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ON THE DISTRIBUTION OF BUBBLES OF THE BROWNIAN SHEET[1]

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Let W be a real-valued, two-parameter Brownian sheet. Let us define $N(t;h)$ to be the total number of bubbles of W in $[0,t]^2$, whose maximum height is greater than h. Evidently, $\lim_{h\downarrow 0} N(t;h) = \infty$ and $\lim_{t\uparrow\infty} N(t;h) = \infty$. It is the goal of this paper to provide fairly accurate estimates on $N(t;h)$ both as $t \to \infty$ and as $h \to 0$. Loosely speaking, we show that there are of order h^{-3} many such bubbles as $h \downarrow 0$ and t^3 many, as $t \uparrow \infty$.

1. Introduction. Throughout, we let $W \triangleq \{W(s,t);\ (s,t) \in \mathbb{R}_+^2\}$ be the standard real-valued, two-parameter Brownian sheet. That is to say, W is a centered linear Gaussian process with the following covariance structure: for all $s,t,u,v \geq 0$,

$$\mathbb{E}W(s,u)W(t,v) = (s \wedge t)(u \wedge v).$$

It is well known that the level sets of W have a complex fine structure. To explain this better, let us next define the zero level set, $\mathfrak{z}$, of W:

$$\mathfrak{z} \triangleq \{(s,t) \in \mathbb{R}_+^2 : W(s,t) = 0\} = W^{-1}(\{0\}).$$

Then by Adler ([1], Theorem 8.9.4), the Hausdorff dimension of $\mathfrak{z}$ is almost surely $3/2$. See [1] for refinements and further references. A trivial but important consequence of this development is that $\mathfrak{z}$ is almost surely uncountable. On the other hand, W is known to have continuous sample paths (see Adler [1, 2, 11 and 14], for example). Therefore it is not at all surprising that there are infinitely many "bubbles" of W which sit above zero. (Of course, this also holds for negatively valued bubbles, by symmetry.) We are following Dalang and Walsh [4] in their definition of a bubble. Namely, $B \subseteq \mathbb{R}_+^2$ is a so-called bubble if the interior of B is nonempty and for all $(s,t) \in \partial B$, $W(s,t) = 0$, while for all $(s,t) \in \operatorname{int} B$, $|W(s,t)| > 0$.

Recently there has been substantial progress in the analysis of a typical bubble. See [8, 4, 5, and 10] for instance. All of the aforementioned papers deal with the local structure of a typical Brownian sheet bubble. In contrast to the aforementioned works, the results of this paper are global in nature. Indeed, the following question is to be our guiding light: How many small bubbles are there? By the argument of the previous paragraph, this question may at first seem superfluous since there are an infinity of such bubbles.

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However, there is a better way to make sense of the question. To do so, let us first define $N(t;h)$ to be the number of bubbles in $[0,t]^2$ whose height exceeds h. Sample path regularity of W insures us that $N(t;h) < \infty$, for each $t, h > 0$. On the other hand, since $\mathfrak{z}$ is uncountable, it should not come as a surprise that almost surely, $N(t;h) \to \infty$, as $h \to 0$. It is therefore natural to ask about the rate at which $N(t;h)$ tends to infinity. A partial answer to the above question is the main concern of this paper which we state below as our first theorem.

THEOREM 1.1. *For each $t > 0$, with probability* 1,

$$\lim_{h \downarrow 0} h^{\nu} N(h;t) = \begin{cases} 0, & \text{if } \nu > 3, \\ \infty, & \text{if } \nu < 3. \end{cases}$$

The behavior of $h^{\nu}N(h;t)$ at the critical case, that is, when $\nu = 3$, is as of yet unknown and seems to be a rather delicate problem.

An immediate corollary of Theorem 1.1 is the following important (though simple) result, which says that roughly speaking there are h^{-3} many bubbles of height more than h in the square $[0,t]^2$. More precisely, we state the following corollary.

COROLLARY 1.2. *For each fixed $t > 0$, with probability* 1,

$$\lim_{h \downarrow 0} \frac{\ln N(h;t)}{\ln h^{-1}} = 3.$$

A scaling argument applied to Theorem 1.1 would heuristically suggest our next result. We will state it without a proof, as the proof is similar (though not identical) to the one we shall provide for Theorem 1.1.

THEOREM 1.3. *For each $h > 0$, with probability* 1,

$$\lim_{t \uparrow \infty} t^{-\nu} N(h;t) = \begin{cases} 0, & \text{if } \nu > 3, \\ \infty, & \text{if } \nu < 3. \end{cases}$$

Finally, we mention the following "large time" analogue of Corollary 1.2, which corresponds to Theorem 1.3.

COROLLARY 1.4. *For each fixed $h > 0$, with probability* 1,

$$\lim_{t \uparrow \infty} \frac{\ln N(h;t)}{\ln t} = 3.$$

These results should be compared with the analogous work for a one-parameter Brownian motion. It is now known that using excursion theory, one can show that there are of order h^{-1} many excursions whose height exceeds h. See Revuz and Yor ([12], Exercise XII.(2.10).2°) for the precise statement. The sharpest known result in the case of ordinary Brownian motion appears in [9] (Theorem 1.4) together with a host of further references.

Throughout this paper, K denotes a universal constant, whose value may change from line to line. In the few cases where K depends on something interesting, this dependence is specifically noted for the sake of clarity.

A few words about the organization of the paper are in order. Section 2 contains a proof for the upper bound for $N(h;t)$. This handles the case $\nu > 3$. Using some of the technical estimates of Section 3, the lower bound (i.e., case $\nu < 3$) will appear in Section 4. Finally, Section 5 contains the driving heuristic behind the proofs as well as additional remarks.

2. Proof of the upper bound of Theorem 1.1. It suffices to prove the upper bound when $t = 1$, that is,

$$\lim_{h \downarrow 0} h^{\nu} N(h; 1) = 0 \quad \text{a.s. for all } \nu > 3.$$

To arrive at the result for general $t > 0$, one can either use a scaling argument, or go through the proof and replace 1 by t everywhere. With this reduction in mind, let us define for each $t \in (0, 1]$,

$$W_t(s) \triangleq W(s,t), \qquad 0 \le s \le 1.$$

For each fixed $t \in (0, 1]$, W_t is a Brownian motion with infinitesimal variance t. This is best seen by checking the covariance structure. As a result, W_t has a process of local times, $\{L_t^x(s);\ x \in \mathbb{R}^1,\ s \in [0,1]\}$, which can be obtained via Tanaka's formula [12 and 13] as follows:

$$L_t^x(s) = t^{-1}\left(|W_t(s) - x| - |x| - \int_0^s \operatorname{sgn}(W_t(r) - x) W_t(dr)\right). \tag{2.1}$$

From now on, we shall write $L_t(s) \triangleq L_t^0(s)$ for brevity. The normalization, $1/t$, comes from the fact that the quadratic variation of $s \mapsto W_t(s)$ is given by $[W_t]_s = ts$. As a result, $L_t^x(s)$ is the occupation density of W_t: for all Borel functions $f \ge 0$,

$$\int_0^s f(W_t(r))\, dr = \int_{-\infty}^{\infty} f(x) L_t^x(s)\, dx.$$

A consequence of Walsh [13] is the following result:

LEMMA 2.1. *There exists an a.s. continuous modification of the process,* $\{L_t(s);\ s \in [0,1],\ t \in (0,1]\}$.

Furthermore, a scaling argument will prove the next lemma.

LEMMA 2.2. *For each* $t > 0$ *fixed,* $\{\sqrt{t} L_t(s);\ s \in [0,1]\}$ *is standard Brownian local time at zero. Indeed,* $\{(\sqrt{t} L_t(s), t^{-1/2} W_t(s));\ s \in [0,1]\}$ *has the same finite-dimensional distributions as* $\{(L_1(s), W_1(s));\ s \in [0,1]\}$, *and* $\{W_1(s);\ s \in [0,1]\}$ *is standard Brownian motion.*

A consequence of Lemma 2.2 is the following fact which was first observed in [13]:

$$L_t(1) \to_{\mathbb{P}} \infty \quad \text{as } t \to 0. \tag{2.2}$$

Let us define, for the remainder of this section, the following random variables: $T_t(0; h) \triangleq 0$ and, for all $k = 0, 1, 2, \dots,$

$$\begin{aligned} T_t(2k+1; h) &\triangleq \inf\{s > T_t(2k; h) \colon W_t(s) = h\}, \\ T_t(2k+2; h) &\triangleq \inf\{s > T_t(2k+1; h) \colon W_t(s) = 0\}, \\ U_t(h) &\triangleq \max\{k \geq 1 \colon T_t(2k+1; h) \leq 1\}. \end{aligned} \tag{2.3}$$

Since $\{W_t(s);\ s \in [0,1]\}$ is Brownian motion with variance t, the T_t's are the $[0, h]$-crossing times and $U_t(h)$ is the total number of upcrossings of $[0, h]$ by the Brownian motion $\{W_t(s);\ s \in [0,1]\}$. Here and throughout, upcrossings are meant in exactly the same way as in martingale theory. We shall sometimes refer to $[T_t(2k; h), T_t(2k+2; h)]$ as a (t, h)-blip. Notice that the total number of all (t, h)-blips in $[0,1]$ is simply $U_t(h)$. With this in mind, we shall next state and prove a technical estimate.

LEMMA 2.3. *For each $\eta \in (0,1)$, there exists a $K = K(\eta) > 0$ such that for all $t \in (\eta, 1]$, $h \in (0,1)$ and $a > 1$,*

$$\mathbb{P}\Big(\big|hU_t(h) - tL_t(1)\big| \geq a\sqrt{htL_t(1)}\Big) \leq \exp(-Ka).$$

PROOF. Since $\{W_1(s);\ s \in [0,1]\}$ is standard Brownian motion, applying the proof of Lemma 4.2 of [9], we can show that for all t, h and a as given in the statement of the theorem,

$$\mathbb{P}\Big(\big|ht^{-1/2}U_1(ht^{-1/2}) - L_1(1)\big| \geq \sqrt{ht^{-1/2}L_1(1)}\,a\Big) \leq \exp(-Ka).$$

By the scaling Lemma (Lemma 2.2), for each fixed $t > 0$,

$$\big\{\big(U_1(ht^{-1/2}), L_1(1)\big);\ h > 0\big\}$$

has the same finite-dimensional distributions as

$$\big\{\big(U_t(h), t^{1/2}L_t(1)\big);\ h > 0\big\}.$$

After some algebra, we see that for all t, h and a, as in the statement of the lemma,

$$\mathbb{P}\Big(\big|hU_t(h) - tL_t(1)\big| \geq \sqrt{htL_t(1)}\,a\Big) \leq \exp(-Ka).$$

This proves the result. □

From now on, let us fix some $\eta \in (0,1)$, $\nu > 3$ and define $h_n \triangleq 2^{-n}$. Let

$$\mathbb{F}_n \triangleq \big\{jh_n^{\nu-1} \colon 1 \leq j \leq h_n^{-\nu+1}\big\}.$$

LEMMA 2.4. *With probability* 1,

$$\limsup_{n\to\infty} h_n^{\nu} \sum_{t\in\mathbb{F}_n\cap[\eta,1]} U_t(h_n) < \infty.$$

PROOF. Applying Lemma 2.3 with $h \triangleq h_n$ and $a \triangleq n^2$, we see that as $n\to\infty$,

$$\begin{aligned}
&\mathbb{P}\Big(\big|h_nU_t(h_n) - tL_t(1)\big| \geq \sqrt{h_n tL_t(1)}\,n^2, \text{ for some } t\in\mathbb{F}_n\cap[\eta,1]\Big)\\
&\quad\leq \#\mathbb{F}_n\cdot\exp(-Kn^2)\\
&\quad\leq K\cdot 2^{n\nu}\exp(-Kn^2)\\
&\quad\leq K\cdot n^{-2},
\end{aligned}$$

which sums. By the Borel–Cantelli lemma, with probability 1,

$$h_nU_t(h_n) \leq tL_t(1) + \sqrt{h_n tL_t(1)}\,n^2$$

simultaneously for all $t\in\mathbb{F}_n\cap[\eta,1]$, eventually.

Therefore, with probability 1, the following must eventually hold for any $\varepsilon > 0$:

$$\begin{aligned}
h_n \sum_{t\in\mathbb{F}_n\cap[\eta,1]} U_t(h_n) &\leq \sum_{t\in\mathbb{F}_n\cap[\eta,1]} tL_t(1) + h_n^{1/2}n^2 \sum_{t\in\mathbb{F}_n\cap[\eta,1]} \sqrt{tL_t(1)}\\
&\leq (1+\varepsilon)\#\mathbb{F}_n\left(\int_\eta^1 tL_t(1)\,dt + h_n^{1/2}n^2\int_\eta^1\sqrt{tL_t(1)}\,dt\right)\\
&\qquad\text{(by Riemann approximation and Lemma 2.1)}\\
&\leq (1+\varepsilon)h_n^{-\nu+1}\left(\int_\eta^1 tL_t(1)\,dt + h_n^{1/2}n^2\sqrt{\int_\eta^1 tL_t(1)\,dt}\right)\\
&\qquad\text{(by the Cauchy-Schwarz inequality)}\\
&\leq (1+2\varepsilon)h_n^{-\nu+1}\int_\eta^1 tL_t(1)\,dt\\
&\leq (1+2\varepsilon)h_n^{-\nu+1}\int_0^1 tL_t(1)\,dt.
\end{aligned}$$

The fact that the above integral is a.s. finite is straightforward, since by Lemma 2.2,

$$\mathbb{E}L_t(1) = \mathbb{E}L_1(1)\cdot t^{-1/2}$$

and $L_1(1)$ is standard Brownian motion local time at zero before time 1. □

Now we can proceed to prove the promised upper bound. Clearly, it is sufficient to prove that for all $\nu > 3$,

$$\lim_{h\to 0} h^{\nu}N(h;1) = 0, \quad \text{a.s.}$$

For $\eta \in (0,1)$, let $N^{(\eta)}(h)$ denote the total number of bubbles, B, such that (i) $B \cap [0,1] \times [\eta, 1] \neq \varnothing$ and (ii) there exists some $(s,t) \in B \cap [0,1] \times [\eta, 1]$ such that $W(s,t) \geq h$. It is then enough to prove that with probability 1,

$$\lim_{n\to\infty} h_n^{\nu} N^{(\eta)}(h_{n+1}) = 0 \quad \text{a.s.} \tag{2.4}$$

Indeed, if (2.4) holds, then for all $h_{n+1} \leq h \leq h_n$,

$$h^{\nu} N^{(\eta)}(h) \leq h_n^{\nu} N^{(\eta)}(h_{n+1}) \to 0 \quad \text{a.s.}$$

Therefore, with probability 1, $h^{\nu}N^{(\eta)}(h) \to 0$, for all rational $\eta \in (0,1)$. By subsequencing, we see that we can find $\eta = \eta(h) \leq h^4$, so that $h^{\nu}N^{(\eta)}(h) \to 0$. By the modulus of continuity of W (see below for another application of this argument, plus more details), with probability 1,

$$N^{(\eta)}(h) = N(h;1) \quad \text{eventually,}$$

for such a (possibly random) sequence, $\eta(h)$. This proves the desired result. It therefore remains to prove (2.4).

Suppose $B \subseteq [0,1]^2$ is a bubble of height more than h_{n+1} with $B \cap [0,1] \times [\eta, 1] \neq \varnothing$. This implies that there exists some $(s_0, t_0) \in B$ such that $W(s_0,t_0) \geq h_{n+1}$. Pick $t_n \in \mathbb{F}_n \cap [\eta/2, 1]$ closest to t_0 (with some arbitrary convention for breaking ties.) Since $|t_0 - t_n| \leq h_n^{\nu-1}$, by the modulus of continuity of W (see [11]), for any $r \in ((\nu-1)^{-1}, 2^{-1})$, if n were large enough (possibly random),

$$\begin{aligned} W(s_0, t_n) &\geq W(s_0, t_0) - h_n^{r(\nu-1)} \\ &\geq h_{n+1} - h_n^{r(\nu-1)} \\ &\geq h_n, \end{aligned}$$

eventually. In words, for all n large enough, any bubble, B, of height more than h_{n+1} with $B \cap [0,1] \times [\eta, 1] \neq \varnothing$, gives rise to a (t, h_n)-blip, for some $t \in \mathbb{F}_n \cap [\eta/2, 1]$. Hence,

$$\begin{aligned} N^{(\eta)}(h_{n+1}) &\leq \sum_{t \in \mathbb{F}_n \cap [\eta/2,1]} \#((t, h_n)\text{-blips}) \quad \text{(eventually)} \\ &= \sum_{t \in \mathbb{F}_n \cap [\eta/2,1]} U_t(h_n). \end{aligned}$$

By Lemma 2.4,

$$\limsup_{n\to\infty} h_n^{\nu} N^{(\eta)}(h_{n+1}) < \infty.$$

Since $\nu > 3$ is arbitrary, the above lim sup is actually a limit and the limit is zero. This proves the promised (2.4) and hence the upper bound for Theorem 1.1 □

3. Some estimates. In this section, we state and prove relevant technical estimates for the random objects defined below. These objects (as well as the accompanying estimates of this section) are used in the next section to finish the proof of Theorem 1.1.

From now on, let $\delta \in (1,2)$ be arbitrary though fixed. Define $\tilde{T}_t(0;h) \triangleq 0$ and for all $k = 0, 1, \ldots,$

$$\tilde{T}_t(2k+1;h) \triangleq \inf\{s > \tilde{T}_t(2k;h) + h^\delta/2 \colon W_t(s) = h\},$$
$$\tilde{T}_t(2k+2;h) \triangleq \inf\{s > \tilde{T}_t(2k+1;h) + h^\delta/2 \colon W_t(s) = 0\}, \tag{3.1}$$
$$\tilde{U}_t(h;s) \triangleq \max\{k \geq 0 \colon \tilde{T}_t(2k+1;h) \leq s\}.$$

Hence, $\tilde{T}_t$'s are approximate upcrossing times and, correspondingly, $\tilde{U}_t$'s are the approximate upcrossing numbers. Throughout this and the next section, let us define for every $\eta \in (0,1)$, the approximate version, $\tilde{\mathbb{F}}_\eta(h)$, of $\mathbb{F}_n$, defined before the statement of Lemma 2.4. Namely, having fixed some $\delta \in (1,2)$ above, we define

$$\tilde{\mathbb{F}}_\eta(h) \triangleq \{jh^\delta \colon 1 \leq j \leq h^{-\delta}\} \cap [\eta, 1]. \tag{3.2}$$

We begin with a basic probability estimate.

LEMMA 3.1. *For all $j \geq 1$, $h, t > 0$ and $x > 1$,*

$$\mathbb{P}\Big(L_t\big(\tilde{T}_t(j;h)\big) - L_t\big(\tilde{T}_t(j-1;h)\big) \geq x\Big)$$
$$\leq 2\exp(-th^{-\delta}x^2/4) + \exp(-x/(2h)).$$

PROOF. It is nice to have an auxiliary Wiener space at hand on which the probability calculations can be carried out with some ease. To this end, let us construct an auxiliary Wiener space as follows: Ω_0 denotes the space of all real-valued continuous functions on $\mathbb{R}^1_+$ endowed with uniform convergence on compacta; $\mathbb{Q}$ is Wiener measure on Ω_0 and $\mathbb{Q}^x$ will denote the usual shift. Brownian motion, β, on this space is nothing other than the coordinate maps given by $\beta(t)(\omega) \triangleq \omega(t)$ for all $\omega \in \Omega_0$ and all $t \geq 0$. Furthermore, one has the measurable shift functional, $\vartheta\colon \Omega_0 \mapsto \Omega_0$, given by $(\beta \circ \vartheta(t))(s)(\omega) \triangleq \beta(t + s)(\omega)$ for all $\omega \in \Omega_0$. (Our notation is a minor adaptation of the standard one; see, e.g., [12].) Now we can proceed with the proof.

Since $\{W_t(s);\ s \in [0,1]\}$ is a Brownian motion with variance t, by the strong Markov property and Lemma 2.2,

$$\mathbb{P}\Big(L_t\big(\tilde{T}_t(2k;h)\big) - L_t\big(\tilde{T}_t(2k-1;h)\big) \geq x\Big) = \mathbb{Q}^{h\sqrt{t}}\big(l(\tau_h) \geq \sqrt{t}\,x\big),$$

where

$$\tau_h \triangleq \inf\{s > h^\delta/2 \colon \beta(s) = 0\}$$

and l is the local time of β at zero. Therefore,

$$\begin{aligned}
&\mathbb{P}\Big(L_t\big(\tilde{T}_t(2k;h)\big) - L_t\big(\tilde{T}_t(2k-1;h)\big) \ge x\Big)\\
&\quad = \mathbb{Q}^{h\sqrt{t}}\big(l(h^\delta/2) \ge \sqrt{t}\,x\big)\\
&\quad \le \mathbb{Q}^0\big(l(h^\delta/2) \ge \sqrt{t}\,x\big) \quad \text{(by the strong Markov property)}\\
&\quad = \mathbb{Q}^0\big(l(1) \ge \sqrt{2}\,th^{-\delta/2}x\big) \quad \text{(by scaling)}\\
&\quad \le \exp(-th^{-\delta}x^2),
\end{aligned} \tag{3.3}$$

since, as mentioned before, $l(1)$ has the same law as $|\beta(1)|$.

On the other hand,

$$\begin{aligned}
&\mathbb{P}\Big(L_t\big(\tilde{T}_t(2k+1;h)\big) - L_t\big(\tilde{T}_t(2k;h)\big) \ge x\Big)\\
&\quad = \mathbb{Q}^0\Big(l(h^\delta/2) + l\big(\hat{\tau}_{h\sqrt{t}}\big)\circ\vartheta(h^\delta/2) \ge x\sqrt{t}\Big),
\end{aligned}$$

where for all $a \in \mathbb{R}^1$,

$$\hat{\tau}_a \triangleq \inf\{s \ge 0 \colon \beta(s) = a\}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\Big(L_t\big(\tilde{T}_t(2k+1;h)\big) - L_t\big(\tilde{T}_t(2k;h)\big) \ge x\Big)\\
&\quad \le \mathbb{Q}^0\big(l(h^\delta/2) \ge x\sqrt{t}/2\big) + \mathbb{Q}^0\Big(l\big(\hat{\tau}_{h\sqrt{t}}\big)\circ\vartheta(h^\delta/2) \ge x\sqrt{t}/2\Big)\\
&\quad = \mathbb{Q}^0\big(l(1) \ge x\sqrt{t}\,h^{-\delta/2}/\sqrt{2}\big) \quad \text{(by scaling)}\\
&\qquad + \mathbb{Q}^0\Big(\mathbb{Q}^{\beta(h^\delta/2)}\Big(l\big(\hat{\tau}_{h\sqrt{t}}\big) \ge x\sqrt{t}/2\Big)\Big) \quad \text{(by the strong Markov property)}\\
&\quad \le \mathbb{Q}^0\big(l(1) \ge x\sqrt{t}\,h^{-\delta/2}/\sqrt{2}\big)\\
&\qquad + \mathbb{Q}^0\Big(l\big(\hat{\tau}_{h\sqrt{t}}\big) \ge x\sqrt{t}/2\Big) \quad \text{(by the strong Markov property)}.
\end{aligned}$$

Now $l(\hat{\tau}_h)$ is exponentially distributed with mean h. This is a standard result; see, for example, [9]. Since $l(1)$ has Gaussian tails,

$$\begin{aligned}
&\mathbb{P}\Big(L_t\big(\tilde{T}_t(2k+1;h)\big) - L_t\big(\tilde{T}_t(2k;h)\big) \ge x\Big)\\
&\quad \le \exp(-tx^2h^{-\delta}/4) + \exp(-x/(2h)).
\end{aligned}$$

This, together with (3.3), proves the lemma. □

A key observation is the following, which is basically telescoping an appropriate sum:

$$\begin{aligned}
L_t(s) \le &\sum_{k=1}^{\tilde{U}_t(h;s)+1} \Big(L_t\big(\tilde{T}_t(2k;h)\big) - L_t\big(\tilde{T}_t(2k-1;h)\big)\Big)\\
&+ \sum_{k=1}^{\tilde{U}_t(h;s)+1} \Big(L_t\big(\tilde{T}_t(2k+1;h)\big) - L_t\big(\tilde{T}_t(2k;h)\big)\Big).
\end{aligned} \tag{3.4}$$

With this observation, we can state and prove the following which estimates $\tilde{U}_t$'s (i.e., the approximate number of upcrossings of $[0, h]$ by the process W_t) in terms of the local time of the process $W_t(s)$. To make things more precise, recall that we have fixed some $\delta \in (1, 2)$. We then have the following lemma.

LEMMA 3.2. *For any $\eta \in (0,1)$, $\alpha \in (0, \delta/2)$ and for all $s \in (0,1]$, with probability 1, there exists an (a.s.) finite $n_0 = n_0(\omega)$, such that for all $n \geq n_0$,*

$$\tilde{U}_t(h_n; s) \geq h_n^{-\alpha} L_t(s) - 1 \quad \textit{for all } t \in \tilde{\mathbb{F}}_\eta(h_n).$$

REMARK 1. The appearance of the -1 in the above comes from the fact that we as yet are essentially unable to prove that with probability 1,

$$\inf_{t \in [\eta, 1]} L_t(1) > 0.$$

This is a recurring problem that is dealt with (perhaps in disguise) in a variety of ways in this paper.

PROOF OF LEMMA 3.2. By Lemma 3.1, for all $n = 1, 2, \ldots,$

$$\begin{aligned}
&\mathbb{P}\Big(L_t\big(\tilde{T}_t(j; h_n)\big) - L_t\big(\tilde{T}_t(j-1; h_n)\big) \geq h_n^{-\alpha} \text{ for some } j \leq h_n^{-4} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and some } t \in \tilde{\mathbb{F}}_\eta(h_n)\Big) \\
&\quad \leq \sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} \sum_{j=1}^{\lfloor h_n^{-4} \rfloor + 1} \mathbb{P}\Big(L_t\big(\tilde{T}_t(j; h_n)\big) - L_t\big(\tilde{T}_t(j-1; h_n)\big) \geq h_n^{-\alpha}\Big) \\
&\quad \leq 2 \sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} h_n^{-4}\Big(2\exp\big(-t h_n^{-(\delta+2\alpha)}/4\big) + \exp\big(-h_n^{-(1+\alpha)}/2\big)\Big) \\
&\quad \leq 2 \# \tilde{\mathbb{F}}_\eta(h_n) \cdot h_n^{-4}\Big(2\exp\big(-\eta h_n^{-(\delta+2\alpha)}/4\big) + \exp\big(-h_n^{-(1+\alpha)}/2\big)\Big) \\
&\quad \leq K h_n^{-(\delta+4)}\Big(2\exp\big(-\eta h_n^{-(\delta+2\alpha)}/4\big) + \exp\big(-h_n^{-(1+\alpha)}/2\big)\Big),
\end{aligned}$$

which sums in n since $h_n \triangleq 2^{-n}$. Hence, by the Borel–Cantelli lemma, almost surely,

$$\sup_{j \leq h_n^{-4}} \Big(L_t\big(\tilde{T}_t(j; h_n)\big) - L_t\big(\tilde{T}_t(j-1; h_n)\big)\Big) \leq h_n^{\alpha} \quad \text{for all } t \in \tilde{\mathbb{F}}_\eta(h_n), \tag{3.5}$$

eventually. On the other hand, $\tilde{U}_t(h_n; s) \leq U_t(h_n)$, for all $n \geq 1$ and all $s, t \in (0,1]$. Hence, by Lemma 2.3 and the Borel–Cantelli Lemma,

$$\tilde{U}_t(h_n; s) \leq 2h_n^{-1} L_t(1) \quad \text{for all } t \in \tilde{\mathbb{F}}_\eta(h_n),$$

eventually. By Lemma 2.1 and (3.2),

$$\sup_{t \in \tilde{\mathbb{F}}_\eta(h_n)} L_t(1) \leq \sup_{t \in [\eta, 1]} L_t(1) < \infty,$$

almost surely. Consequently, with probability 1,

$$\sup_{t \in \tilde{\mathbb{F}}_\eta(h_n)} \tilde{U}_t(h_n; s) \le h_n^{-4} \quad \text{eventually.} \tag{3.6}$$

Hence, by (3.4) and (3.5), almost surely,

$$L_t(s) \le 2\big(\tilde{U}_t(h_n; s) + 1\big) h_n^{\alpha} \quad \text{for all } t \in \tilde{\mathbb{F}}_\eta(h_n),$$

eventually. Since $\alpha \in (0, \delta/2)$ was arbitrary, the result follows. □

A consequence of Lemma 3.2 applies to the Brownian motions $W_t(\cdot + a)$ is the following simple corollary.

COROLLARY 3.3. *For any $\eta \in (0,1)$ and $\alpha \in (0, \delta/2)$, there exists an (a.s.) finite $n_0 = n_0(\omega)$ such that for all $n \ge n_0$,*

$$\tilde{U}_t(h_n; 1) - \tilde{U}_t(h_n; a) \ge h_n^{-\alpha}(L_t(1) - L_t(a)) - 1,$$

simultaneously for all $t \in \tilde{\mathbb{F}}_\eta(h_n)$ and all rational $a \in (0,1)$.

As mentioned in Remark 1, positivity of the local times is a key issue in this paper. Our next lemma provides a necessary estimate to this end.

LEMMA 3.4. *There exists a $K \in (1, \infty)$, such that for all $a \in (0,1)$,*

$$\sup_{0 \le t \le 1} \mathbb{P}\big(L_t(1) - L_t(a) \le a^{1/5}\big) \le K \cdot a^{1/5}.$$

PROOF. For any $t \in (0,1]$ and all $a \in (0,1)$,

$$\begin{aligned}
\mathbb{P}\big(L_t(1) - L_t(a) \le a^{1/5}\big) &= \mathbb{P}\big(L_1(1) - L_1(a) \le \sqrt{t}\, a^{1/5}\big) \quad \text{(Lemma 2.2)} \\
&\le \mathbb{P}\big(L_1(1) - L_1(a) \le a^{1/5}\big) \\
&\le \mathbb{P}\big(L_1(a) \ge a^{1/5}\big) + \mathbb{P}\big(L_1(1) \le 2a^{1/5}\big) \\
&= \mathbb{P}\big(L_1(1) \ge a^{3/10}\big) + \mathbb{P}\big(L_1(1) \le 2a^{1/5}\big) \\
&\le K \cdot \exp(-K^{-1} a^{-3/5}) + K \cdot a^{1/5},
\end{aligned}$$

by another application of the scaling lemma (Lemma 2.2) together with the following consequence of Paul Lévy's theorem, which we have already used a number of times: $L_1(1) =_D |W_1(1)|$. This result follows immediately. □

From now on, throughout the following two sections, we define without further mention, $a_n \triangleq n^{-10}$. We next show that we indeed have positivity of local times, $L_t(1)$, simultaneously for "most values of" t. More precisely we have the following lemma.

LEMMA 3.5. *Suppose* $\tilde{\mathbb{F}}_n \subseteq [0,1]$ *is finite, discrete and* $\lim_{n\to\infty} \#\tilde{\mathbb{F}}_n = \infty$. *Then almost surely,*

$$\lim_{n\to\infty} \frac{\sum_{t\in\tilde{\mathbb{F}}_n} 1\big(L_t(1) - L_t(a_n) \le a_n^{1/5}\big)}{\#\tilde{\mathbb{F}}_n} = 0.$$

PROOF. By Lemma 3.4, one can take expectations to see that

$$\begin{aligned}\mathbb{E} \sum_{t\in\tilde{\mathbb{F}}_n} 1\big(L_t(1) - L_t(a_n) \le a_n^{1/5}\big) &\le \#\tilde{\mathbb{F}}_n \sup_{0\le t\le 1} \mathbb{P}\big(L_t(1) - L_t(a_n) \le a_n^{1/5}\big)\\ &\le K\cdot \#\tilde{\mathbb{F}}_n a_n^{1/5}\\ &= K\cdot \#\tilde{\mathbb{F}}_n n^{-2}.\end{aligned}$$

Hence, for any $\varepsilon > 0$, Chebychev's inequality implies the following:

$$\mathbb{P}\left(\sum_{t\in\tilde{\mathbb{F}}_n} 1\big(L_t(1) - L_t(a_n) \le a_n^{1/5}\big) \ge \varepsilon \#\tilde{\mathbb{F}}_n \right) \le K\varepsilon^{-1} n^{-2},$$

which sums. By the Borel–Cantelli lemma, the result follows. □

We conclude the section by stating the following version of Bernstein's inequality for binomial random variables. A proof can be found in a number of sources; see, for example, Theorem 8.1.1 of [7], for Petrov's refinement of Cramér's strong large deviation result.

LEMMA 3.6. *Let* S_n^p *be a binomial random variable with parameters n and* $p \in (0,1)$, *respectively. Then for any* $c \in (0,p)$,

$$\mathbb{P}(S_n^p \le cn) \le (1 + o(1)) \cdot \sqrt{\frac{2p(1-p)}{n\pi}} \cdot (p-c) \cdot \exp\left(\frac{-n(p-c)}{2p(1-p)} \right).$$

4. Proof of the lower bound of Theorem 1.1. This section uses all of the notation and variables of Section 3. Most notably, $\delta \in (1,2)$ remains fixed.

Suppose $(a,t) \in \mathbb{R}_+^2$. We shall next define the boundary of the box around (a,t) of side h and an associated object. To this end, suppose $a, t > h > 0$. Then we can define

$$\partial_t(a;h) \triangleq \bigcup_{-h\le x\le h} \{(a+x, t\pm h)\} \cup \{(a\pm h, t+x)\}, \tag{4.1a}$$

$$\partial_t^R(a;h) \triangleq \{(a+h, t+y) : 0 \le y \le h\}. \tag{4.1b}$$

In words, $\partial_t(a;h)$ is the square of side $2h$, centered at the point (a,t), while $\partial_t^R(a;h)$ is the upper right edge of it.

Following Section 2, let us define approximate (t,h)-blips by

$$B_t^k(h) \triangleq \big[\tilde{T}_t(2k;h), \tilde{T}_t(2k+2;h)\big]. \tag{4.2}$$

Notice that by the definition of $\tilde{T}_t$'s, the length of $B_t^k(h)$ exceeds h^δ. Furthermore, $W(t, \tilde{T}_t(2k; h)) = W(t, \tilde{T}_t(2k + 2; h)) = 0$, whereas $W(t, \tilde{T}_t(2k + 1; h)) = h$.

We shall divide the approximate blips into two categories: good and bad. We say that $B_t^k(h)$ is "good," if for all $(s, t) \in \partial_t(\tilde{T}_t(2k + 1; h); h^2)$, $W(s, t) \leq 0$. Otherwise, we say that $B_t^k(h)$ is "bad." The following lemma motivates this division of the approximate blips.

LEMMA 4.1. *There exists some $h_0 > 0$ depending on δ such that for all $h \in (0, h_0)$ and every $\eta \in (0, 1)$,*

$$N(h; 1) \geq \sum_{t \in \tilde{\mathbb{F}}_\eta(h)} \sum_{k=0}^{\tilde{U}_t(h;1)} 1\big(B_t^k(h) \text{ is good}\big).$$

PROOF. Suppose $B_t^k(h)$ is good. Then by definition, $W \leq 0$ everywhere on the box boundary, $\partial_t(\tilde{T}_t(2k + 1; h); h^2)$, whereas $W = h$ at the center of the same box. Hence, by sample path continuity, we have isolated a bubble in the interior of the box, $\partial_t(\tilde{T}_t(2k + 1; h); h^2)$. Furthermore, since $\delta \in (1, 2)$ is fixed, for all h small enough (how small depends only on the magnitude of δ), the bubble thus obtained is unique to the approximate (t, h)-blip, $B_t^k(h)$. This is the statement of the lemma. □

We next recall an observation of Dalang and Walsh (see [4]): temporarily define $S \triangleq \inf\{s > 0: W(s, 1) = 1\}$. Then we have the following local decomposition of Brownian sheet near the point, $(S, 1)$: for all $u, v \geq 0$,

$$W(S + u, 1 + v/S) = 1 + X(u) + Y(v) + Z(u, v/S), \tag{4.3}$$

where X and Y are independent Brownian motions and Z is a Brownian sheet. (Z is independent of X and Y, but we will not need this fact.) The key ingredient to the proof of the above result is that one can think of W as integrated white noise. More precisely, by enlarging the probability space if need be, one can represent W as

$$W(s, t) = \mathbb{W}((0, s] \times (0, t]), \quad \text{for all } s, t \geq 0,$$

where $\mathbb{W}$ is white noise thought of as an $L^2(\mathbb{P})$-measure. See Čentsov [3] and Walsh [14] for this and more. Once this is established, (4.3) follows from the strong Markov property of W at time $(S, 1)$. For other results on the Markovian character of Brownian sheet, see [6].

Similar to [4], one can prove the following multidecomposition result for Brownian sheet.

LEMMA 4.2. *For each* $h \in (0, 1/4)$, $k = 0, 1, \dots,$ $t \in (0,1)$ *and* $0 \le u, v \le h^2$,

$$W\big(\tilde{T}_t(2k+1;h) + u, t + v\big) = h + X_t^k(u;h) + Y_t^k\big(v\tilde{T}_t(2k+1;h);h\big) + Z_t^k\big(u, v\tilde{T}_t(2k+1;h);h\big),$$

where:

(a) $\{X_t^k(\cdot\,;h);\ k \ge 0\}$ *is a sequence of i.i.d. Brownian motions with variance* t.

(b) $\{(Y_t^k(\cdot\,;h);\ t \ge 0);\ k \ge 0\}$ *are independent sequences of Brownian motions.*

(c) $Z_t^k(\cdot,\cdot\,;h)$ *is a Brownian sheet.*

Moreover, for each $h \in (0,1/4)$ *and* $t \in (0,1)$, *the X's and Y's are independent. Finally, the processes X, Y and Zs have the following explicit white noise representations*:

$$X_t^k(u;h) = \mathbb{W}\big(\big(\tilde{T}_t(2k+1;h), \tilde{T}_t(2k+1;h) + u\big] \times (0,t]\big),$$

$$Y_t^k(v;h) = \mathbb{W}\big(\big(0, \tilde{T}_t(2k+1;h)\big] \times \big(t, t + v/\tilde{T}_t(2k+1;h)\big]\big),$$

$$Z_t^k(u,v;h) = \mathbb{W}\big(\big(\tilde{T}_2(2k+1;h), \tilde{T}_t(2k+1;h) + u\big] \times \big(t, t + v/\tilde{T}_t(2k+1;h)\big]\big).$$

The proof is an easy adaptation of that of (4.3) and will be omitted.

Our next goal is to show that the Brownian sheet parts, Z_t^k, of Lemma 4.2 are uniformly negligible. (Incidentally, this is similar to the development of [4].) Recalling that $h_n \triangleq 2^{-n}$, we shall achieve our goal via the following almost sure estimate.

LEMMA 4.3. *For each* $\eta \in (0,1)$, *with probability* 1,

$$\lim_{n \to \infty} h_n^{-1} \max_{0 \le k \le h_n^{-4}} \max_{t \in \tilde{\mathbb{F}}_\eta(h_n)} \sup_{0 \le u, v \le h_n^2} \big|Z_t^k(u,v;h_n)\big| = 0.$$

PROOF. By scaling and standard Gaussian estimates (cf. the estimates of Orey and Pruitt [11], for example), there exists $K \in (0,\infty)$ such that for all $x > 1$ and all $n > 1$.

$$\begin{aligned}
&\mathbb{P}\Bigg(\max_{\substack{k \le h_n^{-4} \\ t \in \tilde{\mathbb{F}}_\eta(h_n)}} \sup_{0 \le u, v \le h_n^2} \big|Z_t^k(u,v;h_n)\big| \ge x\Bigg) \\
&\quad \le \#\tilde{\mathbb{F}}_\eta(h_n) \cdot h_n^{-4} \cdot \mathbb{P}\Big(\sup_{u,v \le h_n^2} \big|Z_1^1(u,v;1)\big| \ge x\Big) \\
&\quad = h_n^{-(4+\delta)} \mathbb{P}\Big(\sup_{u,v \le 1} \big|Z_1^1(u,v;1)\big| \ge x h_n^{-2}\Big) \\
&\quad \le \exp\big(-Kx^2 h_n^{-4}\big).
\end{aligned}$$

Fix $\varepsilon > 0$ small and let $x \triangleq \varepsilon h_n$. The above estimate together with the Borel–Cantelli lemma shows that almost surely,

$$\lim_{n\to\infty} h_n^{-1} \cdot \max_{\substack{k \le h_n^{-4} \\ t \in \tilde{\mathbb{F}}_\eta(h_n)}} \sup_{u,v \le h_n^2} \left| Z_t^k(u,v;h_n) \right| \le \varepsilon.$$

Taking $\varepsilon \downarrow 0$ along a countable sequence proves the lemma. □

Now we can proceed to prove the lower bound for Theorem 1.1. Arguing as in the upper bound, it suffices to prove that for all $\nu \in (0,3)$, with probability 1,

$$\liminf_{h \downarrow 0} h^\nu N(h;1) = \infty. \tag{4.4}$$

Recall the definition of a good approximate blip. Suppose that we could prove that for every $\varepsilon, \eta \in (0,1)$, with probability 1,

$$\liminf_{n\to\infty} h_n^{\alpha+\delta-\varepsilon} \sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} \sum_{k=0}^{\tilde{U}_t(h_n;1)} 1\big(B_t^k(h_n) \text{ is good}\big) = \infty. \tag{4.5}$$

We recall that $\delta \in (1,2)$ and $\alpha \in (0, \delta/2)$ were fixed and arbitrary. By Lemma 4.1, (4.5) implies (4.4) and hence the theorem. Recall the definition of ∂_t^R from (4.1). We will prove the following: for every $\varepsilon, \eta \in (0,1)$, with probability 1,

$$\liminf_{n\to\infty} h_n^{\alpha+\delta-\varepsilon} \sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} \sum_{k=0}^{\tilde{U}_t(h_n;1)} 1\big(B_t^k(h_n) \text{ is } R\text{-good}\big) = \infty. \tag{4.6}$$

Here and throughout, an approximate blip, $B_t^k(h)$ is said to be R-good if for all (u,v) in $\partial_t^R(\tilde{T}_t(2k+1;.h); h^2)$, $W(u,v) \le 0$. Evidently, a good approximate blip, $B_t^k(h)$, is R-good; the converge is typically false. As such, (4.6) is strictly weaker than (4.5). However, once we know how to prove (4.6), no new ideas are needed to prove (4.5). We have decided to only prove (4.6) merely because its proof is substantially cleaner to write. To extend the proof of (4.6) given below, one needs an analogue of Lemma 4.2 for the cases where u and v are not both positive. That, in turn, requires the analogue of (4.3) in the case where u and v are not both positive. This appears in Dalang and Walsh [4] and the end result is that X and Y in (4.3) are replaced by two independent locally Brownian processes. We leave the numerous details of the proof of (4,5) to the interested reader and proceed to prove the slightly weaker version, (4.6). For more words on this extension, see Section 5.4 below.

Let us fix some $\alpha_0, \beta_0 > 0$ such that $\alpha_0 > 2 + \beta_0$. We will strive to show that for every $\eta \in (0,1)$, there exist some $p_0, q_0 \in (0,1)$ such that the following eventually hold, with probability 1:

$$\min_{\substack{t \in \tilde{\mathbb{F}}_\eta(h_n): \\ L_t(1) - L_t(a_n) \geq a_n^{1/5}}} \sum_{\{k:\, a_n \leq \tilde{T}_t(2k+1;\, h_n) \leq 1\}} 1\big(X_t^k(h_n^2; h_n) \leq -\alpha_0 h_n\big) \geq p_0\big(h_n^{-\alpha} a_n^{1/5} - 1\big), \tag{4.7a}$$

$$\min_{1 \leq k \leq h_n^{-4}} \sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} 1\Big(\sup_{0 \leq v \leq h_n^2} Y_t^k(v; h_n) \leq \beta_0 h_n\Big) \geq q_0 h_n^{-\delta}. \tag{4.7b}$$

Our claim is that (4.7) implies (4.6). We begin by noting that $(u, v) \in \partial_t^R(\tilde{T}_t(2k+1; h); h^2)$ if and only if $u = \tilde{T}_t(2k+1; h) + h^2$ and $v \in [t, t+h^2]$. With this in mind, by Lemma 4.2, for all $(u, v) \in \partial_t^R(\tilde{T}_t(2k+1; h); h^2)$,

$$\begin{aligned} W(u,v) &= h + X_t^k(h^2; h) + Y_t^k\big(v\tilde{T}_t(2k+1; h); h\big) \\ &\quad + Z_t^k\big(h^2, v\tilde{T}_t(2k+1; h); h\big), \quad 0 \leq v \leq h^2. \end{aligned}$$

Using Lemma 4.3, we see that simultaneously over all $t \in \tilde{\mathbb{F}}_\eta(h_n)$ and all k such that $\tilde{T}_t(2k+1; h_n) \leq 1$,

$$\begin{aligned} W(u,v) &\leq 2h_n + X_t^k(h_n^2; h_n) + Y_t^k\big(v\tilde{T}_t(2k+1; h_n); h_n\big) \\ &\leq 2h_n + X_t^k(h_n^2; h_n) + \sup_{0 \leq v \leq h_n^2} Y_t^k(v; h_n), \end{aligned}$$

eventually. So for every pair (k, t) satisfying:

(a) $\tilde{T}_t(2k+1; h_n) \in [0, 1]$,
(b) $t \in \tilde{\mathbb{F}}_\eta(h_n)$,
(c) $X_t^k(h_n^2; h_n) \leq -\alpha_0 h_n$,
(d) $\sup_{0 \leq v \leq h_n^2} Y_t^k(v; h_n) \leq \beta_0 h_n$,

we have

$$W(u,v) \leq 0 \quad \text{for all } (u,v) \in \partial_t^R\big(\tilde{T}_t(2k+1; h_n); h_n^2\big), \tag{4.8}$$

since $\alpha_0 > 2 + \beta_0$. In other words, if (a) through (d) hold, then $B_t^k(h_n)$ is R-good. Since $\#\{k: \tilde{T}_t(2k+1; h_n) \leq 1\} = U_t(h_n; 1)$, by (3.6),

$$\sup_{t \in \tilde{\mathbb{F}}_\eta(h_n)} \#\big\{k : \tilde{T}_t(2k+1; h_n) \leq 1\big\} \leq h_n^{-4},$$

eventually. Moreover, by Lemma 3.5,

$$\sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} 1\big(L_t(1) - L_t(a_n) \leq a_n^{1/5}\big) \leq \tfrac{1}{2} q_0 h_n^{-\delta} \quad \text{eventually.}$$

Therefore, by (4.7), for all n sufficiently large,

$$\sum_{t\in\tilde{\mathbb{F}}_\eta(h_n)}\ \sum_{k:\tilde{T}_t(2k+1;h_n)\le 1} 1\big(B_t^k(h_n)\text{ is }R\text{-good}\big)$$

$$\ge \sum_{t\in\tilde{\mathbb{F}}_\eta(h_n)}\ \sum_{a_n\le\tilde{T}_t(2k+1;h_n)\le 1} 1\big(L_t(1)-L_t(a_n)\ge a_n^{1/5}\big)1\big(B_t^k(h_n)\text{ is }R\text{-good}\big)$$

$$\ge \sum_{t\in\tilde{\mathbb{F}}_\eta(h_n)} 1\big(L_t(1)-L_t(a_n)\ge a_n^{1/5}\big)$$

$$\times\Bigg[\sum_{a_n\le\tilde{T}_t(2k+1;h_n)\le 1} 1\big(X_t^k(h_n^2;h_n)\le -\alpha_0 h_n\big) \times 1\Big(\sup_{0\le v\le h_n^2} Y_t^k(v;h_n)\le \beta_0 h_n\Big)\Bigg]$$

$$\ge \tfrac{1}{2}p_0q_0\big(h_n^{-\alpha}a_n^{1/5}-1\big)h_n^{-\delta}$$

$$= \tfrac{1}{2}p_0q_0\big(n^{-2}-2^{-n\alpha}\big)\cdot h_n^{-(\alpha+\delta)}.$$

This proves (4.6) and the proof is complete. It therefore suffices to prove (4.7).

An elementary coupling argument shows that for each fixed $t\in\tilde{\mathbb{F}}_\eta(h_n)$ such that $L_t(1)-L_t(a_n)\ge a_n^{1/5}$,

$$\sum_{k:\,a_n\le\tilde{T}_t(2k+1;h_n)\le 1} 1\big(X_t^k(h_n^2;h_n)\le -\alpha_0 h_n\big) \times 1\big(\tilde{U}_t(h_n;1)-\tilde{U}_t(h_n;a_n)\ge h_n^{-\alpha}(L_t(1)-L_t(a_n))-1\big),$$

is stochastically bounded above by

$$\sum_{1\le k\le h_n^{-\alpha}a_n^{1/5}-1} 1\big(X_t^k(h_n^2;h_n)\le -\alpha_0 h_n\big).$$

Since

$$p_1\triangleq \mathbb{P}\big(X_t^k(h_n^2;h_n)\le -\alpha_0 h_n\big)=\mathbb{P}\big(X_1^1(1;1)\le -\alpha_0\big)\in(0,1),$$

by Lemmas 3.6 and 4.2, for all $p_0\in(0,p_1)$, there exists some $k=K(p_0,p_1)>1$ such that

$$\mathbb{P}\Bigg(\sum_{1\le k\le h_n^{-\alpha}a_n^{1/5}-1} 1\big(X_t^k(h_n^2;h_n)\le -\alpha_0 h_n\big)\le p_0\big(h_n^{-\alpha}a_n^{1/5}-1\big)\Bigg)$$

is bounded above by $K\cdot\exp(-K^{-1}h_n^{-\alpha}a_n^{1/5})$. Hence,

$$\mathbb{P}\Bigg(\min_{\substack{t\in\tilde{\mathbb{F}}_\eta(h_n)\\ L_t(1)-L_t(a_n)\ge a_n^{1/5}}}\ \sum_{k:\,a_n\le\tilde{T}_t(2k+1;h_n)\le 1} 1\big(X_t^k(h_n^2;h_n)\le -\alpha_0 h_n\big) \le p_0\big(h_n^{-\alpha}a_n^{1/5}-1\big)\Bigg)$$

$$\leq K \cdot \#\tilde{\mathbb{F}}_\eta(h_n) \cdot \exp\big(-K^{-1} h_n^{-\alpha} a_n^{1/5}\big)$$
$$\leq K \cdot \exp\big(K^{-1}(n\delta - e^{n\alpha} n^{-2})\big)$$
$$\leq n^{-2} \quad \text{eventually.}$$

By the Borel–Cantelli lemma, (4.7a) follows.

The proof of (4.7b) is along similar lines. By scaling and the support theorem,

$$q_1 \triangleq \mathbb{P}\Big(\sup_{0 \leq v \leq h_n^2} Y_t^k(v; h_n) \leq \beta_0 h_n\Big) = \mathbb{P}\Big(\sup_{0 \leq v \leq 1} Y_1^1(1;1) \leq \beta_0\Big) \in (0,1).$$

Hence for all $q_0 \in (0, q_1)$ there exists some $k = K(q_0, q_1) > 1$ such that

$$\mathbb{P}\bigg(\sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} 1\Big(\sup_{0 \leq v \leq h_n^2} Y_t^k(v; h_n) \leq \beta_0 h_n\Big) \leq q_0 \#\tilde{\mathbb{F}}_\eta(h_n)\bigg)$$
$$\leq K \cdot \exp\big(-K^{-1} \#\tilde{\mathbb{F}}_\eta(h_n)\big)$$
$$\leq K \cdot \exp(-K^{-1} e^{n\delta}).$$

Hence by (3.6),

$$\mathbb{P}\bigg(\min_{k \leq h_n^{-4}} \sum_{t \in \tilde{\mathbb{F}}_\eta(h_n)} 1\Big(\sup_{0 \leq v \leq h_n^2} Y_t^k(v; h_n) \leq \beta_0 h_n\Big) \leq q_0 \#\tilde{\mathbb{F}}_\eta(h_n)\bigg)$$

is eventually bounded above by

$$K \cdot h_n^{-4} \exp(-K^{-1} e^{n\delta}) \leq K \cdot \exp\big(K^{-1}(4n - e^{n\delta})\big) \leq n^{-2},$$

eventually. By the Borel–Cantelli lemma, (4.7b) follows. This proves Theorem 1.1. □

5. Concluding remarks.

5.1. *Multidimensional time.* We begin this section with a brief discussion of the Brownian sheet with multidimensional time, that is, a centered real-valued Gaussian process indexed by $\mathbb{R}_+^N$ whose covariance is given by

$$\mathbb{E}W(\mathbf{t})W(\mathbf{s}) = \prod_{i=1}^{N} (t_i \wedge s_i),$$

where $\mathbf{t} \triangleq (t_1, \dots, t_N)$ and $\mathbf{s} \triangleq (s_1, \dots, s_N)$ are in $\mathbb{R}_+^N$. Having defined bubbles as in Section 1, let us define $N(t; h)$ to be the total number of W in $[0, t]^N$ with height exceeding $h > 0$. Our proof of Theorem 1.1 goes to show that almost surely, $\ln N(t; h) \sim (2N - 1)\ln h^{-1}$, as $h \to 0$.

5.2. *Heuristics.* Although the proof of Theorem 1.1 may seem complicated, the driving ideas are rather simple. As in Section 2, define $U_t(h)$ to be the number of upcrossings of $[0, h]$ made by $\{W_t(s);\ 0 \leq s \leq 1\}$. Then by picking a very small η in Lemma 2.3 and using standard methods involving the easy half of the Borel–Cantelli lemma, one more or less has the following:

$$\begin{aligned} h^3 \sum_{t \in \mathbb{G}(h)} U_t(h) &\simeq h^2 \sum_{t \in \mathbb{G}(h)} tL_t(1) \\ &\simeq h^2 \# \mathbb{G}(h) \int_0^1 tL_t(1)\, dt \\ &= \int_0^1 tL_t(1)\, dt \\ &\triangleq l, \end{aligned}$$

where $\mathbb{G}(h) \triangleq \{jh^2 \colon 1 \le j \le h^{-2}\}$ is the analogue of $\mathbb{F}$ and $\tilde{\mathbb{F}}$ of Sections 2 through 4. Although l is not the local time of the Brownian sheet in $[0,1]^2$ at zero as one might expect, the above shows that the total number of (t,h)-blips for all $\mathbb{G}(h)$ is of order h^{-3}. Once this is established, it is not hard to convince oneself, via scaling, that the number of (t,h)-blips for all $t \in \mathbb{G}(h)$ should be roughly the same as the total number of Brownian sheet bubbles. Sections 2 through 4 are devoted to a rigorous proof of a weaker (i.e., correct at the logarithmic level) version of this statement.

5.3. *Open problems.* The heuristic given above strongly suggests that $h^3 N(t;h)$ should have an asymptotic limit as $h \to 0$. More precisely, I believe there exists a constant, $c \in (0,\infty)$, such that with probability 1,

$$\lim_{h \to 0} h^3 N(h;t) = c \int_0^t sL_s(t)\, ds. \tag{5.1}$$

Even though $\int_0^t sL_s(t)\,ds$ is not the local time of $\{W(u,v);\ (u,v) \in [0,t]^2\}$ at zero, (5.1) seems to be the correct analogue of Paul Lévy's celebrated upcrossing theorem. A strong form of the latter together with further results and references appears in [9].

Another interesting but perhaps more technical open problem yet arises from our proof of Theorem 1.1. A source of technical difficulty in our proof was positivity of local times. More precisely, we had to introduce Lemmas 3.4 and 3.5 (and all of the subsequent applications of these lemmas) since we did not have a handy proof of the following question: Is it true that almost surely,

$$L_t(1) > 0 \quad \text{simultaneously for all } t \in (0,1]?$$

In other words, in the notation of Walsh [13], is it true that the local times along lines of W at zero are positive at time one? A rather simple argument involving scaling and Kolmogorov's 0–1 law can be used to show that it is sufficient to prove (2.2) with convergence in probability replaced by almost sure convergence. This is a surprisingly delicate and (at least to me) interesting problem which I have not been able to solve.

5.4. *On the proof of* (4.5). We conclude with a few remarks on how to extend the proof of (4.6) to that of (4.5). The essential step that needs to be made is the extension of (4.3) to the cases where u and v are both necessarily positive. This extension is obtained by the strong Markov property and path decompositions of one-dimensional Brownian motion and appears in [4]. Indeed, in the notation of (4.3), one has the following:

$$\begin{aligned} W(S-u, t+v/S) &= 1 - b(u) + Y(v) - Z'(u, v/S), \\ W(S+u, 1/(1+v/S)) &= 1 + X(u) - Y'(v) + Z''(u, v/S) \\ &\quad - (v/S)W(S+u, 1/(1+v/S)), \\ W(S-u, t/(1+v/S)) &= 1 - b(u) - Y'(v) - Z''(u, v/S) \\ &\quad - (v/S)W(S-u, 1/(1+v/S)), \end{aligned}$$

where X, Y, X' and Y' are independent Brownian motions and b is a three-dimensional Bessel process absorbed upon its last hit on one and is independent of X, X', Y and Y'. Furthermore, Z, Z' and Z'' are Brownian sheets. The proof of the above can be used, together with the strong Markov property, to prove the obvious extension of Lemma 4.2, thus giving the local structure of the sheet near each box around our approximate blips in terms of processes $X_t^k(\cdot; h)$, $X_t'^k(\cdot; h)$, $b_t^k(\cdot; h)$, $Y^k(\cdot; h)$ and $Y_t'^k(\cdot; h)$ which are defined in the spirit of Lemma 4.2. One can then extend (a) through (d) prior to (4.8) to include these processes as well. [Nothing new is needed; all of the processes in question are locally Brownian and therefore, the same estimates leading to (a) through (d) work for h small enough.] To finish, we point out one final technical point. Namely, the above extension of Lemma 4.2 also yields a number of Brownian sheets that a priori cannot be ignored [(a)–(d) say nothing about these]. Therefore, Lemma 4.3 needs to be extended to show that none of these Brownian sheets contributes. Since the estimates in the proof of Lemma 4.3 are exponential and there are only a polynomial number of Brownian sheets that we need to worry about, the proof of Lemma 4.3 goes through with no essential changes.

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