

## INTERMEDIATE PHASE FOR THE CONTACT PROCESS ON A TREE<sup>1</sup>

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We show that in between the two critical values for the contact process on trees, there are infinitely many extremal nontranslation invariant stationary distributions. This conclusion is reminiscent of results of Grimmett and Newman for percolation on the product of a tree with the integers where there is an intermediate phase with infinitely many infinite clusters.

**1. Introduction.** Let  $\mathcal{T}$  be a homogeneous tree in which  $\kappa \geq 3$  branches emanate from each vertex of  $\mathcal{T}$ . The contact process on  $\mathcal{T}$  is a Markov process in which the state at time  $t$ ,  $\xi_t \subset \mathcal{T}$ , indicates the collection of sites occupied by particles. Particles die at rate 1 and vacant sites become occupied at rate  $\lambda$  times the number of occupied neighbors. For more about the contact process on  $\mathbb{Z}^d$ , see Liggett (1985) or Durrett (1988), (1993). Let  $o$  be a distinguished vertex of the tree, which we call the origin. Let  $\xi_t^x$  denote the contact process starting with one particle at  $x$  at time 0 and let  $|\xi_t^x|$  be the total number of particles. We define the following critical values:

$$\lambda_1 = \inf\{\lambda: P_\lambda(|\xi_t^o| \geq 1, \forall t > 0) > 0\},$$
$$\lambda_2 = \inf\{\lambda: P_\lambda(o \in \xi_t^o \text{ infinitely often}) > 0\}.$$

In words,  $\lambda_1$  is the critical value corresponding to the global survival of the contact process and  $\lambda_2$  corresponds to the local survival. Of course,  $\lambda_1 \leq \lambda_2$ . On  $\mathbb{Z}^d$ ,  $\lambda_1 = \lambda_2$ . See Bezuidenhout and Grimmett (1990). Pemantle (1992) proved that  $\lambda_1 < \lambda_2$  for the contact process on a homogeneous tree with  $\kappa \geq 4$ . We believe, as many others do, that  $\lambda_1 < \lambda_2$  when  $\kappa = 3$ . However, current bounds on the critical values are not sufficiently precise to separate them.

In this paper we will investigate the stationary distributions for the contact process on a tree. First, note that  $\delta_0$ , the measure concentrating on the configuration with no particles, is always a stationary distribution. It is also known that if we start the contact process with one particle on each site

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of  $\mathcal{T}$ , then the law of the contact process converges weakly to a stationary distribution,  $\xi_\infty^1$ , which is called the upper invariant measure since it is the largest stationary distribution in the natural partial ordering.

Using the self-duality of the contact process, it is easy to see that

$$P(x \in \xi_\infty^1) = P(\xi_t^o \neq \emptyset \text{ for all } t \geq 0),$$

so  $\xi_\infty^1 \neq \delta_0$  for  $\lambda > \lambda_1$  and  $\xi_\infty^1 = \delta_0$  for  $\lambda < \lambda_1$ . Using the continuity of the survival probability of the contact process [proved by Pemantle (1992) for  $\kappa \geq 4$  and by Morrow, Schinazi and Zhang (1992) for  $\kappa = 3$ ] we also get  $\xi_\infty^1 = \delta_0$  for  $\lambda = \lambda_1$ .

The results above easily imply that the stationary distribution is unique for  $\lambda \leq \lambda_1$ . Our next result implies that for  $\lambda > \lambda_1$  the only translation invariant stationary distributions are  $\delta_0$  and  $\xi_\infty^1$ .

**THEOREM 1.** *If the initial configuration,  $\xi_0$ , is translation invariant and assigns 0 probability to the empty configuration, then the distribution of the contact process on the tree converges weakly to the upper invariant measure  $\xi_\infty^1$  as  $t \rightarrow \infty$ .*

Theorem 1 is essentially due to Harris (1976) since his argument for  $\mathbb{Z}^d$ , as explained for instance in Durrett (1993), extends with minor modifications to trees.

Bezuidenhout and Grimmett (1990) have proved that  $\delta_0$  and  $\xi_\infty^1$  are the only stationary distributions for the contact process on  $\mathbb{Z}^d$ . As the next result shows, the situation is quite different on the tree.

**THEOREM 2.** *For  $\lambda \in (\lambda_1, \lambda_2)$  there are infinitely many extremal stationary distributions for the contact process on the tree.*

To construct the new stationary distributions we define, for  $x \neq o$ , the cone generated by  $x$ ,  $\Gamma(x)$ , to be the set of all  $y$  for which the self-avoiding path from  $o$  to  $y$  contains  $x$ . If we let  $\xi_t^{\Gamma(x)}$  denote the contact process with 1's on  $\Gamma(x)$  and let  $t \rightarrow \infty$ , then for  $\lambda \in (\lambda_1, \lambda_2)$  we get an extremal nontranslation invariant stationary distribution  $\xi_\infty^{\Gamma(x)}$ .

To prove the existence of the limit, we compactify the tree by adding its boundary (defined precisely in Section 2), and show that if  $\lambda \in (\lambda_1, \lambda_2)$  and  $A$  is a finite set, then  $\lim_{t \rightarrow \infty} \xi_t^A = I_\infty^A \subset \partial\mathcal{T}$  and  $P(\xi_\infty^{\Gamma(x)} \cap A \neq \emptyset) = P(I_\infty^A \cap \Gamma(x) \neq \emptyset)$ . The first step in proving the extremality of the measures we construct is to note:

**PROPOSITION 1.** *Suppose  $\xi_t$  has (additive) dual  $\zeta_t$ . If  $\mu$  is a stationary distribution for  $\xi_t$ , then  $h(A) = \mu(\xi \cap A \neq \emptyset)$  is a harmonic function for  $\zeta_t$ .*

Similar results for the exclusion process and voter model have been used by Liggett [(1973); see Corollary 1.2 on page 434] and Holley and Liggett [(1975); see (5.11) on page 659]. For any Markov process, it is well known [see Dynkin (1978) or Revuz (1984)] that

**PROPOSITION 2.** *There is a 1-1 correspondence between harmonic functions with  $0 \leq h \leq 1$  and shift invariant random variables  $0 \leq Z \leq 1$ , given by  $h(A) = E_A Z$ .*

Since our new stationary measures correspond to the indicator function of  $l_\infty \cap \Gamma(x) \neq \emptyset$  and indicator random variables are extreme points of random variables with  $0 \leq Z \leq 1$ , the desired result follows.

The last construction generalizes easily to show that if  $B$  is a closed subset of  $\partial\mathcal{T}$ , then

$$\mu(\xi \cap A \neq \emptyset) = P(l_\infty^A \cap B \neq \emptyset)$$

defines an extremal stationary distribution. It is natural, if somewhat naive, to guess that this gives all of the stationary distributions:

**CONJECTURE 1.** *The shift invariant  $\sigma$ -field is generated by  $\lim_{t \rightarrow \infty} \xi_t$ .*

Turning to  $\lambda > \lambda_2$ , we let  $\tau^A = \inf\{t: \xi_t^A = \emptyset\}$  and make the following conjecture:

**CONJECTURE 2.** *For  $\lambda > \lambda_2$  the complete convergence theorem holds*

$$\xi_t^A \Rightarrow P(\tau^A < \infty) \delta_0 + P(\tau^A = \infty) \xi_\infty^1$$

and hence  $\delta_0$  and  $\xi_\infty^1$  are the only stationary distributions.

In the first version of this paper, we quoted Pemantle's (1992) proof for large  $\lambda$  and other evidence for this conjecture, but Zhang's (1994) proof of Conjecture 2 makes that discussion obsolete.

It is natural to guess that when the contact process survives, the origin  $o$  is positive recurrent when  $\lambda > \lambda_2$  and null recurrent when  $\lambda = \lambda_2$ . Somewhat surprisingly, Zhang (1994) has proved this is wrong.

**THEOREM 3.** *When  $\lambda = \lambda_2$ ,  $P(o \in \xi_t^o \text{ i.o.}) = 0$ , and the conclusion of Theorem 2 holds.*

The rest of the paper is devoted to the proof of Theorem 2. Before turning to that task, we would like to thank Tom Liggett for pointing out an error in our original version of Proposition 1 and providing us with the simple proof given here and the associated references.

**2. Proof of Theorem 2.** We begin by recalling the graphical construction of the contact process [for more details see Durrett (1988)]. For each  $x \in \mathcal{S}$  and  $0 \leq k \leq \kappa$ , we define independent Poisson processes,  $\{T_n^{x,k}, n \geq 1\}$ . The processes  $\{T_n^{x,0}, n \geq 1\}$  have rate 1. At their arrival times we mark a  $\delta$  at  $x$  to indicate that a death occurs if  $x$  is occupied. The processes  $\{T_n^{x,k}, n \geq 1\}$  for  $k \geq 1$  have rate  $\lambda$ . At their arrival times we draw an arrow from the  $k$ th neighbor of  $x$  to  $x$  to indicate that if the neighbor is occupied, then there will be a birth at  $x$ .

To construct the process from this ‘‘percolation substructure,’’ we say that there is a path from  $(x, s)$  to  $(y, t)$ , and write  $(x, s) \rightarrow (y, t)$  if there is a sequence of times  $s = s_0 < s_1 < s_2 < \dots < s_n < s_{n+1} = t$  and spatial locations  $x_0 = x, x_1, \dots, x_n = y$  so that for  $i = 1, 2, \dots, n$  there is an arrow from  $x_{i-1}$  to  $x_i$  at time  $s_i$  and the vertical segments  $\{x_i\} \times (s_i, s_{i+1})$  for  $i = 0, 1, \dots, n$  do not contain any  $\delta$ 's. To construct the contact process with initial configuration  $A$ , we let  $y \in \xi_t^A$  if there is a path from  $(x, 0)$  to  $(y, t)$  for some  $x$  in  $A$ .

Extending a definition given in the Introduction, let  $\Gamma(o)$  be the connected component containing the origin  $o$  when we delete one of the arcs incident to  $o$ . Let  $\eta_t^o$  denote the contact process starting from one particle at  $o$  restricted to  $\Gamma(o)$ , that is, no births are allowed outside of  $\Gamma(o)$ . Morrow, Schinazi and Zhang (1992) have shown (see the proof of Theorem 2 there)

$$(2.1) \quad \text{if } \lambda > \lambda_1, \text{ then } \alpha \equiv P(|\eta_t^o| > 0, \text{ for all } t > 0) > 0.$$

Observe that

$$(2.2) \quad P(\xi_t^o \cap \Gamma(o) \neq \emptyset \text{ for all } t) \geq P(|\eta_t^o| > 0, \text{ for all } t) > 0.$$

**LEMMA 1.** *For any finite  $A$  and any site  $x$ ,  $1_{(\xi_t^A \cap \Gamma(x) \neq \emptyset)}$  converges a.s. as  $t \rightarrow \infty$ .*

**PROOF.** If  $y \in \Gamma(x)$ , then

$$(2.3) \quad P(\xi_t^y \cap \Gamma(x) \neq \emptyset \text{ for all } t) \geq P(\xi_t^y \cap \Gamma(y) \neq \emptyset \text{ for all } t) \geq \alpha.$$

To see the first inequality note that if  $y \in \Gamma(x)$ , then  $\Gamma(y) \subset \Gamma(x)$ . The second inequality is a consequence of (2.2) and translation invariance. From (2.3), it follows that on  $H = \{\xi_t^A \cap \Gamma(x) \neq \emptyset \text{ i.o.}\}$  we eventually find a particle, so that if we only consider its children in  $\Gamma(x)$ , it starts a process that lives forever. To argue this formally let  $G = \liminf_{t \rightarrow \infty} \{\xi_t^A \cap \Gamma(x) \neq \emptyset\}$  and  $\mathcal{F}_t$  be the  $\sigma$ -field generated by  $\{\xi_s^A, s \leq t\}$ . The martingale convergence theorem implies

$$1_G = \lim_{t \rightarrow \infty} E(1_G | \mathcal{F}_t),$$

but the Markov property implies that the right-hand side is greater than or equal to  $\alpha$  infinitely often on  $H$ , so  $H \subset G$ . On the other hand, it is clear that  $G \subset H$ , so  $H = G$  and the proof is complete.  $\square$

LEMMA 2. *The law of  $\xi_t^{\Gamma(x)}$  converges weakly to  $\xi_\infty^{\Gamma(x)}$ , a stationary distribution.*

PROOF. Self duality implies  $P(\xi_t^{\Gamma(x)} \cap A \neq \emptyset) = P(\xi_t^A \cap \Gamma(x) \neq \emptyset)$ . The right-hand side converges as  $t \rightarrow \infty$  by Lemma 1 and the bounded convergence theorem.  $\square$

To show that when  $\lambda \in (\lambda_1, \lambda_2)$ ,  $\xi_\infty^{\Gamma(x)}$  is nontranslation invariant and hence is not a combination of  $\delta_0$  and  $\xi_\infty^1$ , we observe that by self-duality, monotonicity, and translation invariance, if  $y \in \Gamma(x)$ , then

$$\begin{aligned}
 (2.4) \quad P(y \in \xi_t^{\Gamma(x)}) &= P(\xi_t^y \cap \Gamma(x) \neq \emptyset) \\
 &\geq P(\xi_t^y \cap \Gamma(y) \neq \emptyset) \\
 &\geq P(|\eta_t^y| \geq 0, \text{ for all } t) = \alpha > 0
 \end{aligned}$$

if  $\lambda > \lambda_1$ . Using self-duality again, we have that if  $y \notin \Gamma(x)$ ,

$$(2.5) \quad P(y \in \xi_t^{\Gamma(x)}) = P(\xi_t^y \cap \Gamma(x) \neq \emptyset) \leq P(x \in \xi_s^y \text{ for some } s).$$

However, if  $\lambda < \lambda_2$ , it is easy to see that the r.h.s. of (2.5) goes to zero as the distance between  $x$  and  $y$  increases to infinity [see Lemma 6.4 in Pemantle (1992)].

To see that all the limits  $\xi_\infty^{\Gamma(x)}$  are distinct, we note that (2.4) implies that  $P(y \in \xi_t^{\Gamma(x)})$  is bounded below by  $\alpha$  on  $\Gamma(x)$ , but (2.5) implies  $P(y_n \in \xi_t^{\Gamma(x)})$  goes to 0 if  $y_n \notin \Gamma(x)$  and the distance from  $y_n$  to  $x$  goes to  $\infty$ .

To prove that the stationary distributions we have constructed are extremal, we use the following result (stated in the Introduction).

PROPOSITION 1. *Suppose  $\xi_t$  has additive dual  $\zeta_t$ . If  $\mu$  is a stationary distribution for  $\xi_t$ , then  $h(A) = \mu(\xi \cap A \neq \emptyset)$  is a harmonic function for  $\zeta_t$ .*

PROOF. Suppose  $\xi_0$  has distribution  $\mu$  and  $\zeta_0 = A$ . Stationarity of  $\mu$  and the duality of  $\xi$  and  $\zeta$  imply

$$\begin{aligned}
 P(\xi_0 \cap A \neq \emptyset) &= P(\xi_t \cap \zeta_0 \neq \emptyset) \\
 &= P(\xi_0 \cap \zeta_t \neq \emptyset) \\
 &= \sum_B P_A(\zeta_t = B) P(\xi_0 \cap B \neq \emptyset),
 \end{aligned}$$

so  $h(A) = P(\xi_0 \cap A \neq \emptyset)$  is a harmonic function for  $\zeta_t$ .  $\square$

Define the boundary of the tree,  $\partial\mathcal{T}$ , to be the collection of infinite self-avoiding paths starting at  $o$ . Define a metric on the tree  $\mathcal{T}$  by  $d(x, y) = 2^{-|x \wedge y|}$ , where the  $|x \wedge y|$  is the distance from  $o$  to the closest point  $x \wedge y$  on the self-avoiding path from  $x$  to  $y$ . The notation  $x \wedge y$  comes from the fact that if we write  $\mathcal{T}$  as the finite words from  $\{1, 2, \dots, d\}$  with  $d$  allowed only in the first position, then  $x \wedge y$  is the initial segment that is common to both words. The last definition extends the metric to  $\bar{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$ , making  $\bar{\mathcal{T}}$  a closed set and  $\partial\mathcal{T}$  the boundary of  $\mathcal{T}$ .

LEMMA 3. *With probability 1,  $\lim_{t \rightarrow \infty} \xi_t^A = l_\infty^A$  exists and is a closed subset of  $\partial\mathcal{T}$ .*

PROOF. By Lemma 1,  $1_{(\xi_t^A \cap \Gamma(x) \neq \emptyset)}$  converges a.s. to  $\phi(x, \omega) \in \{0, 1\}$ . The limit set  $l_\infty^A$  is the set of all infinite paths on which  $\phi(x, \omega)$  is always 1. To see that the limit set is closed, let  $I(x) \subset \partial\mathcal{T}$  be the set of infinite self-avoiding paths that start at  $o$  and end in  $\Gamma(x)$ . Let  $X_n(\omega)$  be the set of  $x$  with  $\phi(x, \omega) = 1$  and  $|x| = n$  (i.e., their distance from  $o$  is  $n$ );  $l_n = \bigcup_{x \in X_n} I(x)$  is closed and  $l_\infty^A = \bigcap_n l_n$ .  $\square$

Putting things together we have:

LEMMA 4. *All the  $\xi_\infty^{\Gamma(x)}$  are extremal stationary distributions.*

PROOF. Combining Propositions 1 and 2 with Lemma 3, we see that  $\xi_\infty^{\Gamma(x)}$  corresponds to  $Z$  equal to the indicator function of  $\{l_\infty \cap \Gamma(x) \neq \emptyset\}$ , which is an extreme point of the random variables with  $0 \leq Z \leq 1$  and hence of the shift-invariant random variables satisfying those inequalities.  $\square$

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