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THE RADIAL PART OF A Γ-MARTINGALE AND A NON-IMPLOSION THEOREM

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An upper bound is given for the behaviour of the radial part of a Γ-martingale, generalizing previous work of the author on the radial part of Riemannian Brownian motion. This upper bound is applied to establish an integral curvature condition to determine when Γ-martingales cannot "implode" in finite intrinsic time, answering a question of Emery and generalizing work of Hsu on the C_0-diffusion property of Brownian motion.

1. Introduction. This paper has a two-fold purpose. First, it shows how to provide an upper bound on the behaviour of the radial part of a Γ-martingale, generalizing the result of Kendall (1987), but using a new proof which clarifies the relationship of the result to convexity. Second, it uses this upper bound to answer a question of Emery about when Γ-martingales are prevented from "imploding" in from ∞ in finite intrinsic time.

The importance of the class of manifold-valued random processes known as Γ-*martingales* arises from its use in the probabilistic approach to harmonic maps [see Kendall (1988, 1990) and Picard (1991)] and from its close relationship to the notion of convexity [see Emery (1985, 1989), Emery and Mokobodzki (1991) and Kendall (1991, 1992a, b)]. Recall that a (continuous sample-path) random process X on a complete Riemannian manifold $\mathbb{M}$ (or more generally a manifold with connection, but we will not deal with this case here) is said to be a Γ-martingale if its composition with a C^2 map φ yields a submartingale at least when X belongs to regions of convexity of φ. Thus the class of Γ-martingales generalizes the notion of Brownian motion on a complete Riemannian manifold (in which "convexity" above is replaced by "subharmonicity").

Stochastic differential geometry shows how one may construct important auxiliary processes from a Γ-martingale (or more generally a manifold-valued semimartingale). Here we give a very brief review of these constructions: see Kendall (1988) [corrections in Huang and Kendall (1991)] for a full exposition using the notation of this paper and Emery (1989) for an approach based on convexity. Using the horizontal subspace $H_{\pi(\xi)} \leq T_\xi O(\mathbb{M})$ at the frame ξ of the orthonormal frame bundle $O(\mathbb{M})$ (supplied by the Levi–Civita connection),

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one defines the *stochastic development* M and the *stochastic parallel transport* Ξ for a Γ-martingale X via

$$d_S \Xi = H_\Xi \, d_S X, \qquad d_S X = \Xi \, d_S M, \tag{1.1}$$

and M turns out to be a local martingale (motivating the terminology "Γ-martingale"). In fact one may use (1.1) to *define* the notion of a Γ-martingale. In particular the *intrinsic time* of X is defined by $\int \operatorname{trace} d[X,X] = \int \operatorname{trace} d[M,M]$.

The theory of Γ-martingales generalizes linear martingale theory in an interesting way. In Darling (1983) and Zheng (1983) there is proved a Γ-martingale convergence theorem, while the notion of Γ-martingale convergence is used to characterize important domains in Kendall (1992a). In Kendall (1990) and Picard (1991), Γ-martingales are used to generalize the classic probabilistic solution of the Dirichlet problem to an important class of nonlinear elliptic systems, the Euler–Lagrange systems for harmonic maps, thus fulfilling the promise implicit in the analogy between Γ-martingales and Brownian motion noted above.

Emery (1985) reviews various Γ-martingale issues and propounds a number of conjectures. In particular, he raises the question: given a manifold $\mathbb{M}$, is it possible to build a Γ-martingale X restricted to the time set $(0,\infty)$ such that $X(0+) = \infty$ with probability 1? (The Alexandrov compactification is used to define ∞.) [Work in He and Zheng (1984) shows that $X(0+) = \infty$ is the only alternative to existence of the limit $X(0+)$ in $\mathbb{M}$.] What curvature conditions on $\mathbb{M}$ ensure that $\mathbb{P}[X(0+) \neq \infty] = 1$ if $\mathbb{P}[\int_0 \operatorname{trace} d[X,X] < \infty] = 1$? Since we are studying a phenomenon which in some sense is the opposite of explosion, we describe the case $X(0+) = \infty$ as *implosion*. So Emery's question is summarized as:

When is it possible for there to be a Γ*-martingale which "implodes" from* ∞ *at time* 0?

(Note that Emery also deals with the case of *explosion*—whether X can reach ∞ in finite intrinsic time—in the above paper.)

Apart from its intrinsic interest to the student of Γ-martingale theory, the possibility of implosion may turn out to be relevant to the probabilistic theory of harmonic maps referred to above. These considerations motivate this paper, which sets out to explore the circumstances in which implosion will not occur.

The analogous question for Brownian motion on $\mathbb{M}$ (which is after all a Γ-martingale with $d[M,M] = \underline{1} \times dt$) has been recently dealt with in a very full way by Hsu (1989), building on work of Azencott (1974), Dodziuk (1983) and Yau (1978). Brownian motion in $\mathbb{M}$ cannot implode precisely when $\mathbb{M}$ possesses the *Feller* or C_0*-diffusion property* (which is to say, the heat semigroup of $\mathbb{M}$ preserves the class of continuous functions vanishing at ∞). This is an exercise in the path-space approach to diffusion theory, which we leave to the interested reader. Hsu showed that Brownian motion in $\mathbb{M}$

possesses the C_0-diffusion property when the Ricci curvatures of $\mathbb{M}$ are bounded below by a radial function $-\kappa^2(r)$ such that $\int^\infty \kappa(r)^{-1}\,dr = \infty$. Since the class of Γ-martingales is controlled by sectional curvatures rather than averaged Ricci curvatures, it is natural to conjecture that Γ-martingales in $\mathbb{M}$ cannot implode when the *sectional* curvatures are bounded below by $-\kappa^2(r)$ with $\int^\infty \kappa(r)^{-1}\,dr = \infty$. We prove this conjecture in this paper.

To prove his results for Brownian motion, Hsu used the full stochastic differential analysis of the radial part of Brownian motion which is to be found in Kendall (1987). We see below that the Γ-martingale non-implosion result follows from an analogous result for Γ-martingales, which we state and prove as Theorem 2.3 in Section 2. Because the class of Γ-martingales is very wide, it is not possible to give as detailed an analysis of the Γ-martingale radial part as is given in the Brownian case. However, by way of compensation the method of proof given here is much simpler, being based on Toponogov's theorem from geometry [used to give a new proof of the preparatory geometrical Lemma 2.1, originally proved by Wu (1979)] and a convexity argument. Note also that Corollary 2.4 gives as full an analysis in a moderately restricted Γ-martingale case as is available in the Brownian case. (Examples 2.2 and 2.5 use Γ-martingales on the cylinder to show that a full analysis of the general Γ-martingale case would have to be more complicated.)

Given the radial part analysis of Section 2, the non-implosion proof is a straightforward extension of Hsu's approach. It is given in Section 3. The basic step is to bound the rate of escape of Γ-martingales from geodesic balls, given in Lemma 3.1. The results of Section 2 are required because we cannot assume that the geodesic balls are free of cut locus. The non-implosion criterion is established in Theorem 3.2. For the sake of completeness we also establish Theorem 3.3, a Γ-martingale version of a Brownian result, which says that if the sectional curvatures are nonpositive and the manifold is simply connected, then implosion can never happen regardless of whether or not there exist lower bounds on the sectional curvatures.

The paper is concluded by three examples in Section 4. In particular, these examples show that the presence of positive curvature makes implosion possible only because of the resulting presence of cut locus and *not* because of the intensity of positive curvature.

2. The radial part of a Γ-martingale. In this section we establish an upper bound (Theorem 2.3) on the behaviour of the radial part of a Γ-martingale in a general complete Riemannian manifold. We prepare for this by proving Lemma 2.1, which implies that the study of the cut locus of a point on a manifold may be viewed as part of the study of convex functions on that manifold. In fact, all cut loci arise (at least locally) as the combined discontinuity sets of the first and second derivatives of certain convex functions. The function $y \mapsto -\mathrm{dist}(y, x)$ may be transformed into a convex function in the neighborhood of a point $z \neq x$ simply by adding a large enough multiple of $y \mapsto \frac{1}{2}\,\mathrm{dist}(y, z)^2$. (Note that a convex function which arises in this manner is of a special form, because its first- and second-derivative discontinuity set must be closed. I do not know if there are other special features of such convex functions.)

Using this lemma we are able, in Theorem 2.3, to bound the behaviour of the radial part of a Γ-martingale X, even when X visits the cut locus of the reference point from which the radial part is computed. Since Brownian motion is an example of a Γ-martingale, it follows that Theorem 2.3 generalizes the main part of Theorem 1.1 of Kendall (1987). [In fact, as shown in Corollary 2.4, the whole of that theorem follows rapidly from the present Theorem 2.3, and so this paper provides an alternative and more general approach to Kendall (1987).] Note that Lemma 2.1 is due to Wu [(1979), Theorem 3]. We include a new short proof based on Toponogov's theorem for the sake of completeness of exposition [see Greene and Wu (1974) for similar geometrical applications of Toponogov's theorem].

LEMMA 2.1. *Suppose that* $\mathbb{M}$ *is a complete Riemannian manifold with sectional curvatures bounded from below by a constant strictly greater than* $-K^2 < 0$. *Then the function*

$$y \mapsto -\operatorname{dist}(y, x) + \frac{K}{2}\coth(K\operatorname{dist}(z, x))\operatorname{dist}(y, z)^2 \tag{2.1}$$

is convex in a neighborhood of z. *Furthermore, if* $\mathbb{M}$ *is compact, then there is* $\varepsilon > 0$, *such that the neighborhood may be chosen to be a geodesic ball of radius* ε *and centered at* z *for all* z *with* $\operatorname{dist}(z, x) \geq \frac{1}{2}\inf\{\operatorname{dist}(u, C(u)): u \in \mathbb{M}\}$.

PROOF. Suppose $-\tilde{K}^2$ is the infimum of zero and the sectional curvatures of $\mathbb{M}$, so that $-\tilde{K}^2 > -K^2$ (recall that $-K^2$ is a *strict* lower bound). Let $\gamma: [-\varepsilon, \varepsilon] \to \mathbb{M}$ be a geodesic with $\gamma(0) = y$.

We use Toponogov's theorem [an exposition of this result is given in Cheeger and Ebin (1975), Theorem 2-2B]. Let $\tilde{\mathbb{M}}$ be the two-dimensional simply connected Riemannian manifold of constant curvature $-\tilde{K}^2$, with Riemannian distance $\widetilde{\operatorname{dist}}$. (Thus $\tilde{\mathbb{M}}$ is the hyperbolic plane of negative curvature $-\tilde{K}^2$.) Let $\tilde{x}, \tilde{y}, \tilde{\gamma} \in \tilde{\mathbb{M}}$ correspond to $x, y, \gamma \in \mathbb{M}$ in the sense that:

(i) $\tilde{x}, \tilde{y} \in \mathbb{M}$ and $\widetilde{\operatorname{dist}}(\tilde{x}, \tilde{y}) = \operatorname{dist}(x, y)$,

(ii) $\tilde{\gamma}: [-\varepsilon, \varepsilon] \to \tilde{\mathbb{M}}$ is a geodesic with $\tilde{\gamma}(0) = \tilde{y}$ and is such that the angle in $\tilde{\mathbb{M}}$ between $\tilde{\gamma}$ and the geodesic from $\tilde{y}$ to $\tilde{x}$ is the same as the angle in $\mathbb{M}$ between γ and the geodesic from y to x.

Then Toponogov's theorem asserts that for all t,

$$\operatorname{dist}(\gamma(t), x) \leq \widetilde{\operatorname{dist}}(\tilde{\gamma}(t), \tilde{x}) \quad \text{for all } t.$$

Note that Toponogov's theorem requires no conditions on the cut locus in $\mathbb{M}$. In its full generality it *does* require conditions on the cut locus in $\tilde{\mathbb{M}}$, but this cut locus is empty and so these conditions are satisfied vacuously.

Since $\text{dist}(y, x) = \widetilde{\text{dist}}(\tilde{y}, \tilde{x})$, it follows that for positive $s, t \leq \varepsilon$,

$$(2.2)\quad \begin{aligned} &\frac{1}{st}\left[\frac{s}{t+s}\text{dist}(\gamma(t), x) - \text{dist}(y, x) + \frac{t}{t+s}\text{dist}(y(-s), x)\right] \\ &\leq \frac{1}{st}\left[\frac{s}{t+s}\widetilde{\text{dist}}(\tilde{\gamma}(t), \tilde{x}) - \widetilde{\text{dist}}(\tilde{y}, \tilde{x}) + \frac{t}{t+s}\widetilde{\text{dist}}(\tilde{\gamma}(-s), \tilde{X})\right]. \end{aligned}$$

Now because $s, t \in (0, \varepsilon]$, it follows that as $\varepsilon \to 0$, the right-hand side of (2.2) converges to

$$\tilde{K}\coth\left(\tilde{K}\widetilde{\text{dist}}(\tilde{Y}, \tilde{x})\right)\cos^2\alpha = \tilde{K}\coth\left(\tilde{K}\,\text{dist}(y, x)\right)\cos^2\alpha,$$

where α is the angle between $\tilde{\gamma}$ and the geodesic from $\tilde{y}$ to $\tilde{x}$ (equivalently, between γ and the geodesic from y to x). This follows from the computation on the hyperbolic plane $\tilde{\mathbb{M}}$ of

$$\left[\frac{d^2}{dt^2}\widetilde{\text{dist}}(\tilde{\gamma}(t), \tilde{x})\right]_{t=0} = \tilde{K}\coth\left(\tilde{K}\widetilde{\text{dist}}(\tilde{y}, \tilde{x})\right)\cos^2\alpha.$$

Since the Hessian at $y = z$ of $y \mapsto \frac{1}{2}\text{dist}(y, z)^2$ is the identity tensor (recall that the Hessian is the second covariant derivative), it therefore follows from (2.2), from continuity arguments and from the choice of $-K^2 < -\tilde{K}^2$, that

$$\begin{aligned} &\frac{s}{t+s}\text{dist}(\gamma(t), x) - \text{dist}(y, x) + \frac{t}{t+s}\text{dist}(\gamma(-s), x) \\ &\quad\leq \frac{K}{2}\coth(K\,\text{dist}(z, x))\left[\frac{s}{t+s}\text{dist}(\gamma(t), z)^2 - \text{dist}(y, z)^2\right. \\ &\qquad\qquad \left. + \frac{t}{t+s}\text{dist}(\gamma(-s), z)^2\right] \end{aligned}$$

for sufficiently small ε and for y sufficiently close to z.

Furthermore, if $\mathbb{M}$ is compact, then uniform continuity arguments apply to show the above inequality will hold for some fixed $\varepsilon > 0$, for all y such that $\text{dist}(y, z) < \varepsilon$, for all z with $\text{dist}(z, x) \geq \frac{1}{2}\inf\{\text{dist}(u, C(u)): u \in \mathbb{M}\}$. □

We note a corollary of this proof: It suffices to set $-K^2$ to be a strict lower bound on only the sectional curvatures within an open set containing one of the minimal geodesics from x to z. Indeed it may be possible to further reduce the multiplying constant $K\coth(K\,\text{dist}(z, x))$ by building a special-purpose generalization of Toponogov's theorem to allow comparison with a manifold of nonconstant negative radial curvatures. However, we do not pursue this here.

We now turn to consider the radial part of a Γ-martingale. Note that in complete contrast to the Brownian case of Theorem 1.1 of Kendall (1987), it is possible for a Γ-martingale to spend *all* its time on the cut locus of a reference point. (Because the cut locus is of measure zero, a Brownian motion is almost sure to spend only a Lebesgue-null set of time on it.) There follows the simplest nontrivial example of this phenomenon.

EXAMPLE 2.2. Consider a Γ-martingale X which moves according to a real-valued Brownian motion along an axis $\{n\} \times \mathbb{R}$ of an intrinsically flat cylinder $S^1 \times \mathbb{R}$. This axis is precisely the cut locus of the point $(-n, 0)$. Hence the Γ-martingale X spends all its time on the cut locus of $(-n, 0)$.

It is also necessary to take account of the ability of Γ-martingales to visit prescribed points, again in contrast to the Brownian motion case. Example 2.2 shows this also [X visits the point $(n, 0)$]. Thus a stochastic differential equation for the radial part of a general Γ-martingale must contain a term corresponding to possible visits to the reference point.

THEOREM 2.3. *Suppose that* $\mathbb{M}$ *is a complete Riemannian manifold with sectional curvatures bounded from below by* $-K^2$, *and that* X *is a* Γ-*martingale in* $\mathbb{M}$. *Let* $R = \operatorname{dist}(X, x)$ *denote the radial part of* X, *computed using a fixed reference point* x *in* $\mathbb{M}$. *Then* R *is a semimartingale and its Doob–Meyer decomposition is given in stochastic differential form by*

$$d_I R = d_I N + d\Lambda, \tag{2.3}$$

where N *is a real-valued continuous local martingale and* Λ *is a predictable process of locally bounded variation. Furthermore,* N, Λ *satisfy*

$$\begin{aligned} d[N, N] &\leq \operatorname{trace} d[X, X], \\ 1_{(R>0)}\, d\Lambda &\leq \frac{K}{2}\coth(KR) \operatorname{trace} d[X, X]. \end{aligned} \tag{2.4}$$

REMARK. The stochastic differential inequalities of (2.4) are interpreted using integration against nonnegative bounded predictable functions.

REMARK. As may be seen from the proof below, the inequalities of (2.4) correspond to a comparison with the hyperbolic plane of curvature $-K^2$.

REMARK. It follows from Azema and Yor (1978) that $\int 1_{(R=0)}\, d\Lambda$ is the local time of R at 0.

PROOF OF THEOREM 2.3. By a stopping time argument we may reduce the problem to two cases:

(i) X never hits the cut locus $C(x)$ of the reference point x.
(ii) X stays at least $\frac{1}{2} \operatorname{dist}(x, C(x))$ away from x.

Case (i): This is standard apart from the behavior at $R = 0$. We sketch the details here.

First note that $y \mapsto \operatorname{dist}(y, x)$ is smooth off $C(x) \cap \{x\}$ and $C(x)$ is not hit in this case. Hence from Itô's lemma it follows that R is a semimartingale except perhaps in an arbitrarily small neighborhood of x.

However, $y \mapsto \operatorname{dist}(y, x)$ is convex in a sufficiently small neighborhood of x, since $\operatorname{dist}(\gamma(t), x)$ is a convex function $|t|$ of t if $\gamma(0) = x$ and has positive second derivative at $t = 0$ if $\gamma(0)$ is close to, but not equal to x [this follows, e.g., from Kendall (1988), (3.13)]. Hence $R = \operatorname{dist}(X, x)$ is a submartingale and hence a semimartingale in a small neighborhood of x.

Hence in this case a pasting argument shows R is a semimartingale for all time and so the decomposition (2.3) holds.

The first bound of (2.4) follows from Emery's characterization of $\int \operatorname{trace} d[X, X]$ as a limiting quadratic variation, since the triangle inequality shows that

$$\sum (R(t_{n+1}) - R(t_n))^2 \le \sum \operatorname{dist}(X(t_{n+1}), X(t_n))^2$$

and the left-hand side converges in probability to $[R, R] = [N, N]$ while the right-hand side converges in probability to $\int \operatorname{trace} d[X, X]$ [see Emery (1989), Proposition 3.23]. The second bound of (2.4) follows from standard comparison arguments in stochastic differential geometry, as described, for example, in Kendall [(1988), Theorem 2 (geometric form of Itô's lemma); estimates in Section 3].

Case (ii): All follows once we establish a local supermartingale property for

$$R - \frac{K}{2}\int \coth(KR) \operatorname{trace} d[X, X], \tag{2.5}$$

for then R is a semimartingale and so the decomposition (2.3) holds. The first bound of (2.4) follows as in case (i), and the second bound is immediate from the local supermartingale property of (2.5).

So the proof is completed by establishing that (2.5) is a local supermartingale. Note that a limiting argument allows us to suppose that $-K^2$ is a strict lower bound on the curvatures, while a localization argument allows us to suppose that $\mathbb{M}$ is compact. Let $\varepsilon \in (0, \frac{1}{2}\inf\{\operatorname{dist}(u, C(u)) \colon u \in \mathbb{M}\})$ also satisfy the requirements of Lemma 2.1. Then

$$y \mapsto -\operatorname{dist}(y, x) + \frac{K}{2}\coth(K \operatorname{dist}(z, x)) \operatorname{dist}(y, z)^2$$

is convex in ball (z, ε) for all z with $\operatorname{dist}(z, x) \ge \frac{1}{2} \operatorname{dist}(x, C(x))$.

Now define stopping times $0 = T_0 < T_1 < \cdots$ by

$$T_{n+1} = \inf\{t > T_n \colon \operatorname{dist}(X(t), X(T_n)) = \varepsilon\}.$$

Then the convexity noted above means that the following is a supermartingale:

$$R - \frac{K}{2}\int \coth(K \operatorname{dist}(X(\tau_n), x)) d_I\big(\operatorname{dist}(X, X(\tau_n))^2\big),$$

where

$$\tau_n(t) = T_n \quad \text{for } t \in (T_n, T_{n+1}]$$

and

$$\tau_n(0) = 0.$$

Using localization, we may suppose that

$$\int_0^\infty \text{trace}\, d[X, X] \le A$$

for some fixed constant A.

To conclude the proof it suffices to show that as $\varepsilon \to 0$, so

$$\mathbb{E}\left[\frac{K}{2}\int \coth(K\,\text{dist}(X(\tau_n), x))\,d_f\big(\text{dist}(X, X(\tau_n))^2\big)\right] \tag{2.6}$$
$$\to \mathbb{E}\left[\frac{K}{2}\int \coth(KR)\,\text{trace}\, d[X, X]\right],$$

but as $\varepsilon \to 0$, so

$$\frac{K}{2}\coth(K\,\text{dist}(X(\tau_n, x))) \to \frac{K}{2}\coth(KR)$$

uniformly. Set

$$h_n = \frac{K}{2}[\coth(K\,\text{dist}(X(\tau_n), x)) - \coth(KR)] \tag{2.7}$$

so that $\|h_n\|_\infty = \text{ess sup}_\omega \sup_t \|h_n(t)\| \to 0$ as $n \to \infty$.

An Itô analysis of $R_n^2 = \text{dist}(X, X(\tau_n))^2$ over the predictable intervals of constancy of τ_n shows

$$d_I(R_n^2) = 2R_n\, d_I M_n + d[M_n, M_n] + dH_n, \tag{2.8}$$

where M_n is a local martingale given by

$$d_I M_n = [\text{grad}_1 \text{dist}(X, X(\tau_n)) \cdot \Xi]\, d_I M$$

for Ξ the stochastic parallel transport and M the stochastic development of X, and where H_n is a predictable process of locally bounded variation satisfying

$$K^+ R_n \cot(K^+ R_n)(\text{trace}\, d[X, X] - d[M_n, M_n]) \tag{2.9}$$
$$\le dH_n \le KR_n \coth(KR_n)(\text{trace}\, d[X, X] - d[M_n, M_n]),$$

where $+(K^+)^2$ is an *upper* bound on the sectional curvatures of $\mathbb{M}$ and the corresponding lower bound (left-hand inequality) holds at least when $\varepsilon < \pi K^+$. (Note that $\text{trace}\, d[X, X] \ge d[M_n, M_n] \ge 0$.) These estimates follow from Kendall [(1988), Theorem 2 (geometric form of Itô's lemma); estimates in Section 3].

Hence,

$$\frac{K}{2}\int \coth(K\,\text{dist}(X(\tau_n), x))\,d(R_n^2) - \frac{K}{2}\int \coth(KR)\,\text{trace}\, d[X, X]$$
$$= \frac{K}{2}\int \coth(K\,\text{dist}(X(\tau_n), x))(2R_n d_I M_n + d[M_n, M_n] + dH_n)$$
$$- \frac{K}{2}\int \coth(KR)\,\text{trace}\, d[X, X].$$

The left-hand side is bounded [as dist(X, x) > $\frac{1}{2}$ dist($x, C(x)$) > 0, the first integrand on the left-hand side is piecewise time constant, and $\int$ trace $d[X, X]$ is bounded by localization] and so a localization argument may be employed to show that the local martingale term has zero expectation. Hence

$$\left|\mathbb{E}\left[\frac{K}{2}\int \coth(K\operatorname{dist}(X(\tau_n), x))d_I(R_n^2) - \frac{K}{2}\int \coth(KR)\operatorname{trace} d[X, X]\right]\right|$$

$$= \left|\mathbb{E}\left[\frac{K}{2}\int \coth(K\operatorname{dist}(X(\tau_n), x))(d[M_n, M_n] + dH_n) - \frac{K}{2}\int \coth(KR)\operatorname{trace} d[X, X]\right]\right|$$

$$\leq \|h_n\|_\infty \cdot \mathbb{E}\left[\int (d[M_n, M_n] + dH_n)\right] + \mathbb{E}\left[\frac{K}{2}\int \coth(KR)\cdot |d[M_n, M_n] + dH_n - \operatorname{trace} d[X, X]|\right] \tag{2.10}$$

$$\leq \|h_n\|_\infty \cdot \mathbb{E}\left[\int (1 + KR_n \coth(KR_n))\operatorname{trace} d[X, X]\right] + \mathbb{E}\left[\frac{K}{2}\int \coth(KR)\cdot |d[M_n, M_n] + dH_n - \operatorname{trace} d[X, X]|\right].$$

The first term on the right converges to zero since

$$\|h_n\|_\infty \to 0,$$

$$R_n \leq \operatorname{diam}(\mathbb{M}),$$

$$\int_0^\infty \operatorname{trace} d[X, X] \leq A < \infty.$$

The second term converges to zero because $R \geq \frac{1}{2}\operatorname{dist}(x, C(x)) > 0$ and because of the estimates on dH_n together with the bound $R_n \leq \varepsilon$. This establishes the convergence required in (2.6) and so concludes the proof. □

We now consider a special case of Theorem 2.3 from which Theorem 1.1 of Kendall (1987) follows as a direct consequence.

COROLLARY 2.4. *Suppose in the situation of Theorem 2.3 that the Γ-martingale X spends almost no time in $C(x)$ [in the sense that $\{t: X(t) \in C(x)\}$ is almost surely a Lebesgue null set] and that the underlying filtration is such that all continuous martingales M have bracket processes $[M, M]$ which are absolutely continuous as (random) measures on the time set $[0, \infty)$. Then the system (2.3) and (2.4) of stochastic differential equations and inequalities may be further improved to*

$$d_I R = [\operatorname{grad}, \operatorname{dist}(X, x)\cdot \Xi]\, d_I M + \tfrac{1}{2}[\operatorname{Hess}_1 \operatorname{dist}(X, x)\cdot(\Xi, \Xi)]\, d[M, M] + dL_0^R - dL, \tag{2.11}$$

where:

(i) *The gradient* $\mathrm{grad}_1\,\mathrm{dist}$ *and the Hessian* $\mathrm{Hess}_1\,\mathrm{dist}$ *of the function* $y \mapsto \mathrm{dist}(y, x)$ *are set to zero for* $y \in C(x)$.

(ii) Ξ *is the stochastic parallel transport of* X *and* M *is its stochastic development.*

(iii) L_0^R *is the local time of* R *at* 0.

(iv) L *is a nondecreasing process which increases only on the time set* $\{t\colon X(t) \in C(x)\}$.

REMARK. Since X is a Γ-martingale, its stochastic development M is a local martingale.

REMARK. The notation L is chosen to suggest the notion of "local time on the cut locus." However it should be noted that a true local time on $C(x)$ should be defined in terms only of X and intrinsic geometric properties of $C(x)$, and that heuristics suggest L will then be an integral of a predictable "averaged gradient" with respect to this local time. (The same point applies to the Brownian motion case.) Cranston, March and I now have results which make rigorous sense of these heuristics.

PROOF OF COROLLARY 2.4. By Itô's formula in a geometric guise [see, e.g., Kendall (1988), Theorem 2 (geometric form of Itô's lemma)], the difference

$$R - \int [\mathrm{grad}_1\,\mathrm{dist}(X, x) \cdot \Xi]\, d_I M - \tfrac{1}{2}\int [\mathrm{Hess}_1\,\mathrm{dist}(X, x) \cdot (\Xi, \Xi)]\, d[M, M]$$

must be constant off the set $\{t\colon X(t) = x \text{ or } X(t) \in C(x)\}$. The difference integrated over the set $\{t\colon X(t) = x\}$ yields L_0^R by definition of local time of a nonnegative semimartingale; see Azema and Yor (1978). Consider the difference integrated over $\{t\colon X(t) \in C(x)\}$. This must have zero martingale part, since its bracket increases only on the Lebesgue-null time set $\{t\colon X(t) \in C(x)\}$ and yet by hypothesis is absolutely continuous as a (random) measure. However, by Theorem 2.3, if

$$\begin{aligned} -L &= \int_{X(t) \in C(x)} \big(dR - [\mathrm{grad}_1\,\mathrm{dist}(X, x) \cdot \Xi]\, d_I M \\ &\qquad - \tfrac{1}{2}[\mathrm{Hess}_1\,\mathrm{dist}(X, x) \cdot (\Xi, \Xi)]\, d[M, M]\big) \\ &= \int_{X(t) \in C(x)} dR = \int_{X(t) \in C(x)} d\Lambda, \end{aligned}$$

then

$$\begin{aligned} -dL &= 1_{(X(t) \in C(x))}\, d\Lambda \le 1_{(X(t) \in C(x))} \cdot \frac{K}{2}\coth(KR)\,\mathrm{trace}\, d[X, X] \\ &= 1_{(X(t) \in C(x))} \cdot \frac{K}{2}\coth(KR)\,\mathrm{trace}\, d[M, M] = 0 \end{aligned}$$

as $\{t: X(t) \in C(x)\}$ is null and $\int \text{trace}\, d[M, M]$ is an absolutely continuous measure. Hence L is nondecreasing, as required in (iv) above. □

To see that some restriction is required on the underlying filtration, consider the following example.

EXAMPLE 2.5. Let $x = (n, 0) \in S^1 \times \mathbb{R}$ be a point on a cylinder with cut locus $C(x) = \{n\} \times \mathbb{R}$. Consider (two-dimensional) Brownian motion X on $S^1 \times \mathbb{R}$ perturbed whenever it crosses $C(x)$ by a real Brownian motion increment along $C(x) = \{n\} \times \mathbb{R}$ run according to the clock supplied by

$$c \times \textit{local time of } X \textit{ on } C(x).$$

In this case the radial part has a decomposition according to (2.11) except that L is no longer a process of locally bounded variation, but has a nonzero martingale part with bracket process absolutely continuous with respect to the local time of X on $C(x)$. Moreover, calculation using Itô's formula, using the planar process obtained by the map

$$\begin{aligned} &S^1 \times \mathbb{R} \to [-2, 2] \times \mathbb{R},\\ &(x, y) \mapsto (((\text{signed dist}(x, n) + 2) \bmod 4) - 2, y), \end{aligned}$$

shows that the part of L which is of locally bounded variation will only be nondecreasing when c is sufficiently small.

REMARK. It would be very useful to have an extension of the more explicit formula (2.11) to the general context of Theorem 2.3. However, Example 2.5 indicates that such an extension would require a careful and deep analysis of the relationship between $d[X, X]$ and the geometry of the cut locus. Fortunately such depth is not required for the purposes of this paper, and further investigation of this relationship is left as an open problem.

3. Non-implosion conditions for Γ-martingales. It is shown in this section that if $\mathbb{M}$ has sectional curvatures which are bounded below by a radial bound which does not decay too rapidly to $-\infty$, then Γ-martingales on $\mathbb{M}$ cannot "implode" from infinity. That is to say, if X is a Γ-martingale over time set $(0, \infty)$ on such a manifold $\mathbb{M}$ and if $\int_0 \text{trace}\, d[X, X] < \infty$, then $X(0+)$ must exist in $\mathbb{M}$. The condition on the sectional curvatures of $\mathbb{M}$ is an integral condition which directly generalizes the integral condition on Ricci curvatures shown by Hsu (1989) to be sufficient for the Feller or C_0-diffusion property for Brownian motion on $\mathbb{M}$. The Feller or C_0-diffusion property is equivalent to non-implosion for Brownian motion and indeed the methods and results of this section are a direct extension of Hsu (1989), bearing the same relationship to Section 2 above as is borne by Hsu (1989) to Kendall (1987).

First note that He and Zheng (1984) show that the condition $\int_0 \text{trace}\, d[X, X] < \infty$ forces the existence of $X(0+)$ in the Alexandrov compactification of $\mathbb{M}$, for a Γ-martingale X. Thus it suffices to show that if $\lim_{t\downarrow 0} \operatorname{dist}(X(t), x) = \infty$, then $\int_0 \operatorname{trace} d[X, X] = \infty$.

We commence by proving a preparatory lemma concerning the rate at which a Γ-martingale escapes from a unit ball in a Riemannian manifold. This lemma is a direct generalization of Hsu and March [(1985), Lemma 4], which concerns the Brownian case and whose general form can itself be traced back to a result in Prat (1971). The Hsu–March lemma is central to Hsu's treatment of the Feller or C_0-diffusion property of Brownian motion on $\mathbb{M}$ in Hsu (1989), which as we have observed forms the basis for this section. Note that Darling (1992) provides estimates on escape of Γ-martingales which in a sense are much more general than those given below, but which require as a condition that the unit ball in question should not intersect the cut locus of its center. [The Brownian results of Prat (1971) and Hsu and March (1985) are subject to a similar limitation, but extend immediately using the results of Kendall (1987).]

Incidentally, it is reasonable to conjecture that the results of Darling (1992) should remain correct in the presence of cut locus, since the results of Section 2 above imply the cut locus should only reduce the rate of escape of a Γ-martingale. This intuition might be sustained either by proving a Γ-martingale radial part comparison theorem (based on a successful investigation following up the remark at the end of Section 2 above) or by suitable modification of the method of proof in Darling (1992). However, this is not pursued here.

LEMMA 3.1. *Suppose that $\mathbb{M}$ is a complete Riemannian manifold which has sectional curvatures bounded from below by $-K^2 \leq 1$. Suppose further that X is a Γ-martingale in $\mathbb{M}$. Let*

$$T = \inf\{t > 0: \operatorname{dist}(X(t), X(0)) = 1\}.$$

Then the probability that T is small relative to the intrinsic time of X can be bounded above: There are universal positive constants C_1, C_2 such that

$$\mathbb{P}\left[\int_0^T \operatorname{trace} d[X, X] < C_1/K \text{ and } T < \infty\right] \leq \frac{1}{\sqrt{K}} \exp(-C_2 K). \tag{3.1}$$

REMARK. The method of proof used below exploits coupling ideas as an alternative to the shift-operator approach used in the Brownian case by Hsu and March (1985). The basic idea is to compare first a process of constant drift reflected at a level $\varepsilon > 0$, then to the sum of a reflected martingale and a semideterministic increasing process and then to reflected real-valued Brownian motion. These crude comparisons suffice for the purposes of this lemma.

REMARK. In the Brownian case $\int_0^t \text{trace}\, d[X, X] = mt$ for $m = \dim \mathbb{M}$, so that the lemma does indeed generalize Hsu and March [(1985), Lemma 4].

PROOF OF LEMMA 3.1. From Theorem 2.3, if $R = \text{dist}(X, X(0))$, then

$$d_I R = d_I N + d\Lambda \tag{3.2}$$

for N a real-valued local martingale begun at 0 and Λ a real-valued process with paths of locally bounded variation. Moreover

$$\begin{aligned} d[N, N] &\le d\sigma = \text{trace}\, d[X, X], \\ 1_{(R>0)}\, d\Lambda &\le \frac{K}{2}\coth(KR)\, d\sigma \end{aligned} \tag{3.3}$$

and $R(0) = 0$ and $\int 1_{(R=0)}\, d\Lambda$ is the local time of R at 0. We seek a distributional lower bound on

$$\sigma(T) = \int_0^T \text{trace}\, d[X, X]. \tag{3.4}$$

Consider first the comparison of R with $\tilde{R}$, where

$$\begin{aligned} d_I \tilde{R} &= d_I N + \frac{K}{2}\coth(K\varepsilon)\, d\sigma + dL_\varepsilon^{\tilde{R}}, \\ \tilde{R}(0) &= \varepsilon, \end{aligned} \tag{3.5}$$

for fixed $\varepsilon \in (0, 1)$ and $L_\varepsilon^{\tilde{R}}$ the local time of $\tilde{R}$ at level ε [thus $L_\varepsilon^{\tilde{R}}$ is the smallest nondecreasing predictable process such that the solution to (3.5) is never less than ε]. The difference $\tilde{R} - R$ is a process of locally bounded variation:

$$d_I(\tilde{R} - R) = \left(\frac{K}{2}\coth(K\varepsilon)\, d\sigma - d\Lambda\right) + dL_\varepsilon^{\tilde{R}}. \tag{3.6}$$

By the bound on $d\Lambda$ in (3.3) the term in large brackets integrates to a process which is nondecreasing on the time set $\{t\colon R(t) \ge \varepsilon\}$, while $L_\varepsilon^{\tilde{R}}$ is nondecreasing by definition. Consequently the difference $\tilde{R} - R$ is nondecreasing on the time set $\{t\colon R(t) \ge \varepsilon\}$. However, by construction $\tilde{R} \ge \varepsilon$ and so $\tilde{R} - R \ge 0$ on the time set $\{t\colon R(t) > 0\}$. From these two facts and from $\tilde{R}(0) - R(0) = \varepsilon > 0$, it follows that $\tilde{R} \ge R$ always up to time T. Hence, if $\tilde{T} = \inf\{t\colon \tilde{R}(t) = 1\}$, then

$$\tilde{T} \le T, \qquad \sigma(\tilde{T}) \le \sigma(T)$$

and it suffices to provide a lower bound for $\sigma(\tilde{T})$.

Now consider the process $\tilde{R}$ defined by

$$\begin{aligned} d_I \tilde{R} &= d_I\left(N + L_0^N\right) + \frac{K}{2}\coth(K\varepsilon)\, d\sigma, \\ \tilde{R}(0) &= \varepsilon, \end{aligned} \tag{3.7}$$

where L_0^N is the local time at 0 of the local martingale N, so that $\tilde{R}$ is driven by the nonnegative process $N + L_0^N$. Note that the system (3.7) can be integrated:

$$\tilde{R} = N + L_0^N + \frac{K}{2}\coth(K\varepsilon)\sigma + \varepsilon.$$

Then the difference $\hat{R} - \tilde{R}$ satisfies

$$(3.8) \qquad \begin{aligned} d_I(\hat{R} - \tilde{R}) &= dL_0^N - dL_\varepsilon^{\tilde{R}}, \\ (\hat{R} - \tilde{R})(0) &= 0. \end{aligned}$$

Again the difference is a process of locally bounded variation, since dL_0^N is the differential of the local time of the martingale N at 0. The difference is nondecreasing except perhaps when $\tilde{R} = \varepsilon$, but since $\hat{R}$ is bounded below by

$$\hat{R}(t) \geq \frac{K}{2}\coth(K\varepsilon)\sigma(t) + \varepsilon,$$

it follows that $\hat{R} \geq \tilde{R}$ always. Hence if $\hat{T} = \inf\{t\colon \hat{R}(t) = 1\}$, then

$$\hat{T} \leq \tilde{T}, \qquad \sigma(\hat{T}) \leq \sigma(\tilde{T})$$

and it suffices to provide a lower bound for $\sigma(\hat{T})$.

The proof is completed by considering the following sequence of identities and inequalities, of which the last two correspond to a comparison (via random time change) with reflected real-valued Brownian motion. Fix $\delta > 0$ and set

$$a(K, \varepsilon, \delta) = \frac{1 - \varepsilon - \delta}{(K/2)\coth(K\varepsilon)}.$$

Then

$$\begin{aligned} &\mathbb{P}\big[\sigma(\hat{T}) < a(K, \varepsilon, \delta) \text{ and } \hat{T} < \infty\big] \\ &\quad \leq \mathbb{P}\big[\hat{R}(t) = 1 \text{ for some } t \text{ such that } \sigma(t) < a(K, \varepsilon, \delta)\big] \\ &\quad \leq \mathbb{P}\big[N(t) + L_0^N(t) = \delta \text{ for some } t \text{ such that } \sigma(t) < a(K, \varepsilon, \delta)\big] \end{aligned}$$

because if $\hat{R}(t) = 1$ and $\sigma(t) < a(K, \varepsilon, \delta)$, then $1 < N(t) + L_0^N(t) + \varepsilon + (1 - \varepsilon - \delta)$. Hence

$$\begin{aligned} &\mathbb{P}\big[\sigma(\hat{T}) < a(K, \varepsilon, \delta) \text{ and } \hat{T} < \infty\big] \\ &\quad \leq \mathbb{P}\big[N(t) + L_0^N(t) = \delta \text{ for some } t \text{ such that } [N, N](t) < a(K, \varepsilon, \delta)\big] \\ &\quad \leq \mathbb{P}\big[|B(u)| = \delta \text{ for some } u \text{ such that } u < a(K, \varepsilon, \delta)\big] \end{aligned}$$

since $d[N,N] \leq \delta\sigma$ and the random time change based on $[N,N]$ defines a real-valued Brownian motion reflected at 0, namely, $|B([N,N])| = N + L_0^N$. Consequently, we may use the reflection principle to obtain the estimate

$$\mathbb{P}\left[\sigma(\hat{T}) < a(K,\varepsilon,\delta) \text{ and } \hat{T} < \infty\right] \leq 2 \times \sqrt{\frac{2}{\pi}} \int_{\delta/\sqrt{a(K,\varepsilon,\theta)}}^{\infty} \exp\left(\frac{-w^2}{2}\right) dw.$$

We may now use the standard estimate

$$\int_v^{\infty} \exp\left(\frac{-w^2}{2}\right) dw \leq \frac{1}{v}\exp\left(\frac{-v^2}{2}\right) \quad \text{for } v > 0,$$

to deduce the following [with $\alpha = 4(1-\varepsilon-\delta)/\coth(K\varepsilon) = 2a(K,\varepsilon,\delta)K$]:

$$\begin{aligned}\mathbb{P}\left[\sigma(\hat{T}) < \frac{\alpha}{2K} \text{ and } \hat{T} < \infty\right] &\leq 2 \times \sqrt{\frac{\alpha}{\pi\delta^2}}\frac{1}{\sqrt{K}}\exp\left(-\frac{K\delta^2}{\alpha}\right) \\ &\leq \frac{1}{\sqrt{K}}\exp\left(-K\left(\frac{\delta^2}{\alpha} - \left(0 \vee \frac{1}{2}\log\frac{4\alpha}{\pi\delta^2}\right)\right)\right)\end{aligned}$$

since $K \geq 1$.

For fixed $\delta \in (0,1)$ and for all positive $\alpha < \max\{4(1-u-\delta)\tanh(u) \colon 0 \leq u \leq 1-\delta\}$, we can solve for $\varepsilon \in (0, 1-\delta)$ in

$$\alpha = 4(1-\varepsilon-\delta)\tanh(K\varepsilon),$$

for all $K \geq 1$. Moreover, for all sufficiently small α, we have

$$\frac{\delta^3}{\alpha} - \left(0 \vee \frac{1}{2}\log\frac{4\alpha}{\pi\delta^2}\right) > 0.$$

Fixing such a δ and such a sufficiently small α, we deduce the result with

$$C_1 = \frac{\alpha}{2}, \qquad C_2 = \frac{\delta^2}{\alpha} - \left(0 \vee \frac{1}{2}\log\frac{4\alpha}{\pi\delta^2}\right). \qquad \square$$

We turn now to the equation of Γ-martingale implosion, which is a matter of modifying the argument of Hsu (1989) to apply to the Γ-martingale situation and the results proved above.

THEOREM 3.2. *Suppose that* $\mathbb{M}$ *is a complete Riemannian manifold which has sectional curvatures bounded below as follows: For a fixed reference point* $z \in \mathbb{M}$, *the sectional curvatures at any* $y \in \mathbb{M}$ *are bounded below by* $-\kappa^2(\operatorname{dist}(y,x))$, *where* $\kappa^2(r)$ *is an increasing function of* r, $\kappa^2 \geq 1$ *everywhere, and*

$$\int^{\infty} \frac{dr}{\kappa(r)} = \infty. \tag{3.9}$$

Suppose further that X is a Γ-martingale in $\mathbb{M}$ over the time interval $(0,\infty)$. *If* $X(0+)=\infty$, *then* $\int_0 \operatorname{trace} d[X,X] = \pm\infty$.

REMARK. If for any Γ-martingales X in $\mathbb{M}$ over the time interval $(0,\infty)$ there is the implication "$X(0+)=\infty$ implies $\int_0 \operatorname{trace} d[X,X] = +\infty$," then we say *no Γ-martingales in $\mathbb{M}$ may implode*. This is the Γ-martingale generalization of the Feller or C_0-diffusion property for Riemannian Brownian motion. Note the strong resemblance between (3.9) and the Ricci curvature integral condition for Brownian motion [Hsu (1989)].

REMARK. Without loss of generality, we may suppose that $\lim_{r\to\infty}\kappa(r)=\infty$.

PROOF OF THEOREM 3.2. Let X be a Γ-martingale in $\mathbb{M}$ over the time set $(0,\infty)$ with $X(0+)=\infty$. Consider the following stopping times:

$$T_n = \inf\{t>0: \operatorname{dist}(X(t),x)=n\},$$
$$S_n = \inf\{t>T_n: \operatorname{dist}(X(t),X(T_n))=1\},$$

under the usual convention that infima of empty sets are equal to ∞. Since $X(0+)=\infty$, we have

$$\cdots T_n \le S_n \le T_{n-1} \le S_{n-1} \le \cdots \le T_1 \le S_1.$$

By Lemma 3.1, for all n,

$$\mathbb{P}\left[\int_{T_n}^{S_n} \operatorname{trace} d[X,X] \le \frac{C_1}{\kappa(n+1)} \text{ and } S_n<\infty \mid \mathscr{F}_{T_n}, T_n \text{ is finite}\right] \le \exp(-C_2\kappa(n)). \tag{3.10}$$

We shall show $\mathbb{P}[\int_0 \operatorname{trace} d[X,X]\le t]=0$, for all t. Fix t and for given m define $n=n(m)$ by

$$\sum_{r=m}^{m+n-1}\frac{1}{\kappa(r+1)} < \frac{t}{C_1} \le \sum_{r=m}^{m+n}\frac{1}{\kappa(r+1)}$$

(this is possible because $\sum_{r=1}^{\infty} 1/(\kappa(r)) \ge \int_0^\infty dr/(\kappa(r))=\infty$).
Hence

$$\begin{aligned}
&\mathbb{P}\left[\int_0^{S_m} \operatorname{trace} d[X,X] \le t \text{ and } S_m<\infty\right]\\
&\quad\le \sum_{r=m}^{m+n}\mathbb{P}\left[\int_{T_r}^{S_r}\operatorname{trace} d[X,X] \le \frac{C_1}{\kappa(r+1)}, S_r \text{ finite}\right]\\
&\quad\le \sum_{r=m}^{m+n}\exp(-C_2\kappa(r+1)).
\end{aligned}$$

Now choose m large enough so that $\kappa(m+1) > C_2^{-1}$. Since κ is increasing, $\kappa \geq 1$ and $ae^{-a} \geq be^{-b}$, for $1 \leq a \leq b$, we have

$$\exp(-C_2\kappa(r+1)) \leq \frac{\kappa(m+1)}{\kappa(r+1)}\exp(-C_2\kappa(m+1))$$

and hence for large enough m,

$$\begin{aligned}\mathbb{P}\left[\int_0^{S_m}\text{trace } d[X,X] \leq t \text{ and } S_m < \infty\right] &\leq \kappa(m+1)\exp(-C_2\kappa(m+1))\sum_{r=m}^{m+n}\frac{1}{\kappa(r+1)} \\ &\leq \kappa(m+1)\exp(-C_2\kappa(m+1))\left(\frac{t}{C_1}+\frac{1}{\kappa(m+n+1)}\right).\end{aligned}$$

Since $\kappa(r) \to \infty$ as $r \to \infty$ we see

$$\lim_{m\to\infty}\mathbb{P}\left[\int_0^{S_m}\text{trace } d[X,X] \leq t \text{ and } S_m < \infty\right] = 0.$$

It follows that $X(0+) = \infty$ implies

$$\mathbb{P}\left[\int_0 \text{trace } d[X,X] = \infty\right]1,$$

as required. □

For the sake of completeness, we note the following Γ-martingale version of a result due to Azencott (1974), Dodziuk (1983) and Yau (1978).

THEOREM 3.3. *If* $\mathbb{M}$ *is a complete simply connected Riemannian manifold which has sectional curvatures everywhere nonpositive and if* X *is a* Γ-*martingale in* $\mathbb{M}$ *over the time interval* $(0,\infty)$ *with* $X(0+) = \infty$, *then* $\int_0 \text{trace } d[X,X] = \infty$.

PROOF (Sketch). Consider $R = \text{dist}(X, x)$, which is a nonnegative local submartingale on every time interval $[\varepsilon,\infty) \subset (0,\infty)$. Apply a reverse martingale convergence argument to deduce $R(0+) < \infty$ if $\int_0 d[R,R] \leq \int_0 \text{trace } d[X,X]$ is finite. □

4. Examples. In this section we give examples of a negatively curved (but not simply connected) complete Riemannian manifold supporting an imploding Γ-martingale (Example 4.1), a simply connected (but not entirely negatively curved) complete Riemannian manifold supporting an imploding Γ-martingale (Example 4.2) and a complete Riemannian manifold supporting an imploding Γ-martingale, but on which Brownian motion does not implode (Example 4.3). These examples delimit to some extent the range of possibilities, but still leave some questions open. For instance, Example 4.3 is three-dimensional: Is there a two-dimensional example? Additionally, can one construct a Riemannian manifold for which the Ricci curvatures have a constant lower bound, but which support an imploding Γ-martingale?

The basic construction used in these examples is that of a surface of revolution. Given $f\colon \mathbb{R} \to (0, \infty)$, define $\mathbb{M}^f$ to be the surface of revolution

$$y^2 + z^2 = f^2(x).$$

The metric for $\mathbb{M}^f$ viewed as a surface embedded in $\mathbb{R}^3$ is given by

$$ds^2 = dx^2 + f^2(x)\, d\theta^2, \tag{4.1}$$

where $y = x \cos \theta$, $z = x \sin \theta$. The curvature is

$$-f''(x)/f(x). \tag{4.2}$$

It is convenient in the following text to consider the stochastic process (X, Θ) [in (x, θ)-coordinates] given by

$$\begin{aligned} d_I X &= \frac{f'}{2f}(X)\, dt, \\ d_I(\Theta) &= f(X)^{-1}\, d_I W, \end{aligned} \tag{4.3}$$

for W a real-valued Brownian motion, with initial conditions to be specified below. This process is actually a Γ-martingale under the metric (4.1) with a deterministic radial part. To see it is a Γ-martingale, note that

$$J = (0, f(x))$$

defines a Jacobi vector field along the radial geodesic $\{(x, 0)\colon x \geq 0\}$. Hence if γ is a geodesic starting at $(x, 0)$, normal to this radial geodesic, then

$$\left[\frac{d^2}{du^2} \operatorname{dist}(0, \gamma(u))\right]_{u=0} = \frac{f'(x)}{f(x)}. \tag{4.4}$$

However, it then follows that with (X, Θ) given by (4.3) above and for any smooth $\phi\colon \mathbb{M}^f \to \mathbb{R}$, the process

$$\phi(X, \Theta) - \frac{1}{2} \int \left[\frac{d^2}{du^2} \phi(\gamma^{(X, \Theta)}(u))\right]_{u=0} dt$$

is a local martingale, where $\gamma^{(x,\theta)}$ is a geodesic at (x, θ) normal to the radial geodesic. It follows that (X, Θ) is a Γ-martingale [use, for example, Kendall (1988), Theorem 2 (geometric form of Itô's lemma)].

EXAMPLE 4.1. This example is due to Emery (personal communication). Consider $\mathbb{M}^f$ where

$$f(x) = \begin{cases} \exp(-2x^3/3), & \text{for } x \geq 1, \\ (3-2x)e^{-2/3}, & \text{for } x < 1. \end{cases} \tag{4.5}$$

This defines a C^2 surface (modification of this example to the C^∞ smooth case is straightforward).

The curvature is given by

$$-f''(x)/f(x) = \begin{cases} -4x(x^3-1), & \text{if } x \geq 1, \\ 0, & \text{if } x < 0, \end{cases} \tag{4.6}$$

and so this example is a likely candidate for support of an imploding Γ-martingale (compare Theorem 3.2). That this is indeed the case follows by consideration of the limiting Γ-martingale given by (4.3) as $X(0) \to \infty$. For $t \in (0, 1]$, the equations (4.3) yield

$$X(t) = \frac{1}{t}, \tag{4.7}$$

while Θ is given by a time-inhomogeneous diffusion on S^1 with zero drift and infinitesimal variance given by $\exp(4t^{-3}/3)$.

In fact, *Brownian motion* on $\mathbb{M}^f$ can implode as well. This is most easily seen by using an unpublished observation due to T. K. Carne, that if $\mathbb{M}$ is a complete Riemannian manifold, if $\mathbb{N} = \mathbb{M} \times S^k$ and if $\mathbb{N}$ is given the metric

$$ds^2 = dm^2 + g^2(m)\, d\theta^2 \tag{4.8}$$

for a C^2 function $g\colon \mathbb{M} \to (0, \infty)$, then Brownian motion on $\mathbb{N}$ projects down to a process which is Brownian motion on $\mathbb{M}$ plus a superimposed drift $(k/2)\operatorname{grad}\log g$. (This is proved by computation of infinitesimal generators.) In application to $\mathbb{M}^f$ above it shows that the x-coordinate of Brownian motion on $\mathbb{M}^f$ is real-valued Brownian motion with a superimposed drift $-x^2(d/dx)$ solving the stochastic differential equation

$$dX = dB - X^2\, dt$$

for a real-valued Brownian motion B. This permits implosive solutions as is easily seen by considering $Y = X^{-1}$, since

$$dY = -Y^2\, dB + \left(\tfrac{1}{2} + Y^3\right) dt$$

has a nonnegative (possibly explosive) solution with $Y(0) = 0$, $Y|_{(0,\infty)} > 0$ [nonnegativity and positivity following by using the time change $d\tau = Y^4\, dt$

and comparing with $dZ = dA + \frac{1}{2}Z^{-4}\,dt > dA + \frac{1}{2}Z^{-1}\,dt$ if $Z \in (0,1)$, $dA = -Y^2\,dB$].

It is left as an exercise for the reader to decide why it is no paradox for the universal cover of $\mathbb{M}^f$ to support no imploding Γ-martingales.

The above example gives implosion when the lower curvature integral condition of Theorem 4.2 is contravened. A simple modification shows that the nonpositive curvature condition of Theorem 4.3 is necessary:

EXAMPLE 4.2. Consider the modification $\mathbb{M}^*$ of $\mathbb{M}^f$ which is the surface of revolution of

$$f(x) = \begin{cases} \exp(-2x^3/3), & \text{for } x \geq 0, \\ \sqrt{1-x^2}, & \text{for } -1 \leq x < 0. \end{cases} \tag{4.9}$$

Again this defines a C^2 surface, but modification to the C^∞ smooth case is straightforward.

Since $\mathbb{M}^*$ agrees with $\mathbb{M}^f$ over $x \geq 1$, implosion of a Γ-martingale (or indeed of Brownian motion) can be arranged. Clearly $\mathbb{M}^*$ is simply connected and clearly the region of positive curvature permits explosion NOT through any intensity of positive curvature, but simply by permitting sufficient influence from the cut locus.

One might require an example in which a Γ-martingale imploded, but Brownian motion did not. This is provided by a simple modification of Example 4.1:

EXAMPLE 4.3. Consider $\mathbb{M}^f$ as in Example 4.1. Construct a manifold $\mathbb{N}$ based on $\mathbb{M}^f$ using Carne's prescription (4.8) with

$$g(x,\theta) = \begin{cases} \exp\left(\dfrac{2x^3}{3}\right), & \text{for } x \geq 1, \\ \dfrac{1}{3-2x}\exp\left(\dfrac{2}{3}\right), & \text{for } x < 1. \end{cases} \tag{4.10}$$

Successive projections $\mathbb{N} \to \mathbb{M}^f \to \mathbb{R}$ using Carne's idea will project Brownian motion on $\mathbb{N}$ to ordinary real-valued Brownian motion on $\mathbb{R}$ with *no drift*. Hence Brownian motion on $\mathbb{N}$ cannot implode. (Note that we are applying Carne's idea to the projection $\mathbb{M}^f \to \mathbb{R}$ of a Riemannian Brownian motion plus drift; however, a Girsanov argument deals with the drift.) Hence Brownian motion on $\mathbb{N}$ has the Feller or C_0-diffusion property. On the other hand, $\mathbb{M}^f$ is embedded in $\mathbb{N}$ as a connected component of a fixed-point set of an isometry and so is a totally geodesic subset. Hence the implosive Γ-martingale of Example 4.1 is carried over into an implosive Γ-martingale on $\mathbb{N}$.

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Note added in proof: Since this paper was accepted for publication there have been further advances in the study of the radial part of Brownian motion and semimartingales. See, for example, Cranston, Kendall and March (1993), *Probab. Theory Related Fields* **96** 353–368; and Le and Barden (1995), *Probab. Theory Related Fields* **101** 133–146.

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