The Annals of Probability
1995, Vol. 23, No. 3, 1389–1413

CONDITIONAL PROPAGATION OF CHAOS AND A CLASS OF QUASILINEAR PDE'S[1]

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We consider conditional propagation of chaos and use it to solve a class of quasilinear equations of parabolic type. In addition, we construct a class of continuous stochastic processes associated with the above nonlinear equations. Our method imposes fewer smoothness conditions on the coefficients and allows a degenerate nonlinear weight before a divergence form operator. We hope this probabilistic approach will introduce a better microscopic picture for understanding some Stefan type problems.

1. Introduction. We are interested in the following equation in $R^d (d \le 3)$:

$$b(x,u(x,t))\frac{\partial}{\partial t}u(x,t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a^{ij}(x,u(x,t))\frac{\partial}{\partial x_j}u(x,t)\right] \tag{1}$$

with the initial condition $u(x,0) = u_0(x)$. This class of PDE's has important applications in the study of heat conduction, where some phase transitions are involved [29]. The famous Stefan problem [5, 6] can be reduced to its special case where $a^{ij}(\cdot,\cdot)$ is a constant matrix and $b(\cdot,\cdot)$ is discontinuous (see [13], (9.6), page 497). Let us denote

$$p(x,u) = u^{-1}\left(\int_0^u b(x,v)\,dv\right), \qquad \overline{a}^{ij}(x,u) = p^{-1}(x,u)a^{ij}(x,u).$$

Then

$$b(x,u) = \frac{\partial}{\partial u}[up(x,u)] = p(x,u) + u\frac{\partial}{\partial u}p(x,u).$$

Thus, (1) becomes

$$\begin{aligned} &u(x,t)\frac{\partial}{\partial t}p(x,u(x,t)) + p(x,u(x,t))\frac{\partial}{\partial t}u(x,t) \\ &\quad = \sum_{ij}^{d} \frac{\partial}{\partial x_i}\left[p(x,u(x,t))\overline{a}^{ij}(x,u(x,t))\frac{\partial}{\partial x_j}u(x,t)\right]. \end{aligned} \tag{2}$$

We construct in this paper a continuous process with density function

$$u(x,t)p(x,u(x,t))$$

Received March 1994; revised September 1994.
[1]Research supported by NSF Grant DMS-92-04038.
AMS 1991 *subject classifications*. Primary 60G46; secondary 58G32, 60F05, 58G11.
Key words and phrases. Weak convergence, diffusion processes, quasilinear PDE.

with respect to Lebesgue measure which solves (2) with given initial value

$$h(x) = u_0(x)p(x, u_0(x)). \tag{3}$$

Once the process is constructed, we also get a solution to (1). When $p(x,u)$ is a constant, (2) is reduced to the usual divergence form operator. The conditions we impose on the coefficients follow:

CONDITION A. The function $b(x,u)$ is measurable in (x,u) and $b(\cdot,u)$ is continuous at the point $u = 0$ in u. There is a constant $1 < \lambda < \infty$ and a continuous function $\gamma(x) > 0$ with $\int \gamma(x)\,dx \le \lambda^{-1/2}$ such that

$$0 < b(\cdot,\cdot), \qquad \lambda^{-1/2}\gamma(x) < a^{ij}(\cdot,\cdot) < \lambda^{1/2}\gamma(x) \quad \text{a.e.} \tag{4}$$

and

$$\lambda^{-1/2}\gamma(x) < \frac{\int_0^u b(x,v)\,dv}{u} < \lambda^{1/2}\gamma(x), \qquad \left|1 - \frac{\int_0^u b(x,v)\,dv}{u b(x,u)}\right| \le 1 \quad \text{a.e. } x. \tag{5}$$

CONDITION B. The function $a^{ij}(x,u)$ is continuous in u and measurable in (x,u).

CONDITION C. We have $0 \le h(x) \le \lambda$, $\int h(x)\,dx = 1$ and

$$\int h(x)\log h(x)\,dx < \infty, \qquad \int \log(\gamma(x))h(x)\,dx > -\infty.$$

REMARK 1. The restriction on the weight b is quite weak. In fact, $b(\cdot,\cdot)$ may be unbounded and discontinuous when $u \ne 0$. However, under those assumptions, we may deduce immediately that $p(x,u)$ and $\bar{a}^{ij}(x,u)$ are continuous in u and

$$\lambda^{-1} < \bar{a}^{ij}(\cdot,\cdot) < \lambda, \qquad \lambda^{-1/2}\gamma(x) < p(\cdot,\cdot) < \lambda^{1/2}\gamma(x). \tag{6}$$

From our assumptions, we deduce also that γ has a strictly positive lower bound on each bounded set.

REMARK 2. Since $u_0 p(x,u_0)$ is a strictly increasing and continuous function with zero initial value in u_0, $u_0(x)$ and $h(x)$ are in one-to-one correspondence, so the conditions imposed on $h(\cdot)$ can be easily translated to conditions on $u_0(\cdot)$.

Denote by $u = \xi(x,v)$ the inverse function of $v = up(x,u)$ and define

$$q(x,v) = p(x,\xi(x,v)), \qquad g^{ij}(x,v) = \bar{a}^{ij}(x,\xi(x,v)).$$

Then (2) becomes

$$
\begin{aligned}
&q(x,up(x,u))\frac{\partial}{\partial t}u(x,t)+u\frac{\partial}{\partial t}q(x,up(x,u))\\
&\quad=\sum_{ij}^{d}\frac{\partial}{\partial x_i}\left[q(x,up(x,u))g^{ij}(x,up(x,u))\frac{\partial}{\partial x_j}u(x,t)\right].
\end{aligned}
\tag{7}
$$

Our method mainly follows from Kac [10, 11] and McKean [17, 18]. It can be described simply as follows. Consider K diffusion processes in R^d with the same coefficients depending on their instant frequency function of those K processes. When $K \to \infty$, we expect that the frequency function around a point x at time t tends to the common d-dimensional marginal density function $u(x,t)$ which solves (7). In the last decade, there have been a lot of fruitful studies along this direction [8, 9, 19, 21, 23, 24]. There are two key steps in that approach: (1) the tightness of laws of constructed diffusions when $K \to \infty$ and (2) the law of large numbers (the density of frequency function converges to the marginal density). In this paper, for the sake of handling the density functions more easily, we do not use the empirical process [9]. So our tightness results are separated into two stages: tightness results for diffusions with differentiable coefficients and frequency function when $K \to \infty$ and tightness results for limit processes obtained from the first stage when the frequency function tends to the density function and coefficients become nondifferentiable.

We give first an integrated gradient estimate which will be carried through Section 5 to get a second stage tightness result and the convergence of density functions. The tightness of the laws of the corresponding diffusion processes follows from that estimate. So the main problem is just to identify the limit processes. We use the law of large numbers to do that. The standard path is to prove the asymptotic independence of the d-dimensional diffusion processes, which is called the propagation of chaos. We usually need quite strong technical assumptions in order to prove rigorously the propagation of chaos. However, if we use the conditional independence of interchangeable random variables given their permutable σ-field [2], we can get a conditional propagation of chaos result immediately. Finally, we use again a tightness result for marginal distributions to get the desired limit process. We do not know much about the limit process yet. When the coefficients and the solution $u(\cdot,\cdot)$ are all smooth, it is not difficult to see that the process constructed here is a diffusion process in the sense of Stroock and Varadhan. When $b(\cdot,\cdot)$ is constant, the theory of (time-dependent) Dirichlet forms [7, 22, 16] gives us an efficient tool to handle the process. However, the process we construct in this article has a time-dependent weight in front of the divergence form. Further study of the limit process may help us to understand better, in a pathwise sense, the Stefan problem.

2. A gradient estimate. To simplify our notation, we assume $d = 1$ in the following discussions unless otherwise specified until Section 5. The reader may check easily that our arguments do not depend on the dimension. So let $g(\cdot,\cdot) = g^{ij}(\cdot,\cdot)$ and $q(\cdot,\cdot)$ be two measurable functions. Let us impose an additional condition for the moment (we will get rid of it in Section 5). Denote by C_b^2 the class of continuous functions with bounded derivatives up to the second order. Assume that

$$q(x,v) \in C_b^2, \qquad g(x,v) \in C_b^2. \tag{8}$$

Take $0 < \varepsilon < 1$. Define $\kappa_\varepsilon(x)$ as a smooth function in R^1 such that:

1. $\kappa_\varepsilon(x) = 1$ when $|x| \le \varepsilon$ and $0 \le \kappa_\varepsilon(x) \le 1$, $\forall\ x$.
2. $\kappa_\varepsilon(x) = 0$ when $|x| \ge \varepsilon + \varepsilon^2$.
3. $\kappa_\varepsilon(x)$ is a decreasing function of $|x|$.

Denote $V(\varepsilon) = \int \kappa_\varepsilon(x)\,dx$,

$$g_i^{(K)}(x) = g\left(x_i, \frac{\sum_{k=1}^K \kappa_\varepsilon(x_k - x_i)}{V(\varepsilon)K}\right) \tag{9}$$

and

$$q^{(K)}(x) = \prod_{i=1}^K q\left(x_i, \frac{\sum_{k=1}^K \kappa_\varepsilon(x_k - x_i)}{V(\varepsilon)K}\right). \tag{10}$$

Let $W_t^{(K)}$ be a K-dimensional standard Brownian motion. Suppose that $X_t^{(K)} = (X_{t,1}^{(K)}, \dots, X_{t,K}^{(K)})$ is the solution of the following stochastic differential equation

$$dX_{t,i} = \sqrt{2g_i^{(K)}(X_t)}\,dW_{t,i}^{(K)} + \frac{1}{q^{(K)}(X_t)}\frac{\partial}{\partial x_i}\left[q^{(K)}(X_t)g_i^{(K)}(X_t)\right]dt \tag{11}$$

with the initial density $h(x_1)\cdots h(x_K)$ with respect to Lebesgue measure. Denote by $H(K,y,s,x)$ its transition density with respect to the invariant measure $q^K(x)\,dx$. Then

$$H(K,x,t) = \int H(K,y,t,x)h(y_1)\cdots h(y_K)\,dy_1\cdots dy_K$$

gives the density function of X_t with respect to the invariant measure. The generator is given by

$$\mathscr{L}^{(K)} = \frac{1}{q^{(K)}(x)}\sum_{i=1}^K \frac{\partial}{\partial x_i}\left[q^{(K)}(x)g_i^{(K)}(x)\frac{\partial}{\partial x_i}\right].$$

Therefore,

$$\frac{\partial}{\partial t}H(K,x,t) = \mathscr{L}^{(K)}H(K,x,t). \tag{12}$$

Since $X_t^{(K)}$ may be approximated by a sequence of differences of forward martingales and backward martingales with uniformly bounded brackets, X_t is conservative (see [26] for details), so

$$\frac{\partial}{\partial t}\int H(K,x,t)q^{(K)}(x)\,dx = 0.$$

Therefore,

$$\begin{aligned}
&\int H(K,x,0)\log H(K,x,0)q^{(K)}(x)\,dx \\
&\qquad - \int H(K,x,T)\log H(K,x,T)q^{(K)}(x)\,dx \\
&\quad = -\int_0^T \int \frac{\partial}{\partial t}[H(K,x,T-s)\log H(K,x,T-s)]q^{(K)}(x)\,dx\,ds \\
&\quad = -\int\int_0^T \left[\log H(K,x,T-s)\frac{\partial}{\partial t}H(K,x,T-s)q^{(K)}(x)\,dx\right]ds \\
&\qquad -\int\int_0^T\left[\frac{\partial}{\partial t}H(K,x,T-s)\right]ds\,q^{(K)}(x)\,dx \qquad (13)\\
&\quad = -\int_0^T\int \log H(K,x,T-s)\mathscr{L}^{(K)}H(K,x,T-s)q^{(K)}(x)\,dx\,ds \\
&\quad = \sum_i \int_0^T\int H^{-1}(K,x,T-s)\frac{\partial}{\partial x_i}H(K,x,T-s)g_i^{(K)}(x) \\
&\qquad\qquad \times q^{(K)}(x)\frac{\partial}{\partial x_i}H(K,x,T-s)\,dx\,ds.
\end{aligned}$$

By the fact that $x\log x \ge x-1$ and $\int(H(K,x,T)q^{(K)}(x)\,dx) = 1$ we get

$$\begin{aligned}
H(K,x,T)\log H(K,x,T) &\ge \left(1+\log\left(\int H(K,x,T)q^{(K)}(x)\,dx\right)\right) \\
&\qquad \times\left(H(K,x,T)-\int H(K,x,T)q^{(K)}(x)\,dx\right) \\
&= H(K,x,T)-\int H(K,x,T)q^{(K)}(x)\,dx.
\end{aligned}$$

Integrating both sides with respect to $q^{(K)}(x)\,dx$,

$$\begin{aligned}
&\int H(K,x,T)\log(H(K,x,T))q^{(K)}\,dx \\
&\quad \ge \int H(K,x,T)q^{(K)}(x)\,dx - \int H(K,x,T)q^{(K)}(x)\,dx\left(\int q^{(K)}(x)\,dx\right) \\
&\quad \ge 0.
\end{aligned}$$

The last inequality is from the fact that

$$\int q^{(K)}(x)\,dx \le \int \prod_i \sup_u p(x_i,u)\,dx_i \le \prod_i \int \lambda^{1/2}\gamma(x_i)\,dx_i \le 1.$$

Thus, by (13),

$$\begin{aligned}
&\sum_{i=1}^{K}\int_0^T \int H^{-1}(K,x,T-s)\frac{\partial}{\partial x_i}H(K,x,T-s)g_i^{(K)}q^{(K)}(x)\\
&\qquad \times \frac{\partial}{\partial x_i}H(K,x,T-s)\,dx\,ds\\
&\quad \le \int H(K,x,0)\log H(K,x,0)q^{(K)}(x)\,dx\\
&\quad = \int \prod_i h(x_i)\log((q^{(K)}(x))^{-1}\prod_i h(x_i))\,dx_1\cdots dx_K\\
&\quad = -K\int\cdots\int \log\biggl(q\biggl(x_1,\frac{\sum_{k=1}^K \kappa_\varepsilon(x_k-x_1)}{V(\varepsilon)K}\biggr)\biggr)h(x_1)\cdots h(x_K)\,dx_1\cdots dx_K\\
&\qquad + K\int h(x_1)\log h(x_1)\,dx\\
&\quad \le -K\int \log\Bigl(\lambda^{-1/2}\gamma(x_1)\Bigr)h(x_1)\,dx_1 + K\int h(x_1)\log h(x_1)\,dx\\
&\quad \le \frac{K}{2}\log\lambda - K\int \log(\gamma(x_1))h(x_1)\,dx_1 + K\int h(x_1)\log h(x_1)\,dx_1.
\end{aligned}$$

By symmetry,

$$\begin{aligned}
&E\biggl[\int_0^T \bigl(H(K,X_s^{(K)},s)\bigr)^{-2}\frac{\partial}{\partial x_i}H g_i^{(K)}\frac{\partial}{\partial x_i}H\,ds\biggr]\\
&\quad = \int_0^T\int H^{-1}(K,x,T-s)\frac{\partial}{\partial x_i}H(K,x,T-s)g_i^{(K)}q^{(K)}(x)\\
&\qquad \times \frac{\partial}{\partial x_i}H(K,x,T-s)\,dx\,ds\\
&\quad \le \int h(x_1)\log h(x_1)\,dx_1 + \frac{1}{2}\log\lambda - \int \log(\gamma(x_1))h(x_1)\,dx_1.
\end{aligned} \tag{14}$$

3. Tightness results. It is well known that the Markov property will be preserved for time-reversed process. The following lemma gives the generator for the time-reversed process. Its homogeneous version was in the very essential part of Nelson's stochastic machanics [20, 27].

LEMMA 1. *Let $g = (g^{ij}(x,t))$ and $q(x,t)$ be twice differentiable functions. Suppose that X_t is a Markov diffusion process with time-dependent generator*

$$\mathscr{L}_t f = \sum_{i,j} q^{-1}(x,t)\frac{\partial}{\partial x_i}\left[q(x,t)g^{ij}(x,t)\frac{\partial}{\partial x_j}f\right]$$

and initial probability $H(x,0)q(x,0)\,dx$. Then the instantaneous distribution of X_t is given by $H(x,t)q(x,t)\,dx$, which satisfies the following equation:

$$\frac{\partial}{\partial t}H + H\frac{\partial}{\partial t}\log q(x,t) = \mathscr{L}_t H. \tag{15}$$

Moreover, the time-reversed process X_{T-t} has the generator

$$\overline{\mathscr{L}}_t f = \mathscr{L}_{T-t} f + 2\sum_{i,j}\frac{\partial}{\partial x_i}\log H(x,T-t)g^{ij}(x,T-t)\frac{\partial}{\partial x_j}f. \tag{16}$$

PROOF. The existence of the density function $H(\cdot,t)$ is well known. Equation (15) follows easily from integration by parts. So let us just prove (16). Denote by $(\sqrt{g}^{ij})$ the square-root matrix of (g^{ij}). Then it is easy to check that the law of X_t is given by the stochastic differential equation

$$dX_{t,i} = \sum_j 2^{1/2}\sqrt{g}^{ij}(X_t,t)\,dW_{t,j} + \sum_j q^{-1}(X_t,t)\frac{\partial}{\partial x_j}[q(X_t,t)g^{ij}(X_t,t)]\,dt.$$

For any ξ, $\eta \in C_0^2$ and $t > s$,

$$\begin{aligned}\lim_{t\to s}&(t-s)^{-1}E[(\xi(X_t)-\xi(X_s))(\eta(X_t)-\eta(X_s))]\\ &= 2\sum_{i,j}E\left(\frac{\partial}{\partial x_i}\eta(X_s)g^{ij}(X_s,s)\frac{\partial}{\partial x_j}\xi(X_s)\right).\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{t\to s}&(t-s)^{-1}E[\xi(X_{T-s})(\eta(X_{T-t})-\eta(X_{T-s}))]\\ &= \lim_{t\to s}\{(t-s)^{-1}E[\xi(X_{T-t})(\eta(X_{T-t})-\eta(X_{T-s}))]\\ &\qquad - (t-s)^{-1}E[(\xi(X_{T-t})-\xi(X_{T-s}))(\eta(X_{T-t})-\eta(X_{T-s}))]\}\\ &= -E[\xi(X_{T-s})\mathscr{L}_{T-s}\eta(X_{T-s})]\\ &\quad - 2\sum_{i,j}E\left(\frac{\partial}{\partial x_i}\eta(X_{T-s})g^{ij}(X_{T-s})\frac{\partial}{\partial x_j}\xi(X_{T-s})\right).\end{aligned}$$

However, from the integration by parts formula,

$$
\begin{aligned}
&\sum_{i,j} E\left[\frac{\partial}{\partial x_i}\eta(X_{T-s})g^{ij}(X_{T-s})\frac{\partial}{\partial x_j}\xi(X_{T-s})\right] \\
&\quad = \sum_{i,j}\int\left(\frac{\partial}{\partial x_i}\eta(x)g^{ij}(x)\frac{\partial}{\partial x_j}\xi(x)\right)H(x,T-s)q(x,T-s)\,dx \\
&\quad = -\sum_{i,j}\int \xi(x)\left(\frac{\partial}{\partial x_i}\eta(x)g^{ij}(x)\frac{\partial}{\partial x_j}H(x,T-s)\right)q(x,T-s)\,dx \\
&\qquad -\sum_{i,j}\int \xi(x)\frac{\partial}{\partial x_j}\left[q(x,T-s)g^{ij}\frac{\partial}{\partial x_i}\eta(x)\right]H(x,T-s)\,dx \\
&\quad = -\sum_{i,j} E\left[\xi(X_{T-s})\left(\frac{\partial}{\partial x_i}\eta(X_{T-s})g^{ij}(X_{T-s})\frac{\partial}{\partial x_j}\log H(X_{T-s},T-s)\right)\right] \\
&\qquad - E(\xi(X_{T-s})\mathscr{L}_{T-t}\eta(X_{T-s})),
\end{aligned}
$$

so we get the conclusion. □

Now let us continue the discussion of the previous section. Since we assume in (8) that the coefficients are continuously differentiable for each fixed K, we have the decomposition

$$
\begin{aligned}
&X_{t,i}^{(K)} - X_{0,i}^{(K)} \\
&\quad = \int_0^t \frac{\sum_{j=1}^K \kappa'_\varepsilon(X_{s,i}-X_{s,j})}{KV(\varepsilon)}(q^{(K)})^{-1}\left(X_{s,i},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right) \\
&\qquad\quad \times \frac{\partial}{\partial u}\left[q^{(K)}\left(X_{s,i},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right)\right. \\
&\qquad\qquad\quad \left.\times g^{(K)}\left(X_{i,s},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right)\right]ds \\
&\qquad + \int_0^t (q^{(K)})^{-1}\left(X_{s,i},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right) \\
&\qquad\quad \times \frac{\partial}{\partial x}\left[q^{(K)}\left(X_{s,i},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right)\right. \\
&\qquad\qquad\quad \left.\times g^{(K)}\left(X_{s,i},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right)\right]ds \\
&\qquad + \int_0^t \sqrt{2g^{(K)}\left(X_{s,i},\frac{\sum_{j=1}^K \kappa_\varepsilon(X_{s,i}^{(K)}-X_{s,j}^{(K)})}{KV(\varepsilon)}\right)}\,dW_s^i.
\end{aligned}
\tag{17}
$$

For the time-reversed process we have by Lemma 1,

$$
\begin{aligned}
& X^{(K)}_{T-t,i} - X^{(K)}_{T,i} \\
&\quad = 2\int_0^t H^{-1} g^{(K)}\left(X^{(K)}_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{V(\varepsilon)K}\right)\frac{\partial}{\partial x_i} H\,ds \\
&\qquad + \int_0^t \frac{\sum_{j=1}^K \kappa'_\varepsilon(X_{T-s,i} - X_{T-s,j})}{KV(\varepsilon)} (q^{(K)})^{-1} \\
&\qquad\quad \times \left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right) \\
&\qquad\quad \times \frac{\partial}{\partial u}\left[q^{(K)}\left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right)\right. \\
&\qquad\qquad \left.\times\, g^{(K)}\left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right)\right] ds \\
&\qquad + \int_0^t (q^{(K)})^{-1}\left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right) \\
&\qquad\quad \times \frac{\partial}{\partial x}\left[q^{(K)}\left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right)\right. \\
&\qquad\qquad \left.\times\, g^{(K)}\left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right)\right] ds \\
&\qquad + \int_0^t \sqrt{2 g^{(K)}\left(X_{T-s,i}, \frac{\sum_{j=1}^K \kappa_\varepsilon\big(X^{(K)}_{T-s,i} - X^{(K)}_{T-s,j}\big)}{KV(\varepsilon)}\right)}\, d\overline{W}^i_s.
\end{aligned}
\tag{18}
$$

From (17) and (18) we get, under the initial probability $H(K, x, 0) q^{(K)}(x)\,dx$, a new decomposition (see also [14, 15]):

$$
\begin{aligned}
X^{(K)}_{t,i} - X^{(K)}_{0,i} &= -\int_0^t H^{-1} g^{(K)}\left(X^{(K)}_{s,i}, \frac{\sum_{k=1}^K \kappa_\varepsilon\big(X^{(K)}_{s,k} - X^{(K)}_{s,i}\big)}{V(\varepsilon)K}\right)\frac{\partial}{\partial x_i} H\,ds \\
&\quad + \frac{1}{2}\int_0^t \sqrt{2 g^{(K)}(X_s)}\,dW_{s,i} - \frac{1}{2}\int_{T-t}^T \sqrt{2 g^{(K)}(X_{T-s})}\,d\overline{W}_{s,i} \\
&= -\int_0^t H^{-1} g^{(K)}\left(X^{(K)}_{s,i}, \frac{\sum_{k=1}^K \kappa_\varepsilon\big(X^{(K)}_{s,k} - X^{(K)}_{s,i}\big)}{V(\varepsilon)K}\right)\frac{\partial}{\partial x_i} H\,ds \\
&\quad + \frac{1}{2} M^{(K)}_{t,i} - \frac{1}{2}\big(\overline{M}^{(K)}_{T,i} - \overline{M}^{(K)}_{T-t,i}\big),
\end{aligned}
\tag{19}
$$

where W_t is a Brownian motion for the forward filtration, $\overline{W}_t$ is a Brownian motion for the backward filtration, $M^{(K)}_{t,i}$ is a martingale for the forward filtration and $\overline{M}^{(K)}_{t,i}$ is a martingale for the backward filtration. We can easily verify that

$$E\left[\int_0^T \left|\frac{d[M_i^{(K)}, M_i^{(K)}]_t}{dt}\right|^2 dt\right] = 2E\left[\int_0^T |g_i^{(K)}(X_s^{(K)})|^2\, ds\right] \le 2\lambda T$$

and

$$E\left[\int_0^T \left|\frac{d[\bar{M}_i^{(K)}, \bar{M}_i^{(K)}]_t}{dt}\right|^2 dt\right] = 2E\left[\int_0^T |g_i^{(K)}(X_{T-s}^{(K)})|^2\, ds\right] \le 2\lambda T.$$

So the bracket processes of the two martingales that appear in (19) have uniform L^2 bounded derivatives. We may also find from (14) that the finite variational part of (19) has uniform L^2 bounded derivatives. Thus we deduce that (see [27, 12, 25]) all three parts of (19) are tight sequences in law on $C([0,T] \to R)$. Therefore, $\{X_{t,i}^{(K)}\}_i$ is also tight on $C([0,T] \to R^\infty)$ for fixed ε. Furthermore, we deduce [28] that $\{X_{t,i}^{(K)}, M_{t,i}^{(K)}, \bar{M}_{t,i}^{(K)}\}_i$ is tight on $C([0,T] \to R^\infty \times R^\infty \times R^\infty)$.

Denote the derivative of κ_ε as κ'_ε. Then

$$\begin{aligned}
&\frac{1}{K}\sum_{j=1}^{K} \kappa_\varepsilon(X_{t,i}^{(K)} - X_{t,j}^{(K)}) - \frac{1}{K}\sum_{j=1}^{K} \kappa_\varepsilon(X_{0,i}^{(K)} - X_{0,j}^{(K)}) \\
&\quad = \frac{1}{2K}\sum_j \int_0^t \kappa'_\varepsilon(X_{s,i}^{(K)} - X_{s,j}^{(K)})\, d(M_{s,i}^{(K)} - M_{s,j}^{(K)}) \\
&\qquad - \frac{1}{2K}\sum_j \int_{T-t}^T \kappa'_\varepsilon(X_{T-s,i}^{(K)} - X_{T-s,j}^{(K)})\, d(\bar{M}_{s,i}^{(K)} - \bar{M}_{s,j}^{(K)}) \\
&\qquad - \frac{1}{K}\sum_j \int_0^t [\kappa'_\varepsilon(X_{s,i}^{(K)} - X_{s,j}^{(K)})] H^{-1} g_i^{(K)} \frac{\partial}{\partial x_i} H\, ds \\
&\qquad + \frac{1}{K}\sum_j \int_0^t [\kappa'_\varepsilon(X_{s,i}^{(K)} - X_{s,j}^{(K)})] H^{-1} g_i^{(K)} \frac{\partial}{\partial x_j} H\, ds.
\end{aligned}$$

By (14),

$$\begin{aligned}
&E\left[\int_0^T \left|\kappa'_\varepsilon(X_{s,i}^{(K)} - X_{s,j}^{(K)}) H^{-1} g^{(K)} \frac{\partial}{\partial x_j} H\right|^2 dt\right] \\
&\quad \le \|g^{(K)}\|_\infty \left(\int h(x_1) \log h(x_1)\, dx_1 + \frac{1}{2}\log\lambda \right. \\
&\qquad \left. - \int \log(\gamma(x_1)) h(x_1)\, dx_1\right) \sup(|\kappa'_\varepsilon|^2).
\end{aligned} \tag{20}$$

So the law of $(1/K)\sum_{j=1}^K \kappa_\varepsilon(X_{t,i}^{(K)} - X_{t,j}^{(K)})$ is also tight (when $K \to \infty$) and the limit processes are still continuous [27]. A similar conclusion holds for $(\sum_{j=1}^K \kappa'_\varepsilon(X_{s,i} - X_{s,j}))/(KV(\varepsilon))$. Let us take a weakly convergent subsequence, still denoted as

$$\left\{\left\{X_{i,t}^{(K)},\ M_{t,i}^{(K)},\ \overline{M}_{T-t,i}^{(K)},\ \frac{\sum_{j=1}^{K}\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)}{KV(\varepsilon)},\ \frac{\sum_{j=1}^{K}\kappa_\varepsilon'\big(X_{s,i}-X_{s,j}\big)}{KV(\varepsilon)}\right\}_i\right\}_K,$$

and denote its limit in law as

$$\{X_{t,i},\ M_{t,i},\ \overline{M}_{T-t,i}\ N_{t,i},\ N_{t,i}'\}_i.$$

Using Skorohod's theorem, we may realize the convergent subsequence and its limit on the same probability space such that with probability 1 the convergence happens uniformly in $t \in [0,T]$. So we pass to the weak limit through the integral with respect to time (see [12] or [27] for details) in (17) and get

$$(21)\qquad \begin{aligned} X_{t,i}-X_{0,i} &= \int_0^t q^{-1}N_{s,i}'\frac{\partial}{\partial u}[qg(X_{s,i},N_{s,i})]\,ds \\ &\quad + \int_0^t q^{-1}\frac{\partial}{\partial x}[qg(X_{s,i},N_{s,i})]\,ds + \int_0^t \sqrt{2g(X_{s,i},N_{s,i})}\,dW_s. \end{aligned}$$

Similarly we deduce by (14) and (18) that there is a process Q_s such that (see [27] for details)

$$(22)\qquad \begin{aligned} X_{T-t,i}-X_{T,i} &= 2\int_0^t Q_{s,i}\,ds + \int_0^t q^{-1}N_{T-s,i}'\frac{\partial}{\partial u}[qg(X_{T-s,i},N_{T-s,i})]\,ds \\ &\quad + \int_0^t q^{-1}\frac{\partial}{\partial x}[qg(X_{T-s,i},N_{T-s,i})]\,ds \\ &\quad + \int_0^t \sqrt{2g(X_{T-s,i},N_{T-s,i})}\,d\overline{W}_s. \end{aligned}$$

and, by (19),

$$(23)\qquad \begin{aligned} X_{T-t,i}-X_{T,i} &= \tfrac{1}{2}\int_0^t \sqrt{2g(X_{s,i},N_{s,i})}\,dW_s \\ &\quad - \tfrac{1}{2}\int_0^t \sqrt{2g(X_{T-s,i},N_{T-s,i})}\,d\overline{W}_s - \int_0^t Q_{s,i}\,ds. \end{aligned}$$

From [27], we know that the inequality (14) is preserved through the weak limit, so

$$(24)\qquad \begin{aligned} E\left[\int_0^T |Q_s|^2\,ds\right] &\le \|g\|_\infty\bigg(\int h(x_1)\log h(x_1)\,dx_1 \\ &\quad + \tfrac{1}{2}\log\lambda - \int \log(\gamma(x_1))h(x_1)\,dx_1\bigg). \end{aligned}$$

4. Conditional law of large numbers. From symmetry of $\{X_{t,i}\}_i$, when $J < K$,

$$E\left[\left|\frac{\sum_{j=1}^{J}\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)}{JV(\varepsilon)}-\frac{\sum_{k=1}^{K}\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,k}^{(K)}\big)}{KV(\varepsilon)}\right|^2\right]$$

$$\begin{aligned}
&= \frac{1}{J^2K^2V^2(\varepsilon)}\bigg\{K^2\sum_{j=2}^{J}\sum_{k=2}^{J}E\big[\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,k}^{(K)}\big)\big]\\
&\qquad + K^2\sum_{j=1}^{J}E\big[\kappa_\varepsilon(0)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)\big]\\
&\qquad + J^2\sum_{j=2}^{K}\sum_{k=2}^{K}E\big[\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,k}^{(K)}\big)\big]\\
&\qquad + J^2\sum_{k=1}^{K}E\big[\kappa_\varepsilon(0)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,k}^{(K)}\big)\big]\\
&\qquad - 2JK\sum_{j=2}^{J}\sum_{k=2}^{K}E\big[\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,k}^{(K)}\big)\big]\\
&\qquad - 2JK\sum_{k=1}^{K}E\big[\kappa_\varepsilon(0)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,k}^{(K)}\big)\big]\\
&\qquad - 2JK\sum_{j=1}^{J}E\big[\kappa_\varepsilon(0)\kappa_\varepsilon\big(X_{t,i}^{(K)}-X_{t,j}^{(K)}\big)\big]\bigg\}\\
&= \frac{1}{J^2K^2V^2(\varepsilon)}\{(K^2J+J^2K-2JK(J+K))E\big[\kappa_\varepsilon(0)\kappa_\varepsilon\big(X_{t,1}^{(K)}-X_{t,2}^{(K)}\big)\big]\\
&\qquad + (K^2(J-1)+J^2(K-1)-2JK(J-1))\\
&\qquad \times E\big[\big(\kappa_\varepsilon\big(X_{t,1}^{(K)}-X_{t,2}^{(K)}\big)\big)^2\big]\\
&\qquad + (K^2(J-1)(J-2)+J^2(K-1)(K-2)\\
&\qquad\qquad -2JK(J-1)(K-2))\\
&\qquad \times E\big[\kappa_\varepsilon\big(X_{t,1}^{(K)}-X_{t,2}^{(K)}\big)\kappa_\varepsilon\big(X_{t,1}^{(K)}-X_{t,3}^{(K)}\big)\big]\}.
\end{aligned}$$

The right-hand side gives a uniform estimate. Letting $K\to\infty$, we deduce that there is a function $C(\varepsilon,J)$ such that $C(\varepsilon,J)\to 0$ when $J\to\infty$ and

$$E\left[\left|N_{t,i}-\frac{\sum_{j=1}^{J}\kappa_\varepsilon(X_{t,i}-X_{t,j})}{JV(\varepsilon)}\right|^2\right]\le C(\varepsilon,J). \tag{25}$$

Now let us identify the limit process (21). Denote $\mathscr{X}=C([0,T]\to R^{3d})$. Then $\{Y_k\}_k=\{(X_{t,k})_t\ (M_{t,k})_t\ (\overline{M}_{t,k})_t\}_k$ are random variables taking their values in the Polish space $\mathscr{X}$. Recall [2, 3] that $\{Y_k\}_k$ are *interchangeable* if the joint distribution on every finite subset of n of these random variables depends only upon n and not the particular subset, $n \geq 1$. A mapping $\pi = (\pi_1, \pi_2, \dots)$ from the set $\mathscr{N}$ of all positive integers onto itself is called a finite permutation if π is one-to-one and $\pi_k = k$ for all but a finite number of integers. Let Q denote the set of all finite permutations π and let $\mathscr{B}^\infty$ be the class of Borel subsets of $\mathscr{X}^\infty$ and $Y = \{Y_k\}_k$. Define $\pi Y = \{Y_{\pi_k}\}$ for $\pi = \{\pi_k\}$. Then

$$\mathscr{S} = \{Y^{-1}(B)\colon\ B \in \mathscr{B}^\infty,\ P[Y^{-1}(B) \,\triangle\, (\pi Y)^{-1}(B)] = 0,\ \forall\ \pi \in Q\}$$

is called the *σ-algebra of permutable events* of Y.

THEOREM 1. *Suppose that $\{Y_k\}_k$ are interchangeable. Then they are conditionally independent and identically distributed given $\mathscr{S}$. Furthermore, there exists a regular conditional distribution, say P^ω, for $Y = \{Y_k\}_k$ given $\mathscr{S}$ such that for each $\omega \in \Omega$ the coordinate random variables Y_k of the probability space $(\mathscr{X}^\infty,\ \mathscr{B}^\infty,\ P^\omega)$ are independent and identically distributed.*

PROOF. The proof is just the same as Theorem 7.3.2 and Corollary 7.3.5 of [2]. The only necessary change is to replace the real line R in the original proof by $\mathscr{X}$. □

Moreover, we have the following lemma.

LEMMA 2. *Let f be a measurable function on $\mathscr{X}$. If the $f(Y_i)$ are mutually independent, then $\{f(Y_i)\}_i$ is independent of $\mathscr{S}$.*

PROOF. Denote $Z_i = f(Y_i)$. Then the $\{Z_i\}$ are independent, so by the strong law of large numbers, $K^{-1}\sum_{i=1}^K I_A(Z_i) \to P(Z_i \in A)$ a.s. If $Q \in \mathscr{S}$, then

$$\begin{aligned} E(QI_A(Z_i)) &= \lim_{K\to\infty} \frac{1}{K} E\biggl(\sum_{j=1}^K QI_A(Z_j)\biggr) \\ &= E[QP(Z_i \in A)] \\ &= P(Q)P(Z_i \in A). \end{aligned}$$ □

Under the conditional probability, we deduce from the strong law of large numbers that there is a function $v(\varepsilon, x, t)$ such that

$$\frac{\sum_{i=1}^K \kappa_\varepsilon(x - X_{t,i})}{KV(\varepsilon)} \to v(\varepsilon, x, t) = \frac{E^\omega(\kappa_\varepsilon(x - X_{t,i}))}{V(\varepsilon)}. \tag{26}$$

Thus, by (25),

$$N_{t,i} = v(\varepsilon, X_{t,i}, t) = V^{-1}(\varepsilon) \int \kappa_\varepsilon(\xi_{t,i} - x) u(x,t) q(x, v(\varepsilon, X_{t,i}, t))\, dx. \tag{27}$$

By Lemma 2, $\{\overline{W}_{t,i}\}_i$ are independent of $\mathscr{S}$, so from (22) we have, under P^{ω},

$$
\begin{aligned}
X_{T-t,i} - X_{T,i} \\
&= \int_0^t q^{-1} v'(\varepsilon, X_{T-s,i}, T-s) \frac{\partial}{\partial u}[qg(X_{T-s,i}, v(\varepsilon, X_{T-s,i}, T-s))]\, ds \\
&\quad + \int_0^t q^{-1} \frac{\partial}{\partial x}[qg(X_{T-s,i}, v(\varepsilon, X_{T-s,i}, T-s))]\, ds + 2\int_0^t Q_{T-s,i}\, ds \\
&\quad + \int_0^t \sqrt{2g(X_{T-s,i}, N_{T-s,i})}\, d\overline{W}_s.
\end{aligned} \tag{28}
$$

We are interested in finding the precise form of Q_s. From interchangeability, it is sufficient to consider the case $i = 1$. By (21) we deduce that, under P^{ω},

$$
\begin{aligned}
X_{t,1} - X_{0,1} &= \int_0^t q^{-1} \frac{\partial}{\partial x}[q(X_{s,1}, v(\varepsilon, X_{s,1}, s)) g(X_{s,1}, v(\varepsilon, X_{s,1}, s))]\, ds \\
&\quad + \int_0^t \sqrt{2g(X_{s,1}, v(\varepsilon, X_{s,i}, s))}\, dW_s.
\end{aligned} \tag{29}
$$

From Lemma 1, the time-reversed process should have the decomposition

$$
\begin{aligned}
X_{T-t,1} - X_{T,1} &= \int_0^t q^{-1} \frac{\partial}{\partial x}[q(X_{T-s,1}, v(\varepsilon, X_{T-s,1}, T-s)) \\
&\qquad \times g(X_{T-s,1}, v(\varepsilon, X_{T-s,1}, T-s))]\, ds \\
&\quad + 2\int_0^t (u^{\omega})^{-1} g \frac{\partial}{\partial x} u^{\omega}(X_{T-s,1}, T-s)\, ds \\
&\quad + \int_0^t \sqrt{2g(X_{T-s,1}, v(\varepsilon, X_{T-s,1}, t))}\, d\overline{W}_s,
\end{aligned}
$$

where $u^{\omega}(\cdot,\cdot)$ is the density function.

By comparing the above equation with (28), we deduce that

$$
Q_{T-t} = (u^{\omega})^{-1} g \frac{\partial}{\partial x} u^{\omega}(X_{T-t,1}, T-t), \tag{30}
$$

so from (24),

$$
\begin{aligned}
&E\left\{E^{\omega}\left[\int_0^T \left|(u^{\omega})^{-1} g \frac{\partial}{\partial x} u^{\omega}(X_{T-s,1}, T-s)\right|^2\right]\right\} \\
&\quad \le \|g\|_{\infty}\left(\int h(x_1)\log h(x_1)\, dx_1 + \frac{1}{2}\log\lambda \right. \\
&\qquad \left. - \int \log(\gamma(x_1)) h(x_1)\, dx_1\right).
\end{aligned} \tag{31}
$$

Thus we can find at least an $\omega(\varepsilon)$ such that (28) and (29) hold and

$$
\begin{aligned}
&E^{\omega}\left[\int_0^T\left|(u^{\omega})^{-1}g\frac{\partial}{\partial x_i}u^{\omega}(X_{T-s,1})\right|^2\right] \\
&\quad\le \|g\|_{\infty}\left(\int h(x_1)\log h(x_1)\,dx_1+\frac{1}{2}\log\lambda-\int\log(\gamma(x_1))h(x_1)\,dx_1\right).
\end{aligned}\tag{32}
$$

The right-hand side is independent of ε. We also have the forward and backward decomposition as in [14]:

$$
\begin{aligned}
X_{t,1}-X_{0,1}&=\frac{1}{2}M_{t,1}-\frac{1}{2}(\overline{M}_{T,1}-\overline{M}_{T-t,1}) \\
&\quad-\int_0^t(u^{\omega})^{-1}g\frac{\partial}{\partial x}u^{\omega}(X_{T-s,1},T-s)\,ds.
\end{aligned}\tag{33}
$$

Its density function satisfies

$$
\begin{aligned}
&\frac{\partial}{\partial t}u^{\omega}(x,t)+u^{\omega}(x,t)\frac{\partial}{\partial t}\log q(x,v(\varepsilon,x,t)) \\
&\quad=q^{-1}(x,v(\varepsilon,x,t))\frac{\partial}{\partial x}\left[g(x,v(\varepsilon,x,t))q(x,v(\varepsilon,x,t))\frac{\partial}{\partial x}u^{\omega}(x,t)\right].
\end{aligned}\tag{34}
$$

Applying the integration by parts formula, we deduce for any pair $0\le t_1<t_2<\infty$ and any $f\in C_0^1$,

$$
\begin{aligned}
&\int f(x)u^{\omega}(x,t_2)q(x,v(\varepsilon,x,t_2))\,dx \\
&\qquad-\int f(x)u^{\omega}(x,t_1)q(x,v(\varepsilon,x,t_2))\,dx \\
&\quad=\int_{t_1}^{t_2}\int\frac{\partial}{\partial x}f(x)q(x,v(\varepsilon,x,s))g(x,v(\varepsilon,x,s)) \\
&\qquad\times\frac{\partial}{\partial x}u^{\omega}(x,s)\,dx\,ds.
\end{aligned}\tag{35}
$$

We select $\omega(\varepsilon)$ and write $u^{\omega}(x,t)$ as $u(x,t)$ in the following discussion.

5. Second stage tightness result. For each pair $a^{ij}(x,u)$ and $b(x,u)$ satisfying Conditions A–C, we may find a sequence of $a_m^{ij}(x,u)$ and a sequence of $b_m(x,u)$ satisfying the following conditions:

CONDITION D. Conditions A–C hold for $a_m^{ij}(x,u)$ and $b_m(x,u)$. Furthermore,

$$
\left|1-\frac{\int_0^u b_m(x,v)\,dv}{ub_m(x,u)}\right|\le 1-\frac{1}{m}\quad\text{a.e. } x.\tag{36}
$$

CONDITION E. $a_m^{ij}(x,u)$ and $b_m(x,u)$ approach $a^{ij}(x,u)$ and $b(x,u)$ in the following sense: For each $r > 0$,

$$\int_{|x|<r} \sup_u \left|\bar{a}_m^{ij}(x,u) - \bar{a}^{ij}(x,u)\right| dx + \int_{|x|<r} \sup_u |p_m(x,u) - p(x,u)|\, dx \to 0.$$

CONDITION F. The induced $g_m^{ij}(x,v)$ and $q_m(x,v)$ satisfy (8).

Let us first fix m and drop the subscript m to simplify the notation. Then the following lemma is true.

LEMMA 3. *Use the notation of the Introduction. Assume there is a constant* $c > 0$ *such that*

$$\left|1 - \frac{\int_0^u b(x,v)\,dv}{ub(x,u)}\right| \le 1 - c \quad \textit{for a.e. } x. \tag{37}$$

Then

$$\left|\frac{\partial q}{\partial v}\right| \le (1-c)\frac{q(x,v)}{v}. \tag{38}$$

PROOF. From the definitions in the Introduction,

$$v = \xi(x,v)q(x,v). \tag{39}$$

Take the derivative with respect to v on both sides:

$$1 = \xi(x,v)\frac{\partial}{\partial v}q(x,v) + q(x,v)\frac{\partial}{\partial v}\xi(x,v).$$

So

$$\frac{\partial}{\partial v}q(x,v) = \frac{1 - q(x,v)(\partial/\partial v)\xi(x,v)}{\xi(x,v)}.$$

Let us differentiate on both sides of the following equality with respect to u:

$$\xi(x, up(x,u)) = u.$$

Then we get

$$\frac{\partial \xi}{\partial v}\left(u\frac{\partial p}{\partial u} + p\right) = 1.$$

Thus

$$\frac{\partial \xi}{\partial v} = \left(u\frac{\partial p}{\partial u} + p\right)^{-1} = \frac{1}{b(x,u)}.$$

Therefore,

$$\left|\frac{\partial}{\partial v}q\right| = \left|\left(1 - \frac{p(x,\xi(x,v))}{b(x,\xi(x,v))}\right)/(\xi(x,v))\right| \le \frac{1-c}{\xi(x,v)}. \qquad \square$$

Take $\varepsilon = 1/n$ in the previous section. We only consider the marginal distribution of $X_{t,1}$. Then we get a sequence of diffusions $\{X_t^{(n)}\}$ with corresponding density functions $\{u_n(x,t)\}$ that solve (34). Let us show that sequence is still tight and its density function gives the solution to (7).

Let $0 < r < \infty$ be a constant. Denote $B(r) = \{x;\ |x| < r\}$. By (32) and the fact that $q(x,t)$ is bounded from below on each $B(r)$ by a positive constant, we have

$$C_r = \sup_n \sum_{i=1}^d \int_{B(r)} \int_0^T \left|\frac{\partial}{\partial x_i} u_n^{1/2}(x,s)\right|^2 ds\, dx < \infty. \tag{40}$$

Define on $[0,T] \times B(r)$ a Hilbert space $\mathscr{H}^{(r)}$ with the norm

$$\|f\|^{(r)} = \sqrt{\sum_{i=1}^d \int_{B(r)} \int_0^T \left|\frac{\partial}{\partial x_i} f(x,s)\right|^2 ds\, dx + \int_{B(r)} \int_0^T |f(x,s)|^2\, ds\, dx}. \tag{41}$$

Then we have the following lemma. We do not need the additional assumption (8) in the proof of its first conclusion, and we do not need the assumption (36) in the proof of (42). Since its proof is standard (see [4]) and quite long, we leave it to the Appendix.

LEMMA 4. *There is a subsequence $u_{n(k)}(x,t)$ and $u(x,t)$ such that for each $r > 0$, $u_{n(k)}^{1/2}(x,t)$ converges weakly in $\mathscr{H}^{(r)}$ to $u^{1/2}(x,t)$ with $\int_0^T \int |u|^{5/2}\, dx\, dt < \infty$ and*

$$\int_0^T \int_{B(r)} \left|\sqrt{u_{n(k)}(x,t)} - \sqrt{u(x,t)}\right|^5 dx\, dt \to 0. \tag{42}$$

If we assume further that (37) *holds, then*

$$\int_0^T \int_{B(r)} \left| v\left(\frac{1}{n(k)}, x, t\right) - u(x,t)p(x,u(x,t))\right| dx\, dt \to 0. \tag{43}$$

Now let us explicitly write the subscript m. For each fixed m, we have a limit function $u^{(m)}(x,t) = \lim_{n\to\infty} u_n^{(m)}(x,t)$ according to the previous lemma.

Passing to the limit in (35),

$$\begin{aligned}
&\int f(x)u^{(m)}(x,t_2)p_m(x,u^{(m)}(x,t_2))\,dx \\
&\qquad - \int f(x)u^{(m)}(x,t_1)p_m(x,u^{(m)}(x,t_1))\,dx \\
&\quad = \int f(x)u^{(m)}(x,t_2)q_m(x,u^{(m)}(x,t_2)p_m(x,u^{(m)}))\,dx \\
&\qquad - \int f(x)u^{(m)}(x,t_1)q_m(x,u^{(m)}(x,t_1)p_m(x,u^{(m)}))\,dx \\
&\quad = \sum_{i,j=1}^{d}\int_{t_1}^{t_2}\int \frac{\partial}{\partial x_i} f(x)q_m(x,u^{(m)}(x,s)p_m(x,u^{(m)})) \\
&\qquad \times g_m^{ij}(x,u^{(m)}(x,s)p_m(x,u^{(m)}))\frac{\partial}{\partial x_j}u^{(m)}(x,s)\,dx\,ds \qquad (44)\\
&\quad = \sum_{i,j=1}^{d}\int_{t_1}^{t_2}\int \frac{\partial}{\partial x_i} f(x)p_m(x,u^{(m)}(x,s))\overline{a}_m^{ij}(x,u^{(m)}(x,s)) \\
&\qquad \times \frac{\partial}{\partial x_j}u^{(m)}(x,s)\,dx\,ds.
\end{aligned}$$

From (40) and Lemma 4, it is easy to see that

$$\sup_m \sum_{i=1}^{d}\int_{B(r)}\int_0^T \left|\frac{\partial}{\partial x_i}(u^{(m)}(x,s))^{1/2}\right|^2 ds\,dx < \infty.$$

Repeating the proof for the first part of Lemma 4 (the only difference is that we use perturbed p_m to replace q), we deduce that $u^{(m)}$ has a convergent subsequence. By passing to the limit in (44), we deduce

$$\begin{aligned}
&\int f(x)u(x,t_2)p(x,u(x,t_2))\,dx - \int f(x)u(x,t_1)p(x,u(x,t_1))\,dx \\
&\quad = \sum_{i,j=1}^{d}\int_{t_1}^{t_2}\int \frac{\partial}{\partial x_i} f(x)p(x,u(x,s))\overline{a}^{ij}(x,u(x,s)) \qquad (45)\\
&\qquad \times \frac{\partial}{\partial x_j}u(x,s)\,dx\,ds.
\end{aligned}$$

Furthermore, we deduce from (33) and (31) that (see [27] and [14]) there is a measurable process $G_s = (G_{s,j})$ with $E[\int_0^T |G_s|^2\,ds] < \infty$ such that

$$X_{t,j} - X_{0,j} = \int_0^t G_{s,j}\,ds + \tfrac{1}{2}(M_{t,j} - M_{0,j}) - \tfrac{1}{2}(\overline{M}_{T,j} - \overline{M}_{T-t,j}), \qquad (46)$$

where M_t is a forward martingale and $\overline{M}_t$ is a backward martingale. We may conclude our discussion with the following theorem.

THEOREM 2. *Under Conditions* A–C, *there is a continuous process* $X_t = (X_{t,j})$ *with decomposition* (46) *such that its density function* $u(x,t)$ *with respect to* $p(x,u(x,t))\,dx$ *satisfies* (1) *and* $u^{1/2} \in \mathscr{H}$.

APPENDIX

PROOF OF LEMMA 4. *Step* 1. From (40), $u_n^{1/2}(\cdot,\cdot)$ is contained in a bounded ball in $\mathscr{H}^{(r)}$. Since $\mathscr{H}^{(r)}$ is reflexive, the bounded ball in $\mathscr{H}^{(r)}$ is weakly compact. So we can find a weakly convergent subsequence, still denoted as $u_n(x,t)$, such that u_n converges weakly to some $u \in \mathscr{H}^{(r)}$.

Step 2. Our next goal is to show for fixed r,

$$\int_0^T \int_{B(r)} |u_{n(k)}(x,t) - u(x,t)|\,dx\,dt \to 0. \tag{47}$$

Denote

$$U(\delta,x,n,t) = \int \kappa_\delta(x-y) q_n\left(y, v\left(\frac{1}{n}, y, t\right)\right) dy$$

and

$$u_n(\delta,x,t) = U^{-1}(\delta,x,n,t) \int \kappa_\delta(x-y) u_n(y,t) q_n\left(y, v\left(\frac{1}{n}, y, t\right)\right) dy.$$

Then we can easily prove (see [1, 14, 25]) by equicontinuity of the processes (33) that

$$|u_n(\delta,x,t) - u_n(\delta,x,s)| \le O(t-s), \tag{48}$$

where $O(t-s) \to 0$ when $(t-s) \to 0$. The quantity $O(t-s)$ is independent of x and n but depends on δ. We have also

$$\begin{aligned}
&|u_n(\delta,x,t) - u_n(x,t)| \\
&\quad = \left| \int \frac{\kappa_\delta(y)}{U(\delta,x,n,t)} u_n(x-y,t) q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) dy \right. \\
&\qquad \left. - \int \frac{\kappa_\delta(y)}{U(\delta,x,n,t)} u_n(x,t) q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) dy \right| \\
&\quad \le \int \frac{\kappa_\delta(y)}{U(\delta,x,n,t)} |u_n(x-y,t) - u_n(x,t)| q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) dy
\end{aligned}$$

$$= \int \frac{\kappa_\delta(y)}{U(\delta,x,n,t)} \left| \int_0^1 \frac{d}{d\theta} u_n(x-\theta y,t)\,d\theta \right| q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) dy$$
$$\le \int \frac{\kappa_\delta(y)}{U(\delta,x,n,t)} \int_0^1 |y| \left| \frac{\partial}{\partial x} u_n(x-\theta y,t) \right| q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) d\theta\,dy$$
$$\le \int \frac{2\delta\kappa_\delta(y)}{U(\delta,x,n,t)} \int_0^1 \left| \frac{\partial}{\partial x} u_n(x-\theta y,t) \right| q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) d\theta\,dy.$$

The last inequality is because $\kappa_\delta(y)|y| \le 2\delta\kappa_\delta(y)$. Thus

$$\int_0^T \int_{B(r)} |u_n(\delta,x,t) - u_n(x,t)|\,dx\,dt$$
$$\le \int_0^T \int_{B(r)} \int \frac{2\delta\kappa_\delta(y)}{U(\delta,x,n,t)} \int_0^1 \left| \frac{\partial}{\partial x} u_n(x-\theta y,t) \right|$$
$$\times q_n\left(x-y, v\left(\frac{1}{n}, x-y, t\right)\right) d\theta\,dy\,dx\,dt$$
$$\le 2\delta\lambda^{1/2} \max_{x\in B(r+\delta)} [\gamma(x)] \int_0^T \int_0^1 \int \frac{2\delta\kappa_\delta(y)}{U(\delta,x,n,t)}$$
$$\times \int_{B(r)} \left| \frac{\partial}{\partial x} u_n(x-\theta y,t) \right| dx\,dy\,d\theta\,dt$$
$$\le 2\delta\lambda^{1/2} \max_{x\in B(r+\delta)} [\gamma(x)] \int_0^T \int_{B(r+\delta)} \left| \frac{\partial}{\partial x} u_n(x,t) \right| dx\,dt \tag{49}$$
$$\le 2\delta\lambda^{1/2} \max_{x\in B(r+\delta)} [\gamma(x)] \int_0^T \left(\int_{B(r+\delta)} \left| \frac{\partial}{\partial x} u_n^{1/2}(x,t) \right|^2 dx \right)^{1/2}$$
$$\times \left(\int_{B(r+\delta)} u_n(x,t)\,dx \right)^{1/2} dt$$
$$\le 2\delta\lambda^{1/2} \max_{x\in B(r+\delta)} [\gamma(x)] \int_0^T \left(\int_{B(r+\delta)} u_n^{-1}(x,t) \left| \frac{\partial}{\partial x} u_n(x,t) \right|^2 dx \right)^{1/2} dt$$
$$\times \lambda^{1/4} \left(\min_{x\in B(r+\delta)} [\gamma(x)] \right)^{-1/2}$$
$$\le 2\delta\lambda^{3/4} \max_{x\in B(r+\delta)} [\gamma(x)] \left(\min_{x\in B(r+\delta)} [\gamma(x)] \right)^{-1/2} C_{r+\delta},$$

where the constant $C_{r+\delta}$ is from (40).

We may also easily verify that for each fixed t and δ, $u_n(\delta,x,t)$ together with its derivative are bounded uniformly in n. So by (48), (49) and the Arzela–Ascoli theorem, there is a subsequence which converges uniformly to some continuous $u(\delta, x, t)$. If we still denote this subsequence as $u_n(\cdot,\cdot)$, then

$$
\begin{aligned}
&\int_0^T \int_{B(r)} |u_m(x,t) - u_n(x,t)|\,dx\,dt \\
&\quad \le \int_0^T \int_{B(r)} |u_n(x,t) - u_n(\delta,x,t)|\,dx\,dt \\
&\qquad + \int_0^T \int_{B(r)} |u_n(\delta,x,t) - u_m(\delta,x,t)|\,dx\,dt \\
&\qquad + \int_0^T \int_{B(r)} |u_m(\delta,x,t) - u_m(x,t)|\,dx\,dt \\
&\quad \le 4\delta\lambda^{3/4} \max_{x\in B(r+\delta)}[\gamma(x)]\Big(\min_{x\in B(r+\delta)}[\gamma(x)]\Big)^{-1/2} C_{r+\delta} \\
&\qquad + \int_0^T \int_{B(r)} |u_n(\delta,x,t) - u_m(\delta,x,t)|\,dx\,dt
\end{aligned}
\tag{50}
$$

When $n \wedge m$ is sufficiently large, the last term is smaller than $6\delta C$. We use a diagonalization argument and Step 3 with $\delta = 1, \frac{1}{2}, \frac{1}{4}, \ldots$ to obtain a subsequence, also denoted $\{u_{n(k)}(\cdot,\cdot)\}$, such that (47) holds.

Step 3. From (47), it is easy to see that $\{\sqrt{u_{n(k)}(\cdot,\cdot)}\}$ converges to $\{\sqrt{u(\cdot,\cdot)}\}$ a.e. Therefore, for each $0 < \varepsilon < 1$,

$$
\int_0^T \int_{B(r)} \left|\sqrt{u_{n(k)}(x,t)} - \sqrt{u(x,t)}\right|^{2-\varepsilon} dx\,dt \to 0. \tag{51}
$$

For each $1 \le q \le \beta$, denote by β the Sobolev conjugate of $2-\varepsilon$. When ε is sufficiently small, $\beta > 5$. Let $\theta > 0$ be the solution of

$$
\frac{1}{q} = \frac{\theta}{2-\varepsilon} + \frac{1-\theta}{\beta}.
$$

By Hölder's inequality,

$$
\begin{aligned}
&\int_0^T \int_{B(r)} \left|\sqrt{u_{n(k)}(x,t)} - \sqrt{u(x,t)}\right|^q dx\,dt \\
&\quad = \int_0^T \int_{B(r)} \left|\sqrt{u_{n(k)}(x,t)} - \sqrt{u(x,t)}\right|^{q\theta + q(1-\theta)} dx\,dt \\
&\quad \le \left[\int_0^T \int_{B(r)} \left|\sqrt{u_{n(k)}(x,t)} - \sqrt{u(x,t)}\right|^{2-\varepsilon} dx\,dt\right]^{q\theta/(2-\varepsilon)} \\
&\qquad \times \left[\int_0^T \int_{B(r)} \left|\sqrt{u_{n(k)}(x,t)} - \sqrt{u(x,t)}\right|^{\beta} dx\,dt\right]^{(1-\theta)/\beta}
\end{aligned}
$$

By the Gagliardo–Nirenberg–Sobolev inequality ([4], page 138), the second factor on the right-hand side is bounded, so we deduce (42) from (51).

Step 4. Now we are going to show (43) under the additional assumption (37). We still denote $n(k)$ as n. Take derivatives on both sides of the equation:

$$v\left(\frac{1}{n},x,t\right)=V^{-1}\left(\frac{1}{n}\right)\int\kappa_{1/n}(x-y)u_n(t,y)q\left(y,v\left(\frac{1}{n},y,t\right)\right)dy. \tag{52}$$

Then we get from the integration by parts formula,

$$\begin{aligned}
&\frac{\partial}{\partial x}v\left(\frac{1}{n},x,t\right)\\
&\quad=V^{-1}\left(\frac{1}{n}\right)\int\kappa_{1/n}(x-y)\frac{\partial}{\partial y}u_n(t,y)q\left(y,v\left(\frac{1}{n},y,t\right)\right)dy\\
&\qquad+V^{-1}\left(\frac{1}{n}\right)\int\kappa_{\varepsilon}(x-y)u_n(t,y)\frac{\partial}{\partial y}q\left(y,v\left(\frac{1}{n},y,t\right)\right)dy\\
&\qquad+V^{-1}\left(\frac{1}{n}\right)\int\kappa_{\varepsilon}(x-y)\frac{\partial}{\partial y}u_n(t,y)\frac{\partial}{\partial v}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\frac{\partial v}{\partial y}\,dy.
\end{aligned}$$

So

$$\begin{aligned}
&\left|\frac{\partial}{\partial x}v\left(\frac{1}{n},x,t\right)\right|\\
&\quad\le V^{-1}\left(\frac{1}{n}\right)\int\kappa_{1/n}(x-y)\left|\frac{\partial}{\partial y}u_n(t,y)\right|q\left(y,v\left(\frac{1}{n},y,t\right)\right)dy\\
&\qquad+V^{-1}\left(\frac{1}{n}\right)\int\kappa_{1/n}(x-y)u_n(t,y)\left|\frac{\partial}{\partial y}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|dy\\
&\qquad+V^{-1}\left(\frac{1}{n}\right)\int\kappa_{1/n}(x-y)u_n(t,y)\left|\frac{\partial}{\partial v}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|\left|\frac{\partial v}{\partial y}\right|dy.
\end{aligned}$$

Thus, take the integral with respect to $I_{B(r)}V^{-1}(1/n)u_n(x,t)q(x,v(1/n,x,t))\times dx$ on both sides:

$$\begin{aligned}
&\int_{B(r)}u_n(x,t)q\left(x,v\left(\frac{1}{n},x,t\right)\right)\left|\frac{\partial}{\partial x}v\left(\frac{1}{n},x,t\right)\right|dx\\
&\quad\le\int_{B(r)}v(\varepsilon,y,t)\left|\frac{\partial}{\partial y}u_n(t,y)\right|q\left(y,v\left(\frac{1}{n},y,t\right)\right)dy\\
&\qquad+\int_{B(r)}v\left(\frac{1}{n},y,t\right)u(t,y)\left|\frac{\partial}{\partial y}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|dy\\
&\qquad+\int_{B(r)}u_n(y,t)v\left(\frac{1}{n},y,t\right)\left|\frac{\partial}{\partial v_n}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|\left|\frac{\partial v}{\partial y}\right|dy\\
&\quad\le\int_{B(r)}v\left(\frac{1}{n},y,t\right)\left|\frac{\partial}{\partial y}u_n(t,y)\right|q\left(y,v\left(\frac{1}{n},y,t\right)\right)dy
\end{aligned}$$

$$+\int_{B(r)} v\left(\frac{1}{n},y,t\right)u(t,y)\left|\frac{\partial}{\partial y}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|dy$$
$$+\int_{B(r)} u_n(y,t)(1-c)q\left(y,v\left(\frac{1}{n},y,t\right)\right)\left|\frac{\partial}{\partial y}v\left(\frac{1}{n},y,t\right)\right|dy.$$

The last inequality is from Lemma 3. That is,

$$c\int_{B(r)} u_n(x,t)q\left(x,v\left(\frac{1}{n},x,t\right)\right)\left|\frac{\partial}{\partial x}v\left(\frac{1}{n},x,t\right)\right|dx$$
$$\le \int_{B(r)} v\left(\frac{1}{n},y,t\right)\left|\frac{\partial}{\partial y}u_n(t,y)q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|dy$$
$$+\int_{B(r)} v\left(\frac{1}{n},y,t\right)u_n(t,y)\left|\frac{\partial}{\partial y}q\left(y,v\left(\frac{1}{n},y,t\right)\right)\right|dy.$$

The right-hand side is uniformly bounded in n by Steps 1–3. Now we have

$$\int_0^T\int_{B(r)}\left|\frac{\partial}{\partial x}\left[u_n(x,t)v\left(\frac{1}{n},x,t\right)\right]\right|dx\,dt$$
$$\le \int_0^T\int_{B(r)}\left|u_n(x,t)\frac{\partial}{\partial x}v\left(\frac{1}{n},x,t\right)\right|dx\,dt$$
$$+\int_0^T\int_{B(r)}\left|v\left(\frac{1}{n},x,t\right)\frac{\partial}{\partial x}u_n(x,t)\right|dx\,dt$$
$$\le \int_0^T\int_{B(r)}\left|u_n(x,t)\frac{\partial}{\partial x}v\left(\frac{1}{n},x,t\right)\right|dx\,dt$$
$$+\int_0^T\int_{B(r)}\left|v\left(\frac{1}{n},x,t\right)\frac{\partial}{\partial x}\sqrt{u_n(x,t)}\right|dx\,dt.$$

Thus we deduce that $\partial/\partial x[v(1/n,x,t)u_n(x,t)]$ is bounded in L_1. Repeating Steps 1–3, there is a subsequence $v(1/n(k),x,t)u_{n(k)}(x,t)$ converging in L_1 ([4], page 146) to some $\eta(x,t)$. Thus we deduce that

$$v\left(\frac{1}{n(k)},x,t\right)\to\frac{\eta(x,t)}{u(x,t)}\quad\text{a.e.}$$

whenever $u(x,t)\neq 0$. Therefore,

$$v\left(\frac{1}{n(k)},x,t\right)=\int\kappa_{1/(n(k))}(x-y)u_{n(k)}(y,t)q\left(y,v\left(\frac{1}{n(k)},y,t\right)\right)dy$$
$$\to u(x,t)q\left(x,\frac{\eta(x,t)}{u(x,t)},t\right).$$

Since $v(1/n,\cdot,t)$ is bounded in L_1, it is easy to see that, for almost all (x,t), $\eta(x,t)=0$ when $u(x,t)=0$. Hence

$$\eta(x,t)=u^2(x,t)q\left(x,\frac{\eta(x,t)}{u(x,t)}\right).$$

Therefore,

$$\eta(x,t) = u^2(x,t)p(x,u(x,t)),$$

as $v = \xi(x,v)q(x,v)$ has a unique solution $\xi(x,v)$. □

Acknowledgments. The author sincerely thanks R. Carmona, Z. Q. Chen, K. L. Chung, M. Schechter, D. Stroock, H. Tucker, R. J. Williams and an anonymous referee for helpful comments and suggestions on a preliminary version of this paper.

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