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RECONSTRUCTION OF BAND LIMITED PROCESSES FROM IRREGULAR SAMPLES[1]

BY CHRISTIAN HOUDRÉ

Georgia Institute of Technology

The problem of recovering, say, a band-limited weakly stationary process from a set of its irregularly spaced samples is studied. For rather general sampling sequences some sufficient conditions ensuring mean square or pathwise reconstruction are obtained. For the cases of regular samples with either finitely many missing ones and/or finitely many irregular ones, a necessary and sufficient condition is presented. Some elements of the proofs involve classical results on nonharmonic Fourier series as well as more recent results on frames.

1. Introduction. The main object of the work presented here is to study the reconstruction of a process from a discrete set of its samples. The processes under study generally have a band-limited spectral representation, while reconstruction is achieved via a series expansion, giving a so–called irregular sampling theorem. The results presented below can be read in at least two different ways. First, as statistical results they provide a potential solution to the so called "missing data" problem: Statistical interpolation from sparse or missing data can be achieved under a density condition. Second, as information-theoretic results, they provide irregular sampling theorems for, say, deterministic signals corrupted by random "noise." We do expect both interpretations to be fruitfully exploited in applications.

The regular sampling theorem variously attributed to (among others) Cauchy, Kotel'nikov, Shannon and Whittacker, has been the subject of many studies with theoretical or applied flavors. These many contributions are well reflected in the rather comprehensive works of Butzer, Splettstösser and Stens [5] or Higgins [9]. However, for path reconstruction or interpolation of stochastic processes, rather few results, mainly dealing with regular sampling points, are available. Under a stochastic model assumption, the

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usual approach in the literature is to obtain a mean square convergent series representation which follows from the deterministic sampling results. When interpolating with samples from a path of a process, reconstruction "on average" might be inadequate and path reconstruction has to be considered.

As already mentioned, the results on the almost sure convergence in the sampling expansion of band-limited processes are limited: Mainly, there are three papers all dealing with uniformly spaced samples and second-order processes. First, Belayev [2] obtained, for stationary processes and under an oversampling assumption, exact reconstruction via the cardinal series. Second, Piranashvili [21] extended Belayev's result to include some classes of second-order nonstationary processes. Third, Gaposhkin [8] gave, as a consequence to a more general theorem, a necessary and sufficient condition for path reconstruction of stationary processes. Gaposhkin's result does not require oversampling, the sampling points are uniform and so the recovery is also achieved via the cardinal series.

In the work below, both the assumptions of stationarity of the process and of regularity of the samples are relaxed. No moment condition is needed. Conditions for the path recovery of some classes of nonstationary band-limited processes using irregularly spaced samples are given. Various related problems and extensions are also considered and a truncation error analysis is provided.

Let us now give a more detailed description of the content of the paper: In the next section we set the framework, introduce some notation, terminology and essentially recall known results. In Section 3 and under the assumption that the sampling sequence $\{t_k\}_{k\in\mathbb{Z}}$ is such that $\sup_{k\in\mathbb{Z}} |t_k - (k/d)| < +\infty$ and $\inf_{k\neq n} |t_k - t_n| > 0$, some sufficient conditions for reconstruction (both in L^α and pathwise) are provided. The exponential kernels in the spectral representation are also replaced by more general ones. Then, for sampling sequences $\{t_k\}_{k\in\mathbb{Z}} = \{k/d\}_{k\in\mathbb{Z}-\{k_1,\dots,k_r\}} \cup \{s_1, s_2, \dots, s_\ell\}, d > 0$, and for processes band-limited to $(-\gamma, \gamma)$, $0 < \gamma < \pi d$, an explicit interpolating series, generalizing the cardinal series expansion, is presented. A necessary and sufficient condition for path interpolation via this series representation is then given. For band-limited stationary Gaussian or harmonizable stable processes, the criterion is always satisfied. To end the section, the rate of convergence of the sampling expansion is studied and a truncation error analysis provided.

2. Preparation.

Let $(\Omega, \mathscr{B}, P)$ be a probability space. For $0 \le \alpha \le 2$, let $L^\alpha(\Omega, \mathscr{B}, P)$ [$L^\alpha(P)$ for short] be the corresponding space of complex-valued random variables equipped for $0 < \alpha \le 2$ with the (quasi-) norms $(\mathscr{E}|\cdot|^\alpha)^{1/\alpha} = \|\cdot\|_\alpha$ ($\mathscr{E}$ is expectation), while on $L^0(P)$ the topology is the one induced by convergence in probability (metrized in the usual fashion). The main class of processes considered here has a spectral representation, namely, $X_t = \int_{\mathbb{R}} e^{it\lambda}\, dZ(\lambda)$, $t \in \mathbb{R}$, where the random measure Z induces a bounded linear operator from $C_0(\mathbb{R})$ to $L^\alpha(P)$, $0 \le \alpha \le 2$. Using the terminology of [10] and [11] (where the reader can find more details, examples and references), these processes are (bounded, continuous) (α,∞)-bounded. Essential to our approach is the following result (again see [10] and [11]).

LEMMA 2.1. *Let* $X = \{X_t\}_{t\in\mathbb{R}}$, $X_t \in L^\alpha(\Omega,\mathscr{B},P)$, *be* (α,∞)-*bounded,* $0 \le \alpha \le 2$, *with random measure* Z_X. *Then there exists a probability space* $(\tilde\Omega,\tilde{\mathscr{B}},\tilde P)$ *with* $L^2(P) \subset L^2(\tilde P)$, *a stationary process* $Y = \{Y_t\}_{t\in\mathbb{R}}$, $Y_t \in L^2(\tilde P)$, *and a random variable* $\Lambda \in L^{2\alpha/(2-\alpha)}(P)$ *such that* $X_t = \Lambda Q Y_t$, $t \in \mathbb{R}$, *where* Q *is the orthogonal projection from* $L^2(\tilde P)$ *to* $L^2(P)$.

In Lemma 2.1, Z_Y the random measure of Y, is orthogonally scattered; hence there exists a finite positive measure F (a *dominating measure*) such that

$$\mathscr{E}\left|\int_{\mathbb{R}} f\,dQ\,Z_Y\right|^2 \le \|Q\|^2\mathscr{E}\left|\int_{\mathbb{R}} f\,dZ_Y\right|^2 = \|Q\|^2\int_{\mathbb{R}}|f|^2\,dF \tag{2.1}$$

for all $f \in L^2(F) = \{f\colon \mathbb{R} \longrightarrow \mathbb{C}, \int_{\mathbb{R}}|f|^2\,dF < +\infty\}$. Furthermore, examining the proofs in [10] and [11] will convince the reader that Z_Y can be chosen in such a way that its support coincides with the support of Z_X. In particular, if X is band-limited to (a,b), that is, if $Z_X = 0$, a.s. P, outside of (a,b), then so is Y, that is, $Z_Y = 0$, a.s. P, outside of (a,b).

Now that the probability material needed hereafter has been given, let us briefly state some elements of the theory of frames. Frames were introduced by Duffin and Schaeffer [7] in their study of nonharmonic Fourier series and have, in recent times, regained popularity in connection with wavelet theory (see Meyer [19] and Young [22] for more recent work and references).

Let H be a complex separable Hilbert space, with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. A sequence $\{x_k\}$ in H is a *frame* if there exist two positive constants A and B (the frame bounds) such that $A\|x\|^2 \le \sum_k|\langle x,x_k\rangle|^2 \le B\|x\|^2$, for all $x \in H$. It is readily verified that frames are complete and that $\|x_k\|^2 \le B$, for all k. A frame is *tight* if $A = B$ and a frame which fails to be a frame by the removal of any one of its elements is called *exact*. Exact frames and bounded unconditional bases are identical. Associated with a frame $\{x_k\}$ is the (well defined) frame operator T given via $Tx = \sum_k\langle x,x_k\rangle x_k$, $x \in H$. The operator T is positive, invertible with $A \le T \le B$, that is, $A\langle x,x\rangle \le \langle Tx,x\rangle \le B\langle x,x\rangle$, for all $x \in H$. If T^{-1} denotes the inverse of the frame operator, then $\{T^{-1}x_k\}$ is itself a frame (the dual frame) with bounds $1/B$ and $1/A$ and, in particular, $\sum_k|\langle x,T^{-1}x_k\rangle|^2 < +\infty$. Furthermore, when the frame is exact, the sequences $\{x_k\}$ and $\{T^{-1}x_k\}$ are biorthonormal, that is, $\langle x_k,T^{-1}x_n\rangle = \delta_{k,n}$, for all k,n. Since T is invertible, $x = \sum_k\langle x,x_k\rangle T^{-1}x_k = \sum_k\langle x,T^{-1}x_k\rangle x_k$ unconditionally for every $x \in H$. Of these two expansions (known, respectively, as the frame expansion and the dual frame expansion), the second is the most useful since the expansion is with respect to the elements of the frame. An inexact frame is not a base. However, the dual frame expansion is somehow unique in that its expansion has minimal energy: if $x = \sum_k c_kx_k$, then $\sum_k|c_k|^2 = \sum_k|\langle x,T^{-1}x_k\rangle|^2 + \sum_k|c_k - \langle x,T^{-1}x_k\rangle|^2$.

To conclude this brief review of frames in Hilbert space, we make a simple but rather useful observation (for $C = 2/A + B$, the statement below is well known and already present in [7]).

LEMMA 2.2. *Let C be any positive constant such that* $\max\{|1 - CA|, |1 - CB|\} = K < 1$ *(in particular, let $C = 1/B$). Then CT is invertible; the $T^{-1} = C\sum_{n=0}^{\infty}(I - CT)^n$ in the uniform operator topology.*

PROOF. Since $A \leq T \leq B$ and since $I - CT$ is Hermitian, $\|I - CT\| \leq \max\{|1 - CA|, |1 - CB|\} = K$. Hence, whenever $K < 1$, $CT = I - (I - CT)$ is invertible and its well known series expansion gives the conclusion. □

The observation that a C above could be chosen only depending on B is rather useful ($C = 2/A + B$ is taken in [7]) because, in general, one lacks a knowledge of estimates of A, while estimates of B are known and easier to get. The reason is that B corresponds to the boundedness of the operator T, while A corresponds to its more intractable invertibility, which is often an existential result. Another advantage of having such freedom on C is the possibility of enhancing the convergence of the series by choosing the appropriate C. Of course, $C = 2/A + B$ is the optimal constant.

Recall (again see [7]) that a sequence of reals $\{t_k\}_{k\in\mathbb{Z}}$ has *uniform density* $d > 0$ if $D = \sup_{k\in\mathbb{Z}} |t_k - k/d| < +\infty$ and if $\delta = \inf_{k\neq n} |t_k - t_n| > 0$. For the class of nonharmonic Fourier series corresponding to these sequences, Duffin and Schaeffer developed a convergence theory extending the work of Levinson ([18], Chapter IV) and of Paley and Wiener [20]. From their fundamental paper, we extract (combining elements of Theorems I and IV there) the following result, which for the sake of convenience is only stated for sequences with uniform density 1.

LEMMA 2.3. *Let $\{t_k\}_{k\in\mathbb{Z}}$ have uniform density* 1 *and let $0 < \gamma < \pi$. Then $\{\exp(it_k\cdot)\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\gamma, \gamma)$ and there exists a unique (see Remark* 2.4*) sequence $\{h_k\}_{k\in\mathbb{Z}} \subset L^2(-\gamma, \gamma)$ such that for any $g \in L^2(-\gamma, \gamma)$,*

$$g(\lambda) = \lim_{n\to\infty} \sum_{-n}^{n} \left(\frac{1}{2\gamma} \int_{-\gamma}^{\gamma} \overline{h_k(x)} g(x)\, dx \right) \exp(it_k\lambda),$$

in $L^2(-\gamma, \gamma)$. Moreover, if $g \in L^2(-\pi, \pi)$,

$$\sum_{k=-\infty}^{+\infty} \left(\int_{-\gamma}^{\gamma} \overline{h_k(x)} g(x)\, dx \right) \exp(it_k\cdot)$$

does converge in $L^2(-\pi, \pi)$ and the corresponding ordinary and nonharmonic Fourier series are uniformly equiconvergent to zero over any $[-\pi + \varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$.

REMARK 2.4. The nonharmonic Fourier expansion of a function g is far from being unique (as will become even clearer with our next lemma). The uniqueness in the statement corresponds to the uniqueness given by the dual frame expansion. An important feature of the above lemma is the fact that for any given (*fixed*) γ,

$$\sum_{k=-\infty}^{+\infty}\left(\int_{-\gamma}^{\gamma}\overline{h_k(x)}g(x)\,dx\right)\exp(it_k\lambda),$$

which a priori is only convergent for $-\gamma < \lambda < \gamma$ [since $\{\exp(it_k\cdot)\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\gamma,\gamma)$], does in fact converge over the bigger interval $(-\pi,\pi)$ whenever $g \in L^2(-\pi,\pi)$. Furthermore,

$$\lim_{n\to\infty}\left\{\sum_{-n}^{n}\hat{g}(k)\exp(ik\lambda) - \sum_{-n}^{n}\left(\frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\overline{h_k(x)}g(x)\,dx\right)\exp(it_k\lambda)\right\} = 0$$

uniformly for $\lambda \in [-\pi+\varepsilon,\pi-\varepsilon]$, $\varepsilon > 0$. Another important aspect of Lemma 2.3 is that the removal of a finite number of points does not affect the results (a sequence with uniform density 1 remains of this type after deleting a finite number of its points).

In general, γ above cannot be replaced by π. However, for sequences $\{t_k\}_{k\in\mathbb{Z}}$ such that $D = \sup_{k\in\mathbb{Z}}|t_k - k| < \frac{1}{4}$, in which case $\delta = \inf_{k\neq n}|t_k - t_n| > 0$, such a replacement is possible. This result (stated below) which is due to Levinson ([18], Chapter IV) preceded Duffin and Schaeffer's work and has its origins in the work of Paley and Wiener [20]:

Let $D = \sup_{k\in\mathbb{Z}}|t_k - k| < \frac{1}{4}$. Then $\{\exp(it_k\cdot)\}_{k\in\mathbb{Z}}$ is complete in $L^2(-\pi,\pi)$ and there exists a unique biorthonormal sequence $\{h_k\}_{k\in\mathbb{Z}} \subset L^2(-\pi,\pi)$ such that for any function in $L^2(-\pi,\pi)$, the ordinary and the nonharmonic Fourier series are uniformly equiconvergent over any $[-\pi+\varepsilon,\pi-\varepsilon]$, $\varepsilon > 0$. Moreover, the h_k are given via

$$\Psi_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{h_k(x)}e^{ixt}\,dx = \frac{G(t)}{G'(t_k)(t-t_k)},$$

where

$$G(t) = (t-t_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right)\left(1-\frac{t}{t_{-n}}\right), \qquad t\in\mathbb{R}.$$

The bound $\frac{1}{4}$ is tight and is a sufficient condition for $\{\exp(it_k\cdot)\}_{k\in\mathbb{Z}}$ to form a bounded unconditional basis of $L^2(-\pi,\pi)$ (see Kadec [15]). Hence, the Paley–Wiener and Levinson theories are only concerned with the theory of exact frames. The series expansion for $D < \frac{1}{4}$ is more constrained. It fails to hold by the removal of a single sampling point. The functions Ψ_k are called Lagrange interpolating functions since

$$\Psi_k(t_n) = \begin{cases} 0, & \text{for } k \neq n,\\ 1, & \text{for } k = n.\end{cases}$$

When $t_k = k$, $G(t) = \sin \pi t/\pi$, but, in general, no closed form expression is available for G. Still when $t_k = k$, we have $h_k(x) = e^{ikx}$ and $\Psi_k(t) = (\sin \pi(t-k))/(\pi(t-k))$.

Lemma 2.3 would be highly unsatisfactory without more explicit knowledge (outside of the case $D < \frac{1}{4}$) of the sampling coefficients $(1/2\gamma)\int_{-\gamma}^{\gamma} \overline{h_k(x)} g(x)\,dx$. A determination of these coefficients is the purpose of our next lemma. This result is the dual of a method and proof (replacing also B below by $2/A + B$) given for functions in the Paley–Wiener class PW_γ by Benedetto and Heller [3]. The proof is in [13] and so is omitted here.

LEMMA 2.5. *Let $\{t_k\}_{k\in\mathbb{Z}}$ have uniform density 1, and let $B \geq (1+|e^{\gamma D}-1|)^2$, $0 < \gamma < \pi$. Then,*

$$\frac{1}{2\gamma}\int_{-\gamma}^{\gamma} \overline{h_k(x)} g(x)\,dx = B^{-1}\sum_{n=0}^{\infty}\sum_{p=0}^{n}(-B)^{p-n}\binom{n}{p}\Phi(n-p, t_k, \gamma),$$

where

$$\Phi(0, t_k, \gamma) = \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} \exp(-it_k x) g(x)\,dx$$

and where for $q \geq 1$,

$$\Phi(q, t_k, \gamma) = \sum_{m_1=-\infty}^{+\infty}\sum_{m_2=-\infty}^{+\infty}\cdots\sum_{m_{q-1}=-\infty}^{+\infty}\sum_{m_q=-\infty}^{+\infty}\frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\exp(-it_{m_1}x)g(x)\,dx$$
$$\times\frac{\sin\gamma(t_{m_1}-t_{m_2})\cdots\sin\gamma(t_{m_{q-1}}-t_{m_q})\sin\gamma(t_{m_q}-t_k)}{\gamma(t_{m_1}-t_{m_2})\cdots\gamma(t_{m_{q-1}}-t_{m_q})\gamma(t_{m_q}-t_k)}.$$

In Duffin and Schaeffer's proof of the frame property, the lower frame bound A is only given in an existential way. Since estimates on A seem to be lacking, Lemma 2.5 is rather useful. The expression for $\int_{-\gamma}^{\gamma} \overline{h_k(x)} g(x)\,dx$ is horrible and since the series

$$B^{-1}\sum_{n=0}^{\infty}\sum_{p=0}^{n}(-B)^{p-n}\binom{n}{p}\Phi(n-p, t_k, \gamma) = B^{-1}\sum_{n=0}^{\infty}\sum_{p=0}^{n}(-B)^{-p}\binom{n}{p}\Phi(p, t_k, \gamma)$$

is not unconditionally convergent, no more simplification can be expected. However, the finite approximation formula is quite manageable:

$$B^{-1}\sum_{n=0}^{N}\sum_{p=0}^{n}(-B)^{p-n}\binom{n}{p}\Phi(n-p, t_k, \gamma) = B^{-1}\sum_{p=0}^{N}(-B)^{-p}\Phi(p, t_k, \gamma)\sum_{n=p}^{N}\binom{n}{p}.$$

When g is an exponential function, for example, for band-limited processes, nicer expressions as well as error estimates are available (see some of the results in the next sections). Again, when $D < \frac{1}{4}$, γ can be replaced by π and the uniqueness of the decomposition ensures that the two expressions for $\int_{-\pi}^{\pi} \overline{h_k(x)} g(x)\,dx$ agree. When the frame is tight, that is, when the frame constants A and B are equal, $T = AI$, $T^{-1} = A^{-1}I$ and $\int_{-\gamma}^{\gamma} \overline{h_k(x)} g(x)\,dx = A^{-1}\int_{-\gamma}^{\gamma} \exp(-it_k x) g(x)\,dx$.

3. Reconstruction. We now present our first results on the reconstruction of processes applying the framework of the previous section.

THEOREM 3.1. *Let $\{t_k\}_{k\in\mathbb{Z}}$ have uniform density d and let X be a process band-limited to $(-\gamma d, \gamma d)$, $\gamma < \pi$, and $(\alpha, +\infty)$-bounded, $0 \le \alpha \le 2$, with associated finite dominating measure F. Let there exist $\varepsilon > 0$ such that F is absolutely continuous over $(-\gamma d, -\gamma d + \varepsilon) \cup (\gamma d - \varepsilon, \gamma d)$ and such that its Radon–Nikodym derivative F' is essentially bounded over these same intervals. Then $X(t) = \sum_{k=-\infty}^{+\infty} \Psi_k(t) X(t_k)$ in $L^\alpha(P)$, uniformly on compact subsets of $\mathbb{R}$, where*

$$\Psi_k(t) = B^{-1} \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-B)^{-p} \binom{n}{p} \Phi_p(t, t_k)$$

and where

$$\Phi_0(t, t_k) = \frac{\sin \gamma d(t - t_k d^{-1})}{\gamma d(t - t_k d^{-1})},$$

with

$$\begin{aligned}\Phi_q(t, t_k) &= \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_{q-1}=-\infty}^{+\infty} \sum_{m_q=-\infty}^{+\infty} \frac{\sin \gamma d(t - t_{m_1} d^{-1}) \sin \gamma(t_{m_1} - t_{m_2})}{\gamma d(t - t_{m_1} d^{-1}) \gamma(t_{m_1} - t_{m_2})} \cdots \\ &\quad \times \frac{\sin \gamma(t_{m_{q-1}} - t_{m_q}) \sin \gamma(t_{m_q} - t_k)}{\gamma(t_{m_{q-1}} - t_{m_q}) \gamma(t_{m_q} - t_k)},\end{aligned}$$

$q \ge 1$, with also $B \ge (1 + |e^{\gamma d D} - 1|)^2$.

PROOF. Without loss of generality, again let $d = 1$. Applying Lemmas 2.3, 2.5 and 2.1 to the functions $g_t(\lambda) = e^{it\lambda}$, $t \in \mathbb{R}$, $-\gamma < \lambda < \gamma$, we get

$$\begin{aligned}X(t) &= \int_{|\lambda|<\gamma} \exp(it\lambda)\, dZ_X(\lambda) = \int_{|\lambda|<\gamma} \lim_{n\to+\infty} \sum_{k=-n}^{n} \Psi_k(t) \exp(it_k\lambda)\, dZ_X(\lambda) \\ &= \Lambda \int_{|\lambda|<\gamma} \lim_{n\to+\infty} \sum_{k=-n}^{n} \Psi_k(t) \exp(it_k\lambda)\, dQ\, Z_Y(\lambda) \\ &= \Lambda \left(\int_{|\lambda|<\gamma-\varepsilon} + \int_{\gamma-\varepsilon<|\lambda|<\gamma} \right) \lim_{n\to+\infty} \Psi_k(t) \exp(it_k\lambda)\, dQ\, Z_Y(\lambda),\end{aligned}$$

where the $\Psi_k(t)$ are given as in the statement. We now wish to interchange the limits and the integrals. First, by the equiconvergence result (Lemma 2.3),

$$\lim_{n\to\infty} \sup_{|\lambda|<\gamma-\varepsilon} \left| \sum_{k=-n}^{n} \left(\frac{\sin \gamma(t-k)}{\gamma(t-k)} \exp(ik\lambda) - \Psi_k(t) \exp(it_k\lambda) \right) \right| = 0.$$

Then

$$\sup_{-\gamma<\lambda<\gamma} \left| \sum_{k=-n}^{n} \frac{\sin \gamma(t-k)}{\gamma(t-k)} e^{ik\lambda} \right| \le C, \qquad t \in K,$$

where K is a compact subset of $\mathbb{R}$ and $n \geq 0$. It thus follows from (2.1) that

$$\int_{|\lambda|<\gamma-\varepsilon} \lim_{n\to+\infty} \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda)\,dQ\,Z_Y(\lambda)$$
$$= \lim_{n\to+\infty} \int_{|\lambda|<\gamma-\varepsilon} \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda)\,dQ\,Z_Y(\lambda),$$

in $L^2(P)$, and this provides the first interchange. For the second interchange, by (2.1) and since $F' \leq C$ (throughout, C is an absolute constant whose value might change from one expression to another) over $(-\gamma d, -\gamma d+\varepsilon) \cup (\gamma d - \varepsilon, \gamma d)$, it follows that

$$\limsup_{n\to+\infty} \mathscr{E} \left| \int_{\gamma-\varepsilon<|\lambda|<\gamma} \left\{ \exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda) \right\} dQ\,Z_Y(\lambda) \right|^2$$
$$\leq \limsup_{n\to+\infty} \int_{\gamma-\varepsilon<|\lambda|<\gamma} \left| \exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda) \right|^2 F'(\lambda)\,d\lambda$$
$$\leq C \limsup_{n\to+\infty} \int_{\gamma-\varepsilon<|\lambda|<\gamma} \left| \exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda) \right|^2 d\lambda$$
$$\leq C \limsup_{n\to+\infty} \int_{|\lambda|<\gamma} \left| \exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda) \right|^2 d\lambda = 0,$$

since $\{\exp(it_k\lambda)\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\gamma,\gamma)$. Hence, $QY(t) = \sum_{k=-\infty}^{+\infty} \Psi_k(t)\, QY(t_k)$ in $L^2(P)$ and $X(t) = \sum_{k=-\infty}^{+\infty} \Psi_k(t)X(t_k)$ in $L^\alpha(P)$. □

REMARK 3.2. Again (see Remark 2.4) when $D = \sup_{k\in\mathbb{Z}} |t_k - k/d| < 1/4d$, γ in Theorem 3.1 can be replaced by π and an alternative expression for Ψ_k is given by

$$\Psi_k(t) = \frac{G(t)}{G'(t_k)(t-t_k)},$$

where

$$G(t) = (t-t_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right)\left(1-\frac{t}{t_{-n}}\right), \qquad t \in \mathbb{R}.$$

For $D < \log 2/\pi$ and X stationary, this result with a similar proof is owing to Beutler [4]. The hypotheses of Theorem 3.1 are, in particular, verified under a guard band assumption. Recent results of Jaffard [14] (based in part on Landau [16]) do provide a characterization of sampling sequences $\{t_k\}_{k\in\mathbb{Z}}$ for which $\{\exp(it_k\cdot)\}_{k\in\mathbb{Z}}$ is a frame. Thus, under an equiconvergence result, Theorem 3.1 remains valid for these more general sequences.

We now examine a first almost sure convergence problem.

THEOREM 3.3. *Let $D = \sup_{k\in\mathbb{Z}} |t_k - k/d| < 1/8d$, $d > 0$, and let X be a process band-limited to $(-\gamma d, \gamma d)$, $\gamma < \pi$, and $(\alpha, +\infty)$-bounded, $0 \le \alpha \le 2$. Then $X(t) = \sum_{k=-\infty}^{+\infty} \Psi_k(t) X(t_k)$ with probability 1, uniformly on compact subsets of $\mathbb{R}$, where*

$$\Psi_k(t) = \frac{G(t)}{G'(t_k)(t - t_k)}$$

and

$$G(t) = (t - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right), \qquad t \in \mathbb{R}.$$

PROOF. As usual, let $d = 1$. We first wish to show that whenever $t \in K$, a compact set of $\mathbb{R}$ and for $\varepsilon > 0$,

$$\sup_{|\lambda| < \pi - \varepsilon} \left| \exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t) \exp(it_k\lambda) \right| = O(n^{4D-1})$$

(with the usual O notation). This quantitative estimate essentially follows from the proof of Theorem XVIII in Levinson [18]. Let us try to dig out some buried estimates there. Before doing so, we note the following two Fourier series estimates which follow from standard arguments:

$$\sup_{|\lambda| < \pi - \varepsilon} \left| \exp(it\lambda) - \sum_{k=-n}^{n} \frac{\sin \pi(t-k)}{\pi(t-k)} \exp(ik\lambda) \right| = O\left(\frac{1}{n}\right)$$

and

$$\sup_{|\lambda| < \pi - \varepsilon} \left| \exp(it\lambda) \int_{\lambda-\pi}^{\lambda+\pi} \exp(-i\tau t) \frac{\sin(n + (1/2))\tau}{1/2\tau} \, d\tau - \exp(it\lambda) \int_{\lambda-\pi}^{\lambda+\pi} \exp(-i\tau t) \frac{\sin(n + 1/2)\tau}{\sin 1/2\tau} \, d\tau \right| = O\left(\frac{1}{n}\right).$$

Hence, using (16.11) in [18],

$$\begin{aligned} &\left| \exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t) \exp(it_k\lambda) \right| \\ &\quad \le \left| \sum_{k=-n}^{n} \Psi_k(t) \exp(it_k\lambda) - \frac{1}{\pi} \int_{-\pi}^{\pi} \exp(it\tau) \frac{\sin(n + 1/2)(\lambda - \tau)}{(\lambda - \tau)} \, d\tau \right| + O\left(\frac{1}{n}\right) \qquad (3.1) \\ &\quad = \left| \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} \exp(it\tau) \, d\tau \lim_{A\to\infty} \int_{-A}^{A} G(u) \exp(-i\tau u) \, du \times \int_{\mathscr{C}} \frac{\exp(i\lambda z)}{G(z)(u - z)} \, dz \right| + O\left(\frac{1}{n}\right), \end{aligned}$$

where $\mathscr{C}$ is the rectangular path with vertices at $n+\frac{1}{2}+iM, -n-\frac{1}{2}+iM, -n-\frac{1}{2}-iM$ and $n+\frac{1}{2}-iM$. Now, using inequality (16.19) in [18], (3.1) with $\mathscr{C}$ replaced by its horizontal lines is majorized by

$$Cn^2(M^2+n^2)\exp(-M(\pi-|\lambda|)) \le Cn^2(M^2+n^2)\exp(-M\varepsilon). \tag{3.2}$$

For $\mathscr{C}$ replaced by its vertical lines, since by (16.08) in [18], $|G(z=x+iy)| \ge C\exp(\pi|y|)(1+|z|)^{-4D}$, and since $|\lambda| < \pi-\varepsilon$ with $u \in K$, we have

$$\begin{aligned}
&\left|\int_{-M}^{M} \frac{\exp(i\lambda(N+1/2+iy))}{G(N+1/2+iy)(u-N-1/2-iy)}\,dy\right| \\
&\quad \le \int_0^M \frac{\exp((-\pi-\lambda)y)(1+|N+1/2+iy|)^{4D}}{|u-N-1/2-iy|}\,dy \\
&\qquad + \int_{-M}^0 \frac{\exp((\pi-\lambda)y)(1+|N+1/2+iy|)^{4D}}{|u-N-1/2-iy|}\,dy \\
&\quad \le \int_0^M \frac{\exp(-\varepsilon y)(1+|N+1/2+iy|)^{4D}}{|u-N-1/2-iy|}\,dy \\
&\qquad + \int_{-M}^0 \frac{\exp(\varepsilon y)(1+|N+1/2+iy|)^{4D}}{|u-N-1/2-iy|}\,dy \\
&\quad \le Cn^{-(1-4D)} + Cn^{-(1-4D)}.
\end{aligned} \tag{3.3}$$

Letting $M \to +\infty$ in (3.2), it follows from (3.1) and (3.3) that

$$\sup_{|\lambda|<\pi-\varepsilon}\left|\exp(it\lambda) - \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda)\right| \le Cn^{-(1-4D)}. \tag{3.4}$$

In order to conclude, and since $X = \Lambda QY$ (Lemma 2.1), it is enough to show that

$$\sum_{n=1}^{\infty} \mathscr{E}|QY(t) - \sum_{k=-n}^{n} \Psi_k(t)QY(t_k)|^2 < +\infty.$$

Now QY is band-limited to $(-\gamma,\gamma)$; hence, (2.1) as well as (3.4) lead to

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathscr{E}\left|QY(t) - \sum_{k=-n}^{n} \Psi_k(t)QY(t_k)\right|^2 \\
&\quad \le C\sum_{n=1}^{\infty}\int_{-\gamma}^{\gamma}\left|\exp(it\lambda) - \sum_{k=-n}^{n}\Psi_k(t)\exp(it_k\lambda)\right|^2 dF(\lambda)
\end{aligned}$$

$$\leq CF(-\gamma,\gamma)\sum_{n=1}^{\infty}\sup_{|\lambda|<\gamma}\left|\exp(it\lambda)-\sum_{k=-n}^{n}\Psi_k(t)\exp(it_k\lambda)\right|^2$$

$$\leq C\sum_{n=1}^{\infty}n^{-2(1-4D)}<\infty,$$

since $t \in K$ and since $8D < 1$. □

It is clear that whenever X has moments of order greater than 2, the condition $D < 1/8$ can be weakened. For example, if X is a zero mean stationary Gaussian process, then we can have $D < (4+\varepsilon)^{-1}$, $\varepsilon > 0$, as is easily seen by replacing in the above proof the second moment by an even moment of appropriate magnitude.

We now briefly sketch, as an example, a result extending Theorem 3.3. This shows how to generalize the various results of Belayev [2] and Piranashvili [21] (as well as some of [12]) to our non-second-order irregular sampling framework.

Let $X(t) = \int_E f(t,\lambda)\,dZ_X(\lambda)$, where E is a Borel set, $t \in \mathbb{R}$, where Z_X extends to a bounded linear operator from $C_0(E)$ to $L^\alpha(P)$ and where $f(\cdot,\lambda)$ extends to an entire function of exponential type such that $\sup_{\lambda\in E}\sup_{t\in\mathbb{R}}|f(t,\lambda)| < +\infty$. Let $c(\lambda) = \limsup_{n\to+\infty}(n!|c_n(\lambda)|)^{1/n}$, where the $c_n(\lambda)$ are given via $f(t,\lambda) = \sum_{n=0}^{+\infty} c_n(\lambda)t^n$.

THEOREM 3.4. *Let* $\sup_{\lambda\in E} c(\lambda) = \sigma < +\infty$, *let* $\rho > \sigma$ *and let* $D = \sup_{k\in\mathbb{Z}}|t_k - k\pi/\rho| < \pi/8\rho$. *Then* $X(t) = \sum_{k=-\infty}^{+\infty}\Psi_k(t)X(t_k)$ *with probability* 1, *uniformly on compact subsets of* $\mathbb{R}$, *where*

$$\Psi_k(t) = \frac{G(t)}{G'(t_k)(t-t_k)}$$

and

$$G(t) = (t-t_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right)\left(1-\frac{t}{t_{-n}}\right), \qquad t \in \mathbb{R}.$$

Alternatively, for $\rho > \sigma$, $\beta < (\rho-\sigma)/q$, q *integer, and for* $D = \sup_{k\in\mathbb{Z}}|t_k - k\pi/\rho| < \pi/2\rho$,

$$X(t) = \sum_{k=-\infty}^{+\infty}\Psi_k(t)X(t_k),$$

with probability 1, *where*

$$\Psi_k(t) = \frac{G(t)}{G'(t_k)(t-t_k)}\frac{\sin^q\beta(t-t_k)}{\beta^q(t-t_k)^q}.$$

PROOF. Without loss of generality, let $\rho = \pi$. Then, as in the proof of Theorem 3.3, we first show that

$$\sup_{\lambda\in A}\left|f(t,\lambda) - \sum_{k=-n}^{n} f(t_k,\lambda)\Psi_k(t)\right| \le C/n^{1-4D}$$

uniformly on compact subsets of $\mathbb{R}$. For any λ such that $c(\lambda) < \pi$ and proceeding as in the proof of Theorem 3.3, we have

$$f(\xi,\lambda) - \sum_{k=-n}^{n} f(t_k,\lambda)\Psi_k(\xi) = G(\xi)\int_{\mathscr{C}} \frac{f(\lambda,z)}{G(z)(z-\xi)}\,dz,$$

where $\mathscr{C}$ is as in (3.1). Since

$$|f(\lambda,z)| \le \sup_{t\in\mathbb{R}} |f(\lambda,t)| e^{c(\lambda)|y|},$$

$z = x + iy$, it follows as in the proof of (3.3) that

$$\begin{aligned}
&\left|\int_{-M}^{M} \frac{f(\lambda, N+1/2+iy)}{G(N+1/2+iy)(\xi - N - 1/2 - iy)}\,dy\right| \\
&\quad\le \sup_{t\in\mathbb{R}} |f(\lambda,t)| \left(\int_0^M \frac{\exp(-(\pi - c(\lambda))y)(1+|N+1/2+iy|)^{4D}}{|\xi - N - 1/2 - iy|}\,dy\right. \\
&\qquad\qquad \left. + \int_M^0 \frac{\exp((\pi - c(\lambda))y)(1+|N+1/2+iy|)^{4D}}{|\xi - N - 1/2 - iy|}\,dy\right) \\
&\quad\le C \sup_{t\in\mathbb{R}} |f(\lambda,t)| (\pi - c(\lambda))^{-1} n^{-(1-4D)},
\end{aligned} \tag{3.5}$$

for ξ in a compact set. Replacing $\mathscr{C}$ by its horizontal lines, and still for ξ in a compact, and still proceeding as in Theorem 3.3, the integral

$$G(\xi)\int_{\mathscr{C}} \frac{f(\lambda,z)}{G(z)(z-\xi)}\,dz$$

is majorized by

$$CP(n,m)\exp(-M(\pi - c(\lambda))), \tag{3.6}$$

where $P(n,M)$ is a polynomial. Letting $M \to +\infty$ in (3.6) and using (3.5), we thus get

$$\sup_{\lambda\in E}\left|f(t,\lambda) - \sum_{k=-n}^{n} f(t_k,\lambda)\Psi_k(t)\right| \le C(\pi - \sigma)^{-1} n^{-(1-4D)}. \tag{3.7}$$

Since $Z_X = \Lambda Q Z_Y$ and since $\sup_{\lambda\in E}\sup_{t\in\mathbb{R}} |f(t,\lambda)| < +\infty$, we have

$$X(t) = \int_E f(t,\lambda)\,dZ_X(\lambda) = \Lambda Q \int_E f(t,\lambda)\,dZ_Y$$

and, as previously,

$$\sum_{n=1}^{\infty} \mathscr{E}\left|QY(t) - \sum_{k=-n}^{n} \Psi_k(t)QY(t_k)\right|^2 \le C\sum_{n=1}^{\infty} n^{-2(1-4D)} < +\infty$$

allows us to conclude. To obtain the alternative expression, it is enough to show that

$$\sup_{\lambda\in E}\left|f(t,\lambda) - \sum_{k=-n}^{n} f(t_k,\lambda)\Psi_k(t)\right| \le Cn^{-q-1+4D}.$$

This estimate is essentially obtainable like the previous one. The major difference comes from the fact that $|\sin^q \beta(z-\xi)| \le Ce^{q\beta|y|}$. This is controlled as previously by using $\beta < (\rho - \sigma)/q$. □

REMARK 3.5. When $|f(t,z)| \le C(1+|z|^p)e^{\sigma|y|}$, $p \le q$, p integer, it follows from using the alternative expression that

$$\sup_{\lambda\in E}\left|f(t,\lambda) - \sum_{k=-n}^{n} f(t_k,\lambda)\Psi_k(t)\right| \le Cn^{p-q-1+4D}.$$

Hence, for $D < \pi/4\rho$, $X(t) = \sum_{k=-\infty}^{+\infty}\Psi_k(t)X(t_k)$ with probability 1. Second-order processes band-limited in the sense of Zakai (see Cambanis and Masry [6] for more details) can also be represented as $\int_A f(t,\lambda)\,dZ_X(\lambda)$, where $|f(t,z)| \le C(1+|z|^p)e^{\sigma|y|}$ (see Lee [17]). Hence, the above results generalize and extend the regular sampling results of [6] and [17]. It is clear that for sequences with uniform density 1 or whenever $D < 1/4d$, Theorem 3.1 can also be extended to the more general framework presented in the above theorem.

Although the potential extensions of the previous results are numerous (generalized random processes, sampling with derivatives, sampling for time-limited processes, sampling for second-order processes whose covariance is a tempered function, etc.), none of these will be pursued here. Instead we will study the "simpler" case of regular sampling with either finitely many missing ones and/or finitely many irregular ones.

The results presented below can be interpreted as statistical "missing data" results. For irregularly spaced data satisfying a uniform density condition $d > 0$, and for, say, a weakly stationary process X, the part of the process whose spectrum is contained in $(-\gamma d, \gamma d)$, $0 < \gamma < \pi$, can be exactly recovered [in $L^2(P)$, or pathwise under the necessary and sufficient condition given below]. The mean square error committed with the band-limited assumption is majorized by $\mathscr{E}|\int_{|\lambda|\ge\gamma d} e^{it\lambda}\,dZ(\lambda)|^2 = \int_{|\lambda|\ge\gamma d} dF(\lambda)$. The results allow interpolation, using finite data, of sets containing mixtures of regular and irregular points with gaps.

Throughout the rest of the section, and unless otherwise stated, we assume that the sampling sequence is of the form $\{t_k\}_{k\in\mathbb{Z}} = \{k/d\}_{k\in\mathbb{Z}-\{k_1,\dots,k_r\}} \cup \{s_1, s_2, \dots, s_\ell\}$, $d > 0$. For such sequences, we present another lemma which permits us to find a closed form expression for the interpolating function Ψ_k. Other expressions could be obtained by replacing $g(x)$ by e^{itx} in the statement of Lemma 2.5. However, in that case, the computations become rather cumbersome and difficult to handle. In our special case, it is possible to exploit the fact that only finitely many sampling points differ from the integers to provide a more direct approach. The proof of this lemma (stated for $d = 1$) is in [13] and so is ommited here [below $\langle\cdot,\cdot\rangle_\gamma$ is the inner product in $L^2(-\gamma,\gamma)$].

LEMMA 3.6. *Let* $\{t_k\}_{k\in\mathbb{Z}} = \{k, s_1, s_2, \dots, s_\ell\}_{k\in\mathbb{Z}-\{k_1,\dots,k_r\}}$ *and let* $0 < \gamma < \pi$. *Then the statement of Lemma* 2.3 *continues to hold with*

$$\begin{aligned} \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} & \overline{h_k(x)}g(x)\,dx \\ &= \frac{\gamma}{\pi}\langle g(\cdot), \exp(it_k\cdot)\rangle_\gamma \\ &\quad + \left(\frac{\gamma}{\pi}\right)^2 \sum_{p=1}^{r}\sum_{q=1}^{r}\langle g(\cdot), \exp(ik_p\cdot)\rangle_\gamma \langle \exp(ik_q\cdot), \exp(it_k\cdot)\rangle_\gamma V_{q,p} \\ &\quad - \left(\frac{\gamma}{\pi}\right)^2 \sum_{p=1}^{\ell}\sum_{q=1}^{\ell}\langle g(\cdot), \exp(is_p\cdot)\rangle_\gamma \langle \exp(is_q\cdot), \exp(it_k\cdot)\rangle_\gamma V_{q,p}, \end{aligned} \tag{3.8}$$

where $V_{q,p}$ *is the* (q,p)*–entry of the inverse of the matrix*

$$\left(I - \frac{\gamma \sin\gamma(f_p - f_q)}{\pi\gamma(f_p - f_q)}\right)_{p,q=1,\dots,r+\ell} \quad \textit{with } f_p = \begin{cases} k_p, & p = 1,\dots,r, \\ s_p, & p = r+1,\dots,r+\ell. \end{cases}$$

In the case of no missing observation and regular sampling, $\gamma = \pi$, the correcting term in (3.8) vanishes and the above lemma is just the (dual of the) classical regular sampling theorem. The correcting factor in (3.8) can be rewritten in matrix form $(\gamma/\pi)^2\xi^*(I-(\gamma/\pi)G)^{-1}\eta$, where G is the Gramian matrix $(\langle \exp(if_p\cdot), \exp(if_q\cdot)\rangle_\gamma)_{p,q}$, where ξ^* is the row vector $(\langle \exp(if_p\cdot), \exp(it_k\cdot)\rangle_\gamma)_p$ and where η is the column vector $(\langle \pm g(\cdot), \exp(if_q\cdot)\rangle_\gamma)_q$, ($+g$ for $q = 1,\dots,r$, $-g$ for $q = r+1,\dots,r+\ell$). Hence, for an increasing (but finite) number of missing observations, recursive computational procedures to obtain the correcting factor in (3.8) are possible.

It is assumed from now on that the process $X = \{X_t\}_{t\in\mathbb{R}}$ *is (bounded, continuous)* (α,∞)*-bounded,* $0 \le \alpha \le 2$, *and is also band-limited to* $(-\gamma,\gamma)$, $0 < \gamma < \pi d$, $d > 0$.

With this last assumption, we can now prove another stochastic result and consider convergence of the interpolating series in L^α and with probability 1.

THEOREM 3.7. *Let* X *and* $\{t_k\}_{k\in\mathbb{Z}}$ *be as above. Then* $X(t) = \sum_{k=-\infty}^{\infty}\Psi_k(t)X(t_k)$ *in* $L^\alpha(P)$, *uniformly on any compact subset of* $\mathbb{R}$, *where* *the Ψ_k are given by*

$$\begin{aligned}\Psi_k(t) = \frac{\gamma}{\pi d}\Bigg\{ & \frac{\sin\gamma(d^{-1}t_k - t)}{\gamma(d^{-1}t_k - t)} \\ & + \frac{\gamma}{\pi d}\sum_{p=1}^{r}\sum_{q=1}^{r}\frac{\sin\gamma(d^{-1}k_p - t)}{\gamma(d^{-1}k_p - t)}\frac{\sin\gamma d^{-1}(t_k - k_q)}{\gamma d^{-1}(t_k - t_q)}V_{q,p} \\ & - \frac{\gamma}{\pi d}\sum_{p=1}^{\ell}\sum_{q=1}^{\ell}\frac{\sin\gamma(d^{-1}s_p - t)}{\gamma(d^{-1}s_p - t)}\frac{\sin\gamma d^{-1}(t_k - s_q)}{\gamma d^{-1}(t_k - s_q)}V_{q,p}\Bigg\}.\end{aligned} \tag{3.9}$$

Furthermore, the interpolating series converges with probability 1 *if and only if*

$$\lim_{p\to\infty}\{Z_X(\gamma - 2^{-p}, \gamma) - Z_X(-\gamma, -\gamma + 2^{-p})\} = 0 \quad a.s.\ P.$$

PROOF. We first prove L^α-convergence. Let $d = 1$. Applying Lemma 3.6 to the functions $g_t(\lambda) = e^{it\lambda}$, $t \in \mathbb{R}$, $-\gamma < \lambda < \gamma$, we get

$$X(t) = \int_{|\lambda|<\gamma}\exp(it\lambda)\,dZ_X(\lambda) = \int_{|\lambda|<\gamma}\lim_{n\to\infty}\sum_{k=-n}^{n}\Psi_k(t)\exp(it_k\lambda)\,dZ_X(\lambda),$$

where the expression for the Ψ_k is given via (3.8). We now wish to interchange the limit and the integral. This can be done as follows: Since $e^{it\lambda}\chi_{(-\gamma,\gamma)}(\lambda) \in L^2(-\pi, \pi)$, the equiconvergence result (Lemma 2.3) gives

$$\lim_{n\to\infty}\sup_{-\gamma<\lambda<\gamma}\left|\sum_{k=-n}^{n}\left(\frac{\sin\gamma(t-k)}{\gamma(t-k)}\exp(ik\lambda) - \Psi_k(t)\exp(it_k\lambda)\right)\right| = 0.$$

However, it is classical that

$$\sup_{-\gamma<\lambda<\gamma}\left|\sum_{k=-n}^{n}\frac{\sin\gamma(t-k)}{\gamma(t-k)}e^{ik\lambda}\right| \le C, \qquad t \in K$$

compact subset of $\mathbb{R}$ and $n \ge 0$. Now, since the dominating measure F in (2.1) is finite, we can appeal to bounded convergence to get the result. In fact, it is possible to get this result by directly proving [using (iii) and (iv) of our next lemma] that

$$\sup_{-\gamma<\lambda<\gamma}\left|\sum_{k=-n}^{n}\Psi_k(t)\exp(it_k\lambda)\right| \le C, \qquad t \in K$$

compact subset of $\mathbb{R}$, and $n \ge 0$. This proves the first part of the result. For the a.s. part, more preparation is needed.

REMARK 3.8. The first proof of the L^α-convergence given above applies, as well, to arbitrary sampling sequences with uniform density $d = 1$. Only the interpolating functions Ψ_k have to be changed to those obtained by replacing $g(\cdot)$ by $e^{it\cdot}$ in Lemma 2.5.

Via Theorem 3.7, the value of the missing observations can also be recovered:

$$
\begin{aligned}
X(k_j) = \frac{\gamma}{\pi}\sum_k \Bigg\{ & \frac{\sin\gamma(t_k - k_j)}{\gamma(t_k - k_j)} \\
& + \frac{\gamma}{\pi}\sum_{p=1}^{r}\sum_{q=1}^{r} \frac{\sin\gamma(k_p - k_j)}{\gamma(k_p - k_j)} \frac{\sin\gamma(t_k - k_q)}{\gamma(t_k - t_q)} V_{q,p} \\
& - \frac{\gamma}{\pi}\sum_{p=1}^{\ell}\sum_{q=1}^{\ell} \frac{\sin\gamma(s_p - k_j)}{\gamma(s_p - k_j)} \frac{\sin\gamma(t_k - s_q)}{\gamma(t_k - s_q)} V_{q,p} \Bigg\} X(t_k).
\end{aligned} \tag{3.10}
$$

We now turn to the a.s. result (with $d = 1$) and follow the strategy of proof devised by Gaposhkin [8] in the regular sampling case. The next lemma provides some useful estimates on the kernels $S_n(t,\lambda) = \sum_{k=-n}^{n} \Psi_k(t)\exp(it_k\lambda)$, written for short $S_n(\lambda)$.

LEMMA 3.9. *Let* $S_n(\lambda) = \sum_{k=-n}^{n} \Psi_k \exp(it_k\lambda)$, $\lambda \in (-\gamma,\gamma)$, $t \in K$ *compact subset of* $\mathbb{R}$. *Then,*

(i) $|S_n(\lambda) - S_m(\lambda)| \le C((m-n)/m)$, $2^p \le n < m < 2^{p+1}$.

(ii) $|S_n(\pm\lambda) - S_m(\pm\lambda)| \le C\{(m-n)|\gamma \mp \lambda| + 1/n\}$, $2^p \le n < m < 2^{p+1}$, $\gamma - 1/n \le \pm\lambda < \gamma$, $\lambda \ge 0$.

(iii) $|S_n(\pm\lambda) - e^{\pm i\lambda t} + (e^{\pm i\gamma t} - e^{\mp i\gamma t}/2)| \le C\{n|\gamma \mp \lambda| + 1/n\}$, $n \ge 1$, $\gamma - 1/n \le \pm\lambda < \gamma$, $\lambda \ge 0$.

(iv) $|S_n(\pm\lambda) - e^{\pm i\lambda t}| \le C\{1/(n|\gamma \mp \lambda|) + 1/n\}$, $n \ge 1$, $0 \le \pm\lambda \le \gamma - 1/n$, $\lambda \ge 0$.

PROOF. (i) From the form of $\{t_k\}$ and for p large enough, it is clear that $|\Psi_k(t)| \le C/|k|$; hence, $|S_n(\lambda) - S_m(\lambda)| \le C\sum_{n<|k|\le m} 1/|k| \le C((m-n)/m)$.

(ii) We only consider the case $\gamma - 1/n \le \lambda < \gamma$, since the result for $-\gamma \le \lambda < -\gamma + 1/n$ is similar. From the form of the t_k and of the Ψ_k, it is enough to find similar estimates for

$$
P_n(\lambda) = \sum_{-n}^{n} \frac{\sin\gamma(k-t)}{\gamma(k-t)} e^{ik\lambda} \quad \text{and} \quad \sum_{-n}^{n} \frac{\sin\gamma(k-k_q)}{\gamma(k-k_q)} e^{ik\lambda}
$$

as well as

$$
\sum_{-n}^{n} \frac{\sin\gamma(k-s_q)}{\gamma(k-s_q)} e^{ik\lambda}.
$$

This is done only for P_n, since for the other two sums, it can be done similarly:

$$
\begin{aligned}
P_n(\lambda) &= e^{i\lambda t}\int_{\lambda-\gamma}^{\lambda+\gamma} e^{-i\tau t} \frac{\sin(n+(1/2)\tau)}{\sin(\tau/2)}\, d\tau \\
&= e^{i\lambda t}\int_{\lambda-\gamma}^{\lambda+\gamma} e^{-i\tau t} \frac{\sin(n+(1/2)\tau)}{\tau/2}\, d\tau + O\left(\frac{1}{n}\right).
\end{aligned}
$$

It is thus enough to estimate this last integral which is denoted by $Q_n(\lambda)$. However, now, dividing the domain of integration of $Q_n(\lambda)$ into $(\lambda-\gamma,0]$, $(0,2\lambda]$, $(2\lambda,\lambda+\gamma)$ and since for $\gamma-1/n\le\lambda<\gamma$, the integral over the middle interval is equal to $\pi/2+O(1/n)$, we have

$$Q_n(\lambda)=2e^{i\lambda t}\bigg(\int_{\lambda-\gamma}^{0}e^{-i\tau t}\frac{\sin(n+(1/2)\tau)}{\tau/2}\,d\tau + \int_{2\lambda}^{\lambda+\gamma}e^{-i\tau t}\frac{\sin(n+(1/2)\tau)}{\tau/2}\,d\tau+\frac{\pi}{2}\bigg)+O\Big(\frac{1}{n}\Big). \tag{3.11}$$

Hence from (3.11),

$$\begin{aligned}|Q_n(\lambda)-Q_m(\lambda)| &\le C\bigg(\int_{\lambda-\gamma}^{0}+\int_{2\lambda}^{\lambda+\gamma}\bigg)\bigg|\frac{\sin(n+(1/2))\tau}{\tau}-\frac{\sin(m+(1/2))\tau}{\tau}\bigg|\,d\tau+\frac{C}{n}\\ &\le C\bigg(\int_{\lambda-\gamma}^{0}+\int_{2\lambda}^{\lambda+\gamma}\bigg)|m-n|\,d\tau+\frac{C}{n}\le C|m-n|(\gamma-\lambda)+\frac{C}{n},\end{aligned} \tag{3.12}$$

from which (ii) follows.

(iii) We only prove the inequality for $\lambda>0$:

$$\begin{aligned}&\bigg|S_n(\lambda)-e^{i\lambda t}+\frac{e^{i\gamma t}-e^{-i\gamma t}}{2}\bigg|\\ &\quad\le\bigg|S_n(\lambda)-\frac{e^{i\gamma t}+e^{-i\gamma t}}{2}\bigg|+|-e^{i\lambda t}+e^{i\gamma t}|\\ &\quad\le Cn|\gamma-\lambda|+O\Big(\frac{1}{n}\Big)+C|\gamma-\lambda|,\end{aligned} \tag{3.13}$$

where to obtain the first inequality in (3.13), we use $\lim_{n\to+\infty}S_n(\gamma)=\frac{1}{2}(e^{i\gamma t}+e^{-i\gamma t})$ and also the fact that, as in (ii), to study the asymptotics of $R_n(\gamma)=\sum_{|k|>n}\Psi_k(t)\exp(it_k\gamma)$, it is enough to study

$$\sum_{|k|>n}\frac{\sin\gamma(k-t)}{\gamma(k-t)}e^{ik\gamma},$$

which is a $O(1/n)$.

(iv) Only the case $0\le\lambda\le\gamma-1/n$ is considered. Since $\exp(i\lambda t)-S_n(\lambda)=\sum_{|k|>n}\Psi_k(t)\exp(i\lambda t_k)$, from the Ψ_k we need to show

$$\bigg|\sum_{|k|>n}\frac{\sin\gamma(k-t)}{\gamma(k-t)}e^{ik\lambda}\bigg|\quad\text{and}\quad\bigg|\sum_{|k|>n}\frac{\sin\gamma(k-k_q)}{\gamma(k-k_q)}e^{ik\lambda}\bigg|$$

as well as

$$\bigg|\sum_{|k|>n}\frac{\sin\gamma(k-s_q)}{\gamma(k-s_q)}e^{ik\lambda}\bigg|$$

are dominated by $C(1/(n|\gamma-\lambda|)+1/n)$. These three estimates are similar and we only indicate how to get the first one. Since

$$\left|\sum_{|k|>n} \frac{\sin\gamma(k-t)}{\gamma(k-t)} e^{ik\lambda}\right| = \left|\int_{|x|>n} \frac{\sin\gamma(x-t)}{\gamma(x-t)} e^{ix\lambda}\, dx\right| + O\left(\frac{1}{n}\right),$$

we just need to estimate this last integral. Now, an integration by parts [differentiating $1/(x-t)$] gives (iv). □

We have all the ingredients to prove the a.s. part of Theorem 3.7.

PROOF OF THEOREM 3.7 (a.s.). First, we need to show that

$$\lim_{p\to+\infty} \max_{2^p<n\le 2^{p+1}} |S_n X - S_{2^p} X| = 0, \tag{3.14}$$

that is, that the problem can be reduced to studying the kernels $S_{2^p}X(t) = \sum_{-2^p}^{2^p} \Psi_k(t)X(t_k)$. This is done by using the dyadic decomposition of n. Then, using Lemma 2.2 in [11], and mimicking the proof of Lemma 3.1 there, using the estimates (i), (ii) and (iv) above, (3.14) follows. Once this first reduction of the problem is achieved, we write

$$\begin{aligned} X(t) - S_n X(t) = {} & \{S_{2^p}X(t) - S_n X(t)\} \\ & + \left\{X(t) - S_{2^p}X(t) + \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z_X(\gamma - 2^{-p}, \gamma) \right. \\ & \qquad \left. - \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z_X(-\gamma, -\gamma + 2^{-p})\right\} \\ & - \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z_X(\gamma - 2^{-p}, \gamma) \\ & + \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z_X(-\gamma, -\gamma + 2^{-p}). \end{aligned}$$

To prove the a.s. part of Theorem 3.7 is thus equivalent to showing that with probability 1,

$$\begin{aligned} \lim_{p\to\infty} \left\{X(t) - S_{2^p}X(t) + \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z(\gamma - 2^{-p}, \gamma) \right. & \\ \left. - \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z(-\gamma, -\gamma + 2^{-p})\right\} & = 0, \end{aligned}$$

uniformly for t in a compact of $\mathbb{R}$. It is in turn enough to show that:

$$\begin{aligned} \sum_{p=1}^{\infty} \int_{-\gamma}^{\gamma} \Bigg| e^{i\lambda t} - S_{2^p}(\lambda) + \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} \chi_{(\gamma-2^{-p},\gamma)}(\lambda) & \\ - \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} \chi_{(-\gamma,-\gamma+2^{-p})}(\lambda)\Bigg|^2 dF(\lambda) & < +\infty. \end{aligned}$$

Dividing $|\lambda| < \gamma$ into $0 < |\lambda \mp \gamma| < 2^{-p}$ and $|\lambda \mp \gamma| \geq 2^{-p}$, and providing estimates over these two regions, the result follows. For example, for $|\lambda \mp \gamma| < 2^{-p}$ and if $A_k = \{2^{-k-1} \leq |\lambda \pm \gamma| < 2^{-k}\}$, Lemma 4.7(iii) gives,

$$\begin{aligned}
\sum_{p=1}^{\infty} \int_{0<|\lambda\mp\gamma|<2^{-p}} & \left| e^{\pm i\lambda t} - S_n(\pm\lambda) + \frac{e^{\pm i\gamma t} - e^{\mp i\gamma t}}{2} \right|^2 dF \\
& \leq \sum_{p=1}^{\infty} 2^{2p} \sum_{k=p}^{\infty} \int_{A_k} |\lambda \mp \gamma|^2 \, dF + C \sum_{p=1}^{\infty} \frac{1}{2^{2p}} \\
& \leq C \sum_{p=1}^{\infty} 2^{2p} \sum_{k=p}^{\infty} 2^{-2k} F(A_k) + C \sum_{p=1}^{\infty} \frac{1}{2^{2p}} \\
& \leq C \sum_{p=1}^{\infty} F(A_p) + C \sum_{p=1}^{\infty} \frac{F(A_p)}{2^p} + C < +\infty.
\end{aligned}$$

For $|\lambda \mp \gamma| \geq 2^{-p}$, the estimates are essentially similar to the ones given above using (iv) of Lemma 3.9 instead of (iii). □

The essential moral of Theorem 3.7 is that in a random environment and as far as sampling and reconstruction are concerned, a realization of a process is representative if the end points of the band are filtered out or if the spectrum of the process is smooth enough.

Of course, for regular sampling $\gamma = \pi$ and Theorem 3.7 recovers (and extends to nonstationary processes) the results of [8]. More generally, for sampling sequences whose corresponding interpolating functions Ψ_k satisfy estimates *à la* Lemma 3.9, a.s. convergence can be similarly obtained.

We now give some corollaries which state that for particular cases of X, simplifications occur. We still assume in these results that X and $\{t_k\}$ are as in Theorem 3.7. The proofs of the next three corollaries are omitted since they are similar to the proofs of Corollary 3.5, Corollary 3.4 and Theorem 3.6 in [11].

The first corollary applies, in particular, to stationary Gaussian processes or to "harmonizable" $S\alpha S$ processes.

COROLLARY 3.10. *Let Z_X have independent increments. Then, with probability* 1, $X(t) = \sum_{k=-\infty}^{+\infty} \Psi_k(t) X(t_k)$.

When the dominating measure F in Lemma 2.1 has some degree of smoothness, reconstruction is always possible. In particular, this applies to band-limited second-order stationary processes with continuous spectral density.

COROLLARY 3.11. *Let $dF = F' \, d\theta$, $F' \in L^{1+\varepsilon}(-\gamma, \gamma)$, $\varepsilon > 0$. Then, with probability* 1, $X(t) = \sum_{k=\infty}^{+\infty} \Psi_k(t) X(t_k)$.

The next corollary essentially gives the best possible behavior of F in order to always get exact reconstruction [$F' \in L^1(-\gamma, \gamma)$ is not enough].

Later in this section, we present an example of a stationary process (whose F violates the condition given below) such that with probability 1, $\limsup_{n\to\infty} |\sum_{k=n}^{n} \Psi_k(t)X(t_k)| = +\infty$. Below, log is the base 2 logarithm.

COROLLARY 3.12. *Let $1 < \alpha$ and let there exist a positive Borel measure F on $(-\gamma,\gamma)\times(-\gamma,\gamma)$ such that:*

(i) $\mathscr{E}|Z_X(B)|^\alpha \le F(B\times B)$, *$B$ a Borel set in $(-\gamma,\gamma)$.*

(ii) $$\iint_{0<|\lambda\pm\gamma|<\delta,\ 0<|\mu\pm\lambda|<\delta} \left(\log\log\frac{1}{(\gamma^2-\lambda^2)}\log\log\frac{1}{(\gamma^2-\mu^2)}\right)^{\alpha/2} dF(\lambda,\mu) < +\infty,$$

for some $0 < \delta < 2$. Then with probability 1, $X(t) = \sum_{k=-\infty}^{+\infty} \Psi_k(t)X(t_k)$.

Other corollaries are possible, for example, the guard-band case, that is, if X is band-limited to $(-\gamma+\varepsilon,\gamma-\varepsilon)$, $\varepsilon > 0$, and if we use the Ψ_k of (3.9), then convergence with probability 1 holds. A version of Theorem 3.7 can also be given by replacing $(-\gamma,\gamma)$ by a bounded Borel set.

To finish these notes, we study various problems related to the previous results and thus also complement various results in [8]. We start by studying the rate of convergence in the interpolation formula. We complete the section with a truncation error analysis. Other types of errors occur when using a sampling series representation (aliasing, jitter, round-off, etc.) they are not considered here and should deserve another study.

First, we show below that with the help of a mild enhancing factor, the remainder in the interpolation formula always converges to zero with probability 1.

COROLLARY 3.13. *Under the hypotheses of Theorem 3.7,*

$$\lim_{n\to+\infty}\frac{1}{\log\log n}\sum_{|k|>n}\Psi_k(t)X(t_k) = 0 \quad a.s.\ P,$$

uniformly on compact subsets of $\mathbb{R}$.

PROOF. First, $\sum_{|k|>n}\Psi_k(t)X(t_k)$ is well defined [it converges in $L^\alpha(P)$]. Now, using Theorem 3.7, we need to show that for $2^p < n \le 2^{p+1}$,

$$\lim_{n\to+\infty}\frac{1}{\log\log n}\{Z_X(\gamma-2^{-p},\gamma) - Z_X(-\gamma,-\gamma+2^{-p})\} = 0$$

a.s. P. However, $2^q < p \le 2^{q+1}$,

$$Z_X(\gamma-2^{-p},\gamma) = Z_X(\gamma-2^{-2^q},\gamma) - Z_X[\gamma-2^{-p},\gamma-2^{-2^q})$$

and similarly for $Z_X(-\gamma,-\gamma+2^{-p})$. To prove the result, it is thus enough to prove that

$$\sum_{q=1}^{\infty}\frac{1}{q^2}\mathscr{E}|QZ_Y(\gamma-2^{-2^q},\gamma) - QZ_Y(-\gamma,-\gamma+2^{-2^q})|^2 < +\infty$$

and that

$$\sum_{q=1}^{\infty} \frac{1}{q} \mathscr{E} \max_{2^q < p \le 2^{q+1}} |QZ_Y(\gamma - 2^{-p}, \gamma - 2^{-2^q}) - QZ_Y(-\gamma + 2^{-2^q}, -\gamma + 2^{-p})|^2 < +\infty.$$

The proofs of these two results follow as in the proofs of Theorem 3.7 and Corollary 3.12 and so are omitted. □

The enhancing factor log log is essentially minimal. This is the conclusion of our next proposition. As often in building this kind of counterexample, essential use is made of Rademacher–Menchov systems of divergence. We also note that the result below provides a band-limited stationary process for which $\limsup_{n\to+\infty} |\sum_{k=-n}^{n} \Psi_k(t) X(t_k)| = +\infty$ a.s. P. This is in sharp contrast to the L^α-convergence results of Theorem 3.7 and Remark 3.8. For the proof, we again refer to techniques of [8].

PROPOSITION 3.14. *Let $\{t_k\}$ be as in Theorem 3.7 and let $\{a_n\}_{n\ge1}$ be a nondecreasing sequence of positive reals diverging to infinity and such that:*

(i) $a_n = o(\log\log n)$.
(ii) *There exists $C > 0$ such that $a_{2^{n+1}} \le C a_{2^n}$, n large enough.*

Then there exists a probability space $(\Omega_1, \mathscr{B}_1, P_1)$ and a band-limited stationary process defined on that space such that $\limsup_{n\to+\infty}(1/a_n) \cdot \sum_{k=-n}^{n} \Psi_k(t) X(t_k) = +\infty$ a.s. P.

We are now ready to start a truncation error analysis, that is, we wish to estimate the order of the remainder if only a finite sum is used to approximate X. Since we are concerned with the paths of the process, these estimates have to hold with probability 1. The theorem below is again optimal since, as in our previous result, if $g = o(f)$ replaces f below, a Rademacher–Menchov system of divergence will give a counterexample. As in Corollary 3.12, we could also replace the exponent 2 below by α.

THEOREM 3.15. *Let X and $\{t_k\}_{k\in\mathbb{Z}}$ be as in Theorem 3.7, let F be a dominating measure in (2.1) and let $T_1 < T_2 \in \mathbb{R}$. Let f: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an increasing function such that:*

(i) *There exists $C < 2$ such that $f^2(2n) \le C f^2(n)$, n large enough.*
(ii) $\int_{-\gamma}^{\gamma} f^2(1/(\gamma^2 - \lambda^2))\, dF(\lambda) < +\infty$.

Then

$$\lim_{n\to+\infty} f(n) \sup_{T_1 \le t \le T_2} \left| \sum_{|k|\ge n} \Psi_k(t) X(t_k) \right| = 0$$

with probability 1, *if and only if*

$$\lim_{p\to+\infty} f(2^p)\{Z_X(\gamma - 2^{-p}, \gamma) - Z_X(-\gamma, -\gamma + 2^{-p})\} = 0$$

with probabiliy 1.

PROOF. Using the decomposition results of the previous section, we need to show that with probability 1,

$$\lim_{p\to+\infty} f(2^p) \max_{2^p<n\le 2^{p+1}} |S_n X - S_{2^p} X| = 0. \tag{3.15}$$

and that

$$\begin{aligned}\lim_{p\to+\infty} f(2^p)\bigg| X(t) - S_{2^p}X + \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z_X(\gamma - 2^{-p}, \gamma) \\ - \frac{e^{i\gamma t} - e^{-i\gamma t}}{2} Z_X(-\gamma, -\gamma + 2^{-p})\bigg| = 0.\end{aligned} \tag{3.16}$$

To do so, it is enough to mimic the proof of Theorem 3.7. First, since $C < 2$, we can choose v in Lemma 2.2 in [11] so that $1 < v < 2$, and $(vC_1/2) < 1$. Furthermore, since $f^2(2n) \le Cf^2(n)$, it follows that $f^2(2^p n) \le C^p f^2(n)$ and $f^2(2^p) \le f^2(1)C^p$. Using these results, to prove that (3.15) holds we need, in turn, to show that (also using the notation of Lemma 3.1 in [11])

$$\sum_{p=1}^{\infty} f^2(2^p) \sum_{k=1}^{p} v^k 2^k \max_{(\varepsilon_1,\dots,\varepsilon_k)\in\{0,1\}^k} \int_{-\gamma}^{\gamma} |S_{a_k}(\lambda) - S_{a_{k-1}}(\lambda)|^2 \, dF(\lambda) < +\infty.$$

The methods of the previous section as well as the ones presented above do provide such a result and similarly for (3.16). □

Let us note that if there exist $C_0 > 1$ such that $C_0 f^2(n) \le f^2(2n)$, n large enough, then under (i) and (ii) of Theorem 3.15,

$$\lim_{n\to+\infty} f(n) \sup_{T_1\le t\le T_2} \left| \sum_{|k|\ge n} \Psi_k(t) X(t_k) \right| = 0$$

with probability 1. This is so because this added condition with C_0 ensures that

$$\sum_{p=1}^{\infty} f^2(2^p) \mathscr{E} |QZ_Y(\gamma - 2^{-p}, \gamma) - QZ_Y(-\gamma, -\gamma + 2^{-p})|^2 < +\infty.$$

Still under the conditions of Theorem 3.15, and whenever f there is such that $\int_1^{+\infty} f^{-2}(\lambda)\, d\lambda < +\infty$, then

$$\sup_{T_1\le t\le T_2} \left| \sum_{|k|\ge n} \Psi_k(t) X(t_k) \right| = o(n \log^{1+\varepsilon} n) \quad \text{a.s. } P.$$

More generally, if $\sum_{n=1}^{+\infty} (n\phi(n))^{-2} < +\infty$, then

$$\sup_{T_1\le t\le T_2} \left| \sum_{|k|\ge n} \Psi_k(t) X(t_k) \right| = o(\phi(n)/n) \quad \text{a.s. } P.$$

Finally, Theorem 3.15 can also be slightly sharpened replacing f^2 by f^α, $1 < \alpha < 2$, using the methods of Corollary 3.12 and the above ones.

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SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332
houdre@math.gatech.edu