The Annals of Probability
1995, Vol. 23, No. 2, 552–585

THE ASYMPTOTIC BEHAVIOR OF LOCALLY SQUARE INTEGRABLE MARTINGALES[1]

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Let M be a locally square integrable martingale with predictable quadratic variance $\langle M\rangle$ and let $\Delta M = M - M_-$ be the jump process of M. In this paper, under the various restrictions on ΔM, the different increasing rates of M in terms of $\langle M\rangle$ are obtained. For stochastic integrals $X = B \cdot M$ of the predictable process B with respect to M, the a.s. asymptotic behavior of X is also discussed under restrictions on the rates of increase of B and the restrictions on the conditional distributions of ΔM or on the conditional moments of ΔM. This is applied to some simple examples to determine the convergence rates of estimators in statistics.

1. Introduction. Let $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions and let $M = \{M_t, t \geq 0\}$ be a locally square integrable martingale based on $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$. We denote by $\langle M\rangle$ the quadratic variation of M. If

$$\lim_{t\to\infty} \langle M\rangle_t = \infty \quad \text{a.s.,} \tag{1.1}$$

then it is well known that

$$\lim_{t\to\infty} \frac{M_t}{\sqrt{\langle M\rangle_t \log^{1+\delta}\langle M\rangle_t}} = 0 \quad \text{a.s.} \ \forall\, \delta > 0. \tag{1.2}$$

Lepingle (1976) proved that if $|\Delta M| = |M - M_-| \leq c$ for some constant c and (1.1) holds, then

$$\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} = 1 \quad \text{a.s.,} \tag{1.3}$$

where $\operatorname{LLg} x = \log(\log(x \vee e^e))$. Xu (1990) lightened the restriction of $|\Delta M| \leq c$ and proved that if (1.1) holds and

$$|\Delta M_t| \leq H(t)\sqrt{\frac{\langle M\rangle_t}{\operatorname{LLg}\langle M\rangle_t}}, \qquad \limsup_{t\to\infty} H(t) \leq k, \tag{1.4}$$

Received September 1993; revised June 1994.
[1]The project supported by the National Natural Science Foundation of China.
AMS 1991 *subject classifications*. Primary 60F15, 60G44; secondary 60H05, 62M09.
Key words and phrases. Strong law of large numbers, law of the iterated logarithm, locally square integrable martingale, stochastic integral.

where $H = \{H(t)\}$ is a predictable process and $k \geq 0$ is an arbitrary constant, then

$$\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} \leq 1 + \varepsilon(k) \quad \text{a.s.,} \tag{1.5}$$

where $\varepsilon(k)$ is a finite constant depending on k with $\varepsilon(0) = 0$. For the discrete parameter martingale, similar results were obtained earlier by Stout (1970) and Fisher (1986). From (1.2) to (1.5) it is easy to see that the asymptotic behavior of M strongly depends on the magnitude of ΔM. From (1.2) it is clear that

$$\lim_{t\to\infty} \frac{|\Delta M_t|}{\sqrt{\langle M\rangle_t \log^{1+\delta}\langle M\rangle_t}} = 0 \quad \text{a.s.} \tag{1.6}$$

Now how about the intermediate cases between (1.4) and (1.6)? In Section 2 of this paper, we will give various rates of increase of M_t as $t \to \infty$ under the different restrictions on ΔM. By the way, we also get (1.5) even if the k in (1.4) and (1.5) is a random variable.

For a discrete parameter martingale $X = \{X_n, \mathscr{F}_n, n \geq 0\}$ with $\mathsf{E}X_n^2 < \infty$, if we put

$$b_n^2 = \mathsf{E}_{n-1}[(X_n - X_{n-1})^2], \qquad \varepsilon_n = \frac{X_n - X_{n-1}}{b_n}, \qquad n \geq 1,$$

where $\mathsf{E}_{n-1}[\cdot] = \mathsf{E}[\cdot \mid \mathscr{F}_{n-1}]$ is the conditional expectation with respect to $\mathscr{F}_{n-1}$, then X has the following representation as a weighted partial sum of martingale differences:

$$X_n = X_0 + \sum_{k=1}^{n} b_k \varepsilon_k. \tag{1.7}$$

Its continuous parameter version is just the stochastic integral of the predictable process $B = \{B(t)\}$ with respect to a locally square integrable martingale $M = \{M_t\}$:

$$X_t = \int_0^t B(s)\, dM_s. \tag{1.8}$$

Stochastic integral (1.8) and weighted partial sum (1.7) are met frequently in the statistics of processes; their asymptotic behavior is related to time series analysis and the statistics of processes. For a sequence of independent random variables and deterministic weight coefficients $\{b_n\}$, Chow and Teicher (1978) and Teicher (1979) discussed the a.s. asymptotic behavior of (1.7). Lai and Wei (1982) proved that if $\{\varepsilon_n\}$ is a martingale difference sequence and

$$\mathsf{E}_{n-1}[\varepsilon_n^2] = 1, \qquad \sup_n \mathsf{E}_{n-1}[|\varepsilon_n|^{2+\delta}] < \infty \quad \text{for some } \delta > 0 \text{ a.s.,}$$

then

$$\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{\langle X\rangle_t \log\langle X\rangle_t}} < \infty \quad \text{a.s.}$$

Recently Zhang (1992) also gave a result on the law of the iterated logarithm for the martingale difference sequence. In Section 3 we shall discuss the a.s. asymptotic behavior of stochastic integral $X = B \cdot M$ under restrictions on rates of increase of B and the restrictions on the conditional distributions of ΔM or on the conditional moments of ΔM. Even in the discrete parameter case, our results not only include the above-mentioned results, but also give some new results.

Finally, in Section 4, we give some simple examples to explain the applications of these results to determine the asymptotic behavior of estimators in different statistical problems.

In this paper we will use the usual notations and symbols in the stochastic calculus of semimartingales according to He, Wang and Yan (1992) and Jacod and Shiryaev (1987), unless stated otherwise.

2. The asymptotic behavior of locally square integrable martingales with dominated jumps. For convenience, to describe the asymptotic behavior we need the symbols O, o and some others. For a function g and an increasing positive function a if

$$\limsup_{t\to\infty} \frac{g(t)}{a(t)} \le k,$$

where k is a finite constant, then denote it by $g \le_{\text{ap}} ka$. We also denote $|g| \le_{\text{ap}} ka$ by $g = O(a)$ and write $g = o(a)$ if $\lim_{t\to\infty} g(t)/a(t) = 0$. For a function g, write the extremal function of g by $g^*(t) = \sup_{s\le t} |g(s)|$. If $g^*(t) < \infty \ \forall\, t$, then it is easy to verify that $|g| \le_{\text{ap}} ka$ and $g = o(a)$ are equivalent to $g^* \le_{\text{ap}} ka$ and $g^* = o(a)$, respectively. For a stochastic process G and an increasing process A, we use similar symbols; for example, $G \le_{\text{ap}} KA$ means

$$\limsup_{t\to\infty} \frac{G_t(\omega)}{A_t(\omega)} \le K(\omega) \quad \text{a.s.}$$

It is equivalent to the fact that for each $\varepsilon > 0$ there is a finite random variable $L(\omega,\varepsilon)$ such that

$$G_t(\omega) < (K(\omega)+\varepsilon)A_t(\omega) \qquad \forall\, t > L(\omega,\varepsilon).$$

We also use the symbols $G = O(A)$, $G = o(A)$ and

$$\{G \le_{\text{ap}} KG\} = \left\{\omega\colon \limsup_{t\to\infty} \frac{G_t(\omega)}{A_t(\omega)} \le K(\omega)\right\}.$$

Note that if $G_t^* < \infty \ \forall\, t$ a.s., then $|G| \le_{\text{ap}} KA$ and $G = o(A)$ are equivalent to $G^* \le_{\text{ap}} KA$ and $G^* = o(A)$, respectively.

Let $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions. Denote by $\mathscr{M}^2_{\text{loc}}$ the collection of all locally square integrable martingales based on $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$. For $M \in \mathscr{M}^2_{\text{loc}}$, $\langle M\rangle = \langle M, M\rangle$ is the predictable quadratic variation of M and $\Delta M = \{\Delta M_t = M_t - M_{t-}\}$ is the jump process of M.

To begin with, recall the following inequality for the probability of large deviations for martingales in Shorack and Wellner (1986); it will be one of basic tools of this section.

LEMMA 2.1 [Shorack and Wellner (1986), page 899]. *Let $M \in \mathscr{M}^2_{\text{loc}}$, $|\Delta M| \leq d$ and a, b be positive constants. Then for any stopping time T, the following inequality holds:*

$$\mathsf{P}(M_T^* \geq a,\ \langle M\rangle_T \leq b) \leq 2\exp\left(-\frac{a^2}{2b}\psi\left(\frac{ad}{b}\right)\right), \tag{2.1}$$

where

$$\psi(x) = \frac{2}{x^2}\int_0^x \int_0^y \frac{dz\,dy}{1+z} = \frac{2(1+x)\log(1+x) - 2x}{x^2}, \qquad x > 0. \tag{2.2}$$

From (2.2) it is easy to show that ψ is a decreasing continuous function and

$$\psi(x) \leq 1, \qquad \lim_{x\to 0}\psi(x) = 1,$$

$$\lim_{x\to\infty}\frac{x\psi(x)}{2\log x} = 1. \tag{2.3}$$

In the proofs of the main results, we also need the following simple lemma.

LEMMA 2.2. *Let X, Y be two random variables, f be a strictly continuous increasing function and $A \in \mathscr{F}$. If for all $c \in \mathbb{R}$,*

$$A\{X < c\} \subset \{Y \leq f(c)\} \quad \text{a.s.}, \tag{2.4}$$

then

$$A \subset \{Y \leq f(X)\} \quad \text{a.s.} \tag{2.5}$$

PROOF. From (2.4) we have

$$A(X < r) \subset (Y \leq f(r)) \quad \text{for all rational numbers } r \quad \text{a.s.} \tag{2.6}$$

Denote by f^{-1} the continuous inverse of f. If (2.5) is not true, that is, $\mathsf{P}(A\{X < f^{-1}(Y)\}) > 0$, then there exists a rational r such that

$$\mathsf{P}(A\{X < r < f^{-1}(Y)\}) > 0.$$

It contradicts (2.6), hence (2.5) is true. □

THEOREM 2.3. *Suppose that* $M \in \mathscr{M}^2_{\mathrm{loc}}$ *and*

$$|\Delta M| \le H\sqrt{\frac{\langle M\rangle}{\mathrm{LLg}\langle M\rangle}} \quad a.s., \tag{2.7}$$

where $H = \{H(t)\}$ *is a predictable process. Then*

$$\{\langle M\rangle_\infty = \infty\} \subset \left\{\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \,\mathrm{LLg}\langle M\rangle_t}} \le a(K)\right\} \quad a.s., \tag{2.8}$$

where $\mathrm{LLg}\, x = \log(\log(x \vee e^e))$, $K = \limsup_{t\to\infty} H(t)$, $a(K)$ *is the unique solution of* $a^2\psi(\sqrt{2}aK) = 1$ *for finite* K *and* $a(\infty) = \infty$.

PROOF. At first, suppose that

$$(\Delta M)^*_t \le k\sqrt{\frac{q(t)}{\mathrm{LLg}\, q(t)}} \qquad \forall\, t \text{ a.s.}, \tag{2.9}$$

$$\langle M\rangle_t \le q(t) \qquad \forall\, t \text{ a.s.},$$

where k is a finite constant and $q = \{q(t)\}$ is a predictable increasing process (here and hereafter increasing process means that it is right continuous and with left limits). It will be proved that

$$\{q(\infty) = \infty\} \subset \left\{\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2q(t)\,\mathrm{LLg}\, q(t)}} \le a(k)\right\} \quad \text{a.s.} \tag{2.10}$$

For $p > 1$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$, set

$$T_n = \inf\{t > 0\colon\ q(t) \ge p^{2n}\}.$$

Then T_n is finite on $\{q(\infty) = \infty\}$ and is predictable. From the definition of T_n and (2.9) we have

$$\langle M\rangle_{T_n-} \le q(T_n-) \le p^{2n} \quad \text{a.s.},$$

$$(\Delta M)^*_{T_n-} \le k\sqrt{\frac{q(T_n-)}{\mathrm{LLg}\, q(T_n-)}} \le k\frac{p^n}{\sqrt{\mathrm{LLg}\, p^{2n}}} \overset{\text{def}}{=} d_n \quad \text{a.s.}$$

Put $a_n = a\sqrt{2p^{2n}\,\mathrm{LLg}\, p^{2n}}$, where a is a constant defined below. By Lemma 2.1 we have

$$\begin{aligned} &\mathsf{P}\Big(M^*_{T_n-} > a\sqrt{2p^{2n}\,\mathrm{LLg}\, p^{2n}}\Big) \\ &\quad\le \mathsf{P}\big((M^{T_n-})^*_\infty > a_n,\ \langle M^{T_n-}\rangle_\infty \le p^{2n}\big) \\ &\quad\le 2\exp\left[-\frac{a_n^2}{2p^{2n}}\psi\left(\frac{a_n d_n}{p^{2n}}\right)\right] \\ &\quad= 2\exp\big[-a^2\,\mathrm{LLg}\, p^{2n}\psi(\sqrt{2}ak)\big]. \end{aligned} \tag{2.11}$$

Since

$$a^2\psi(\sqrt{2}ak) = \frac{1}{k^2}\int_0^{\sqrt{2}ak}\int_0^y \frac{dz\,dy}{1+z}$$

is a strictly increasing function of a, then there exists unique $a = a(k) = \inf\{c\colon c^2\psi(\sqrt{2}ck) > 1\}$ such that $a^2\psi(\sqrt{2}ak) = 1$. From the properties of ψ it is easy to verify that $a(k)$ is an increasing continuous function of k and

$$\lim_{k\downarrow 0} a(k) = 1. \tag{2.12}$$

Now take $a > a(k)$. Therefore

$$\alpha \stackrel{\text{def}}{=} a^2\psi(2\sqrt{2}ak) > 1.$$

This and (2.11) yield

$$\mathsf{P}\Big(M^*_{T_n-} > a\sqrt{2p^{2n}\,\mathrm{LLg}\,p^{2n}}\Big) \le 2\exp(-\alpha\,\mathrm{LLg}\,p^{2n}) = \frac{2}{(2n\log p)^{\alpha}}.$$

Thus by the Borel–Cantelli lemma we get

$$\mathsf{P}\Big(M^*_{T_n-} > a\sqrt{2p^{2n}\,\mathrm{LLg}\,p^{2n}}\ \text{i.o.}\Big) = 0. \tag{2.13}$$

For $t \in [T_n, T_{n+1}[$ we have

$$|M_t| \le M^*_{T_{n+1}-}, \qquad q(t) \ge q(T_n) \ge p^{2n},$$

$$\frac{|M_t|}{\sqrt{2q(t)\,\mathrm{LLg}\,q(t)}} \le \frac{M^*_{T_{n+1}-}}{\sqrt{2p^{2n}\,\mathrm{LLg}\,p^{2n}}}.$$

Then from (2.13) we can conclude that

$$\{q(\infty) = \infty\} \subset \left\{\limsup_{t\to\infty}\frac{|M_t|}{\sqrt{2q(t)\,\mathrm{LLg}\,q(t)}} \le pa\right\}.$$

Since a is an arbitrary number greater than $a(k)$ and p is an arbitrary number greater than 1, the following relation holds too:

$$\{q(\infty) = \infty\} \subset \left\{\limsup_{t\to\infty}\frac{|M_t|}{\sqrt{2q(t)\,\mathrm{LLg}\,q(t)}} \le a(k)\right\}. \tag{2.14}$$

Next we will discuss the general case assumed by the theorem. For fixed constants $\varepsilon > 0$ and $k > 0$ put

$$N = 1_{[H\le k+\varepsilon]}\cdot M,$$

here, and hereafter 1_A and $I(A)$ denote the indicator of A. Then we have

$$N \in \mathscr{M}^2_{\mathrm{loc}}, \qquad \langle N\rangle \le \langle M\rangle$$

and from (2.7),

$$|\Delta N| = 1_{[H\le k+\varepsilon]}|\Delta M| \le (k+\varepsilon)\sqrt{\frac{\langle M\rangle}{\mathrm{LLg}\langle M\rangle}} \quad \text{a.s.}$$

Hence (2.10) yields

$$\{\langle M\rangle_\infty = \infty\} \subset \left\{\limsup_{t\to\infty} \frac{|N_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} \le a(k+\varepsilon)\right\} \quad \text{a.s.}$$

However,

$$M - N = 1_{[H>k+\varepsilon]}\cdot M, \qquad \langle M-N\rangle = 1_{[H>k+\varepsilon]}\cdot\langle M\rangle,$$

$$\left\{\limsup_{t\to\infty} H(t) \le k\right\}$$
$$\subset \{\langle M-N\rangle_\infty < \infty\} \subset \left\{\lim_{t\to\infty}(M_t - N_t) \text{ exists and is finite}\right\}.$$

Therefore,

$$\{\langle M\rangle_\infty = \infty\}\left\{\limsup_{t\to\infty} H(t) \le k\right\}$$
$$\subset \left\{\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}}\right.$$
$$\left. = \limsup_{t\to\infty} \frac{|N_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} \le a(k+\varepsilon)\right\} \quad \text{a.s.}$$

Now letting $\varepsilon \downarrow 0$ and using Lemma 2.2, the conclusion (2.8) is established. □

COROLLARY 2.4. *Suppose that* $M \in \mathscr{M}^2_{\text{loc}}$ *and* (2.7) *holds. Then*

$$\{\langle M\rangle_\infty = \infty\}\left\{\lim_{t\to\infty} H(t) = 0\right\} \subset \left\{\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} \le 1\right\} \quad \textit{a.s.} \tag{2.15}$$

PROOF. Recall (2.8), and the conclusion (2.15) comes from (2.12). □

REMARK. From Stout (1970) and Xu (1990) it is easy to show that if M satisfies the "global" assumptions

$$\langle M\rangle_\infty = \infty \quad \text{a.s.},$$

$$|\Delta M_t| \le H(t)\sqrt{\frac{\langle M\rangle_t}{\operatorname{LLg}\langle M\rangle_t}} \quad \forall\, t \ge 0 \text{ a.s.},$$

where $H = \{H(t)\}$ is a predictable process and

$$\lim_{t\to\infty} H(t) = 0 \quad \text{a.s.},$$

then

$$\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} = 1 \quad \text{a.s.}$$

However, it is not known whether the right side of (2.15) can be improved to an equality.

THEOREM 2.5. *Suppose that* $M \in \mathscr{M}^2_{\text{loc}}$ *and for some* $\delta > -1/2$,

$$|\Delta M| \le H\sqrt{\langle M\rangle}\,(\text{LLg}\langle M\rangle)^{\delta},$$

where $H = \{H(t)\}$ *is a predictable process. Then*

$$\begin{aligned}&\{\langle M\rangle_\infty = \infty\}\Big\{\limsup_{t\to\infty} H(t) < \infty\Big\}\\ &\quad\subset \left\{\lim_{t\to\infty} \frac{M_t}{\sqrt{\langle M\rangle_t}\,(\text{LLg}\langle M\rangle)^{\delta+1}} = 0\right\} \quad a.s.\end{aligned} \tag{2.16}$$

PROOF. At first, suppose that there exists a constant $k > 0$ such that for $\delta > -1/2$,

$$(\Delta M)^*_t \le k\sqrt{q(t)}\,(\text{LLg}\,q(t))^{\delta} \qquad \forall\, t \text{ a.s.}, \tag{2.17}$$

$$\langle M\rangle_t \le q(t) \quad \forall\, t \text{ a.s.},$$

where $q = q(t)$ is a predictable increasing process. It will be proved that

$$\{q(\infty) = \infty\} \subset \left\{\lim_{t\to\infty} \frac{M_t}{\sqrt{q(t)}(\text{LLg}\,q(t))^{\delta+1}} = 0\right\} \quad \text{a.s.}$$

For $p > 1$ and $n \in \mathbb{N}$ set

$$T_n = \inf\{t > 0\colon\ q(t) \ge p^{2n}\}.$$

Then T_n is finite on $\{q(\infty) = \infty\}$ and is predictable. From the definition of T_n and (2.17) we have

$$\langle M\rangle_{T_n-} \le q(T_n-) \le p^{2n} \quad \text{a.s.},$$

$$(\Delta M)^*_{T_n-} \le k\sqrt{q(T_n-)}\,(\text{LLg}\,q(T_n-))^{\delta} \le kp^n(\text{LLg}\,p^{2n})^{\delta} \overset{\text{def}}{=} d_n.$$

Put $a_n = \varepsilon p^n(\text{LLg}\,p^{2n})^{\delta+1}$, where $\varepsilon \in (0,1)$ is a constant. By using Lemma 2.1 we have

$$\begin{aligned}
&\mathsf{P}(M^*_{T_n-} > \varepsilon p^n(\text{LLg}\,p^{2n})^{\delta+1})\\
&\quad\le \mathsf{P}((M^{T_n-})^*_\infty > a_n, \langle M^{T_n-}\rangle_\infty \le p^{2n})\\
&\quad\le 2\exp\left[-\frac{a_n^2}{2p^{2n}}\psi\left(\frac{a_n d_n}{p^{2n}}\right)\right]\\
&\quad\le 2\exp\left[-\frac{\varepsilon^2(\text{LLg}\,p^{2n})^{2\delta+2}}{2}\psi(\varepsilon k(\text{LLg}\,p^{2n})^{2\delta+1})\right]\\
&\quad\le 2\exp\left[-\frac{\varepsilon^2(\text{LLg}\,p^{2n})^{2\delta+2}}{2}\,\frac{2(1-\varepsilon)\log(\varepsilon k(\text{LLg}\,p^{2n})^{2\delta+1})}{\varepsilon k(\text{LLg}\,p^{2n})^{2\delta+1}}\right], \quad \text{by (2.3)},\\
&\quad\le \exp[-c'\,\text{LLg}\,p^{2n}\log(\text{LLg}\,p^{2n})] \quad \text{for } n \text{ large enough,}
\end{aligned}$$

where c' is a positive constant. Therefore, by using the Borel–Cantelli lemma it is easy to obtain

$$\mathsf{P}\big(M^*_{T_n-} > \varepsilon p^n (\mathrm{LLg}\, p^{2n})^{\delta+1} \text{ i.o.}\big) = 0$$

and

$$\{q(\infty) = \infty\} \subset \left\{\lim_{t\to\infty} \frac{M_t}{\sqrt{q(t)}\,(\mathrm{LLg}\, q(t))^{\delta}} = 0\right\}.$$

Now for a fixed constant $k > 0$ put

$$N = 1_{[H \le k]} \cdot M.$$

Then the rest of the argument is similar to the proof of Theorem 2.3 and we conclude

$$\begin{aligned}\{\langle M\rangle_\infty = \infty\}\left\{\lim_{t\to\infty} H(t) < \infty\right\} &= \bigcup_{k=1}^{\infty} \{\langle M\rangle_\infty = \infty\}\left\{\lim_{t\to\infty} H(t) \le k\right\} \\ &\subset \left\{\lim_{t\to\infty} \frac{M_t}{\sqrt{2\langle M\rangle_t}\,(\mathrm{LLg}\langle M\rangle)^{\delta}} = 0\right\}. \qquad \square\end{aligned}$$

THEOREM 2.6. *Suppose that* $M \in \mathscr{M}^2_{\mathrm{loc}}$ *and for some* $\delta > 0$,

$$|\Delta M| \le H\sqrt{\langle M\rangle}\,(\log\langle M\rangle)^{\delta},$$

where $H = \{H(t)\}$ *is a predictable process. Then*

$$\{\langle M\rangle_\infty = \infty\} \subset \left\{\limsup_{t\to\infty} \frac{M_t}{\sqrt{\langle M\rangle_t}\,(\log\langle M\rangle)^{\delta}} \le \frac{1}{2\delta} \limsup_{t\to\infty} H(t)\right\} \quad a.s.$$

PROOF. Suppose that there exists a constant $k > 0$ such that for $\delta > 0$,

$$(\Delta M)^*_t \le k\sqrt{q(t)}\,(\log q(t))^{\delta} \qquad \forall\, t \text{ a.s.,} \tag{2.18}$$

$$\langle M\rangle_t \le q(t) \qquad \forall\, t \text{ a.s.,}$$

where $q = q(t)$ is a predictable increasing process. It will be proved that

$$\{q(\infty) = \infty\} \subset \left\{\limsup_{t\to\infty} \frac{M_t}{\sqrt{\langle M\rangle_t}\,(\log\langle M\rangle)^{\delta}} \le \frac{k}{2\delta}\right\}.$$

For $p > 1$ and $n \in \mathbb{N}$ set

$$T_n = \inf\{t > 0\colon\ q(t) \ge p^{2n}\}.$$

Then T_n is finite on $\{q(\infty) = \infty\}$ and is predictable. From the definition of T_n and (2.18) we have

$$\langle M\rangle_{T_n-} \le q(T_n-) \le p^{2n} \quad \text{a.s.,}$$

$$(\Delta M)^*_{T_n-} \le k\sqrt{q(T_n-)}(\log q(T_n-))^{\delta} \le kp^n(\log p^{2n})^{\delta} \stackrel{\mathrm{def}}{=} d_n \quad \text{a.s.}$$

Put $a_n = ap^n(\log p^{2n})^\delta$, where

$$a > \frac{k}{2\delta(1-\varepsilon)} \tag{2.19}$$

is a constant and $0 < \varepsilon < 1$. By using Lemma 2.1, we have

$$\begin{aligned}
&\mathsf{P}\big(M^*_{T_n-} > ap^n(\log p^{2n})^\delta\big)\\
&\quad= \mathsf{P}((M^{T_n-})^*_\infty > a_n,\ \langle M^{T_n-}\rangle_\infty \le p^{2n})\\
&\quad\le 2\exp\left[-\frac{a_n^2}{2p^{2n}}\psi\left(\frac{a_n d_n}{p^{2n}}\right)\right]\\
&\quad\le 2\exp\left[-\frac{a^2(\log p^{2n})^{2\delta}}{2}\psi(ak(\log p^{2n})^{2\delta})\right]\\
&\quad\le c_1\exp\left[-\frac{a^2(\log p^{2n})^{2\delta}}{2}\,\frac{(1-\varepsilon)4\delta\,\mathrm{LLg}\,p^{2n}}{ak(\log p^{2n})^{2\delta}}\right], \quad \text{by (2.3)},\\
&\quad= c_2\exp\left[-\frac{2a\delta(1-\varepsilon)}{k}\,\mathrm{LLg}\,p^{2n}\right]\\
&\quad= \frac{c_2}{(2n\log p)^\alpha} \quad \text{for } n \text{ large enough,}
\end{aligned}$$

where c_1, c_2 are constants and $\alpha = 2a\delta(1-\varepsilon)/k > 1$, which implies

$$\sum_n \mathsf{P}(M^*_{T_n-} > ap^n(\log p^{2n})^\delta) < \infty.$$

Thus by the Borel–Cantelli lemma we get

$$\mathsf{P}(M^*_{T_n-} > ap^n(\log p^{2n})^\delta \text{ i.o.}) = 0$$

and

$$\{q(\infty)=\infty\} \subset \left\{\limsup_{t\to\infty}\frac{|M_t|}{\sqrt{q(t)}\,(\log q(t))^\delta} \le \frac{k}{2\delta}\right\}.$$

Now the rest of the argument is similar to the proof of Theorem 2.3 and we get the conclusion

$$\{\langle M\rangle_\infty=\infty\} \subset \left\{\lim_{t\to\infty}\frac{|M_t|}{\sqrt{2\langle M\rangle_t}\,(\log\langle M\rangle)^\delta} \le \frac{1}{2\delta}\limsup_{t\to\infty}H(t)\right\}. \qquad \square$$

REMARK. From this theorem it is easy to show that if

$$|\Delta M| \le H\sqrt{\langle M\rangle}\,(\log\langle M\rangle)^\delta$$

for some predictable process H and $\delta > 0$, then

$$\begin{aligned}
&\{\langle M\rangle_\infty=\infty\}\{H = O(1)\}\\
&\quad\subset \{\langle M\rangle_\infty=\infty\}\Big\{M = O\Big(\sqrt{\langle M\rangle}\,(\log\langle M\rangle)^\delta\Big)\Big\}\\
&\quad\subset \{\langle M\rangle_\infty=\infty\}\Big\{\Delta M = O\Big(\sqrt{\langle M\rangle}\,(\log\langle M\rangle)^\delta\Big)\Big\}.
\end{aligned}$$

Finally, we mention the discrete-time version of the above theorems. Let $\varepsilon = \{\varepsilon_n, \mathscr{G}_n,\ n \geq 1\}$ be a martingale difference sequence with $\mathsf{E}\varepsilon_n^2 < \infty$, that is,

$$\mathsf{E}_{n-1}(\varepsilon_n) \stackrel{\text{def}}{=} \mathsf{E}(\varepsilon_n \mid \mathscr{G}_{n-1}) = 0 \quad \text{a.s.}$$

Put

$$S_n = \sum_{j=1}^n \varepsilon_j, \qquad s_n^2 = \sum_{j=1}^n \mathsf{E}_{n-1}(\varepsilon_n^2)$$

and

$$M_t = S_{[t]}, \qquad \mathscr{F}_t = \mathscr{G}_{[t]}, \qquad t \geq 0.$$

Then $M = \{M_t, \mathscr{F}_t,\ t \geq 0\} \in \mathscr{M}^2_{\text{loc}}$ and

$$\langle M \rangle_t = s^2_{[t]}.$$

Therefore, from Theorems 2.3–2.6 we have the following statements: If

$$|\varepsilon_n| \leq J_n$$

for some predictable sequence $J = \{J_n\}$, then

$$\begin{aligned} &\{s_\infty^2 = \infty\}\left\{J \leq_{\text{ap}} K\sqrt{s^2/\operatorname{LLg} s^2}\right\} \\ &\quad \subset \left\{\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2s_n^2 \operatorname{LLg} s_n^2}} \leq a(K)\right\} \quad \text{a.s.,} \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\{s_\infty^2 = \infty\}\left\{J = o\left(\sqrt{s^2/\operatorname{LLg} s^2}\right)\right\} \\ &\quad \subset \left\{\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2s_n^2 \operatorname{LLg} s_n^2}} \leq 1\right\} \quad \text{a.s.,} \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\{s_\infty^2 = \infty\}\{J = O(\sqrt{s^2}(\operatorname{LLg} s^2)^\delta)\} \\ &\quad \subset \left\{\lim_{n\to\infty} \frac{S_n}{\sqrt{s_n^2}(\operatorname{LLg} s_n^2)^{\delta+1}} = 0\right\} \quad \text{a.s. for } \delta > -1/2, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\{s_\infty^2 = \infty\}\{J \leq_{\text{ap}} K\sqrt{s^2}(\log s^2)^\delta\} \\ &\quad \subset \left\{\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{s_n^2}(\log s_n^2)^\delta} \leq \frac{K}{2\delta}\right\} \quad \text{a.s. for } \delta > 0, \end{aligned} \tag{2.23}$$

where K is a finite random variable.

It should be noted that if $\{\varepsilon_n\}$ is a sequence of independent random variables, the right side of (2.21) is the upper bound of Kolmogorov's law of the iterated logarithm. Chow and Teicher (1978) and Teicher (1979) obtained results similar to (2.20) and (2.23). In Teicher (1979), the upper bound on the right side of (2.20) is

$$a_2(k) = \frac{1}{\sqrt{2}} \min_{b>0} \left[\frac{1}{b} + \frac{e^{kb} - 1 - kb}{k^2 b} \right].$$

By a direct calculation it may be proved that $a_2(k) = a(k)$. For the martingale difference sequence $\{\varepsilon_n\}$, Stout (1970) and Fisher (1986) first gave an upper bound similar to the right side of (2.20) with a different constant [for larger k, $a(k) = a_2(k)$ is less than that in Fisher (1986)]. Here we improve these results in two respects: (1) we get the continuous parameter martingale version and (2) we get the "local" version, which does not require $s_\infty^2 = \infty$ (or $|\varepsilon_n| \le K\sqrt{s_n^2 \operatorname{LLg} s_n^2}$) almost surely, and here K may be a random variable.

3. The asymptotic behavior of stochastic integrals. Let $M \in \mathscr{M}_{\text{loc}}^2$. Then M has the integral representation

$$M = M^c + x * (\mu^M - \nu^M),$$

where M^c is the continuous local martingale part of M with predictable quadratic variation $\langle M^c \rangle = \langle M^c, M^c \rangle$, μ^M is the jump measure of M, ν^M is the dual predictable projection of μ^M with

$$\int_{\mathbb{R}} x \nu^M(\{t\}, dx) = 0$$

and $(\langle M^c \rangle, \nu^M)$ is called the predictable characteristic of M. It is clear that ν^M has the canonical predictable decomposition [cf. He, Wang and Yan (1992), page 381]

$$\nu^M(\omega, dt, dx) = N_t(\omega, dx) d\langle M \rangle_t,$$

where $N_t(\omega, dx)$ is a transition σ-finite measure from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ to $(\mathbb{R}, \mathscr{B})$ with

$$\int_{\mathbb{R}} x^2 N_t(\omega, dx) = 1 \qquad \forall\, t \in \mathbb{R}_+.$$

For a predictable process B, if B^2 is locally integrable with respect to $\langle M \rangle$, then the stochastic integral $X = B \cdot M$ of B with respect to M is well defined and $X \in \mathscr{M}_{\text{loc}}^2$. Also, X has the integral representation

$$X = X^c + x * (\mu^X - \nu^X),$$

where X^c is the continuous local martingale part of X, μ^X is the jump measure of X and ν^X is the dual predictable projection of μ^X. Meanwhile,

$$\langle X\rangle = B^2\cdot\langle M\rangle,\qquad \langle X^c\rangle = B^2\cdot\langle M^c\rangle, \tag{3.1}$$

$$\begin{aligned}\iint_{[0,t]\times\mathbb{R}} f(s,x)\nu^X(ds,dx) &= \iint_{[0,t]\times\mathbb{R}} 1_{\{B_s\neq 0\}} f(s,B_s x)\nu^M(ds,dx)\\ &= \iint_{[0,t]\times\mathbb{R}} 1_{\{B_s\neq 0\}} f(s,B_s x)N_s(dx)d\langle M\rangle_s,\end{aligned} \tag{3.2}$$

where f is an arbitrary nonnegative measurable function.

DEFINITION. For a family of σ-finite measures $\{N_t,\ t\in I\}$ on $\mathbb{R}$, if there exist a constant k and a finite measure N such that

$$N_t(\{x\colon |x|\geq a\}) \leq kN(\{x\colon |x|\geq a\}) < \infty \qquad \forall\, a>1,\ t\in I,$$

then we say that there exists a majorant measure N for $\{N_t,\ t\in I\}$ and denote it by $(N_t)\prec N$.

The following lemma is evident [cf. Wang (1992)].

LEMMA 3.1. (i) *Suppose that for some $\delta>0$, $\{N_t\}$ satisfies*

$$\sup_t \int_{\mathbb{R}} |x|^{2+\delta} N_t(dx) = C(\omega) < \infty \quad a.s.$$

and

$$N(dx) = 1_{|x|\geq 1}\frac{C(\omega)}{x^{3+\delta}}\,dx.$$

Then

$$(N_t)\prec N \quad and \quad \int x^2 N(dx) < \infty \quad a.s. \tag{3.3}$$

(ii) *If $\{N_t\}\prec N$ and f is a nondecreasing nonnegative function with $f(1)=0$, then*

$$\int_{\mathbb{R}} f(|y|)N_t(dy) \leq k\int_{\mathbb{R}} f(|y|)N(dy) \qquad \forall\, t. \tag{3.4}$$

THEOREM 3.2. *Let $M\in\mathscr{M}^2_{\mathrm{loc}}$, $X=B\cdot M$ and*

$$D_1 = \Big\{\omega\colon \lim_{t\to\infty}\langle M\rangle_t = \lim_{t\to\infty}\langle X\rangle_t = \infty\Big\}, \tag{3.5}$$

$$E_1 = \Big\{\omega\colon \sup_t \int |x|^{2+\delta} N_t(dx) < \infty \quad for\ some\ \delta>0\Big\}. \tag{3.6}$$

Then

(3.7)
$$D_1E_1\left\{\omega\colon\ B^2 = o\left(\frac{\langle X\rangle}{\mathrm{LLg}\langle X\rangle\langle M\rangle^{2/(2+\delta)}\log\langle M\rangle}\right)\right\} \subset \left\{\limsup_{t\to\infty}\frac{|X_t|}{\sqrt{2\langle X\rangle_t\,\mathrm{LLg}\langle X\rangle_t}} \le 1\right\} \quad a.s.,$$

(3.8)
$$D_1E_1\left\{\omega\colon\ B^2 = O\left(\frac{\langle X\rangle}{\mathrm{LLg}\langle X\rangle\langle M\rangle^{2/(2+\delta)}\log\langle M\rangle}\right)\right\} \subset \left\{\limsup_{t\to\infty}\frac{|X_t|}{\sqrt{\langle X\rangle_t\,\mathrm{LLg}\langle X\rangle_t}} < \infty\right\} \quad a.s.,$$

(3.9)
$$D_1E_1\left\{\omega\colon\ B^2 = O\left(\frac{\langle X\rangle\,\mathrm{LLg}^{\gamma}\langle X\rangle}{\langle M\rangle^{2/(2+\delta)}\log\langle M\rangle}\right)\right\} \subset \left\{\lim_{t\to\infty}\frac{|X_t|}{\sqrt{\langle X\rangle_t\,\mathrm{LLg}^{2+\gamma}\langle X\rangle_t}} = 0\right\} \quad a.s.\ for\ \gamma > -1,$$

(3.10)
$$D_1E_1\left\{\omega\colon\ B^2 = O\left(\frac{\langle X\rangle\log^{\gamma}\langle X\rangle}{\langle M\rangle^{2/(2+\delta)}\log\langle M\rangle}\right)\right\} \subset \left\{\limsup_{t\to\infty}\frac{|X_t|}{\sqrt{\langle X\rangle_t\log^{\gamma}\langle X\rangle_t}} < \infty\right\} \quad a.s.\ for\ \gamma \in (0,1]$$

The proof of this theorem will proceed in several steps. We will use the truncation technique; that is, we use the following decomposition of X:

(3.11)
$$\begin{aligned} X &= X^c + x * (\mu^X - \nu^X) \\ &= (X^c + x1_{|x|\le d(t)} * (\mu^X - \nu^X)) + x1_{|x|>d(t)} * (\mu^X - \nu^X) \\ &\stackrel{\text{def}}{=} Y + Z, \end{aligned}$$

where $d = (d(t))_{t\ge 0}$ is a predictable process defined below. Then Y, $Z \in \mathscr{M}^2_{\mathrm{loc}}$ and

(3.12)
$$|\Delta Y_t| \le 2d(t).$$

PROPOSITION 3.3. *Let*

$$E_2 = \left\{\omega\colon\ \{N_t\} \prec N,\ \int x^2 N(dx) < \infty\right\}$$

and let $d = (d(t))_{t\ge 0}$ *be a predictable increasing process. Then*

(3.13)
$$E_2\{B = o(d)\}\{\langle X\rangle_\infty = \infty\} \subset \left\{\lim_{t\to\infty}\frac{\langle Y\rangle_t}{\langle X\rangle_t} = 1\right\} \quad a.s.$$

PROOF. By the definition of Y we have

$$\begin{aligned}\langle Y\rangle &= \langle X^c\rangle + (|x|^2 1_{|x|\le d(t)}) * \nu^X - \sum\big(\Delta[(x1_{|x|\le d(t)}) * \nu^X]\big)^2\\ &\le \langle X\rangle,\end{aligned}$$

where $\sum W$ denotes the summation process of a thin process W. Meanwhile,

$$\begin{aligned}0 &\le \langle X\rangle - \langle Y\rangle\\ &= (|x|^2 1_{[|x|>d(t)]}) * \nu^X + \sum(\Delta[(x1_{[|x|\le d(t)]}) * \nu^X])^2\\ &= (|x|^2 1_{[|x|>d(t)]}) * \nu^X + \sum(\Delta[(x1_{[|x|>d(t)]}) * \nu^X])^2 \quad (\text{by } \Delta[x * \nu^X] = 0)\\ &\le 2(|x|^2 1_{[|x|>d(t)]}) * \nu^X\\ &\le 2\langle X\rangle_T + 2\int_T^{\cdot} b^2(s)\int_{\mathbb{R}} |x|^2 I\left(|x| > \frac{d(s)}{B(s)}\right) N_s(dx)\, d\langle M\rangle_s \quad [\text{by } (3.2)]\\ &\le 2\langle X\rangle_T + 2k\int_0^{\cdot}\int_{\mathbb{R}} |x|^2 I\left(|x| > \frac{d(s)}{B(s)}\right) N(dx)\, d\langle X\rangle_s\\ &\qquad\qquad \text{on } E_2 \ [\text{by } (3.2) \text{ and } (3.4)],\end{aligned}$$

where T is a random variable satisfying

$$\frac{d(s)}{B(s)} > 1 \qquad \forall\, s \ge T,$$

and T is finite on $\{B = o(d)\}$. Thus

$$E_2\{B = o(d)\} \subset \left\{\lim_{s\to\infty}\int_{\mathbb{R}} |x|^2 I\left(|x| > \frac{d(s)}{B(s)}\right) N(dx) = 0\right\}$$

and

$$\begin{aligned}&E_2\{B = o(d)\}\{\langle X\rangle_\infty = \infty\}\\ &\quad\subset \left\{\lim_{t\to\infty}\frac{1}{\langle X\rangle_t}\int_0^t\int_{\mathbb{R}} |x|^2 I\left(|x| > \frac{d(s)}{B(s)}\right) N(dx)\, d\langle X\rangle_s = 0\right\}.\end{aligned}$$

Therefore (3.13) holds. □

PROPOSITION 3.4. *Suppose that* $\varphi = (\varphi(s))_{s\ge 0}$ *is a predictable increasing process and*

$$d^2(t) = B^2(t)\big((\langle M\rangle_t^{2/(2+\delta)} \log\langle M\rangle_t) \vee 1\big). \tag{3.14}$$

Then

$$D_1 E_1\{\langle X\rangle = o(\varphi^2)\} \subset \left\{\lim_{t\to\infty}\frac{Z_t}{\varphi(t)} = 0\right\}. \tag{3.15}$$

PROOF. By using Chebyshev's inequality and (3.2) we have

$$\begin{aligned}1_{|x|>d(s)} * \nu_\infty^X &\le \frac{|x|^{(2+\delta)}}{d^{2+\delta}(s)} * \nu_\infty^X \le \iint_{\mathbb{R}_+\times\mathbb{R}} |x|^{2+\delta}\frac{|B(s)|^{2+\delta}}{d^{2+\delta}(s)} N_s(dx)d\langle M\rangle_s \\ &\le \sup_s \int |x|^{2+\delta} N_s(dx) \int_0^\infty 1\wedge\left(\langle M\rangle_s \log^{1+\delta/2}\langle M\rangle_s\right)^{-1} d\langle M\rangle_s \\ &< \infty \quad \text{a.s. on } E_1.\end{aligned}$$

Note that $D_1\{\langle X\rangle = o(\varphi^2)\} \subset \{\varphi\uparrow\infty\}$ and the jump measure μ^X of X is an integer random measure, $\mu^X(A) = \sum_i I\{(T_i, \Delta X_{T_i}) \in A\}$. If $\sum_i I\{|\Delta X_{T_i}| > d(T_i)\} = 1_{|x|>d(s)} * \mu_\infty^X$ is finite, then $|x|1_{|x|>d(s)} * \mu_\infty^X = \sum_i |\Delta X_{T_i}| I\{|\Delta X_{T_i}| > d(T_i)\}$ is finite too. Therefore

$$\begin{aligned}D_1E_1\{\langle X\rangle = o(\varphi^2)\} &\subset \{1_{|x|>d(s)} * \nu_\infty^X < \infty\}\{\varphi\uparrow\infty\} \\ &\subset \{1_{|x|>d(s)} * \mu_\infty^X < \infty\}\{\varphi\uparrow\infty\} \qquad \text{(cf. [6], page 222)} \\ &\subset \{|x|1_{|x|>d(s)} * \mu_\infty^X < \infty\}\{\varphi\uparrow\infty\} \\ &\subset \left\{\lim_{t\to\infty} \frac{1}{\varphi(t)}(|x|1_{|x|>d(\cdot)} * \mu_t^X) = 0\right\} \quad \text{a.s.}\end{aligned} \tag{3.16}$$

Meanwhile, by the Schwarz inequality we get

$$\begin{aligned}&|x|1_{|x|>d(s)} * \nu_t^X \\ &\quad\le \sqrt{x^2 * \nu_t^X}\left(1_{|x|>d(s)} * \nu_t^X\right)^{1/2} \\ &\quad\le \sqrt{\langle X\rangle_t}\left(\sup_s \int |x|^{2+\delta} N_s(dx) \int_0^\infty 1\wedge\left(\langle M\rangle_s \log^{1+\delta/2}\langle M\rangle_s\right)^{-1} d\langle M\rangle_s\right)^{1/2}\end{aligned}$$

and

$$D_1E_1\{\langle X\rangle = o(\varphi^2)\} \subset \left\{\lim_{t\to\infty} \frac{1}{\varphi(t)}(|x|1_{|x|>d(\cdot)} * \nu_t^X) = 0\right\} \quad \text{a.s.}$$

This, (3.16) and (3.11) yield the conclusion (3.15). □

PROOF OF THEOREM 3.2. At first, note that by Lemma 3.1.1,

$$E_1 \subset E_2 = \left\{\omega\colon \{N_t\} \prec N, \int x^2\,dN < \infty\right\} \quad \text{for some } N.$$

Define $d(t)$ by (3.14). Then $D_1 \subset \{B = o(d)\}$ and from (3.13) we have

$$D_1E_1 \subset \{B = o(d)\}\{\langle X\rangle_\infty = \infty\}E_2 \subset \left\{\lim_{t\to\infty} \frac{\langle Y\rangle_t}{\langle X\rangle_t} = 1\right\}. \tag{3.17}$$

To prove (3.7), put

$$\varphi(t) = \sqrt{2\langle X\rangle_t \,\mathrm{LLg}\langle X\rangle_t}. \tag{3.18}$$

Thus Proposition 3.4 implies that

$$D_1E_1 \subset \left\{ \lim_{t\to\infty} \frac{Z_t}{\sqrt{2\langle X\rangle_t \,\mathrm{LLg}\langle X\rangle_t}} = 0 \right\} \quad \text{a.s.} \tag{3.19}$$

Meanwhile, according to (3.12)

$$|\Delta Y| \le 2d = 2|B|\big((\langle M\rangle^{1/(2+\delta)} \log^{1/2}\langle M\rangle) \vee 1\big).$$

Write

$$G_1 = \left\{\omega\colon\ B^2 = o\left(\frac{\langle X\rangle}{\mathrm{LLg}\langle X\rangle \langle M\rangle^{2/(2+\delta)} \log\langle M\rangle}\right)\right\}.$$

Then from (3.14) and (3.17) we have

$$D_1E_1G_1 \subset D_1E_1\left\{d = o\left(\sqrt{\frac{\langle X\rangle}{\mathrm{LLg}\langle X\rangle}}\right)\right\} \subset \left\{d = o\left(\sqrt{\frac{\langle Y\rangle}{\mathrm{LLg}\langle Y\rangle}}\right)\right\} \quad \text{a.s.}$$

Thus Corollary 2.4 implies

$$D_1E_1G_1 \subset \left\{ \limsup_{t\to\infty} \frac{Y_t}{\sqrt{2\langle Y\rangle_t \,\mathrm{LLg}\langle Y\rangle_t}} \le 1 \right\} \quad \text{a.s.}$$

This, (3.17) and (3.19) yield (3.7).

The proof of (3.8) is similar. Take φ the same as in (3.18) and

$$G_2 = \left\{\omega\colon\ B^2 = O\left(\frac{\langle X\rangle}{\mathrm{LLg}\langle X\rangle \langle M\rangle^{2/(2+\delta)} \log\langle M\rangle}\right)\right\}.$$

Since in this case

$$D_1E_1G_2 \subset \left\{d = O\left(\sqrt{\frac{\langle Y\rangle}{\mathrm{LLg}\langle Y\rangle}}\right)\right\},$$

then from Theorem 2.3 we get

$$D_1E_1G_2 \subset \left\{ \limsup_{t\to\infty} \frac{Y_t}{\sqrt{2\langle Y\rangle_t \,\mathrm{LLg}\langle Y\rangle_t}} < \infty \right\} \quad \text{a.s.}$$

This, (3.17) and (3.19) yield (3.8).

To prove (3.9), put

$$\varphi(t) = \sqrt{\langle X\rangle_t \,\mathrm{LLg}^{2+\gamma}\langle X\rangle_t}.$$

Then instead of Theorem 2.3, use Theorem 2.5 and the method above.

If we put

$$\varphi(t) = \sqrt{\langle X\rangle_t \log^{\gamma}\langle X\rangle_t},$$

then the proof of (3.10) can proceed also in a similar way. □

REMARK. If

$$\mathsf{P}\left(D_1 E_1\left\{B^2 = o\left(\frac{\langle X\rangle}{\mathrm{LLg}\langle X\rangle\langle M\rangle^{2/(2+\delta)}\log\langle M\rangle}\right)\right\}\right) = 1,$$

then

$$\limsup_{t\to\infty}\frac{|X_t|}{\sqrt{2\langle X\rangle_t\,\mathrm{LLg}\langle X\rangle_t}} = 1 \quad \text{a.s.} \tag{3.20}$$

In fact, in this case we may define $d(t)$ by (3.14) and

$$K(t) = \frac{2d(t)}{\sqrt{\langle Y\rangle_t/\,\mathrm{LLg}\langle Y\rangle_t}}.$$

Then $K = \{K(t)\}$ is a predictable process and

$$|\Delta Y| \le K\sqrt{\frac{\langle Y\rangle}{\mathrm{LLg}\langle Y\rangle}}, \qquad \lim_{t\to\infty} K(t) = 0 \quad \text{a.s.}$$

Hence from Xu's result [Xu (1990)] we get

$$\lim_{t\to\infty}\frac{Y_t}{\sqrt{2\langle Y\rangle_t\,\mathrm{LLg}\langle Y\rangle_t}} = 1 \quad \text{a.s.}$$

This, (3.17) and (3.19) yield (3.20).

THEOREM 3.5. *Let* $M \in \mathscr{M}^2_{\mathrm{loc}}$, $X = B\cdot M$ *and*

$$\begin{aligned} E_2 &= \left\{\omega\colon \{N_t\} \prec N,\ \int x^2 N(dx) < \infty\right\},\\ D_2 &= \left\{\omega\colon \lim_{t\to\infty}\langle M\rangle_t = \lim_{t\to\infty}\langle X\rangle_t = \infty,\ \Delta\langle M\rangle = o(\langle M\rangle)\right\}. \end{aligned} \tag{3.21}$$

Then:

(i) *For* $\gamma < 1$,

$$\begin{aligned} &E_2 D_2\left\{B^2 = O\left(\frac{\langle X\rangle\,\mathrm{LLg}^\gamma\langle X\rangle}{\langle M\rangle}\right)\right\}\\ &\qquad \subset \left\{\limsup_{t\to\infty}\frac{|X_t|}{\sqrt{2\langle X\rangle_t\,\mathrm{LLg}\langle X\rangle_t}} \le 1\right\} \quad \textit{a.s.}, \end{aligned} \tag{3.22}$$

(ii) *For* $\gamma \ge 1$ *and* $\beta > \gamma$,

$$\begin{aligned} &E_2 D_2\left\{B^2 = O\left(\frac{\langle X\rangle\,\mathrm{LLg}^\gamma\langle X\rangle}{\langle M\rangle}\right)\right\}\\ &\qquad \subset \left\{\lim_{t\to\infty}\frac{X_t}{\sqrt{\langle X\rangle_t\,\mathrm{LLg}^\beta\langle X\rangle_t}} = 0\right\} \quad \textit{a.s.} \end{aligned} \tag{3.23}$$

The proof of this theorem will proceed in several steps and we will still use the decomposition (3.11) of X. To begin with, we state the following technical proposition, the proof of which is similar to that of a lemma in the book by Chow and Teicher [(1978), Section 10.2, Lemma 3, page 350].

PROPOSITION 3.6. *Let*

$$G_3 = \left\{ \frac{B^2}{\langle X\rangle \log\langle X\rangle \,\mathrm{LLg}\langle X\rangle} = o\left(\frac{1}{\langle M\rangle}\right)\right\}.$$

Then for all $\alpha > 0$ *and* β

$$D_2G_3 \subset \left\{ \frac{\langle M\rangle_t^{\alpha}}{\mathrm{LLg}^{\beta}\langle X\rangle_t} \uparrow \infty \text{ as } t \to \infty \right\},$$

$$D_2G_3 \subset \left\{ \int^{\cdot} \frac{d\langle M\rangle_s}{\langle M\rangle_s^{1-\alpha}\,\mathrm{LLg}^{\beta}\langle X\rangle_s} = O\left(\frac{\langle M\rangle^{\alpha}}{\mathrm{LLg}^{\beta}\langle X\rangle}\right)\right\}.$$

PROPOSITION 3.7. *Let* $d = (d(t))_{t\ge 0}$ *and* $\varphi = (\varphi(t))_{t\ge 0}$ *be two predictable increasing processes with* $\varphi \ge 1$ *and for some* $\alpha \in [1,2]$ *and constant* $i > 0$ *put*

$$H_1(i) = \left\{\lim_{t\to\infty} \varphi(t) = \infty\right\} \cap \{d(t) \ge |B(t)| > 0\ \forall\, t > i\}$$
$$\cap \left\{ \limsup_{|x|\to\infty} \frac{1}{|x|^{2-\alpha}} \int_A \left(\frac{B(s)}{\varphi(s)}\right)^{\alpha} d\langle M\rangle_s < \infty \right\},$$

where $A = \{s \ge i\colon\ (d(s))/|B(s)| < |x|\big\}$. *Then*

$$E_2H_1(i) \subset \left\{ \lim_{t\to\infty} \frac{Z_t}{\varphi(t)} = 0 \right\} \quad a.s. \tag{3.24}$$

PROOF. Note that

$$\left\{ \limsup_{|x|\to\infty} \frac{1}{|x|^{2-\alpha}} \int_A \left(\frac{B(s)}{\varphi(s)}\right)^{\alpha} d\langle M\rangle_s < \infty \right\}$$
$$= \left\{ \sup_{|x|\ge 1} \frac{1}{|x|^{2-\alpha}} \int_A \left(\frac{B(s)}{\varphi(s)}\right)^{\alpha} d\langle M\rangle_s < \infty \right\}.$$

From (3.2) we have

$$\begin{aligned}
&\frac{|x|^{\alpha}}{\varphi^{\alpha}(s)} 1_{[|x|>d(s),s>i]} * \nu_{\infty}^{X} \\
&\quad = \int_i^{\infty} \frac{|B(s)|^{\alpha}}{\varphi^{\alpha}(s)} \int I\left(|x| > \frac{d(s)}{|B(s)|}\right) |x|^{\alpha} N_s(dx)\, d\langle M\rangle_s \\
&\quad \le k \int_{|x|\ge 1} |x|^{\alpha} \left(\int_A \frac{|B(s)|^{\alpha}}{\varphi^{\alpha}(s)} d\langle M\rangle_s \right) N(dx) \quad \text{on } E_2 \text{ [by (3.4)]} \\
&\quad \le kC \int_{\mathbb{R}} x^2 N(dx) < \infty \quad \text{a.s. on } E_2H_1(i),
\end{aligned} \tag{3.25}$$

where C is a finite random variable. Write

$$\begin{aligned} Z &= x1_{|x|>d(t)} * (\mu^X - \nu^X) \\ &= x1_{d(t)<|x|\le\varphi(t)} * (\mu^X - \nu^X) + x1_{|x|>d(t)\vee\varphi(t)} * (\mu^X - \nu^X) \\ &\overset{\text{def}}{=} V + W. \end{aligned} \tag{3.26}$$

Then

$$\begin{aligned} \left\langle \frac{1}{\varphi} \cdot V \right\rangle_\infty &\le \left(\frac{|x|^2}{\varphi^2(s)} 1_{d(s)<|x|\le\varphi(s)} \right) * \nu_\infty^X \\ &\le \left(\frac{|x|^2}{\varphi^2(s)} 1_{d(s)<|x|\le\varphi(s)} \right) * \nu_i^X + \left(\frac{|x|^\alpha}{\varphi^\alpha(s)} 1_{[|x|>d(s),\ s>i]} \right) * \nu_\infty^X < \infty \\ &\qquad \text{a.s. on } E_2H_1(i) \text{ [by (3.25)]}. \end{aligned}$$

Since $(1/\varphi) \cdot V \in \mathscr{M}_{\text{loc}}^2$, then

$$\begin{aligned} E_2H_1(i) &\subset \left\{ \left\langle \frac{1}{\varphi} \cdot V \right\rangle_\infty < \infty \right\} H_1(i) \\ &\subset \left\{ \lim_{t\to\infty} \frac{1}{\varphi} \cdot V_t \text{ exists and is finite} \right\} H_1(i) \\ &\qquad\qquad \text{(by Theorem 8.32 in [6])} \\ &\subset \left\{ \lim_{t\to\infty} \frac{V_t}{\varphi(t)} = 0 \right\} \quad \text{a.s. (by the Kronecker lemma)} \end{aligned} \tag{3.27}$$

Meanwhile, since $\alpha \ge 1$,

$$\left(\frac{|x|}{\varphi(t)} 1_{[|x|>d(t)\vee\varphi(t)]} \right) * \nu_\infty^X \le \left(\frac{|x|^\alpha}{\varphi^\alpha(t)} 1_{[|x|>d(t)\vee\varphi(t)]} \right) * \nu_\infty^X,$$

then, (3.25) implies

$$\begin{aligned} E_2H_1(i) &\subset \left\{ \left(\frac{|x|}{\varphi(t)} 1_{[|x|>d(t)\vee\varphi(t)]} \right) * \nu_\infty^X < \infty \right\} \\ &\subset \left\{ \left(\frac{|x|}{\varphi(t)} 1_{[|x|>d(t)\vee\varphi(t)]} \right) * \mu_\infty^X < \infty \right\} \quad \text{a.s. (cf. [6], page 222).} \end{aligned} \tag{3.28}$$

Also by the Kronecker lemma, from (3.28) we have

$$E_2H_1(i) \subset \left\{ \lim_{t\to\infty} \frac{1}{\varphi(t)} [(x1_{|x|>d(\cdot)\vee\varphi(\cdot)}) * \nu_t^X] = 0 \right\} \quad \text{a.s.,}$$

$$E_2H_1(i) \subset \left\{ \lim_{t\to\infty} \frac{1}{\varphi(t)} [(x1_{[|x|>d(\cdot)\vee\varphi(\cdot)}) * \mu_t^X] = 0 \right\} \quad \text{a.s.}$$

and

$$\lim_{t\to\infty}\frac{W_t}{\varphi(t)} = \lim_{t\to\infty}\frac{1}{\varphi(t)}[(x1_{[|x|>d(\cdot)\vee\varphi(\cdot)]})*\mu_t]$$

(3.29)
$$-\lim_{t\to\infty}\frac{1}{\varphi(t)}[(x1_{[|x|>d(\cdot)\vee\varphi(\cdot)]})*\nu_t]$$

$$=0 \quad \text{a.s. on } E_2H_1(i).$$

Now the conclusion (3.24) comes from (3.26), (3.27) and (3.29). □

PROOF OF THEOREM 3.5. Write

$$G_4=\left\{B^2=O\left(\frac{\langle X\rangle\,\mathrm{LLg}^{\gamma}\langle X\rangle}{\langle M\rangle}\right)\right\}.$$

Then

(3.30)
$$D_2G_4\subset G_3$$

and Proposition 3.6 is applicable on D_2G_4.

(i) With no loss of generality we can assume

$$0\le\gamma<1.$$

Put $\delta=(1-\gamma)/2$ and

$$\varphi^2(t)=(2\langle X\rangle_t\,\mathrm{LLg}\langle X\rangle_t)\vee 1,$$

(3.31)
$$d^2(t)=\frac{\langle X\rangle_t}{\mathrm{LLg}^{1+\delta}\langle X\rangle_t},$$

$$H_4(c,i)=\left\{\omega:\ \frac{d^2(s)}{B^2(s)}\ge\left(\frac{1}{c}\frac{\langle M\rangle_s}{\mathrm{LLg}^{1+\gamma+\delta}\langle X\rangle_s}\right)\vee 1,\ \forall\, s\ge i\right\},$$

where c, $i>0$ are constants. Then

$$D_2\subset\left\{\lim_{t\to\infty}\varphi^2(t)=\infty\right\}.$$

From (3.31), Proposition 3.6 we have $B=o(d)$ a.s. on D_2G_4 and

$$D_2G_4\subset\bigcup_{c,i=1}^{\infty}H_4(c,i).$$

Note that

$$A\overset{\text{def}}{=}\left\{s\ge i:\ \frac{d^2(s)}{B^2(s)}<x^2\right\}$$

$$\subset\left\{s\ge i:\ \frac{\langle M\rangle_s}{\mathrm{LLg}^{1+\gamma+\delta}\langle X\rangle_s}<cx^2\right\}\overset{\text{def}}{=}A_1 \quad \text{a.s. on } D_2H_4(c,i).$$

Set

$$t_1 = t_1(x) = \sup\{s \geq i:\ \langle M\rangle_s \operatorname{LLg}^{-(1+\gamma+\delta)}\langle X\rangle_s < cx^2\}.$$

Then

$$\langle M\rangle_{t_1-} \operatorname{LLg}^{-(1+\gamma+\delta)}\langle X\rangle_{t_1-} \leq cx^2,$$

$$D_2G_4 \subset \left\{\lim_{|x|\to\infty} t_1(x) = +\infty\right\}.$$

Now take $\alpha = 1+\gamma+\delta$. Then $\alpha \in [1,2)$ and

$$\int_A \left(\frac{|B(s)|}{\varphi(s)}\right)^\alpha d\langle M\rangle_s \leq C\int_{A_1} \langle M\rangle_s^{-\alpha/2} \operatorname{LLg}^{-\alpha(1-\gamma)/2}\langle X\rangle_s d\langle M\rangle_s$$

$$\begin{aligned}
(3.32)\qquad &\leq C\langle M\rangle_{t_1-}^{1-\alpha/2} \operatorname{LLg}^{-\alpha(1-\gamma-\delta)/2}\langle X\rangle_{t_1-} \quad \forall\, t_1 > T \text{ (by Proposition 3.6)}\\
&= C\big(\langle M\rangle_{t_1-} \operatorname{LLg}^{-(1+\gamma+\delta)}\langle X\rangle_{t_1-}\big)^{1-\alpha/2} \text{ (from } \alpha = 1+\gamma+\delta)\\
&\leq C|x|^{2-\alpha} \qquad \forall\, |x| > U \text{ a.s. on } D_2H_4(c,i),
\end{aligned}$$

where C, T and U are random variables, but C may vary in different expressions. By (3.32) and (3.24) we get

$$E_2D_2G_4 \subset \bigcup_{c,i} D_2E_2H_4(c,i) \subset \left\{\lim_{t\to\infty} \frac{Z_t}{\varphi(t)} = 0\right\}. \tag{3.33}$$

On the other hand, from (3.12), (3.31) and Proposition 3.3 we have

$$|\Delta Y| \leq 2d \quad \text{a.s.,}$$

$$\begin{aligned}
E_2D_2G_4 &\subset E_2\{B = o(d)\}\{\langle X\rangle_\infty = \infty\}\\
&\subset \left\{\lim_{t\to\infty} \frac{\langle Y\rangle_t}{\langle X\rangle_t} = 1\right\};
\end{aligned}$$

$$E_2D_2G_4 \subset \left\{d = K\sqrt{\frac{\langle Y\rangle}{\operatorname{LLg}\langle Y\rangle}}\right\}, \tag{3.34}$$

where $K = \{K(t)\}$ is a predictable process defined by

$$K(t) = \sqrt{\frac{\langle X\rangle_t \operatorname{LLg}\langle Y\rangle_t}{\langle Y\rangle_t \operatorname{LLg}^{1+\delta}\langle X\rangle_t}}$$

and

$$\lim_{t\to\infty} K(t) = 0 \quad \text{a.s. on } E_2D_2G_4.$$

Therefore, by Corollary 2.4 we have

$$E_2D_2G_4 \subset \left\{\limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{2\langle X\rangle_t \operatorname{LLg}\langle X\rangle_t}} \leq 1\right\} \quad \text{a.s.} \tag{3.35}$$

Now the conclusion (3.22) comes from (3.11), (3.33) and (3.35).

(ii) Put

$$\varphi^2(s) = (\langle X\rangle_s \operatorname{LLg}^{\beta}\langle X\rangle_s) \vee 1,$$
$$d^2(s) = \langle X\rangle_s \operatorname{LLg}^{\beta-2}\langle X\rangle_s,$$
$$\alpha = 2-\beta+\gamma,$$
$$H_5(c,i) = \left\{ \frac{d^2(s)}{B^2(s)} \geq \left(\frac{1}{c}\frac{\langle M\rangle_s}{\operatorname{LLg}^{2-\beta+\gamma}\langle X\rangle_s}\right) \vee 1,\ \forall\, s \geq i \right\},$$

where c, $i > 0$ are constants. With no loss of generality we can assume

$$0 < \beta - \gamma \leq 1.$$

Then

$$1 \leq \alpha < 2.$$

Meanwhile, Proposition 3.6 contains $B = o(d)$ a.s. on D_2G_4 and

$$D_2G_4 \subset \bigcup_{c,i=1}^{\infty} H_5(c,i),$$

$$\begin{aligned} A &\stackrel{\text{def}}{=} \left\{ s \geq i\colon\ \frac{d^2(s)}{B^2(s)} < x^2 \right\} \\ &\subset \left\{ s \geq i\colon\ \frac{\langle M\rangle_s}{\operatorname{LLg}^{\alpha}\langle X\rangle_s} < cx^2 \right\} \stackrel{\text{def}}{=} A_1 \quad \text{a.s. on } D_2H_5(c,i). \end{aligned}$$

Set

$$t_1 = \sup\{s \geq i\colon\ \langle M\rangle_s \operatorname{LLg}^{-\alpha}\langle X\rangle_s < cx^2\}.$$

Then

$$\langle M\rangle_{t_1-} \operatorname{LLg}^{-\alpha}\langle X\rangle_{t_1-} \leq cx^2$$

and

$$\begin{aligned} \int_A \left(\frac{B(s)}{\varphi(s)}\right)^{\alpha} d\langle M\rangle_s &\leq C\int_{A_1} \langle M\rangle_s^{-\alpha/2} \operatorname{LLg}^{\alpha(\gamma-\beta)/2}\langle X\rangle_s d\langle M\rangle_s \\ &\leq C\langle M\rangle_{t_1-}^{1-\alpha/2} \operatorname{LLg}^{-\alpha(2-\alpha)/2}\langle X\rangle_{t_1-} \qquad \forall\, t_1 > T \\ &\qquad\qquad (\text{by Proposition 3.6 and } \alpha = 2-\beta+\gamma) \\ &= C\big(\langle M\rangle_{t_1-} \operatorname{LLg}^{-\alpha}\langle X\rangle_{t_1-}\big)^{1-\alpha/2} \\ &\leq C|x|^{2-\alpha} \qquad \forall\, |x| > U \text{ a.s. on } D_2H_5(c,i), \end{aligned} \tag{3.36}$$

where C, T and U are random variables, but C may vary in different expressions. By (3.36) and (3.24) we get

$$E_2D_2G_4 \subset \bigcup_{c,i} D_2E_2H_5(c,i) \subset \left\{ \lim_{t\to\infty} \frac{Z_t}{\varphi(t)} = 0 \right\} \quad \text{a.s.} \tag{3.37}$$

On the other hand, note that by virtue of Proposition 3.3 and (3.12),

$$|\Delta Y| \le 2d \quad \text{a.s.},$$

$$E_2 D_2 G_4 \subset E_2\{B = o(d)\}\{\langle X\rangle_\infty = \infty\} \subset \left\{ d \le_{\text{ap}} \sqrt{\langle Y\rangle \, \text{LLg}^{\beta-2}\langle Y\rangle} \right\} \quad \text{a.s.};$$

hence, Theorem 2.5 implies

$$E_2 D_2 G_4 \subset \left\{ \limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{\langle X\rangle_t \, \text{LLg}^{\beta}\langle X\rangle_t}} = 0 \right\} \quad \text{a.s.}$$

This, (3.11) and (3.37) yield (3.23). □

THEOREM 3.8. *Let $M \in \mathscr{M}^2_{\text{loc}}$, $X = B\cdot M$ and D_1 and E_2 be defined by* (3.5) *and* (3.21), *respectively. Then*

$$\text{(3.38)} \qquad \begin{aligned} &D_1 E_2 \left\{ B^2 = O\left(\frac{\langle X\rangle \log^\gamma \langle X\rangle}{\langle M\rangle} \right) \right\} \\ &\quad \subset \left\{ \lim_{t\to\infty} \frac{X_t}{\sqrt{\langle X\rangle_t \log^\gamma \langle X\rangle_t}} = 0 \right\} \quad \textit{a.s. for } \gamma \in (0,1], \end{aligned}$$

$$\text{(3.39)} \qquad \begin{aligned} &\{\langle X\rangle_\infty = \infty\} E_2 \left\{ B^2 = O\left(\frac{\langle X\rangle \log\langle X\rangle}{\text{LLg}\langle X\rangle} \right) \right\} \\ &\quad \subset \left\{ \lim_{t\to\infty} \frac{X_t}{\sqrt{\langle X\rangle_t \log \langle X\rangle_t}} = 0 \right\} \quad \textit{a.s.} \end{aligned}$$

PROOF. The proof of this theorem is similar to that of Theorem 3.5 and we shall adhere to the symbols in the proof of Theorem 3.5.

To prove (3.38), put

$$\text{(3.40)} \qquad G_6 = \left\{ B^2 = O\left(\frac{\langle X\rangle \log^\gamma \langle X\rangle}{\langle M\rangle} \right) \right\},$$

$$\varphi^2(t) = (\langle X\rangle_t \log^\gamma \langle X\rangle_t) \vee 1,$$

$$d^2(t) = k^2 \langle X\rangle_t \log^\gamma \langle X\rangle_t,$$

$$\text{(3.41)} \qquad H_6(c,i) = \left\{ \frac{d^2(s)}{B^2(s)} \ge \left(\frac{1}{c} \langle M\rangle_s \right) \vee 1, \ \forall\, s \ge i \right\},$$

where $c, i, k > 0$ are constants. Then $B = o(d)$ a.s. on D_1G_6 and

$$D_1G_6 \subset \bigcup_{c,i=1}^{\infty} H_6(c,i),$$

$$A \overset{\text{def}}{=} \left\{s \geq i\colon \frac{d^2(s)}{B^2(s)} < x^2\right\}$$
$$\subset \{s \geq i\colon \langle M\rangle_s < cx^2\} \overset{\text{def}}{=} A_1 \quad \text{a.s. on } D_1H_6(c,i).$$

Set

$$t_1 = \sup\{s\colon \langle M\rangle_s < cx^2\}.$$

Then

$$\langle M\rangle_{t_1-} \leq cx^2$$

and

$$\text{(3.42)} \qquad \int_A \frac{|B(s)|}{\varphi(s)} d\langle M\rangle_s \leq c' \int_{A_1} \langle M\rangle_s^{-1/2} d\langle M\rangle_s$$
$$\leq c'\langle M\rangle_{t_1-}^{1/2} \leq c''|x| \quad \text{a.s. on } H_6(c,i),$$

where c', c'' are constants depending on c and k. By (3.42) and (3.24) we get

$$\text{(3.43)} \qquad D_1E_2G_6 \subset \bigcup_{c,i} D_1E_2H_6(c,i) \subset \left\{\lim_{t\to\infty} \frac{Z_t}{\varphi(t)} = 0\right\} \quad \text{a.s.}$$

On the other hand, note that by virtue of Proposition 3.3 and (3.12),

$$|\Delta Y| \leq 2d \quad \text{a.s.}$$

$$D_1E_2G_6 \subset E_2\{B = o(d)\}\{\langle X\rangle_\infty = \infty\} \subset \left\{d \leq_{\text{ap}} k\sqrt{\langle Y\rangle \log^\gamma\langle Y\rangle}\right\} \quad \text{a.s.;}$$

hence Theorem 2.6 implies

$$D_1E_2G_6 \subset \left\{\limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{2\langle X\rangle_t \log^\gamma\langle X\rangle_t}} \leq \frac{k}{\gamma}\right\}.$$

Since k may be an arbitrary positive number, letting $k \downarrow 0$ yields

$$D_1E_2G_6 \subset \left\{\limsup_{t\to\infty} \frac{Y_t}{\sqrt{2\langle X\rangle_t \log^\gamma\langle X\rangle_t}} = 0\right\}.$$

This, (3.11) and (3.43) yield (3.38).

To prove (3.39), put

$$G_8 = \left\{B^2 = O\left(\frac{\langle X\rangle \log\langle X\rangle}{\mathrm{LLg}\langle X\rangle}\right)\right\},$$

$$\varphi^2(t) = (\langle X\rangle_t \log\langle X\rangle_t) \vee 1,$$

$$d^2(t) = k^2\langle X\rangle_t \log\langle X\rangle_t,$$

$$H_8(c,i) = \left\{\frac{d^2(s)}{B^2(s)} \geq \left(\frac{1}{c}\,\mathrm{LLg}\langle X\rangle_s\right) \vee 1 \ \forall\, s \geq i\right\},$$

where $c, i, k > 0$ are positive constants. Then

$$\{\langle X\rangle_\infty = \infty\}G_8 \subset \bigcup_{c,i=1}^{\infty} \{\langle X\rangle_\infty = \infty\}H_8(c,i),$$

$$A \overset{\text{def}}{=} \left\{s \geq i\colon\ \frac{d^2(s)}{B^2(s)} < x^2\right\}$$
$$\subset \{s \geq i\colon\ \mathrm{LLg}\langle X\rangle_s < cx^2\} \overset{\text{def}}{=} A_1 \quad \text{a.s. on } \{\langle X\rangle_\infty = \infty\}H_8(c,i).$$

Set

$$t_1 = \sup\{s\colon\ \mathrm{LLg}\langle X\rangle_s < cx^2\}.$$

Then

$$\mathrm{LLg}\langle X\rangle_{t_1-} \leq cx^2$$

and

$$\text{(3.44)} \quad \int_A \frac{B^2(s)}{\varphi^2(s)}\,d\langle M\rangle_s \leq c'\int_{A_1} \frac{d\langle X\rangle_s}{\langle X\rangle_s \log\langle X\rangle_s}$$
$$\leq c'\,\mathrm{LLg}\langle X\rangle_{t_1-} \leq c''x^2 \quad \text{a.s. on } \{\langle X\rangle_\infty = \infty\}H_8(c,i),$$

where c', c'' are constants depending on c and k. By (3.44) and (3.24) we get

$$\text{(3.45)} \quad \begin{aligned}\{\langle X\rangle_\infty = \infty\}E_2G_8 &\subset \bigcup_{c,i} E_2\{\langle X\rangle_\infty = \infty\}H_8(c,i)\\ &\subset \left\{\lim_{t\to\infty} \frac{Z_t}{\varphi(t)} = 0\right\} \quad \text{a.s.}\end{aligned}$$

On the other hand, note that by virtue of Proposition 3.3 and (3.12),

$$|\Delta Y| \leq 2d \quad \text{a.s.},$$

$$\{\langle X\rangle_\infty = \infty\}E_2G_8 \subset E_2\{B = o(d)\}\{\langle X\rangle_\infty = \infty\}$$
$$\subset \left\{d \leq_{\mathrm{ap}} k\sqrt{\langle Y\rangle \log\langle Y\rangle}\right\} \quad \text{a.s.;}$$

hence, Theorem 2.6 implies

$$\{\langle X\rangle_\infty = \infty\}E_2G_8 \subset \left\{\limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{\langle X\rangle_t \log\langle X\rangle_t}} \le k\right\} \quad \text{a.s.}$$

Since k may be an arbitrary positive number, letting $k \downarrow 0$ yields

$$E_2\{\langle X\rangle_\infty = \infty\}G_8 \subset \left\{\lim_{t\to\infty} \frac{Y_t}{\sqrt{\langle X\rangle_t \log\langle X\rangle_t}} = 0\right\} \quad \text{a.s.}$$

This, (3.11) and (3.45) yield (3.39). □

REMARKS. (1) If

$$\mathsf{P}\biggl(E_2D_2\biggl\{B^2 = O\biggl(\frac{\langle X\rangle \operatorname{LLg}^\gamma\langle X\rangle}{\langle M\rangle}\biggr)\biggr\}\biggr) = 1, \tag{3.46}$$

then

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2\langle X\rangle_t \operatorname{LLg}\langle X\rangle_t}} = 1 \quad \text{a.s.} \tag{3.47}$$

In fact, (3.46) and (3.34) imply

$$|\Delta Y| \le 2K\sqrt{\frac{\langle Y\rangle}{\operatorname{LLg}\langle Y\rangle}} \quad \text{a.s.,}$$

$$\lim_{t\to\infty} K(t) = 0 \quad \text{a.s.}$$

Hence Xu's result [Xu (1990)] yields (3.47).

(2) If $B = 1$, then $\langle X\rangle = \langle M\rangle$ and (3.22) becomes

$$E_2\{\langle M\rangle_\infty = \infty, \Delta M = o(M)\} \subset \left\{\limsup_{t\to\infty} \frac{M_t}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} \le 1\right\} \quad \text{a.s.}$$

In particular, if

$$\{N_t\} \prec N, \qquad \int x^2\, dN < \infty \quad \text{a.s.,}$$

$$\langle M\rangle_\infty = \infty, \qquad \Delta M = o(M) \quad \text{a.s.,}$$

then

$$\limsup_{t\to\infty} \frac{M_t}{\sqrt{2\langle M\rangle_t \operatorname{LLg}\langle M\rangle_t}} = 1 \quad \text{a.s.}$$

From the discrete version of this result it is easy to get the Hartman–Wintner law of the iterated logarithm for i.i.d. sequence.

Let $M = \{M_t, t \geq 0\}$ be a process with homogeneous independent increments and

$$\mathsf{E}[M_t] = 0,$$
$$\langle M \rangle_t = \mathsf{E}[M_t^2] = t.$$

If we take $X = B \cdot M$ with

$$B^2(t) = 1_{t \geq 1} \frac{d}{dt}[\exp(\log t \, \mathrm{LLg}^\gamma t)], \qquad \gamma < 1, \tag{3.48}$$

then

$$\langle X \rangle_t \sim \exp(\log t \, \mathrm{LLg}^\gamma t) \quad \text{and} \quad B^2 = O\left(\frac{\langle X \rangle \, \mathrm{LLg}^\gamma \langle X \rangle}{\langle M \rangle} \right), \qquad \gamma < 1,$$

hence, (3.47) holds for $X = B \cdot M$.

Instead of (3.48), if we take B as

$$B^2(t) = 1_{t \geq 1} \frac{d}{dt}[\exp(\log t \, \mathrm{LLg}^\gamma t)], \qquad B^2 = O\left(\frac{\langle X \rangle \, \mathrm{LLg}^\gamma \langle X \rangle}{\langle M \rangle} \right), \qquad \gamma \geq 1,$$

$$B^2(t) = 1_{t \geq 1} \frac{d}{dt}[\exp((\log t)^{1/(1-\gamma)})], \qquad B^2 = O\left(\frac{\langle X \rangle \log^\gamma \langle X \rangle}{\langle M \rangle} \right), \qquad \gamma \in (0, 1),$$

$$B^2(t) = \exp(e^{\sqrt{t}}), \qquad B^2 = O\left(\frac{\langle X \rangle \log \langle X \rangle}{\mathrm{LLg} \langle X \rangle} \right),$$

then (3.23), (3.38) or (3.39) is applicable to get the asymptotic behavior of $X = B \cdot M$, respectively.

Furthermore, suppose

$$\mathsf{E}[M_t^{2+\delta}] < \infty \quad \text{for some } \delta > 0$$

and

$$B^2(t) = \frac{d}{dt}\left[\exp\left(\frac{t^{\delta/(2+\delta)} \, \mathrm{LLg}^\gamma t}{\log t} \right) \right], \quad B^2 = O\left(\frac{\langle X \rangle \, \mathrm{LLg}^\gamma \langle X \rangle}{\langle M \rangle^{2/(2+\delta)} \log \langle M \rangle} \right), \quad \gamma > -1,$$

$$B^2(t) = \frac{d}{dt}\left[\exp(t^{\delta/(2+\delta)} \log^{\gamma-1} t) \right], \quad B^2 = O\left(\frac{\langle X \rangle \log^\gamma \langle X \rangle}{\langle M \rangle^{2/(2+\delta)} \log \langle M \rangle} \right), \quad \gamma \in (0, 1).$$

Statement (3.9) or (3.10) is suitable to get the asymptotic behavior of $X = B \cdot M$ for these B.

Now we mention the relationship between the discrete-time version of the above results and some earlier works. Let $\{\varepsilon_n, \mathscr{G}_n\}$ be a martingale difference sequence with

$$\mathsf{E}_{n-1} \varepsilon_n^2 = 1 \quad \text{a.s.} \tag{3.49}$$

and let $b = \{b_n\}$ be a predictable sequence, that is, $b_n \in \mathscr{G}_{n-1}$. Put

$$(3.50)\qquad \begin{aligned} S_n &= \sum_{j=1}^{n} b_j \varepsilon_j, \qquad s_n^2 = \sum_{j=1}^{n} b_j^2, \\ M_t &= \sum_{j=1}^{[t]} \varepsilon_j, \qquad B(t) = b_{[t]}, \qquad \mathscr{F}_t = \mathscr{G}_{[t]}. \end{aligned}$$

Then $M = \{M_t, \mathscr{F}_t,\ t \geq 0\} \in \mathscr{M}^2_{\mathrm{loc}}$,

$$X_t = B \cdot M_t = \sum_{j=1}^{[t]} b_j \varepsilon_j = S_{[t]}$$

and the sequence of conditional distributions of ε_n with respect to $\mathscr{G}_{n-1}\{N_t\}$ is

$$N_t(A) = \sum_{n=1}^{\infty} 1_{[t=n]} \mathsf{P}(\varepsilon_n \in A \mid \mathscr{G}_{n-1}).$$

Therefore, the discrete-time version of (3.7) is as follows:

THEOREM 3.2$_{\mathrm{d}}$. *Let $\{\varepsilon_n\}$ be a martingale difference sequence with* (3.49), *$\{b_n\}$ be a predictable sequence, S_n, s_n^2 be defined by* (3.50) *and*

$$D_1' = \Big\{\omega\colon \lim_{n\to\infty} s_n^2 = \infty\Big\}, \qquad E_1' = \Big\{\omega\colon \sup_n \mathsf{E}_{n-1}[\varepsilon_n^{2+\delta}] < \infty\Big\}.$$

Then

$$D_1' E_1' \left\{ \frac{b_n^2 \operatorname{LLg} s_n^2}{s_n^2} = o\big(n^{-2/(2+\delta)} (\log n)^{-1}\big) \right\} \subset \left\{ \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2 s_n^2 \operatorname{LLg} s_n^2}} \leq 1 \right\} \quad a.s.$$

In particular,

$$\mathsf{P}\left(D_1' E_1' \left\{ \frac{b_n^2 \operatorname{LLg} s_n^2}{s_n^2} = o\big(n^{-2/(2+\delta)} (\log n)^{-1}\big) \right\} \right) = 1.$$

Then

$$(3.51)\qquad \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2 s_n^2 \operatorname{LLg} s_n^2}} = 1 \quad a.s.$$

Equation (3.51) is just the conclusion of Theorem 2 in Zhang (1992). The discrete versions of the other conclusions (3.8)–(3.10) are similar.

THEOREM 3.5$_{\mathrm{d}}$. *Let $\{\varepsilon_n\}$ be a martingale difference sequence with* (3.49), *let $\{b_n\}$ be a predictable sequence, S_n, s_n^2 be defined by* (3.50) *and*

$$E_2' = \left\{ \omega\colon \{\mathsf{P}(\varepsilon_n \in \cdot \mid \mathscr{G}_{n-1})\} \prec N,\ \int x^2 N(dx) < \infty \right\}.$$

Then:

(i) *For* $\gamma < 1$,

$$D_1'E_2'\{nb_n^2 = O(s_n^2 \operatorname{LLg}^\gamma s_n^2)\} \subset \left\{\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2s_n^2 \operatorname{LLg} s_n^2}} \le 1\right\} \quad a.s.$$

(ii) *For* $\gamma \ge 1$ *and* $\beta > \gamma$,

$$D_1'E_2'\{nb_n^2 = O(s_n^2 \operatorname{LLg}^\gamma s_n^2)\} \subset \left\{\lim_{n\to\infty} \frac{|S_n|}{\sqrt{2s_n^2 \operatorname{LLg}^\beta s_n^2}} = 0\right\} \quad a.s.$$

For an i.i.d. sequence $\{\varepsilon_n\}$ and deterministic $\{b_n\}$, Chow and Teicher (1978) and Teicher (1979) first obtained some of the above results. Here we extended these results to the case of stochastic integrals.

Note that for the discrete-time case,

$$B_t = b_{[t]}, \qquad \langle X\rangle_t = \sum_{k=1}^{[t]} b_k^2.$$

Thus assumption (3.39),

$$B^2 = O\left(\frac{\langle X\rangle \log\langle X\rangle}{\operatorname{LLg}\langle X\rangle}\right),$$

is always satisfied and the "global" version of (3.40) improves Corollary 2 in Lai and Wei (1982) slightly, because from Lemma 3.1, $\{N_t\} \prec N$ with $\int x^2\, dN < \infty$ is a less restrictive hypothesis than $\sup_t \int x^{2+\delta}\, dN_t < \infty$, and the conclusion $S_n = O(\sqrt{s_n^2 \log s_n^2})$ is strengthened.

THEOREM 3.8$_{\text{d}}$. *Let* $\{\varepsilon_n\}$ *be a martingale difference sequence satisfying* (3.49), $\{b_n\}$ *be a predictable sequence and* S_n, s_n^2 *be defined by* (3.50). *Then: For* $\gamma \in (0,1)$,

$$D_1'E_2'\{nb_n^2 = O(s_n^2 \log^\gamma s_n^2)\} \subset \left\{\lim_{n\to\infty} \frac{S_n}{\sqrt{s_n^2 \log^\gamma s_n^2}} = 0\right\} \quad a.s. \text{ for } \gamma \in (0,1)$$

$$D_1'E_2' \subset \left\{\lim_{t\to\infty} \frac{S_n}{\sqrt{s_n^2 \log s_n^2}} = 0\right\} \quad a.s.$$

4. Some examples. In this section we will give some examples which use the asymptotic behavior of martingales to get the convergence rates of some estimators in the statistics of stochastic processes.

The next two examples are borrowed from Zheng (1993) and Fang (1991), respectively.

EXAMPLE 4.1 [Zheng (1993)]. Consider the AR(1) model

$$y_n = \beta y_{n-1} + \varepsilon_n, \qquad n \geq 1, \qquad y_0 = 0,$$

where $\{\varepsilon_n\}$ is a martingale difference sequence such that

$$\mathsf{E}_{n-1}[\varepsilon_n^2] = \sigma^2 > 0, \qquad \sup_n \mathsf{E}_{n-1}[\varepsilon_n^4] < \infty \quad \text{a.s.}$$

Then the least squares estimator $\hat{\beta}_n$ of β is

$$\hat{\beta}_n = \frac{\sum_{j=1}^n y_{j-1} y_j}{\sum_{j=1}^n y_{j-1}^2},$$

$$\tilde{\beta}_n = \hat{\beta}_n - \beta = \frac{\sum_{j=1}^n y_{j-1}\varepsilon_j}{\sum_{j=1}^n y_{j-1}^2}.$$

Note that $\{X_n = \sum_{j=1}^n y_{j-1}\varepsilon_j\} \in \mathscr{M}_{\text{loc}}^2$ and Theorems 3.5_{d} and 3.8_{d} are applicable to it. For the asymptotic behavior of $\hat{\beta}_n$ it suffices to determine the rate of increase of $\sum_{j=1}^n y_j^2$.

If $|\beta| < 1$, it may be proved that

$$0 < \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n y_j^2 = \frac{\sigma^2}{1-\beta^2};$$

hence, applying Theorem 3.2_{d}, (3.7), or Theorem 3.5_{d} (i) yields

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^n y_{j-1}\varepsilon_j|}{\sqrt{2\sum_{j=1}^{n-1} y_j^2 \operatorname{LLg}(\sum_{j=1}^{n-1} y_j^2)}} = 1 \quad \text{a.s.,} \tag{4.1}$$

$$\limsup_{n\to\infty} \sqrt{\frac{\sum_{j=1}^{n-1} y_j^2}{2\operatorname{LLg}(\sum_{j=1}^{n-1} y_j^2)}}\, |\hat{\beta}_n - \beta| = 1 \quad \text{a.s.} \tag{4.2}$$

and

$$\limsup_{n\to\infty} \sqrt{\frac{n}{\operatorname{LLg} n}}\, |\hat{\beta}_n - \beta| < \infty \quad \text{a.s.} \tag{4.3}$$

If $|\beta| = 1$, then, borrowing a result in Donsker and Varadhan (1977), we have

$$\liminf_{n\to\infty} \frac{\operatorname{LLg} n}{n^2} \sum_{j=1}^n y_j^2 > 0;$$

hence, (4.1) and (4.2) hold too and

$$\limsup_{n\to\infty} \frac{n}{\operatorname{LLg} n} |\hat{\beta}_n - \beta| < \infty. \tag{4.4}$$

If $|\beta| > 1$, it may be proved that $\lim_{n\to\infty} \beta^{-2n} \sum_{j=1}^n y_j^2$ exists and

$$0 < \lim_{n\to\infty} \beta^{-2n} \sum_{j=1}^n y_j^2 < \infty.$$

Hence from Theorem 3.8$_d$, (3.39), we have

$$\lim_{n\to\infty} \frac{|\sum_{j=1}^{n} y_{j-1}\varepsilon_j|}{\sqrt{\sum_{j=1}^{n-1} y_j^2 \log(\sum_{j=1}^{n-1} y_j^2)}} = 0$$

and

$$\lim_{n\to\infty} \frac{\hat{\beta}_n - \beta}{\sqrt{n}\beta^n} = 0 \quad \text{a.s.} \tag{4.5}$$

In fact, (4.2) gives the exact random convergence rate of $\hat{\beta}_n$ for $|\beta| \leq 1$ and (4.3)–(4.5) give the convergence order for $\hat{\beta}_n$ in terms of n.

EXAMPLE 4.2 [Fang (1991)]. Let $X = \{X_t,\ t \geq 0\}$ be a Poisson process with $\Lambda_t = \mathsf{E}[X_t] = t^{p+1}/(p+1)$, where p is a parameter. Based on the observation $X = \{X_t,\ 0 \leq t \leq T\}$, the maximum likelihood estimator $\hat{p}(T)$ of p is the unique solution of the equation

$$\int_0^T \log t\, dX_t - \int_0^T t^{\hat{p}(T)} \log t\, dt = 0.$$

By direct calculation, we can prove

$$\lim_{T\to\infty} \hat{p}(T) = p \qquad \text{a.s.},$$

$$\hat{p}(T) - p \sim \frac{\int_0^T \log t\, d(X_t - \Lambda_t)}{\int_0^T t^p \log^2 t\, dt} \qquad \text{a.s.},$$

where $a_T \sim b_T$ means

$$\lim_{T\to\infty} \frac{a_T}{b_T} = 1 \quad \text{a.s.}$$

For the stochastic integral $\int_0^T \log t d(X_t - \Lambda_t)$ we can use Theorem 3.2 or Theorem 3.5 to establish its convergence rate. Since

$$\left\langle \int_0^{\cdot} \log t\, d(X_t - \Lambda_t) \right\rangle_T = \int_0^T \log^2 t\, d\Lambda_t \sim \Lambda_T \log^2 T,$$

hence the integrand $B(t) = \log t$ satisfies the assumptions of Theorem 3.2, (3.37), and Theorem 3.5, (3.22), and therefore

$$\limsup_{T\to\infty} \frac{|\int_0^T \log t\, d(X_t - \Lambda_t)|}{\sqrt{2\Lambda_T (\log^2 T)\, \mathrm{LLg}\, T}} = 1 \quad \text{a.s.}$$

and

$$\limsup_{T\to\infty} \sqrt{\frac{\Lambda_T \log^2 T}{2\, \mathrm{LLg}\, T}}\, |\hat{p}(T) - p| = 1 \quad \text{a.s.}$$

EXAMPLE 4.3. Let $X = \{X_t,\ t \geq 0\}$ be a Gamma process; that is, X be a process with independent increments and

$$\mathsf{E}[\exp(iuX_t)] = \exp\left\{t\int_0^{\infty}(e^{iux} - 1)\nu(dx)\right\},$$

where

$$\nu(dx) = \frac{p}{x}e^{-\vartheta x}\,dx, \qquad x \geq 0,\ p, \vartheta > 0.$$

Based on the observation $\{X_t,\ 0 \leq t \leq 1\}$, take the following $\hat{p}_\varepsilon$ as an estimator of p:

$$\hat{p}_\varepsilon = \frac{N(\varepsilon)}{\log \varepsilon^{-1}},$$

where

$$N(\varepsilon) = \#\{0 \leq t \leq 1\colon\ \Delta X_t = X_t - X_{t-} \geq \varepsilon\}.$$

Basawa and Brockwell (1978) proved that $\hat{p}_\varepsilon \to p$ in probability as $\varepsilon \downarrow 0$ and $\hat{p}_\varepsilon - p$ is asymptotically Gaussian distributed. Now we will give the a.s. convergence rate of $\hat{p}_\varepsilon$. Let μ be the jump measure of X. Then μ is a Poisson random measure and

$$\mathsf{E}[\mu([0,s] \times B)] = s\nu(B) \qquad \forall \text{ Borel sets } B,$$
$$N(\varepsilon) = \mu\big([0,1] \times [\varepsilon, \infty)\big).$$

Note that

$$\begin{aligned}\hat{p}_\varepsilon - p &= \frac{1}{\log \varepsilon^{-1}}N(\varepsilon) - \frac{p}{\log \varepsilon^{-1}}\int_\varepsilon^{\infty}\frac{1}{x}\,dx \\ &= \frac{1}{\log \varepsilon^{-1}}\int_0^1\int_\varepsilon^{\infty} d(\mu - \nu) - \frac{p}{\log^{-1} \varepsilon}\int_\varepsilon^{\infty}\frac{1 - e^{-\vartheta x}}{x}\,dx.\end{aligned}$$

Write

$$Y_s = \int_0^1\int_{1/s}^{\infty} d(\mu - \nu).$$

Since μ is a Poisson random measure, $Y = \{Y_s, s \geq 0\} \in \mathscr{M}^2_{\mathrm{loc}}$ with

$$\langle Y\rangle_s = \int_0^1\int_{1/s}^{\infty} d\nu = p\int_{1/s}^{\infty}\frac{e^{-\vartheta x}}{x}\,dx \sim p\log s \quad \text{as } s \to \infty.$$

Now applying Theorem 3.2, (3.7), or Theorem 3.5, (3.22), we have

$$\limsup_{\varepsilon \to 0}\sqrt{\frac{\log \varepsilon^{-1}}{2p\,\mathrm{LLg}(\log \varepsilon^{-1})}}\,|\hat{p}_\varepsilon - p| = 1 \quad \text{a.s.}$$

Acknowledgment. I am grateful to the referees for a careful reading of the first version of this paper.

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