lim_{n\to\infty} nP[\xi_1/a_n \in (x - \varepsilon, x + \varepsilon)],$$

which can be made arbitrarily small by choosing a small ε_o. We conclude that N has no fixed atoms. One can similarly show that with probability 1, $N([-x, x]^c) \to 0$ as $x \to \infty$. Therefore the support of $P \circ N^{-1}$ is contained in M_0. Finally, by Theorem 6.8 of Kallenberg (1983), the support of λ is contained in that of N and the proof is complete. □

In the proof of Lemma 2.1, we only used the assumption of asymptotic independence of $\{\xi_j\}$ and did not fully take into account the structure of N_n. As a result the characterization is not yet complete. We next show that the canonical measure λ of N has to be of a very specific form.

For $\sigma > 0$, define π_σ: $\mu \to \mu(\sigma^{-1}\cdot)$, $M_0 \to M_0$.

LEMMA 2.2. *Assume that the condition* $\mathscr{A}(\{a_n\})$ *holds for* $\{\xi_j\}$ *and that the weak convergence in Lemma* 2.1 *takes place. Then* $\lambda = \sigma^\alpha\lambda \circ \pi_\sigma$, $\sigma > 0$.

PROOF. It is sufficient to show the result for $\sigma > 1$. For a fixed $\sigma > 1$ let $n' = [\sigma^\alpha n]$. By (2.1) and the proof of Lemma 2.1,

$$E\exp\left\{-\sum_{j=1}^{n} f(\xi_j/a_{n'})\right\} \to (E\exp(-Nf))^{\sigma^{-\alpha}}, \qquad f \in \mathscr{F}.$$

Since $a_{n'}/(\sigma a_n) \to 1$, we have

$$E\exp\left\{-\sum_{j=1}^{n} f(\xi_j/(\sigma a_n))\right\} \to (E\exp(-Nf))^{\sigma^{-\alpha}}, \qquad f \in \mathscr{F}.$$

However, we also have

$$E\exp\left\{-\sum_{j=1}^{n} f(\xi_j/(\sigma a_n))\right\} \to E\exp(-\pi_\sigma^{-1}(N)f), \qquad f \in \mathscr{F}.$$

The result follows from equating the two limiting Laplace transforms. □

For $\mu \in M_0$, let $\mu_+ = \max(0, \text{largest point of } \mu)$, $\mu_- = \min(0, \text{smallest point of } \mu)$ and $x_\mu = \max(\mu_+, \mu_-)$. Define a mapping on M_0 by

$$\Omega: \mu \to (x_\mu, \mu(x_\mu \cdot)).$$

The mapping Ω is continuous with range $(0,\infty) \times \tilde{M}$, where $\tilde{M} = \{\mu \in M: \mu([-1,1]^c) = 0,\ \mu(\{-1\} \cup \{1\}) > 0\}$. Denote by $\mathscr{B}(\tilde{M})$ the Borel σ-field of $\tilde{M}$.

THEOREM 2.3. *Assume that the condition* $\mathscr{A}(\{a_n\})$ *holds for* $\{\xi_j\}$, *and* $N_n \to_d$ *some* $N \neq o$. *Then* N *is infinitely divisible with canonical measure* λ *satisfying* $\lambda(M_0^c) = 0$ *and* $\lambda \circ \Omega^{-1} = \nu \times \mathscr{Q}$, *where* $\mathscr{Q}$ *is a probability measure on* $(\tilde{M}, \mathscr{B}(\tilde{M}))$, $\gamma := \lambda\{\mu: \mu([-1,1]^c) > 0\} \in (0,1]$ *and*

$$\nu(dy) = \gamma\alpha y^{-\alpha-1} I_{(0,\infty)}(y)\,dy.$$

In this case the Laplace transform of N *is*

$$\exp\left\{-\int_0^\infty \int_{\tilde{M}} (1 - \exp(-\mu f(y\cdot)))\mathscr{Q}(d\mu)\nu(dy)\right\}, \qquad f \in \mathscr{F}. \tag{2.3}$$

PROOF. For any fixed $A \in \mathscr{B}(\tilde{M})$, define a measure

$$\mathscr{Q}_A(E) = \lambda \circ \Omega^{-1}(E \times A), \qquad E \in \mathscr{B}(0,\infty).$$

For any $\sigma > 0$, $E \in \mathscr{B}(0,\infty)$, it follows from Lemma 2.2 that

$$\begin{aligned}\mathscr{Q}_A(\sigma E) &= \lambda \circ \Omega^{-1}(\sigma E \times A) = \lambda \circ \pi_\sigma \circ \Omega^{-1}(E \times A)\\ &= \sigma^{-\alpha}\lambda \circ \Omega^{-1}(E \times A) = \sigma^{-\alpha}\mathscr{Q}_A(E).\end{aligned}$$

It is straightforward to show now that $\mathscr{Q}_A = \mathscr{Q}(A) \times \nu$, where $\mathscr{Q}(A)$ is a constant depending on A. Regarding $\mathscr{Q}$ as a measure on $(\tilde{M}, \mathscr{B}(\tilde{M}))$, one obtains the desired decomposition

$$\lambda \circ \Omega^{-1} = \nu \times \mathscr{Q}.$$

Note that $\gamma \neq 0$, since the contrary and Lemma 2.2 imply that $\lambda\{\mu: \mu([-\sigma,\sigma]^c) > 0\} = 0$ for all $\sigma > 0$ and hence that λ is a null measure. Further observe that

$$\gamma = \lambda\{\mu: \mu([-1,1]^c) > 0\} = -\log P[N([-1,1]^c) = 0]$$

is the extremal index of $(|\xi_j|)$. Hence it follows from Leadbetter, Lindgren and Rootzén (1983) that $\gamma \in (0,1]$. Thus

$$\infty > \lambda\{\mu: \mu([-1,1]^c) > 0\} = \lambda \circ \Omega^{-1}((1,\infty) \times \tilde{M}) = \gamma\mathscr{Q}(\tilde{M}),$$

showing that $\mathscr{Q}$ is a probability measure by the choice of γ. □

The representation in (2.3) is not unique. Observe that the function $\mu \to x_\mu$ could be chosen in different ways, resulting in different representations. The simplest modification of the x_μ used here is to multiply it by a positive constant. However, it is possible to use an entirely different function for the purpose of characterization. The particular function used here is bicontinuous, making it possible to view the measures ν and $\mathscr{Q}$ as the limits of the corresponding measures associated with N_n, as will be demonstrated in Theorem 2.5 below.

COROLLARY 2.4 (Cluster representation). *$N =_d \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{P_i Q_{ij}}$, where $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process with intensity measure ν, $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$, $i \ge 1$, are point processes identically distributed according to $\mathscr{Q}$ and all point processes are mutually independent.*

PROOF. The conclusion follows from a comparison of the Laplace transforms of the two point processes. □

REMARK 2.1. It is easy to see that the Poisson process $\sum_{i=1}^{\infty} \delta_{P_i}$ in the above corollary has the representation $\sum_{i=1}^{\infty} \delta_{\gamma^{1/\alpha}\Gamma_i^{-1/\alpha}}$, where $\Gamma_i = \sum_{k=1}^{i} E_k$, with $E_1, E_2, \dots$ denoting iid unit exponential random variables.

REMARK 2.2. In the iid case under (1.1), N_n converges in distribution if and only if the tail balancing condition (1.2) holds. In this case, $\gamma = 1$, $\mathscr{Q}(\{\delta_1\}) = p$ and $\mathscr{Q}(\{\delta_{-1}\}) = q$ so that $P[Q_{i1} = 1] = p$ and $P[Q_{i1} = -1] = q$. Using the representation given in Remark 2.1, the limit point process takes the form

$$N = \sum_{i=1}^{\infty} \delta_{Q_{i1}\Gamma_i^{-1/\alpha}},$$

which is in agreement with the representation given in LePage, Woodroofe and Zinn (1981).

From the representation of the limit point process in the iid case, one readily observes the well-known property that the upper extremes and lower extremes of $\xi_1, \dots, \xi_n$ are asymptotically independent. However, in the dependent case an upper extreme can in general "trigger" lower extremes through the dependence of the sequence and vice versa, so that the extremes on the two ends are potentially dependent in the limit. This was observed by Davis and Resnick (1985) for moving average processes. Under the condition $\mathscr{A}(\{a_n\})$ and $N_n \to_d N$, the upper and lower extremes are asymptotically independent if and only if $\mathscr{Q}\{\mu \in \tilde{M}$: both μ_+ and μ_- are nonzero$) = 0$. In general, by writing the Laplace transform of N as

$$\exp\left\{-\int_0^\infty \int_{\tilde{M}^{(1)}} (1 - \exp(-\mu f(y\,\cdot)))\mathscr{Q}(d\mu)\nu(dy)\right\}$$
$$\times \exp\left\{-\int_0^\infty \int_{\tilde{M}^{(2)}} (1 - \exp(-\mu f(y\,\cdot)))\mathscr{Q}(d\mu)\nu(dy)\right\},$$

where $\tilde{M}^{(1)} = \{\mu \in \tilde{M}: \mu_+ \geq \mu_-\}$ and $\tilde{M}^{(2)} = \{\mu \in \tilde{M}: \mu_+ < \mu_-\}$, one can give the interpretation that it is possible to classify the extremes into two asymptotically independent classes, where in one class the upper extremes are typically larger in magnitude and in the other class the lower extremes are more so.

Note that any probability measure $\mathscr{Q}$ on $\tilde{M}$ makes the expression in (2.3) a valid Laplace transform of an infinitely divisible point process on $\mathbb{R} - \{0\}$. To see this, take $B = [-x, x]^c$ for some $x > 0$ and we have

$$\begin{aligned}\int_0^\infty \int_{\tilde{M}} &(1 - \exp(-\mu(B/y)))\mathscr{Q}(d\mu)\nu(dy) \\ &\leq \int_x^\infty \int_{\tilde{M}} (1 - \exp(-\mu([-1, -x/y) \cup (x/y, 1])))\mathscr{Q}(d\mu)\nu(dy) \\ &\leq \nu(x, \infty) < \infty.\end{aligned}$$

Other criteria could be used to establish the convergence of N_n. For example, the joint weak convergence of the extreme order statistics is obviously equivalent to the weak convergence of N_n. The following result has the advantage of explaining where the various components of N come from.

THEOREM 2.5. *Under the condition $\mathscr{A}(\{a_n\})$ for $\{\xi_j\}$, the following are equivalent:*

(i) *N_n converges in distribution to some $N \neq o$.*

(ii) *For some finite positive constant γ, $k_n P[\bigvee_1^{r_n} |\xi_k| > a_n x] \to \gamma x^{-\alpha}$, $x > 0$, and for some probability measure $\mathscr{Q}$ on $\tilde{M}$, $P[\sum_{j=1}^{r_n} \delta_{\xi_j/(\bigvee_1^{r_n}|\xi_k|)} \in \cdot \,|\bigvee_1^{r_n}|\xi_k| > a_n x] \to_w \mathscr{Q}$, $x > 0$.*

In this case N is infinitely divisible with canonical measure λ confined to M_0 and satisfying

$$\lambda \circ \Omega^{-1} = \nu \times \mathscr{Q},$$

where $\nu(dy) = \gamma\alpha y^{-\alpha-1}\,dy$.

PROOF. First assume that (i) holds and that the canonical measure λ admits the decomposition $\lambda \circ \Omega^{-1} = \nu \times \mathscr{Q}$. Since N has no fixed atoms, $N_n([-x, x]^c) \to_d N([-x, x]^c)$, $x > 0$, or equivalently $\tilde{N}_n([-x, x]^c) \to_d N([-x, x]^c)$, $x > 0$. Thus,

$$\begin{aligned} k_n P\left[\bigvee_1^{r_n} |\xi_k| > a_n x\right] &= \lambda_n(M_x) \\ &= -\log P\left[\tilde{N}_n([-x, x]^c) = 0\right] + o(1) \\ &\to -\log P\left[N([-x, x]^c) = 0\right] \\ &= \lambda(M_x) \\ &= \lambda \circ \Omega^{-1}\left((x, \infty) \times \tilde{M}\right) = \nu(x, \infty) = \gamma x^{-\alpha}, \qquad x > 0. \end{aligned} \tag{2.4}$$

Now fix an $x > 0$ and define the probability measures $P_{n,x}$ and P_x on M_0 by

$$P_{n,x} = \frac{\lambda_n(\cdot \cap M_x)}{\lambda_n(M_x)} \quad \text{and} \quad P_x = \frac{\lambda(\cdot \cap M_x)}{\lambda(M_x)}.$$

We will show that $P_{n,x} \to_w P_x$. Set $B_0 = [-x, x]^c$ and for $1 \le i \le k$, let $B_i = (x_i, \infty)$ or $(-\infty, -x_i)$. Let $(X_{n0}, \dots, X_{nk})$ and $(X_0, \dots, X_k)$ be nonnegative integer-valued random vectors, with distribution functions $P_{n,x} \circ \Phi^{-1}$ and $P_x \circ \Phi^{-1}$, respectively, where Φ is the mapping $M_0 \to \mathbb{Z}_+^{k+1}$ defined by

$$\Phi\colon \mu \to \big(\mu(B_0), \dots, \mu(B_k)\big).$$

By Theorem 4.2 in Kallenberg (1983), weak convergence of $P_{n,x}$ will be implied by

$$(X_{n0}, \dots, X_{nk}) \to_d (X_0, \dots, X_k).$$

Observe that for $s_i \ge 0$,

$$E\left(1 - \exp\left(-\sum_{i=0}^k s_i X_{ni}\right)\right) = \int\left(1 - \exp\left(-\sum_{i=0}^k s_i\, \mu(B_i)\right)\right) P_{n,x}(d\mu)$$

and if $\wedge_{i=1}^k x_i := \wedge_{i=1}^k x_i \ge x$,

$$= \int\left(1 - \exp\left(-\sum_{i=0}^k s_i\, \mu(B_i)\right)\right) \frac{\lambda_n(d\mu)}{\lambda_n(M_x)}$$

which by (2.2), (2.4) and (i),

$$\to \int\left(1 - \exp\left(-\sum_{i=0}^k s_i\, \mu(B_i)\right)\right) \frac{\lambda(d\mu)}{\lambda(M_x)}$$

$$= E\left(1 - \exp\left(-\sum_{i=0}^k s_i X_i\right)\right).$$

Consequently,

$$E \exp\left(-\sum_{i=0}^k s_i X_{ni}\right) \to E \exp\left(-\sum_{i=0}^k s_i X_i\right) \tag{2.5}$$

and $(X_{n0}, \dots, X_{nk}) \to_d (X_{n0}, \dots, X_{nk})$ holds for $\wedge x_i \ge x$. On the other hand, if $\wedge x_i < x$, then since $\exp(-\sum_{i=0}^k s_i x_i) I(x_0 > 0)$ is a bounded continuous function on $\mathbb{Z}_+^{k+1}$, we have from (2.5) with $P_{n,x}$ and P_x replaced by $P_{n, \wedge x_i}$ and $P_{\wedge x_i}$,

$$\int \exp\left(-\sum_{i=0}^k s_i\, \mu(B_i)\right) P_{n,x}(d\mu)$$

$$= \int \exp\left(-\sum_{i=0}^k s_i\, \mu(B_i)\right) I(\mu(B_0) > 0) P_{n, \wedge x_i}(d\mu) \frac{\lambda_n(M_{\wedge x_i})}{\lambda_n(M_x)}$$

$$\to \int \exp\left(-\sum_{i=0}^{k} s_i\,\mu(B_i)\right) I(\mu(B_0) > 0) P_{\wedge x_i}(d\mu)\frac{\lambda(M_{\wedge x_i})}{\lambda(M_x)}$$
$$= \int \exp\left(-\sum_{i=0}^{k} s_i\,\mu(B_i)\right) P_x(d\mu).$$

This establishes the weak convergence of $(X_{n0},\dots,X_{nk})$ and hence the weak convergence of $P_{n,x}$ to P_x.

Since Ω is a continuous mapping,

$$P_{n,x}\circ\Omega^{-1} \to_w P_x\circ\Omega^{-1}$$

on $(0,\infty)\times\tilde{M}$. Using the facts that bivariate convergence implies marginal convergence, $\lambda(M_x) = \nu(x,\infty)$ and $\Omega^{-1}((x,\infty)\times\tilde{M})) = M_x$, we conclude that

$$P_{n,x}\circ\Omega^{-1}((x,\infty)\times\cdot) \to_w P_x\circ\Omega^{-1}((x,\infty)\times\cdot)$$
$$(2.6)\qquad = \frac{\lambda\circ\Omega^{-1}((x,\infty)\times\cdot)}{\gamma x^{-\alpha}} = \mathscr{Q}$$

on $\tilde{M}$. This proves (ii) since the left-hand side of (2.6) is equal to

$$P\left[\sum_{j=1}^{r_n}\delta_{\xi_j/(\vee_1^{r_n}|\xi_k|)}\in\cdot\,\middle|\,\bigvee_1^{r_n}|\xi_k| > a_n x\right].$$

Next assume that (ii) holds. Fix $f\in\mathscr{F}$ and suppose that the support of f is contained in $[-x,x]^c$ for some $x>0$. Then for any $y>0$ and with $P_{n,x}$ as defined earlier,

$$P_{n,x}\circ\Omega^{-1}((y,\infty)\times\cdot) = P_{n,x\vee y}\circ\Omega^{-1}((x\vee y,\infty)\times\cdot)\frac{\lambda_n(M_{x\vee y})}{\lambda_n(M_x)}$$
$$\to_w \mathscr{Q}(\cdot)\frac{\nu(x\vee y,\infty)}{\nu(x,\infty)} \quad \text{on } \tilde{M}.$$

This implies that

$$P_{n,x}\circ\Omega^{-1}(\cdot) \to_w \frac{\nu(\cdot\cap(x,\infty))\times\mathscr{Q}}{\nu(x,\infty)} \quad \text{on } (0,\infty)\times\tilde{M},$$

and since Ω^{-1} is continuous, we have

$$P_{n,x}(\cdot) \to_w \frac{(\nu\times\mathscr{Q})\circ\Omega(\cdot\cap M_x)}{\nu(x,\infty)} \quad \text{on } M_0.$$

Defining $\lambda = (\nu\times\mathscr{Q})\circ\Omega$, the last convergence can be rewritten as

$$P_{n,x}(\cdot) \to_w P_x(\cdot) := \frac{\lambda(\cdot\cap M_x)}{\lambda(M_x)} \quad \text{on } M_0,$$

which implies that

$$\int_{M_x} e^{-\mu f} P_{n,x}(d\mu) \to \int_{M_x} e^{-\mu f} P_x(d\mu).$$

However, since the support of f is contained in $[x, x]^c$ we have

$$\int (1 - e^{-\mu f}) \lambda_n(d\mu) \to \int (1 - e^{-\mu f}) \lambda(d\mu),$$

and because this holds for all $f \in \mathscr{F}$, (i) follows from Lemma 2.1. □

THEOREM 2.6. *Suppose that $N_n \to_d N$ and N has the representation given by Theorem 2.3. Then $\gamma \sum_{i=1}^{\infty} E|Q_i|^{\alpha} \le 1$, where $\sum_{i=1}^{\infty} \delta_{Q_i} \sim \mathscr{Q}$. The equality holds if $\{N_n([-1,1]^c)\}_{n=1}^{\infty}$ is uniformly integrable.*

PROOF. Observe that $N_n([-1,1]^c) = \sum_{j=1}^{n} I_{(a_n,\infty)}(|\xi_j|)$. By Fatou's lemma,

$$EN([-1,1]^c) \le \lim_{n\to\infty} EN_n([-1,1]^c) = 1. \tag{2.7}$$

Clearly $N([-1,1]^c)$ is compound Poisson and the Laplace transform of the compounding distribution is

$$E\int_1^{\infty} \exp\left(-s \sum_{i=1}^{\infty} I_{(1,\infty)}(u|Q_i|)\right) \alpha u^{-\alpha-1}\, du, \qquad s > 0,$$

where the expectation is now taken with respect to $\mathscr{Q}$. Thus the mean of the compounding distribution is $\sum_{i=1}^{\infty} E|Q_i|^{\alpha}$. Since $EN([-1,1]^c) = \gamma \sum_{i=1}^{\infty} E|Q_i|^{\alpha}$, the result follows from (2.7). □

REMARK 2.3. It is possible to extend the above results to the two-dimensional point process

$$N_n^* = \sum_{j=1}^{\infty} \delta_{(j/n, \xi_j/a_n)}.$$

However, under the condition $\mathscr{A}(\{a_n\})$, the weak convergence of N_n^* is equivalent to that of N_n. See Mori (1977).

We next apply Theorem 2.5 to the case where $\{\xi_j\}$ is jointly regularly varying. Let $\mathbf{X} = (X_1, \dots, X_k)$ be a k-dimensional random vector. Then $\mathbf{X}$ is said to be *jointly regularly varying* with index $\alpha > 0$ if there exist a sequence of constants x_n and a random vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ with $P[\|\boldsymbol{\theta}\| = 1] = 1$, where $\|\cdot\|$ denotes a norm on $\mathbb{R}^k$, such that

$$nP[\|\mathbf{X}\| > tx_n, \mathbf{X}/\|\mathbf{X}\| \in \cdot\,] \to_v t^{-\alpha} P[\boldsymbol{\theta} \in \cdot\,], \qquad t > 0.$$

For our application, it is natural to take $\|\cdot\|$ to be the "sup" norm, that is, $\|(x_1, \dots, x_k)\| = \bigvee_{i=1}^{k} |x_i|$, which we will do in the following theorem.

THEOREM 2.7. *Suppose that* $\{\xi_j\}$ *is a stationary sequence of random variables for which all finite-dimensional distributions are jointly regularly varying with index* $\alpha > 0$. *To be specific, let* $\boldsymbol{\theta}^{(m)} = (\theta_i^{(m)}, |i| \le m)$ *be the random vector* $\boldsymbol{\theta}$ *that appears in the definition of joint regular variation of* ξ_i, $|i| \le m$. *Assume that the condition* $\mathscr{A}(\{a_n\})$ *holds for* $\{\xi_j\}$ *and that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\bigvee_{m \le |i| \le r_n} |\xi_i| > ta_n \middle| |\xi_0| > ta_n \right] = 0, \qquad t > 0, \tag{2.8}$$

where a_n *is defined by* (1.4). *Then the limit*

$$\gamma := \lim_{m\to\infty} \frac{E\left(|\theta_0^{(m)}|^\alpha - \bigvee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}{E|\theta_0^{(m)}|^\alpha}$$

exists. If $\gamma = 0$ *then* $N_n \to_d o$; *if* $\gamma > 0$, *then* N_n *converges in distribution to some* N, *where, using the representation* $\lambda \circ \Omega^{-1} = \nu \times \mathscr{Q}$ *described in Theorem* 2.3, $\nu(dy) = \gamma\alpha y^{-\alpha-1}\,dy$ *and* $\mathscr{Q}$ *is the weak limit of*

$$\frac{E\left(|\theta_0^{(m)}|^\alpha - \bigvee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+ I\left(\sum_{|i| \le m} \delta_{\theta_i^{(m)}} \in \cdot\right)}{E\left(|\theta_0^{(m)}|^\alpha - \bigvee_{j=1}^m ||\theta_j^{(m)}|^\alpha\right)_+}$$

as $m \to \infty$, *which exists.*

REMARK 2.4. In many applications $\sum_{i=1}^m \delta_{\theta_i^{(m)}}$ is deterministic for every m. If this condition is added to the assumptions of Theorem 2.7, then clearly the measure $\mathscr{Q}$ in the description of the limit of N_n is a degenerate measure confined to the vague limit of $\sum_{|i| \le m} \delta_{\theta_i^{(m)}}$ as $m \to \infty$.

The following lemmas supply the necessary preliminaries for proving the theorem.

LEMMA 2.8. *Under the assumptions of Theorem* 2.7, *we have*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left| \frac{P[\bigvee_{i=1}^{r_n} |\xi_i| > a_n]}{r_n P[|\xi_1| > a_n]} - P\left[\bigvee_{i=1}^m |\xi_i| \le a_n \middle| |\xi_0| > a_n \right] \right| = 0. \tag{2.9}$$

If, in addition,

$$\liminf_{m\to\infty} \liminf_{n\to\infty} P\left[\bigvee_{i=1}^m |\xi_i| \le a_n \middle| |\xi_0| > a_n \right] > 0, \tag{2.10}$$

then for every set $A \subset \tilde{M}$ *of the form* $A = \{\mu \in \tilde{M}: \mu([-1, -b_i) \cup (b_i, 1]) \ge t_i, 1 \le i \le k\}$ *for some* $k \ge 1$, $t_i \ge 1$, $0 < b_i < 1$, $1 \le i \le k$,

$$\begin{aligned} \lim_{m\to\infty} \limsup_{n\to\infty} \Bigg| & P\left[\sum_{j=1}^{r_n} \delta_{\xi_j / \bigvee_{i=1}^{r_n} |\xi_i|} \in A \middle| \bigvee_{j=1}^{r_n} |\xi_j| > a_n \right] \\ & - P\left[\sum_{|j| \le m} \delta_{\xi_j / \bigvee_{|i| \le m} |\xi_i|} \in A \middle| \bigvee_{j=1}^m |\xi_j| \le a_n \le |\xi_0| \right] \Bigg| = 0. \end{aligned} \tag{2.11}$$

PROOF. Write

$$P\left[\bigvee_{j=1}^{r_n}|\xi_j|>a_n\right]=\sum_{j=1}^{r_n}P\left[|\xi_j|>a_n,\ \bigvee_{i=j+1}^{r_n}|\xi_i|\le a_n\right]$$

which, for each fixed m, can be written as

$$\begin{aligned}\sum_{j=1}^{r_n}P&\left[|\xi_j|>a_n,\ \bigvee_{i=j+1}^{j+m}|\xi_i|\le a_n\right]+R_{n,m,1}\\ &=r_nP[|\xi_1|>a_n]P\left[\bigvee_{i=1}^{m}|\xi_i|\le a_n\middle||\xi_0|>a_n\right]+R_{n,m,1},\end{aligned}$$

where

$$\begin{aligned}|R_{n,m,1}|&\le\sum_{j=1}^{r_n}P\left[|\xi_j|>a_n,\ \bigvee_{i=j+m+1}^{r_n}|\xi_i|>a_n\right]+2mP[|\xi_1|>a_n]\\ &\le r_nP[|\xi_1|>a_n]\left(P\left[\bigvee_{j=m+1}^{r_n}|\xi_j|>a_n\middle||\xi_0|>a_n\right]+2m/r_n\right).\end{aligned}$$

Applying (2.8) and the fact $r_n\to\infty$, we obtain (2.9). Now let A have the form described in the lemma and let $\varepsilon=\bigwedge_{i=1}^{k}b_i$. As before, write

$$\begin{aligned}P&\left[\sum_{j=1}^{r_n}\delta_{\xi_j/\bigvee_{i=1}^{r_n}|\xi_i|}\in A,\ \bigvee_{j=1}^{r_n}|\xi_j|>a_n\right]\\ &=\sum_{j=1}^{r_n}P\left[\sum_{j=1}^{r_n}\delta_{\xi_j/\bigvee_{i=1}^{r_n}|\xi_i|}\in A,\ |\xi_j|>a_n,\ \bigvee_{i=j+1}^{r_n}|\xi_i|\le a_n\right].\end{aligned}$$

The right-hand side is equal to

$$\begin{aligned}\sum_{j=m+1}^{r_n-m}P&\Bigg[\sum_{j=1}^{r_n}\delta_{\xi_j/\bigvee_{i=1}^{r_n}|\xi_i|}\in A,\ |\xi_j|>a_n,\ \bigvee_{i=j+1}^{r_n}|\xi_i|\le a_n,\\ &\qquad\bigvee_{\substack{1\le i\le r_n\\ |i-j|\ge m+1}}|\xi_i|\le\varepsilon a_n\Bigg]+R_{n,m,2},\end{aligned}$$

where

$$|R_{n,m,2}|\le r_nP[|\xi_0|>a_n]\left(P\left[\bigvee_{m+1\le|i|\le r_n}|\xi_i|>\varepsilon a_n\middle||\xi_0|>a_n\right]+2m/r_n\right).$$

By the choice of ε,

$$\sum_{j=m-1}^{r_n-m} P\left[\sum_{j=1}^{r_n} \delta_{\xi_j/\vee_{i=1}^{r_n}|\xi_i|} \in A, |\xi_j| > a_n, \bigvee_{i=j+1}^{r_n} |\xi_i| \le a_n, \bigvee_{\substack{1\le i\le r_n\\ |i-j|\ge m+1}} |\xi_i| \le \varepsilon a_n\right]$$

$$= \sum_{j=m+1}^{r_n-m} P\left[\sum_{|i-j|m} \delta_{\xi_i/\vee_{|i-j|\le m}|\xi_i|} \in A, |\xi_j| > a_n, \bigvee_{i=j+1}^{r_n} |\xi_i| \le a_n, \bigvee_{\substack{1\le i\le r_n\\ |i-j|\ge m+1}} |\xi_i| \le \varepsilon a_n\right]$$

$$= r_n P\left[\sum_{|i|\le m} \delta_{\xi_i/\vee_{|i|\le m}|\xi_i|} \in A, |\xi_0| > a_n, \bigvee_{i=1}^{m} |\xi_i| \le a_n\right] + R_{n,m,3},$$

where $|R_{n,m,3}|$ is bounded the same way as $|R_{n,m,2}|$. As before, using (2.8) and (2.10), we obtain (2.11). □

LEMMA 2.9. *Assume that the conditions of Theorem 2.7 hold. Then for any fixed m,*

$$P\left[\bigvee_{i=1}^{m} |\xi_i| \le a_n \,\middle|\, |\xi_0| > a_n\right] \to \frac{E\left(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}{E|\theta_0^{(m)}|^\alpha} \tag{2.12}$$

and

$$P\left[\sum_{|i|\le m} \delta_{\xi_i/\vee_{|j|\le m}|\xi_j|} \in \cdot \,\middle|\, \bigvee_{i=1}^{m} |\xi_i| \le a_n < |\xi_0|\right] \to_w \frac{E\left(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+ I\left(\sum_{|i|\le m} \delta_{\theta_i^{(m)}} \in \cdot\right)}{E\left(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}. \tag{2.13}$$

PROOF. We first consider (2.13). Let f be a bounded real-valued continuous function on $\tilde{M}$. Since $P[\vee_{|i|\le m}|\theta_i^{(m)}| = 1] = 1$, it suffices to show that

$$\frac{Ef\left(\sum_{|i|\le m} \delta_{\xi_i/\vee_{|j|\le m}|\xi_j|}\right) I\left(\bigvee_{i=1}^{m} |\xi_i| \le a_n < |\xi_0|\right)}{P\left[\vee_{i=1}^m |\xi_i| \le a_n < |\xi_0|\right]} \to \frac{Ef\left(\sum_{|i|\le m} \delta_{\theta_i^{(m)}/\vee_{|j|\le m}|\theta_j^{(m)}|}\right)\left(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}{E\left(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}.$$

Now write

$$\frac{Ef\left(\sum_{|i|\le m}\delta_{\xi_i/\bigvee_{|j|\le m}|\xi_j|}\right)I\left(\bigvee_{i=1}^m|\xi_i|\le a_n<|\xi_0|\right)}{P\left[\bigvee_{i=1}^m|\xi_i|\le a_n<|\xi_0|\right]}$$

$$(2.14)\qquad = \frac{Ef\left(\sum_{|i|\le m}\delta_{\xi_i/\bigvee_{|j|\le m}|\xi_j|}\right)I\left(\bigvee_{i=0}^m|\xi_i|>a_n\right) - Ef\left(\sum_{|i|\le m}\delta_{\xi_i/\bigvee_{|j|\le m}|\xi_j|}\right)I\left(\bigvee_{i=1}^m|\xi_i|>a_n\right)}{P\left[\bigvee_{i=0}^m|\xi_i|>a_n\right]-P\left[\bigvee_{i=1}^m|\xi_i|>a_n\right]}.$$

We first consider the expression $A_n = nEf(\sum_{|i|\le m}\delta_{\xi_i/\bigvee_{|j|\le m}|\xi_j|})I(\bigvee_{i=0}^m|\xi_i| > a_n)$. Since $\|\cdot\|$ is the sup norm, we can write

$$A_n = nEf\left(\sum_{|i|\le m}\delta_{\xi_i/\|\boldsymbol{\xi}\|}\right)I\left(\bigvee_{i=0}^m\frac{|\xi_i|}{\|\boldsymbol{\xi}\|}\|\boldsymbol{\xi}\|>a_n\right),$$

where $\boldsymbol{\xi} = (\xi_i, |i|\le m)$. Since the mapping

$$(x_i, |i|\le m)\to f\left(\sum_{|i|\le m}\delta_{x_i/\bigvee_{|i|\le m}|x_i|}\right)$$

is continuous at each nonzero $(x_i, |i|\le m)$, joint regular variation implies that for each $v>0$,

$$nEf\left(\sum_{|i|\le m}\delta_{\xi_i/\|\boldsymbol{\xi}\|}\right)I(\|\boldsymbol{\xi}\|>va_n)\to CEv^{-\alpha}f\left(\sum_{|i|\le m}\delta_{\theta_i^{(m)}}\right),$$

where C is a constant depending only on m. Using a straightforward argument, we obtain

$$A_n\to CE\left(\bigvee_{i=0}^m|\theta_i^{(m)}|\right)^\alpha f\left(\sum_{|i|\le m}\delta_{\theta_i^{(m)}}\right).$$

The other terms in (2.14) can be handled in a similar fashion whereupon (2.13) follows.

As for (2.12), we have

$$\begin{aligned}nP[|\xi_0|>a_n] &= nP\left[\frac{|\xi_0|}{\|\boldsymbol{\xi}\|}\|\boldsymbol{\xi}\|>a_n\right]\\ &\to CE|\theta_0^{(m)}|^\alpha\\ &=1\end{aligned}$$

and hence $C = 1/E|\theta_0^{(m)}|^\alpha$. □

PROOF OF THEOREM 2.7. Combining (2.9) of Lemma 2.8 and (2.12) of Lemma 2.9, it is easy to see that

$$\frac{E\left(|\theta_0^{(m)}|^\alpha - \bigvee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}{E|\theta_0^{(m)}|^\alpha}$$

is Cauchy in m, and by the same token we obtain

$$\lim_{n\to\infty} k_n P\left[\bigvee_{i=1}^{r_n} |\xi_i| > a_n\right] = \gamma,$$

where $k_n = [n/r_n]$ and γ is as defined by the theorem. If $\gamma = 0$, then it is straightforward to conclude that $N_n \to_d o$. So assume that $\gamma > 0$ from now on. In this case, (2.10) and the first assumption of (ii) of Theorem 2.5 hold. Now write

$$\mathscr{Q}_n(\cdot) = P\left[\sum_{j=1}^{r_n} \delta_{\xi_j/\vee_{i=1}^{r_n}|\xi_i|} \in \cdot \,\middle|\, \bigvee_{j=1}^{r_n} |\xi_j| > a_n\right],$$

$$\mathscr{Q}_n^{(m)}(\cdot) = P\left[\sum_{|j|\le m} \delta_{\xi_j/\vee_{|i|\le m}|\xi_i|} \in \cdot \,\middle|\, \bigvee_{j=1}^{m} |\xi_j| \le a_n < |\xi_0|\right]$$

and

$$\mathscr{Q}^{(m)}(\cdot) = \frac{E\left(|\theta_0^{(m)}|^\alpha - \bigvee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+ I\left(\sum_{|i|\le m} \delta_{\theta_i^{(m)}} \in \cdot\right)}{E\left(|\theta_0^{(m)}|^\alpha - \bigvee_{j=1}^m |\theta_j^{(m)}|^\alpha\right)_+}.$$

Using Kallenberg (1983), Lemma 4.5, it is easy to show that $\mathscr{Q}_n$ is tight. Hence for each subsequence of the set of integers there exists a further subsequence $\{n'\}$ along which $\mathscr{Q}_{n'} \to_w \mathscr{Q}'$ for some probability measure $\mathscr{Q}'$. Thus for each set A of the form specified in Lemma 5.2 such that $\sum_{i=1}^k \mathscr{Q}'(\{\mu: \mu(\partial B_i)\}) = 0$, it follows from Kallenberg (1983), Theorem 4.2, that

$$\lim_{n'\to\infty} \mathscr{Q}_{n'}(A) = \mathscr{Q}'(A). \tag{2.15}$$

On the other hand, it follows from (2.13) of Lemma 2.9 that

$$\mathscr{Q}_n^{(m)}(\cdot) \to_w \mathscr{Q}^{(m)}(\cdot)$$

and together with (2.11) of Lemma 2.8,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left|\mathscr{Q}_n(A) - \mathscr{Q}^{(m)}(A)\right| = 0. \tag{2.16}$$

Thus $\mathscr{Q}^{(m)}(A)$ is Cauchy and it converges to some number $\mathscr{Q}(A)$ and it follows from (2.16) that

$$\lim_{n\to\infty} \mathscr{Q}_n(A) = \mathscr{Q}(A). \tag{2.17}$$

Combining (2.15) and (2.17) gives the conclusion that $\mathscr{Q}(A) = \mathscr{Q}'(A)$ for all A in the form specified above. However, the collection of such A is measure determining, which shows that $\mathscr{Q}(\cdot)$ is a probability measure. Thus $\mathscr{Q}_n \to_w \mathscr{Q}$

by (2.17), from which the second assumption of (ii) of Theorem 2.5 follows. This completes the proof. □

3. Partial sum convergence. In this section, we establish convergence of the partial sums for a class of weakly dependent sequences. Throughout this section, we assume that $\{\xi_j\}$ is a strictly stationary sequence whose marginal distribution function satisfies (1.6). In addition, we assume that the conclusion of Theorem 2.3 holds, namely (see Corollary 2.4),

$$N_n = \sum_{j=1}^{n} \delta_{\xi_j/a_n} \to_d N = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{P_iQ_{ij}}, \tag{3.1}$$

where $\{a_n\}$ is chosen as in (1.4). Note that from Remark 2.1 that the points P_i may be written as $P_i = \gamma^{1/\alpha}\Gamma_i^{-1/\alpha}$, where $\Gamma_i = \sum_{k=1}^{i} E_k$ with $E_1, E_2, \ldots$ iid unit exponentials. Now denote the partial sums of the $\{\xi_j\}$ sequence by

$$S_n = \sum_{j=1}^{n} \xi_j$$

and for any Borel set B in $\mathbb{R}$ define

$$S_nB = a_n^{-1} \sum_{j=1}^{n} \xi_j I_B\left(a_n^{-1}|\xi_j|\right),$$

and

$$SB = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_iQ_{ij} I_B\left(|P_iQ_{ij}|\right).$$

Essentially under condition (3.1), the partial sums, suitably normalized, converge in distribution. This is the content of the following theorem.

THEOREM 3.1. *Let $\{\xi_j\}$ be a strictly stationary sequence satisfying* (1.6) *and* (3.1).

(i) *If $0 < \alpha < 1$, then*

$$a_n^{-1}S_n \to_d S,$$

where $S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_iQ_{ij}$ has a stable distribution.

(ii) *If $1 \le \alpha < 2$ and for all $\delta > 0$,*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\left[|S_n(0,\varepsilon] - ES_n(0,\varepsilon]| > \delta\right] = 0, \tag{3.2}$$

then

$$a_n^{-1}S_n - ES_n(0,1] \to_d S,$$

where S is the distributional limit of

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_iQ_{ij} I_{(\varepsilon,\infty)}\left(|P_iQ_{ij}|\right) - \int_{\varepsilon<|x|\le 1} x\mu(dx)\right) \tag{3.3}$$

as $\varepsilon \to 0$ [μ is the measure in (1.6)], *which exists and has a stable distribution.*

REMARK 3.1. For the case $\alpha > 1$, S_n may be centered by its mean rather than the truncated mean $E\xi_1 I_{(0,a_n]}(|\xi_1|)$. Since

$$\frac{n}{a_n}\left(E\xi_1 - E\xi_1 I_{(0,a_n]}(|\xi|)\right) = \frac{n}{a_n} E\xi_1 I_{(a_n,\infty)}(|\xi_1|) \to (p-q)\frac{\alpha}{\alpha-1},$$

it follows by convergence of types that

$$a_n^{-1}(S_n - nE\xi_1) \to_d S - (p-q)\frac{\alpha}{\alpha-1}.$$

PROOF OF THEOREM 3.1. (i) For every $\varepsilon > 0$, the mapping from M into $\mathbb{R}$ defined by

$$T\colon \sum_{i=1}^{\infty} \delta_{x_i} \to \sum_{i=1}^{\infty} x_i I_{(\varepsilon,\infty)}(|x_i|)$$

is a.s. continuous wrt the limit point process N. Thus, by the continuous mapping theorem,

$$\begin{aligned} S_n(\varepsilon,\infty) &= T(N_n) \\ &\to_d T(N) = S(\varepsilon,\infty). \end{aligned} \tag{3.4}$$

Now let $W_i = \sum_{j=1}^{\infty} Q_{ij}$, so that by Theorem 2.6, $\{W_i\}$ is an iid sequence of random variables with $E|W_i|^{\alpha} < \infty$. This implies that $P_i|W_i|$, $i = 1,\dots,$ are the points of a Poisson process with intensity measure $\gamma E|W_1|^{\alpha}\alpha x^{-\alpha-1}\,dx$ [see Resnick (1986)] and since $\alpha < 1$, these points are summable [see Theorem 2 of Davis (1983)]. It follows that

$$S(\varepsilon,\infty) \to S(0,\infty) = \sum_{i=1}^{\infty} P_i W_i \tag{3.5}$$

a.s. as $\varepsilon \to 0$. Moreover, by Markov's inequality and Karamata's theorem, we have for any $\delta > 0$,

$$\begin{aligned} P\left[|S_n(0,\varepsilon)| > \delta\right] &\le \frac{n}{\delta a_n} E|\xi_1| I_{(0,a_n\varepsilon]}(|\xi_1|) \\ &\sim \frac{\alpha n}{(1-\alpha)\delta a_n} P\left[|\xi_1| > a_n\varepsilon\right] a_n\varepsilon \\ &\to \frac{\alpha}{(1-\alpha)\delta}\varepsilon^{1-\alpha} \quad \text{as } n \to \infty \\ &\to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

whence

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} P\left[|S_n(0,\varepsilon]| > \delta\right] = 0. \tag{3.6}$$

Applying Theorem 4.2 of Billingsley (1968), the convergence

$$S_n(0,\infty) = a_n^{-1} S_n \to_d \sum_{i=1}^{\infty} P_i W_i$$

is immediate.

(ii) The limit in (3.4) also holds in the $\alpha \ge 1$ case and from (1.6) we have

$$ES_n(\varepsilon,1] = na_n^{-1}E\xi_1 I_{(\varepsilon,1]}(|\xi_1|a_n^{-1}) \to b(\varepsilon,1] := \int_{\varepsilon<|x|\le 1} x\mu(dx). \tag{3.7}$$

Consequently,

$$S_n(\varepsilon,\infty) - ES_n(\varepsilon,1] \to_d S(\varepsilon,\infty) - b(\varepsilon,1]. \tag{3.8}$$

Next we show that

$$\tilde{S}(\varepsilon) := S(\varepsilon,\infty) - b(\varepsilon,1] \to_d \text{ some } S \tag{3.9}$$

as $\varepsilon \to 0$. To prove this, it suffices to show that the characteristic functions, $\phi_\varepsilon(t)$, of $\tilde{S}(\varepsilon)$ converge to a function which is continuous at 0. We first show that $\phi_\varepsilon(t)$ is Cauchy as $\varepsilon \to 0$. Write

$$\begin{aligned}\phi_v(t) - \phi_u(t) &= E\exp(it\tilde{S}(v))\big(1-\exp\big(it(\tilde{S}(u)-\tilde{S}(v))\big)\big)\\ &= E\Big[\exp(it\tilde{S}(v))\big(1-\exp\big(it(\tilde{S}(u)-\tilde{S}(v))\big)\big)\\ &\qquad\times I\big(|\tilde{S}(u)-\tilde{S}(v)|>\delta\big)\Big]\\ &\quad+E\Big[\exp(it\tilde{S}(v))\big(1-\exp\big(it(\tilde{S}(u)-\tilde{S}(v))\big)\big)\\ &\qquad\times I\big(|S(u)-\tilde{S}(v)|\le\delta\big)\Big].\end{aligned}$$

Since t is fixed, we can pick a $\delta > 0$, such that

$$\begin{aligned}\Big|E\Big[&\exp(it\tilde{S}(v))\big(1-\exp\big(it(\tilde{S}(u)-\tilde{S}(v))\big)\big)\\ &\times I\big(|\tilde{S}(u)-\tilde{S}(v)|\le\delta\big)\Big]\Big| \le \eta/2,\\ &\qquad 0<u<v<1.\end{aligned} \tag{3.10}$$

Also for any $0 < \varepsilon < 1$,

$$\begin{aligned}&\sup_{0<u<v\le\varepsilon} P\big[|\tilde{S}(u)-\tilde{S}(v)|>\delta\big]\\ &\quad= \sup_{0<u<v\le\varepsilon}\ \lim_{n\to\infty} P[|S_n(u,v]-ES_n(u,v]|>\delta]\\ &\quad\le \sup_{0<u<v\le\varepsilon}\ \limsup_{n\to\infty}\big(P[|S_n(0,u]-ES_n(0,u]|>\delta/2]\\ &\qquad\qquad +P[|S_n(0,v]-ES_n(0,v]|>\delta/2]\big).\end{aligned}$$

By the above relation and (3.2), it is possible to pick an $\varepsilon > 0$ small such that

$$\begin{aligned}\sup_{0<u<v\le\varepsilon}&\Big|E\Big[\exp(it\tilde{S}(v))\big(1-\exp\big(it(\tilde{S}(u)-\tilde{S}(v))\big)\big)\\ &\qquad\times I\big(|\tilde{S}(u)-\tilde{S}(v)|>\delta\big)\Big]\Big|\\ &\le 2\sup_{0<u<v\le\varepsilon} P\big[|\tilde{S}(u)-\tilde{S}(v)|>\delta\big)\big] \le \eta/2.\end{aligned} \tag{3.11}$$

This combined with (3.10) shows that $\phi_\varepsilon(t)$ is Cauchy for each fixed t and hence converges to a function, say $\phi(t)$. Finally, to see that $\phi(t)$ is continuous at 0 it is enough to show that $\phi_\varepsilon(\cdot)$ converges to $\phi(\cdot)$ uniformly on a neighborhood of 0 or, equivalently, that $\phi_\varepsilon(t)$ is Cauchy in the uniform metric on the compact set $|t| \le t_0$. A slight modification of the foregoing argument establishes this fact. [First, choose δ small enough so that (3.10) holds uniformly on $|t| \le t_0$ and then take ε small enough so that (3.11) is valid.] From (3.2), (3.8) and (3.9), it follows, by applying Theorem 4.2 of Billingsley (1968) once again, that

$$S_n(0,\infty) - ES_n(0,1] \to_d S$$

where S has characteristic function ϕ. It remains to show that S has a stable distribution. Notice that $S(\varepsilon,\infty) - b(\varepsilon,1]$ is infinitely divisible with characteristic function given by $\exp[\psi_\varepsilon(t)]$, where

$$\psi_\varepsilon(t) = \int_0^\infty E\left(\exp\left\{it\sum_{j=1}^\infty xQ_{1j}I_{(\varepsilon,\infty)}(|xQ_{1j}|)\right\} - 1\right)\gamma\alpha x^{-\alpha-1}\,dx - itb(\varepsilon,1]$$

$$= \int_{-\infty}^\infty \frac{\exp(itx) - 1 - it\sin x}{x^2} M_\varepsilon(dx) + i\theta_\varepsilon t,$$

$$\theta_\varepsilon = \int_{-\infty}^\infty \frac{\sin x}{x^2} M_\varepsilon(dx) - b(\varepsilon,1]$$

and

$$M_\varepsilon(dx) = x^2\int_0^\infty P\left[\sum_{j=1}^\infty yQ_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|) \in dx\right]\gamma\alpha y^{-\alpha-1}\,dy$$

is the canonical measure in Feller's representation of an infinitely divisible characteristic function [see Feller (1971), XVII.2]. By Theorem 2 of Feller [(1971), page 564], S must also be infinitely divisible, $M_\varepsilon(I) \to M(I)$ for all bounded intervals I and $\theta_\varepsilon \to \theta$, where M is the canonical measure and θ is the constant in the canonical representation of the characteristic function of S. It is straightforward to show that for any $\rho > 0$,

$$M_\varepsilon(0,\rho x) = \rho^{2-\alpha}M_{\varepsilon/\rho}(0,x), \quad M_\varepsilon(-\rho x,0) = \rho^{2-\alpha}M_{\varepsilon/\rho}(-x,0), \quad x > 0.$$

By letting $\varepsilon \to 0$ in these expressions, we find that there exist constants c_- and $c_+ \ge 0$ such that

$$M(-x,0) = c_- x^{2-\alpha}, \qquad M(0,x) = c_+ x^{2-\alpha}, \qquad x > 0.$$

Consequently S has the characteristic function of a stable random variable. □

REMARK 3.2. Since the limit random variable S in Theorem 3.1 is stable, it has characteristic function

$$Ee^{itS} = \begin{cases} \exp\{imt - d|t|^\alpha[1 - i\beta\,\mathrm{sgn}(t)\tan(\pi\alpha/2)]\}, & \text{if } \alpha \ne 1,\\ \exp\{imt - d|t|[1 + i\beta(2/\pi)\mathrm{sgn}(t)\ln|t|]\}, & \text{if } \alpha = 1.\end{cases}$$

The scale and symmetry parameters d and β are given by

$$d = \begin{cases} (c_+ + c_-)\dfrac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\dfrac{\pi\alpha}{2}\right), & \text{if } \alpha \neq 1, \\ (c_+ + c_-)\dfrac{\pi}{2}, & \text{if } \alpha = 1, \end{cases}$$

$$\beta = \frac{c_+ - c_-}{c_+ + c_-},$$

where

$$\begin{aligned} c_+ &= \frac{\alpha}{2-\alpha}\int_1^\infty x^{-2} M(dx) \\ &= \frac{\alpha}{2-\alpha}\lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{j=1}^\infty yQ_{1j} I_{(\varepsilon,\infty)}(y|Q_{1j}|) > 1\right]\gamma\alpha y^{-\alpha-1}\,dy \end{aligned}$$

and

$$\begin{aligned} c_- &= \frac{\alpha}{2-\alpha}\int_{-\infty}^{-1} x^{-2} M(dx) \\ &= \frac{\alpha}{2-\alpha}\lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{j=1}^\infty yQ_{1j} I_{(\varepsilon,\infty)}(y|Q_{1j}|) < -1\right]\gamma\alpha y^{-\alpha-1}\,dy. \end{aligned}$$

The location parameter m is determined by

$$m = \begin{cases} 0, & \text{if } \alpha < 1, \\ \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \dfrac{\sin x}{x^2} M_\varepsilon(dx) - b(\varepsilon, 1], & \text{if } \alpha = 1, \\ \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} x^{-1} M_\varepsilon(dx) - b(\varepsilon, 1], & \text{if } \alpha > 1, \end{cases}$$

where $b(\varepsilon, 1]$ is defined by (3.7). See Feller [(1971), pages 568–570] for details. As seen in the following theorem, one can often compute more explicit values for the parameters c_+, c_- and m.

THEOREM 3.2. *Assume that the conditions of Theorem* 3.1 *hold and use the notation introduced in the proof of Theorem* 3.1 *and Remark* 3.2.

(a) *If* $E(\sum_{j=1}^\infty |Q_{1j}|)^\alpha < \infty$ (*which by virtue of Theorem* 2.6 *is always satisfied if* $\alpha \leq 1$), *then*

$$c_+ = \frac{\alpha}{2-\alpha}\gamma E(W^+)^\alpha$$

and

$$c_- = \frac{\alpha}{2-\alpha}\gamma E(W^-)^\alpha,$$

where $W = \sum_{j=1}^\infty Q_{1j}$.

(b) *If* $\alpha > 1$, *then* $m = (p - q)\alpha/(\alpha - 1)$.
(c) *If* $\alpha = 1$, *then*

$$m = \int_{-\infty}^{\infty}\left(\frac{\sin x - xI_{(0,1)}(|x|)}{x^2}\right)M(dx)$$
$$+ \gamma E\left[\sum_{j=1}^{\infty} Q_{1j}\log\left|\sum_{j=1}^{\infty} Q_{1j}\right|^{-1} I_{(0,1)}\left(\left|\sum_{j=1}^{\infty} Q_{1j}\right|\right)\right]$$
$$+ \lim_{t\to\infty}\gamma E\int_1^t y^{-1}\left[\left(\sum_{j=1}^{\infty} Q_{1j}I_{(1,\infty)}(y|Q_{1j}|)\right)\right.$$
$$\left.\times I_{(0,t)}\left(y\left|\sum_{j=1}^{\infty} Q_{1j}I_{(1,\infty)}(y|Q_{1j}|)\right|\right) - \sum_{j=1}^{\infty} Q_{1j}\right]dy,$$

where each summand is well defined and finite.

REMARK 3.3. While the location parameter m is the same as in the iid case if $\alpha \neq 1$, it is not so if $\alpha = 1$. We have not been able to simplify the expression for m when $\alpha = 1$. In this case, observe that if the limit point process N is Poisson, then the second and third terms on the rhs of m are zero and m reduces to the familiar expression

$$\int_{-\infty}^{\infty}\left(\frac{\sin x - xI_{(0,1)}(|x|)}{x^2}\right)M(dx).$$

PROOF OF THEOREM 3.2(a). Since $\sum_{j=1}^{\infty} yQ_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|) \to yW$ a.s. and W is bounded by $\sum_{j=1}^{\infty}|Q_{1j}|$, it follows by the dominated convergence theorem that

$$c_+ = \frac{\alpha}{2-\alpha}\int_0^{\infty} P[yW^+ > 1]\gamma\alpha y^{-\alpha-1}\,dy = \frac{\alpha}{2-\alpha}E(W^+)^{\alpha}.$$

The argument for c_- is exactly the same. □

PROOF OF THEOREM 3.2(b). It is straightforward to calculate

$$\int_{-\infty}^{\infty} x^{-1}M_\varepsilon(dx) = \frac{\alpha}{\alpha-1}\varepsilon^{1-\alpha}\gamma\sum_{j=1}^{\infty} EQ_{1j}^{\langle\alpha\rangle}$$

and

$$b(\varepsilon,1] = \frac{\alpha}{\alpha-1}(p-q)(\varepsilon^{1-\alpha} - 1),$$

where $x^{\langle\alpha\rangle} = x|x|^{\alpha-1}$. It is enough to show

$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x^2} - x^{-1}\right)M_\varepsilon(dx) \to \int_{-\infty}^{\infty}\left(\frac{\sin x}{x^2} - x^{-1}\right)M(dx), \tag{3.12}$$

since this and the property $\theta_\varepsilon \to \theta$ imply

$$\int_{-\infty}^{\infty} x^{-1} M_\varepsilon(dx) + b(\varepsilon, 1] = \frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}\left(\gamma \sum_{j=1}^{\infty} EQ_{1j}^{\langle\alpha\rangle} - (p-q)\right) + \frac{\alpha}{\alpha-1}(p-q) \to \text{const}$$

as $\varepsilon \to 0$. However, in order for the left-hand side of this equation to have a finite limit, we must have

$$\gamma \sum_{j=1}^{\infty} EQ_{1j}^{\langle\alpha\rangle} = p - q \tag{3.13}$$

from which the conclusion of (b) is immediate.

To prove (3.12), the vague convergence of M_ε to M on $(-\infty, \infty)$ implies that for any $\delta > 0$,

$$\int_{|x|\le\delta}\left(\frac{\sin x}{x^2} - x^{-1}\right) M_\varepsilon(dx) \to \int_{|x|\le\delta}\left(\frac{\sin x}{x^2} - x^{-1}\right) M(dx).$$

So it is enough to establish

$$\lim_{\delta\to\infty} \limsup_{\varepsilon\to 0} \int_{|x|>\delta} |x|^{-1} M_\varepsilon(dx) = 0 \tag{3.14}$$

and

$$\lim_{\delta\to\infty} \int_{|x|>\delta} |x|^{-1} M(dx) = 0.$$

Note that the latter follows easily from the former so we concentrate on (3.14). Write

$$\int_{|x|>\delta} |x|^{-1} M_\varepsilon(dx) = \int_{y=0}^{\infty} f_{\varepsilon,\delta}(y)\gamma\alpha y^{-\alpha-1}\,dy,$$

where

$$f_{\varepsilon,\delta}(y) = \int_{|x|>\delta} |x| P\left[\sum_{j=1}^{\infty} yQ_{1j} I_{(\varepsilon,\infty)}\big(y|Q_{1j}|\big) \in dx\right].$$

Now write

$$f_{\varepsilon,\delta}(y) = f_{\varepsilon,\delta}^{(1)}(y) + f_{\varepsilon,\delta}^{(2)}(y),$$

where

$$f_{\varepsilon,\delta}^{(1)}(y) = \delta P\left[\left|\sum_{j=1}^{\infty} yQ_{1j} I_{(\varepsilon,\infty)}\big(y|Q_{1j}|\big)\right| > \delta\right]$$

and

$$f_{\varepsilon,\delta}^{(2)}(y) = \int_{x=\delta}^{\infty} P\left[\left|\sum_{j=1}^{\infty} yQ_{1j} I_{(\varepsilon,\infty)}\big(y|Q_{1j}|\big)\right| > x\right] dx.$$

Changing variables, we have

$$\int_{y=0}^{\infty} f_{\varepsilon,\delta}^{(1)}(y)\gamma\alpha y^{-\alpha-1}\,dy$$

$$= \delta^{1-\alpha}\int_{y=0}^{\infty} P\left[\left|\sum_{j=1}^{\infty} yQ_{1j}I_{(\varepsilon/\delta,\infty)}(y|Q_{1j}|)\right| > 1\right] c\alpha y^{-\alpha-1}\,dy$$

$$\to \delta^{1-\alpha}\frac{2-\alpha}{\alpha}(c_+ + c_-)$$

as $\varepsilon \to 0$. Similarly,

$$\int_{y=0}^{\infty} f_{\varepsilon,\delta}^{(2)}(y)\gamma\alpha y^{-\alpha-1}\,dy$$

$$= \int_{x=\delta}^{\infty}\left(\int_{y=0}^{\infty} P\left[\left|\sum_{j=1}^{\infty} yQ_{1j}I_{(\varepsilon/x,\infty)}(y|Q_{1j}|)\right| > 1\right]\gamma\alpha y^{-\alpha-1}\,dy\right) x^{-\alpha}\,dx.$$

Since the inner integral converges to $(2-\alpha)/\alpha(c_+ + c_-)$ uniformly in $x > \delta$, the above expression converges to, as $\varepsilon \to 0$,

$$\frac{2-\alpha}{\alpha(\alpha-1)}(c_+ + c_-)\delta^{1-\alpha}.$$

This proves (3.14) and hence (3.12). □

PROOF OF THEOREM 3.2(c). First we show that the relation in (3.13) is also valid in the $\alpha = 1$ case. To see this, recall that the location parameter θ is given by the limit of

$$\theta_\varepsilon = \int_{-\infty}^{\infty}\frac{\sin x}{x^2}M_\varepsilon(dx) - b(\varepsilon, 1].$$

Since $b(\varepsilon, 1] = -(p-q)\log\varepsilon$, we have by the Cauchy criterion that

$$\int_{-\infty}^{\infty}\frac{\sin x}{x^2}M_{2\varepsilon}(dx) - \int_{-\infty}^{\infty}\frac{\sin x}{x^2}M_\varepsilon(dx) \to (q-p)\log 2.$$

The left-hand side of this expression is

$$E\int_{-\infty}^{\infty}\gamma y^{-2}\left\{\sin\left(y\sum_{j=1}^{\infty}Q_{1j}I_{(2\varepsilon,\infty)}(y|Q_{1j}|)\right) - \sin\left(y\sum_{j=1}^{\infty}Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\right)\right\}dy$$

$$= E\int_{-\infty}^{\infty}\gamma y^{-2}\varepsilon^{-1}\left\{\sin\left(\varepsilon y\sum_{j=1}^{\infty}Q_{1j}I_{(2,\infty)}(y|Q_{ij}|)\right)\right.$$

$$\left. -\sin\left(\varepsilon y\sum_{j=1}^{\infty}Q_{ij}I_{(1,\infty)}(y|Q_{ij}|)\right)\right\}dy$$

$$=: E\int_{-\infty}^{\infty} f_\varepsilon(y)\,dy.$$

Now, using the inequality $|\sin(x+y) - \sin(x)| \le |y|$,

$$|f_\varepsilon(y)| \le \gamma y^{-1} \sum_{j=1}^{\infty} |Q_{ij}| I_{(1,2]}(y|Q_{ij}|) =: g(y),$$

where

$$E\int_{-\infty}^{\infty} g(y)\, dy = \gamma \sum_{j=1}^{\infty} E|Q_{1j}| \log 2 < \infty$$

by Theorem 2.6. Since

$$f_\varepsilon(y) \to -\gamma y^{-1} \sum_{j=1}^{\infty} Q_{1j} I_{(1,2]}(y|Q_{1j}|) =: f(y)$$

and

$$E\int_{-\infty}^{\infty} f(y)\, dy = -\gamma \sum_{j=1}^{\infty} EQ_{1j} \log 2,$$

we obtain by dominated convergence,

$$\gamma \sum_{j=1}^{\infty} EQ_{1j} = p - q, \tag{3.15}$$

as asserted.

Next write

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2} M_\varepsilon(dx) - b(\varepsilon, 1] = A_1(\varepsilon) + A_2(\varepsilon) + A_3(\varepsilon),$$

where

$$A_1(\varepsilon) = \int_{-1}^{1} \left(\frac{\sin x}{x^2} - \frac{1}{x} \right) M_\varepsilon(dx),$$

$$A_2(\varepsilon) = \int_{|x|>1} \frac{\sin x}{x^2} M_\varepsilon(dx)$$

and

$$A_3(\varepsilon) = \int_{-1}^{1} \frac{1}{x} M_\varepsilon(dx) - b(\varepsilon, 1].$$

By vague convergence and the fact that $(\sin x)/x^2 - 1/x$ is bounded, we obtain

$$\lim_{\varepsilon \to 0} A_1(\varepsilon) = \int_{-1}^{1} \left(\frac{\sin x}{x^2} - \frac{1}{x} \right) M(dx).$$

We next consider $A_2(\varepsilon)$. Note that by the same argument as above, we obtain for every $\delta > 0$,

$$\int_{1<|x|<\delta} \frac{\sin x}{x^2} M_\varepsilon(dx) \to \int_{1<|x|<\delta} \frac{\sin x}{x^2} M(dx). \tag{3.16}$$

On the other hand,

$$\int_{|x|>\delta}\frac{\sin x}{x^2}M_\varepsilon(dx)=\int_0^\infty \gamma y^{-2}E\Bigg[\sin\Bigg(y\sum_{j=1}^\infty Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\Bigg)$$
$$\times I_{(\delta,\infty)}\Bigg(y\Bigg|\sum_{j=1}^\infty Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\Bigg|\Bigg)\Bigg]\,dy,$$

which is bounded in absolute value by

$$\gamma E\int_0^\infty y^{-2}I_{(\delta,\infty)}\Bigg(y\sum_{j=1}^\infty |Q_{1j}|\Bigg)\,dy=\frac{\gamma\sum_{j=1}^\infty E|Q_{1j}|}{\delta}\to 0\quad\text{as }\delta\to\infty. \tag{3.17}$$

Also

$$\int_{\delta<|x|}\frac{\sin x}{x^2}M(dx)\to 0\quad\text{as }\delta\to\infty. \tag{3.18}$$

By (3.16)–(3.18) we have shown that

$$\lim_{\varepsilon\to 0}A_2(\varepsilon)=\int_{|x|>1}\frac{\sin x}{x^2}M(dx).$$

Finally we consider

$$A_3(\varepsilon)=E\int_\varepsilon^\infty\Bigg[\gamma y^{-2}\Bigg(y\sum_{j=1}^\infty Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\Bigg)I_{(0,1)}\Bigg(y\Bigg|\sum_{j=1}^\infty Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\Bigg|\Bigg)$$
$$-(p-q)y^{-1}I_{(0,1)}(y)\Bigg]\,dy$$

which, by making use of (3.15), can be written as

$$\gamma E\int_\varepsilon^\infty\Bigg[y^{-1}\Bigg(\sum_{j=1}^\infty Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\Bigg)I_{(0,1)}\Bigg(y\Bigg|\sum_{j=1}^\infty Q_{1j}I_{(\varepsilon,\infty)}(y|Q_{1j}|)\Bigg|\Bigg)$$
$$-\sum_{j=1}^\infty Q_{1j}I_{(0,1)}(y)\Bigg]\,dy.$$

Write this expression as

$$B_1(\varepsilon)+B_2(\varepsilon),$$

where

$$B_1(\varepsilon) = \gamma E \int_1^\infty y^{-2} \left(y \sum_{j=1}^\infty Q_{1j} I_{(\varepsilon,\infty)}(y|Q_{1j}|) \right) I_{(0,1)} \left(y \left| \sum_{j=1}^\infty Q_{1j} I_{(\varepsilon,\infty)}(y|Q_{1j}|) \right| \right) dy$$

and

$$B_2(\varepsilon) = \gamma E \int_\varepsilon^1 y^{-1} \left[\left(\sum_{j=1}^\infty Q_{1j} I_{(\varepsilon,\infty)}(y|Q_{1j}|) \right) \times I_{(0,1)} \left(y \left| \sum_{j=1}^\infty Q_{1j} I_{(\varepsilon,\infty)}(y|Q_{1j}|) \right| \right) - \sum_{j=1}^\infty Q_{1j} \right] dy.$$

By dominated convergence, we have

$$\lim_{\varepsilon \to 0} B_1(\varepsilon) = \gamma E \left[\sum_{j=1}^\infty Q_{1j} \log \left| \sum_{j=1}^\infty Q_{1j} \right|^{-1} I_{(0,1)} \left(\left| \sum_{j=1}^\infty Q_{1j} \right| \right) \right]$$

and, after changing variables,

$$B_2(\varepsilon) = \gamma E \int_1^{\varepsilon^{-1}} y^{-1} \left[\left(\sum_{j=1}^\infty Q_{1j} I_{(1,\infty)}(y|Q_{1j}|) \right) \times I_{(0,\varepsilon^{-1})} \left(y \left| \sum_{j=1}^\infty Q_{1j} I_{(1,\infty)}(y|Q_{1j}|) \right| \right) - \sum_{j=1}^\infty Q_{1j} \right] dy.$$

Part (c) of the theorem now follows. □

4. Large deviations. We consider in this section the probabilities of large deviations of $S_n = \sum_{j=1}^n \xi_j$ under the assumption that N_n has the distributional limit described in Corollary 2.4 [or (3.1)]. Our approach is based on probability arguments and the main result, Theorem 4.4, generalizes existing results in a number of ways.

Define

$$S_i(B) = \sum_{j=1}^i \xi_j I_B(|\xi_j|), \qquad i \geq 1, B \subset \mathbb{R}.$$

We begin with two technical lemmas.

LEMMA 4.1. *For $i \geq 1$, $B \subset \mathbb{R}$, $t > 0$ and $0 < \delta \leq 1$,*

$$|P[S_i > t] - P[S_i(B) > \delta t]| \leq P[|S_i(B^c)| > (1-\delta)t].$$

PROOF. The proof is based on the following inclusions:

$$(S_i > t) \subset (S_i(B) > \delta t) \cup (S_i(B^c) > (1-\delta)t)$$

and

$$(S_i(B) > \delta t) \subset (S_i > t) \cup (S_i(B^c) < -(1-\delta)t).$$ □

LEMMA 4.2. *Suppose $N_n \to_d N$, where N has the representation in Corollary 2.4. Then for any $\varepsilon > 0$, $\zeta > 0$ and any sequence r_n such that $r_n \to \infty$, $r_n/n \to 0$, and satisfying (2.1), we have*

$$\lim_{n\to\infty} \frac{P\left[S_{r_n}(\varepsilon a_n,\infty) > \zeta a_n\right]}{r_n P[|\xi_1| > a_n]} = \int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \zeta\right]\nu(du).$$

PROOF. For $f \in \mathscr{F}_s$, we have

$$\begin{aligned}
&k_n E\left(1 - \exp\left(-\sum_{j=1}^{r_n} f\left(\frac{\xi_j}{a_n}\right) I_{(\varepsilon a_n,\infty)}(|\xi_j|)\right)\right)\\
&\quad = k_n P\left[\bigvee_1^{r_n} |\xi_k| > \varepsilon a_n\right]\\
&\qquad \times\left(1 - E\left(\exp\left(-\sum_{j=1}^{r_n} f\left(\frac{\xi_j}{a_n}\right) I_{(\varepsilon a_n,\infty)}(|\xi_j|)\right)\,\middle|\, \bigvee_1^{r_n} |\xi_k| > \varepsilon a_n\right)\right)\\
&\quad \sim \varepsilon^{-\alpha}\gamma\left(1 - E\left(\exp\left(-\sum_{j=1}^{r_n} f\left(\frac{\xi_j}{a_n}\right) I_{(\varepsilon a_n,\infty)}(|\xi_j|)\right)\,\middle|\, \bigvee_1^{r_n} |\xi_k| > \varepsilon a_n\right)\right)
\end{aligned}$$

by (ii) of Theorem 2.5. On the other hand, by Theorems 2.1 and 2.3 and the fact that $\bigvee_{i=1}^\infty |Q_{1i}| = 1$,

$$\begin{aligned}
&k_n E\left(1 - \exp\left(-\sum_{j=1}^{r_n} f\left(\frac{\xi_j}{a_n}\right) I_{(\varepsilon a_n,\infty)}(|\xi_j|)\right)\right)\\
&\quad \to \int_0^\infty E\left(1 - \exp\left(-\sum_{i=1}^\infty f(Q_{1i}u) I_{(\varepsilon,\infty)}(|Q_{1i}|u)\right)\right)\nu(du)\\
&\quad = \varepsilon^{-\alpha}\gamma\left(1 - \int_\varepsilon^\infty E\exp\left(-\sum_{i=1}^\infty f(Q_{1i}u) I_{(\varepsilon,\infty)}(|Q_{1i}|u)\right)\frac{\nu(du)}{\varepsilon^{-\alpha}\gamma}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&E\left(\exp\left(-\sum_{j=1}^{r_n} f\left(\frac{\xi_j}{a_n}\right) I_{(\varepsilon a_n,\infty)}(|\xi_j|)\right)\,\middle|\, \bigvee_1^{r_n} |\xi_k| > \varepsilon a_n\right)\\
&\quad \to \int_\varepsilon^\infty E\exp\left(-\sum_{i=1}^\infty f(Q_{1i}u) I_{(\varepsilon,\infty)}(|Q_{1i}|u)\right)\frac{\nu(du)}{\varepsilon^{-\alpha}\gamma}
\end{aligned}$$

and hence the point process $\sum_{j=1}^{r_n} \delta_{\xi_j/a_n} I_{(a_n,\varepsilon,\infty)}(|\xi_j|)$ converges in distribution conditional on $\bigvee_1^{r_n} |\xi_k| > \varepsilon a_n$. Clearly the mapping $\mu \to \int x\mu(dx)$, $M_0 \to \mathbb{R}$ is almost surely continuous with respect to the limit, so that

$$P\left[S_{r_n}(\varepsilon a_n, \infty) \in a_n \cdot \middle| \bigvee_1^{r_n} |\xi_k| > \varepsilon a_n \right]$$

$$\to_w \int_\varepsilon^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) \in \cdot \right] \frac{\nu(du)}{\varepsilon^{-\alpha}\gamma}.$$

Since the limiting distribution is continuous,

$$\begin{aligned} &P\left[S_{r_n}(\varepsilon a_n, \infty) > \zeta a_n \right] \\ &= P\left[\bigvee_1^{r_n} |\xi_k| > \varepsilon a_n \right] P\left[S_{r_n}(\varepsilon a_n, \infty) > \zeta a_n \middle| \bigvee_1^{r_n} |\xi_k| > \varepsilon a_n \right] \\ &\sim \frac{P\left[\bigvee_1^{r_n} |\xi_k| > \varepsilon a_n \right]}{\varepsilon^{-\alpha}\gamma} \int_\varepsilon^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \zeta \right] \nu(du) \\ &\sim r_n P\left[|\xi_1| > a_n \right] \int_\varepsilon^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \zeta \right] \nu(du), \end{aligned}$$

which completes the proof. □

THEOREM 4.3. *Suppose $N_n \to_d N$, where N has the representation in Corollary* 2.4 *and r_n is any sequence such that $r_n \to \infty$, $r_n/n \to 0$ and satisfying* (2.1). *Also assume that one of the following sets of conditions holds*:

(i) $0 < \alpha < 1$.
(ii) $1 \le \alpha < 2$ *and*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{P\left[\left| S_{r_n}(0, \varepsilon a_n] \right| > a_n \zeta \right]}{r_n P\left[|\xi_1| > a_n \right]} = 0, \qquad \zeta > 0. \tag{4.1}$$

Then

$$\lim_{n \to \infty} \frac{P\left[S_{r_n} > a_n \right]}{r_n P\left[|\xi_1| > a_n \right]} = \lim_{\varepsilon \to 0} \int_0^\infty P\left[\sum_{i=1}^\infty u Q_{1i} I_{(\varepsilon,\infty)}(u|Q_{1i}|) > 1 \right] \nu(du), \tag{4.2}$$

where the limit on the right of (4.2) *necessarily exists in* $[0, \infty)$.

PROOF. First note that (4.1) holds for $0 < \alpha < 1$, since by Chebyshev's inequality and Theorem 2 of Feller [(1971), VIII.9],

$$\begin{aligned}\lim_{n\to\infty} \frac{P\left[\left|S_{r_n}(0, \varepsilon a_n]\right| > \zeta a_n\right]}{r_n P[|\xi_1| > a_n]} &\le \lim_{n\to\infty} \frac{r_n E|\xi_1| I(|\xi_1| \le \varepsilon a_n)}{\zeta a_n r_n P[|\xi_1| > a_n]} \\ &= \lim_{n\to\infty} \frac{\varepsilon a_n P[|\xi_1| > \varepsilon a_n]}{(1-\alpha)\zeta a_n P[|\xi_1| > a_n]} \\ &= \frac{\varepsilon^{1-\alpha}}{(1-\alpha)\zeta} \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

The result will be proved the same way for both (i) and (ii). By Lemma 4.2 and a variant of the argument in Theorem 3.1, it follows from (4.1) that

$$\int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > 1\right] \nu(du)$$

is Cauchy as $\varepsilon \to 0$. Thus the limit on the right of (4.2) exists and is finite. For $\varepsilon > 0$, $0 < \delta < 1$, we have

$$\begin{aligned}&\left| \frac{P[S_{r_n} > a_n]}{r_n P[|\xi_1| > a_n]} - \lim_{\varepsilon\to 0} \int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > 1\right] \nu(du) \right| \\ &\quad \le \left| \frac{P[S_{r_n} > a_n]}{r_n P[|\xi_1| > a_n]} - \frac{P[S_{r_n}(\varepsilon a_n, \infty) > \delta a_n]}{r_n P[|\xi_1| > a_n]} \right| \\ &\quad\quad + \left| \frac{P[S_{r_n}(\varepsilon a_n, \infty) > \delta a_n]}{r_n P[|\xi_1| > a_n]} - \int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \delta\right] \nu(du) \right| \\ &\quad\quad + \left| \int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \delta\right] \nu(du) \right. \\ &\quad\quad \left. - \lim_{\varepsilon\to 0} \int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > 1\right] \nu(du) \right|.\end{aligned}$$

By Lemma 4.1 and (4.1),

$$\begin{aligned}&\left| \frac{P[S_{r_n} > a_n]}{r_n P[|\xi_1| > a_n]} - \frac{P[S_{r_n}(\varepsilon a_n, \infty) > \delta a_n]}{r_n P[|\xi_1| > a_n]} \right| \\ &\quad \le \lim_{n\to\infty} \frac{P\left[\left|S_{r_n}(0, \varepsilon a_n]\right| > (1-\delta) a_n\right]}{r_n P[|\xi_1| > a_n]} \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Also by Lemma 4.2,

$$\lim_{n\to\infty}\left|\frac{P[S_{r_n}(\varepsilon a_n,\infty) > \delta a_n]}{r_n P[|\xi_1| > a_n]} - \int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \delta\right]\nu(du)\right| = 0.$$

Finally it is clear that

$$\lim_{\delta\to 1}\lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > \delta\right]\nu(du)$$
$$= \lim_{\delta\to 1}\delta^{-\alpha}\left(\lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > 1\right]\nu(du)\right)$$
$$= \lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{i=1}^\infty Q_{1i} u I_{(\varepsilon,\infty)}(|Q_{1i}|u) > 1\right]\nu(du).$$

This concludes the proof. □

Theorem 4.3 can be reformulated as follows.

THEOREM 4.4. *Let* $\{t_r > 0\}$ *be such that* $rP[|\xi_1| > t_r] \to 0$. *Define*

$$n(r) = \left[\frac{1}{P[|\xi_1| > t_r]}\right], \qquad r \ge 1.$$

Assume that $N_n \to_d N$, *where* N *has the representation described by Corollary 2.4, and that (2.1) holds with* $n \to \infty$ *along* $\{n(r)\}_{r=1}^\infty$, *where* $r_{n(r)} = r$. *Also assume that one of the following sets of conditions holds.*

(i) $0 < \alpha < 1$.
(ii) $1 \le \alpha < 2$ *and*

$$\lim_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{P[|S_r(0,\varepsilon t_r]| > t_r\zeta]}{rP[|\xi_1| > t_r]} = 0, \qquad \zeta > 0. \tag{4.3}$$

Then

$$\lim_{r\to\infty}\frac{P[S_r > t_r]}{rP[|\xi_1| > t_r]} = \lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{i=1}^\infty u Q_{1i} I_{(\varepsilon,\infty)}(u|Q_{1i}|) > 1\right]\nu(du),$$

where the limit on the right necessarily exists in $[0,\infty)$.

PROOF. Without loss of generality, modify a_n so that $a_{n(r)} = t_r$. Since $n(r)P[|\xi_1| > a_{n(r)}] \to 1$, the modification has no effect on (2.1) or the limiting distribution of N_n. Thus, taking limits along the subsequence $\{n(r)\}_{r=1}^\infty$, the result follows from Theorem 4.3. □

REMARK 4.1. Even though we considered probabilities of large positive deviations, the corresponding result for probabilities of large negative deviations,

$$\lim_{n\to\infty} \frac{P[S_{r_n} < -a_n]}{r_n P[|\xi_1| > a_n]} = \lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{i=1}^\infty uQ_{1i} I_{(\varepsilon,\infty)}(u|Q_{1i}|) < -1\right]\nu(du),$$

is proved in exactly the same way.

REMARK 4.2. Under the assumption of Theorem 3.2(a),

$$\lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_{i=1}^\infty uQ_{1i} I_{(\varepsilon,\infty)}(u|Q_{1i}|) > 1\right]\nu(du) = \gamma E(W^+)^\alpha$$

and

$$\lim_{\varepsilon\to 0}\int_0^\infty P\left[\sum_i uQ_{1i} I_{(\varepsilon,\infty)}(u|Q_{1i}|) < -1\right]\nu(du) = \gamma E(W_-)^\alpha,$$

where $W = \sum_{i=1}^\infty Q_{1i}$.

REMARK 4.3. For $1 \le \alpha < 2$, centering is typically required for (4.3) to hold. See Theorem 4.5 below.

We now specialize to the iid case.

THEOREM 4.5. *Suppose that* $\{\xi_j\}$ *is iid and* (1.1) *and* (1.2) *hold. With* t_r *denoting a sequence satisfying* $rP[|\xi_1| > t_r] \to 0$, *also assume that one of the following sets of conditions holds.*

(i) $0 < \alpha < 1$.
(ii) $1 \le \alpha < 2$ *and* $\lim_{r\to\infty}(r/t_r)E\xi_1 I(|\xi_1| < t_r) = 0$.

Then

$$\lim_{r\to\infty}\frac{P[S_r > t_r]}{rP[|\xi_1| > t_r]} = p. \tag{4.4}$$

PROOF. By Remark 2.2, $N_n \to_d N := \sum_{j=1}^\infty \delta_{Q_{j1}\Gamma_j^{-1/\alpha}}$, where $P[Q_{j1} = 1] = p$ and $P[Q_{j1} = -1] = q$. Thus it is clear that

$$\lim_{\varepsilon\to 0}\int P\left[\sum_{i=1}^\infty uQ_{1i} I_{(\varepsilon,\infty)}(u|Q_{1i}|) > 1\right]\nu(du) = p.$$

We now verify (4.3) for the case $1 \le \alpha < 2$. Since $E\xi_1 I(|\xi_1| \le x)$ is regularly varying at ∞, condition (ii) implies that

$$ES_r(0, \varepsilon t_r] = rE\xi_1 I(|\xi_1| \le \varepsilon t_r] = o(t_r), \qquad \varepsilon > 0.$$

Thus by Chebyshev inequality and Theorem 2 of Feller [(1971), VIII.9],

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{P[|S_r(0,\varepsilon t_r]|>t_r\zeta]}{rP[|\xi_1|>t_r]}\\
&\quad\le \lim_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{P[|S_r(0,\varepsilon t_r]-ES_r(0,\varepsilon t_r]|>t_r\zeta/2]}{rP[|\xi_1|>t_r]}\\
&\quad\le \lim_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{4\operatorname{Var}(S_r(0,\varepsilon t_r])}{t_r^2\zeta^2 rP[|\xi_1|>t_r]}\\
&\quad\le \lim_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{4rE\xi_1^2 I(|\xi_1|\le\varepsilon t_r)}{t_r^2\zeta^2 rP[|\xi_1|>t_r]}\\
&\quad= \lim_{\varepsilon\to 0}\frac{4\alpha}{\zeta^2(2-\alpha)}\varepsilon^{2-\alpha}=0,\qquad \zeta>0.
\end{aligned}$$

The result follows from Theorem 4.4. □

Heyde (1968) shows that if $\{\xi_j\}$ is iid α-stable for $\alpha\in(0,2)$, where $E\xi_1=0$ if $\alpha>1$, then

$$\lim_{n\to\infty}\frac{p[|S_n|>t_n]}{nP[|\xi_1|>t_n]}=1.$$

This result is extended by Theorem 4.5, which has a weaker assumption and a stronger conclusion. Nagaev (1969) derives (4.4) under independence and the assumption $P[\xi_1>x]\in RV_{-\alpha}$, where $2<\alpha<\infty$. See Cline and Hsing (1994) for a unified approach concerning results in this regard in the setting where the ξ_j are iid with dominated varying tail probabilities.

While it is conjectured that the conclusion of Theorem 4.4 holds for $\alpha\ge 2$ under dependence, the approach in this section is not directly applicable since (4.3) does not hold for $\alpha>2$ even in the iid case.

5. Examples and further results. Many of the existing limit results for partial sums of stationary processes with heavy tails are a special case of Theorem 3.1. We consider some of these cases in this section. While we concentrate solely on applications to the point process and partial sum convergences, the corresponding applications to the large deviation results of Section 4 are also easily obtained. As in Section 3, we assume that $\{\xi_j\}$ is a strictly stationary sequence satisfying (1.1) and (1.2).

EXAMPLE 5.1. Suppose $\{\xi_j\}$ satisfies the dependence conditions D and D' of Davis (1983). Then

$$N_n=\sum_{j=1}^{n}\delta_{\xi_j/a_n}\to_d N=\sum_{i=1}^{\infty}\delta_{\Gamma_i^{-1/\alpha}Q_{i1}},\tag{5.1}$$

where $P[Q_{i1}=1]=p$, $P[Q_{i1}=-1]=1-p$, $0\le p\le 1$. Convergence of the normalized partial sums $a_n^{-1}(S_n-b_n)$ contained in Theorems 2 and 3 of

Davis (1983) are now immediate consequences of Theorem 3.1. Note that for $1 \le \alpha < 2$, condition D'' of Davis (1983) implies (3.2).

For $1 \le \alpha < 2$, the point process convergence in (3.1) by itself is not enough to ensure convergence of the normalized partial sums to stable limits. This is demonstrated in the following example.

EXAMPLE 5.2. Let $\{X_j\}$ be a sequence of iid random variables which have mean zero and are in the domain of attraction of a stable distribution with index $1 < \alpha < 2$. Set

$$\xi_j = X_j + Z,$$

where Z is a random variable which has mean zero, finite variance and is independent of $\{X_j\}$. A routine calculation shows that

$$\frac{P[\xi_1 > x]}{P[X_1 > x]} \to 1 \quad \text{and} \quad \frac{P[\xi_1 < -x]}{P[X_1 < -x]} \to 1$$

as $x \to \infty$, that is, the distribution of ξ_1 satisfies (1.1) and (1.2). Let $\{a_n\}$ be the $(1 - n^{-1})$ quantile of the distribution of ξ_1. Since the ξ_j, given $\sigma(Z)$, are iid and the limit point process of $\sum_{j=1}^n \delta_{(X_j+z)/a_n}$ is the same for each fixed value of z, it follows that (5.1) holds for both sequences $\{X_j\}$ and $\{\xi_j\}$ with the same scaling. Moreover,

$$a_n^{-1} \sum_{j=1}^n X_j \to_d S$$

and since $na_n^{-1} \to \infty$, we conclude that

$$a_n^{-1} \sum_{j=1}^n \xi_j = a_n^{-1} \sum_{j=1}^n X_j + na_n^{-1} Z$$

does not converge in distribution. In this case, condition (3.2) fails to hold for the $\{\xi_j\}$ sequence. Note, however, that S_n does converge in distribution with a different scaling to a nonstable limit, namely,

$$n^{-1} \sum_{j=1}^n \xi_j = n^{-1} \sum_{j=1}^n X_j + Z \to_d Z.$$

EXAMPLE 5.3 (Self-norming sums). The results of LePage, Woodroofe and Zinn (1981) and Davis (1983) for self-norming sums can easily be extended to the setting of Section 3. Assume that the conditions of Theorem 3.1 are met and define

$$S_{n,r}^{\#} = \begin{cases} \left\{\sum_{j=1}^n |\xi_j|^r\right\}^{1/r}, & \text{if } 1 \le r < \infty, \\ \bigvee_{j=1}^n |\xi_j|, & \text{if } r = \infty, \end{cases}$$

A straightforward adaptation of the proof of Theorem 3.1 yields the following corollary.

COROLLARY. *Assume that either* $0 < \alpha < 1$ *or that* $1 < \alpha < 2$ *and* $E\xi_1 = 0$. *Define* $(S_r^\#, S)$ *to be the distributional limit, as* $\varepsilon \to 0$, *of*

$$\left(\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\left(\gamma\Gamma_i^{-1}\right)^{1/\alpha}|Q_{ij}|\right)^r\right\}^{1/r},\right.$$

$$\left.\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\gamma\Gamma_i^{-1}\right)^{1/\alpha}Q_{ij}I_{(\varepsilon,\infty)}\left(\left(\gamma\Gamma_i^{-1}\right)^{1/\alpha}|Q_{ij}|\right) - \int_{\varepsilon<|x|\le 1}x\mu(dx)\right)$$

if $\alpha < r < \infty$ *and*

$$\left(\left(\gamma\Gamma_1^{-1}\right)^{1/\alpha}, \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\gamma\Gamma_i^{-1}\right)^{1/\alpha}Q_{ij}I_{(\varepsilon,\infty)}\left(\left(\gamma\Gamma_i^{-1}\right)^{1/\alpha}|Q_{ij}|\right) - \int_{\varepsilon<|x|\le 1}x\mu(dx)\right)$$

if $r = \infty$. *Then* $T_{n,r} \to_d T_r$ *for all* $\alpha < r \le \infty$, *where*

$$T_{n,r} = \frac{S_n}{S_{n,r}^\#},$$

$$T_r = \begin{cases} \dfrac{S}{S_r^\#}, & \text{if } 0 < \alpha < 1,\\[2mm] \dfrac{S - (p-q)\alpha/(\alpha-1)}{S_r^\#}, & \text{if } 1 < \alpha < 2.\end{cases}$$

EXAMPLE 5.4 (k-dependent sequences). Now let $\{\xi_j\}$ be a k-dependent sequence such that $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{k+1})$ is jointly regularly varying in the sense of Section 2. That is, there exist constants x_n and a random vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{k+1})$ such that $P[\|\boldsymbol{\theta}\| = 1] = 1$ and

$$nP\left[\|\boldsymbol{\xi}\| > tx_n, \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \in \cdot\right] \to_v t^{-\alpha}P[\boldsymbol{\theta} \in \cdot]$$

for all $t > 0$. While Theorem 2.7 is directly applicable in this case, we pursue a different route here. First observe that

$$\begin{aligned} nP[|\xi_1| > a_n] &= nP\left[\frac{|\xi_1|}{\|\boldsymbol{\xi}\|}\|\boldsymbol{\xi}\| > a_n\right]\\ &\to CE|\theta_1|^\alpha\\ &= 1 \end{aligned}$$

so that $C = 1/E|\theta_1|^\alpha$. It follows that

$$nP\left[\|\boldsymbol{\xi}\| > a_n, \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \in \cdot\right] \to_v CP[\boldsymbol{\theta} \in \cdot].$$

Next, it is a simple matter to show [see, for example, Theorem 2.1 of Davis and Resnick (1988)] that

$$\tilde{N}_n := \sum_{j=1}^{[n/(k+1)]} \delta_{(\xi_{jk-k+j},\dots,\xi_{jk+j})/a_n} \to_d \sum_{i=1}^{\infty} \delta_{P_i(\theta_{i,1},\dots,\theta_{i,k+1})},$$

where $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process with intensity measure $\alpha y^{-\alpha-1}/((k+1)E|\theta_1|^{\alpha})\,dy$ and $\{(\theta_{i,1},\dots,\theta_{i,k+1}),\ i=1,2,\dots\}$ is a sequence of iid copies of $(\theta_1,\dots,\theta_{k+1})$, which are also independent of the Poisson process. Consequently,

$$N_n = \sum_{j=1}^{n} \delta_{\xi_j/a_n} \to_d \sum_{i=1}^{\infty} \sum_{j=1}^{k+1} \delta_{P_i\theta_{ij}},$$

which has the representation given in Corollary 2.4 with $\sum_{j=1}^{\infty} \delta_{Q_{ij}} = \sum_{j=1}^{k+1} \delta_{\theta_{ij}}$. Applying Theorem 3.1 [condition (3.2) is always valid for k-dependent sequences; see Davis (1983)], the normalized partial sums converge in distribution to S given by

$$S = \begin{cases} \displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{k+1} P_i\theta_{ij}, & \text{if } \alpha < 1, \\ \displaystyle\lim_{\varepsilon \to 0} \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{k+1} P_i\theta_{ij} I_{(\varepsilon,\infty)}(|P_i\theta_{ij}|) - \int_{\varepsilon<|x|\le 1} x\,\mu(dx) \right\}, & \text{if } 1 \le \alpha < 2. \end{cases}$$

Convergence of partial sums in the k-dependent case was also dealt with by Jakubowski and Kobus (1989). Under the assumptions of this example, the limit characteristic function is described in terms of the Lévy measure of the vector $(\xi_1,\dots,\xi_{k+1})$ [see Theorem 5.3 of Jakubowski and Kobus (1989)]. One can easily check that the characteristic function of S has this form.

EXAMPLE 5.5 (Linear processes). We now consider the linear process defined by

$$\xi_j = \sum_{i=0}^{\infty} c_i Z_{j-i},$$

where $\{Z_j\}$ is an iid sequence of random variables with marginal distribution satisfying (1.1) and (1.2) and $\{c_j\}$ is a sequence of constants satisfying the summability condition

$$\sum_{j=0}^{\infty} |c_j|^{\delta} < \infty \quad \text{for some } \delta < \alpha,\ 0 < \delta \le 1.$$

Assume without loss of generality that

$$\sum_{j=0}^{\infty} |c_j|^\alpha = 1.$$

If a_n is the $1 - 1/n$ quantile of the distribution of $|Z_1|$ and hence (1.4) holds [see (4.71) of Resnick (1987)], then by Theorem 2.4(i) of Davis and Resnick (1985),

$$\sum_{j=1}^{n} \delta_{\xi_j/a_n} \to_d \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \delta_{P_i Q_{ij}},$$

where $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process on $(0, \infty)$ with intensity measure $\bigvee_{k=0}^{\infty} |c_k|^\alpha \alpha y^{-\alpha-1}\,dy$ (i.e., $\gamma = \bigvee_{k=0}^{\infty} |c_k|^\alpha$) and $Q_{ij} = \varepsilon_i c_j / \bigvee_{k=0}^{\infty} |c_k|$, with $\{\varepsilon_i\}$ being an iid sequence, independent of the Poisson process with $P[\varepsilon_1 = 1] = p$, $P[\varepsilon_1 = -1] = q$. Applying Theorem 3.1 with $0 < \alpha < 1$, we obtain

$$a_n^{-1} S_n \to_d \left(\sum_{k=0}^{\infty} c_k \right) \sum_{i=1}^{\infty} \varepsilon_i P_i.$$

In order to apply Theorem 3.1(ii) for the case $\alpha \geq 1$, one must check condition (3.2). Due to the complicated dependence structure of the truncated random variables $\xi_j I_{[|\xi_j| \leq a_n \varepsilon]}$, a direct verification of condition (3.2) is not the simplest approach. Instead, we first apply Theorem 3.1(ii) to finite order linear processes and then show how the partial sum convergence of the finite order processes can be extended to infinite order models. For ease of presentation, we confine this discussion to the case $\alpha > 1$ (the case $\alpha = 1$ requires a special argument) and assume, without loss of generality, that $EZ_1 = 0$. Writing

$$a_n^{-1} S_n = U_n + V_n + W_n,$$

where

$$U_n = a_n^{-1} \sum_{j=1}^{n} \sum_{k=0}^{\infty} c_k \left(Z_{j-k} I_{[Z_{j-k} \leq a_n]} - EZ_1 I_{[Z_1 \leq a_n]} \right), \tag{5.2}$$

$$V_n = a_n^{-1} \sum_{j=1}^{n} \sum_{k=0}^{\infty} c_k Z_{j-k} I_{[Z_{j-k} > a_n]} \tag{5.3}$$

and

$$W_n = n a_n^{-1} \left(\sum_{k=0}^{\infty} c_k \right) EZ_1 I_{[Z_1 \leq a_n]} = -n a_n^{-1} \left(\sum_{k=0}^{\infty} c_k \right) EZ_1 I_{[Z_1 > a_n]}, \tag{5.4}$$

we have, by Karamata's theorem,

$$E|V_n| \leq n a_n^{-1} \sum_{k=0}^{\infty} |c_k| E|Z_1| I_{[|Z_1| > a_n]} \to \frac{\alpha}{\alpha - 1} \sum_{k=0}^{\infty} |c_k| \tag{5.5}$$

and

$$|W_n| \leq n a_n^{-1} \sum_{k=0}^{\infty} |c_k| E|Z_1| I_{[|Z_1| > a_n]} \to \frac{\alpha}{\alpha - 1} \sum_{k=0}^{\infty} |c_k|. \tag{5.6}$$

Also,

$$\operatorname{Var}(U_n) = a_n^{-2} \sum_{|k|<n} (n - |k|) \sum_{j=0}^{\infty} c_j c_{j+|k|} \operatorname{Var}\left(Z_1 I_{[|Z_1| > a_n]}\right)$$

$$\le n a_n^{-2} \left(E Z_1^2 I_{[|Z_1| > a_n]}\right) 2 \left(\sum_{j=0}^{\infty} |c_j| \right)^2 \tag{5.7}$$

$$\to \frac{2\alpha}{2-\alpha} \left(\sum_{j=0}^{\infty} |c_j| \right)^2 .$$

Now if $S_n^{(M)}$ denotes the partial sum of the finite order linear process $\xi_{j,M} := \sum_{k=0}^{M} c_k Z_{j-k}$, then by applying Theorem 3.1(ii), we have

$$a_n^{-1} S_n^{(M)} \to_d S^{(M)},$$

where $S^{(M)}$ has a stable distribution with characteristic function given in Remark 3.2 (see also Theorem 3.2) with parameter values

$$d^{(M)} = \Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\gamma^{(M)} \frac{\left|\sum_{k=0}^{M} c_k\right|^\alpha}{\bigvee_{k=0}^{M} |c_k|^\alpha} \left(\sum_{k=0}^{M} |c_k|^\alpha \right)$$

$$= \Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) \left| \sum_{k=0}^{M} c_k \right|^\alpha ,$$

$$\beta^{(M)} = (p-q)\operatorname{sgn}\left(\sum_{k=0}^{M} c_k \right)$$

and

$$m = 0.$$

Clearly, $S^{(M)} =_d (\sum_{k=0}^{M} c_k) S \to_d (\sum_{k=0}^{\infty} c_k)\, S$, where S is stable. Also note that

$$a_n^{-1} |S_n - S_n^{(M)}| \le |U_n| + |V_n| + |W_n|,$$

where the U_n, V_n and W_n are as defined in (5.2)–(5.4) with $c_k = 0$ for $k = 0, \dots, M$. By Markov's inequality and (5.5)–(5.7), we then have for any $\varepsilon > 0$ and M large,

$$\limsup_{n\to\infty} P\left[a_n^{-1}|S_n - S_n^{(M)}| > \varepsilon\right]\varepsilon \le \left(\operatorname{Var}(U_n)\right)^{1/2} + E|V_n| + |W_n|$$

$$\le \left(\frac{2\alpha}{\alpha-1} + \left(\frac{2\alpha}{2-\alpha} \right)^{1/2} \right) \sum_{k=M+1}^{\infty} |c_k|$$

$$\to 0$$

as $M \to \infty$. This implies, by Theorem 4.2 in Billingsley (1968) that

$$a_n^{-1} S_n \to_d \left(\sum_{k=0}^{\infty} c_k \right) S.$$

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