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THE FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR STATIONARY STRONGLY MIXING SEQUENCES

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Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of real-valued mean zero random variables. Let $(\alpha_n)_{n>0}$ be the sequence of strong mixing coefficients. We define the *strong mixing function* $\alpha(\cdot)$ by $\alpha(t) = \alpha_{[t]}$ and we denote by Q the quantile function of $|X_0|$. Assume that

$$(*) \qquad \int_0^1 \alpha^{-1}(t) Q^2(t)\, dt < \infty,$$

where f^{-1} denotes the inverse of the monotonic function f. The main result of this paper is that the functional law of the iterated logarithm (LIL) holds whenever $(X_i)_{i \in \mathbb{Z}}$ satisfies $(*)$. Moreover, it follows from Doukhan, Massart and Rio that for any positive a there exists a stationary sequence $(X_i)_{i \in \mathbb{Z}}$ with strong mixing coefficients α_n of the order of n^{-a} such that the bounded LIL does not hold if condition $(*)$ is violated. The proof of the functional LIL is mainly based on new maximal exponential inequalities for strongly mixing processes, which are of independent interest.

1. Introduction and results. Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of real-valued mean zero random variables with finite variace. As a measure of dependence, we will use the strong mixing coefficients introduced by Rosenblatt (1956). For any two σ-algebras $\mathscr{A}$ and $\mathscr{B}$ in $(\Omega, \mathscr{T}, \mathbb{P})$, let

$$\begin{aligned}\alpha(\mathscr{A}, \mathscr{B}) &= \sup_{(A,B) \in \mathscr{A} \times \mathscr{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \\ &= \sup_{(A,B) \in \mathscr{A} \times \mathscr{B}} |\mathrm{Cov}(\mathbb{I}_A, \mathbb{I}_B)| \le 1/4.\end{aligned}$$

The strong mixing coefficients $(\alpha_n)_{n>0}$ of the sequence $(X_i)_{i \in \mathbb{Z}}$ are defined by $\alpha_n = \sup_{k \in \mathbb{Z}} \alpha(\mathscr{F}_k, \mathscr{G}_{k+n})$, where $\mathscr{F}_k = \sigma(X_i\colon i \le k)$ and $\mathscr{G}_l = \sigma(X_i\colon i \ge l)$. We make the convention that $\alpha_0 = 1/4$. $(X_i)_{i \in \mathbb{Z}}$ is called a strongly mixing sequence if $\lim_{n \to +\infty} \alpha_n = 0$. Examples of such sequences may be found in Davydov (1973), Bradley (1986) and Doukhan (1994).

For stationary strongly mixing sequences, the law of the iterated logarithm (LIL) and the functional LIL may fail to hold when only the variance of the r.v.'s is assumed to be finite [see Davydov (1973)]. Let us recall what is currently known on this topic.

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As far as we know, all the results concerning the functional central limit theorem (CLT) or the functional LIL for stationary strongly mixing sequences are of the following type. Assume that for some adequate function ϕ, $\phi(X_0^2)$ is integrable and that the mixing coefficients satisfy some summability condition (depending of course on ϕ). Then the CLT and the LIL hold.

The first result of this type was Ibragimov's (1962) CLT: he took $\phi(x) = x^r$ with $r > 1$ and gave the summability condition

$$\sum_{n>0} \alpha_n^{1-1/r} < +\infty. \tag{1.1}$$

[By CLT, we mean that the distribution of $n^{-1/2}\sum_{i=1}^n X_i$ is weakly convergent to a (possibly degenerate) normal distribution.] The functional CLT [by functional CLT, we mean that Donsker's normalized polygonal line converges weakly in the Skorohod space $D([0,1])$ to some (possibly degenerate) Wiener measure] was studied by Davydov (1968): he obtained the summability condition $\sum_{n>0} \alpha_n^{1/2-1/(2r)} < +\infty$. Next, Oodaira and Yoshihara (1972) obtained the functional CLT under condition (1.1).

Since a polynomial moment condition is not well adapted to exponential mixing rates, Herrndorf (1985) introduced more flexible moment assumptions. Let $\mathscr{F}$ denote the set of convex and increasing differentiable functions ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and $\lim_{x\to+\infty} x^{-1}\phi(x) = \infty$. Assume that ϕ belongs to $\mathscr{F}$; then Herrndorf obtained the functional CLT under the summability condition

$$\sum_{n>0} \alpha_n \phi^{-1}(1/\alpha_n) < +\infty, \tag{1.2}$$

where ϕ^{-1} denotes the inverse function of ϕ.

As far as we know, the most important result concerning the LIL was obtained by Oodaira and Yoshihara: if the strong mixing coefficients satisfy (1.1) for some $r > 1$ and if $\mathbb{E}|X_0|^{2s} < \infty$ for some $s > r$, then the LIL and the functional LIL hold [see Oodaira and Yoshihara (1971a), Theorem 5 and Oodaira and Yoshihara (1971b), Theorem 1(IV)].

Neither these central limit theorems nor this LIL are known to be optimal. Recently, Doukhan, Massart and Rio (1994) improved on Ibragimov's CLT and Herrndorf's CLT: they obtained a sharp condition on the tail function $t \to \mathbb{P}(|X_0| > t)$ and on the mixing rate implying the CLT and the functional CLT. By sharp condition, we mean that, given some rate of mixing and a tail function violating this condition, one can construct a strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$ with corresponding tail function and mixing rate for which the CLT does not hold. Moreover, they proved that this condition also is sharp for the bounded LIL. So, the aim of this paper is to provide the functional LIL under this sharp condition.

All the previous results rely on covariance inequalities [such as Davydov (1968)] which hold under moment assumptions. The approach of Doukhan, Massart and Rio (1994) to improve on the previous results is based on a new covariance inequality recently established by Rio (1993), which introduced an explicit dependence between the mixing coefficients and the tail function. Let us now introduce some notation that we shall use throughout this paper.

Notation. If (u_n) is a nonincreasing sequence of nonnegative real numbers, we denote by $u(\cdot)$ the rate function defined by $u(t) = u_{[t]}$, where the square brackets designate the integer part. For any nonincreasing function f, let f^{-1} denote the cadlag inverse function of f:

$$f^{-1}(u) = \inf\{t: f(t) \le u\}.$$

For any random variable X with distribution function F, we denote indifferently by Q_X or Q_F the quantile function, which is the inverse of the tail function $t \to \mathbb{P}(|X| > t)$.

The main result. Our way to prove the functional LIL is to obtain a strong invariance principle in the sense of Strassen (1964). Then the functional LIL for the sequence $(X_i)_{i \in \mathbb{Z}}$ follows from the functional LIL for the Brownian motion [see Strassen (1964)]. Since the proof needs the variance inequalities established by Rio (1993) and the central limit theorem of Doukhan, Massart and Rio (1994), we first summarize these results in a single theorem.

THEOREM 1. *Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of real-valued centered random variables such that*

$$M_{2,\alpha} = \int_0^1 \alpha^{-1}(u) Q_{X_0}^2(u)\, du < +\infty. \tag{1.3}$$

Let $S_n = \sum_{i=1}^n X_i$. Then:

(i) *The series $\sum_{t \in \mathbb{Z}} \operatorname{Cov}(X_0, X_t)$ is absolutely convergent to a nonnegative number σ^2 and $n^{-1} \operatorname{Var} S_n$ converges to σ^2.*

(ii) *$S_n / \sqrt{n}$ converges in distribution to the (possibly degenerate) normal distribution $N(0, \sigma^2)$.*

Let us now state our strong invariance principle.

THEOREM 2. *Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of real-valued centered random variables satisfying* (1.3). *Then there exists a sequence $(Y_i)_{i > 0}$ of independent $N(0, \sigma^2)$-distributed random variables such that*

$$\sum_{i=1}^n (X_i - Y_i) = o\left(\sqrt{n \log\log n}\right) \quad a.s.,$$

where σ^2 is defined by Theorem 1(i).

COMMENT. When $\sigma \neq 0$, Theorem 2 and Strassen's functional LIL for the Brownian motion then yield Strassen's functional LIL for the partial sum process

$$\left\{\sigma^{-1}(2n \log\log n)^{-1/2} X_{[nt]}: t \in [0,1]\right\}.$$

We refer the reader to Strassen (1964) for more about this.

Applications. Let us now discuss the scope of (1.3) for strongly mixing sequences. We start by treating the case of bounded random variables.

1. *Bounded random variables.* If X_0 is a bounded r.v., Q is uniformly bounded over $[0,1]$, and (1.3) is equivalent to Ibragimov's condition for the CLT, $\sum_{n>0} \alpha_n < +\infty$ [see Ibragimov and Linnik (1971), Theorems 18.5.3 and 18.5.4].
2. *Conditions on the tail function.* Let ϕ be some element of $\mathscr{F}$. Assume that there exists some positive constant C_ϕ such that the distribution of X_0 satisfies

$$\mathbb{P}(X_0^2 > u) \leq 1/\phi(u/C_\phi).$$

   If $x \to x^{-r}\phi(x)$ is nondecreasing for some $r > 1$, (1.3) holds as soon as the summability condition (1.2) is satisfied. Hence the functional LIL is ensured by a weaker condition on the distribution of X_0 than Herrndorf's moment condition for the CLT, $\mathbb{E}(\phi(X_0^2)) < +\infty$.
3. *Moment conditions.* Assume that $\mathbb{E}(\phi(X_0^2)) < +\infty$ for some $\phi \in \mathscr{F}$. An elementary calculation [see Rio (1993)] shows that (1.3) holds if

$$\sum_{n>0} (\phi')^{-1}(n)\alpha_n < +\infty. \tag{1.4}$$

   This summability condition is weaker than (1.2) [see Rio (1993)]. In particular, when $\phi(x) = x^r$ for some $r > 1$, (1.4) holds if and only if the series $\sum_{k>0} k^{1/(r-1)}\alpha_k$ is convergent, which improves on (1.1).
4. *Exponential mixing rates.* Assume that the mixing coefficients satisfy $\alpha_k = O(a^k)$ for some a in $]0,1[$. Then (1.3) holds if

$$\mathbb{E}(X_0^2 \log^+|X_0|) < +\infty. \tag{1.5}$$

   It is worth noticing that (1.5) cannot be derived from the classical covariance inequalities [see Doukhan (1994) for a review of the previous inequalities].

Theorem 2 is sharp for power-type mixing rates and strongly mixing sequences, as proved by Proposition 3 in Doukhan, Massart and Rio (1994), which we now recall.

THEOREM 3. *Let $a > 1$ be given and let F be any continuous distribution function of a zero-mean real-valued random variable such that*

$$\int_0^1 u^{-1/a} Q_F^2(u)\, du = +\infty.$$

Then there exists a stationary Markov chain $(Z_i)_{i\in\mathbb{Z}}$ of r.v.'s with d.f. F such that:

(i) $0 < \liminf_{n\to+\infty} n^a \alpha_n \le \limsup_{n\to+\infty} n^a\alpha_n < \infty$. *Here $(\alpha_n)_{n>0}$ denotes the sequence of strong mixing coefficients of $(Z_i)_{i\in\mathbb{Z}}$.*

(ii) *Setting $S_n = \sum_{i=1}^n Z_i$, we have*

$$\limsup_{n\to+\infty} \frac{|S_n|}{\sqrt{n\log\log n}} = +\infty \quad a.s.$$

Let us now give an outline of the proof of Theorem 2. Since Skorohod's embedding does not work for strongly mixing sequences, it is necessary to extend Bernstein's inequality for partial sums of independent sequences to strongly mixing sequences [the available Bernstein-type inequalities for strongly mixing sequences of Doukhan, León and Portal (1984) and Bosq (1992) are far from being optimal]. However, there is some loss in these inequalities. Contrary to the independent case, the truncation of the random variables X_n at a level of the order of $(n/LLn)^{1/2}$ does not work: we have to use a *blocking technique*. So, it is essential to obtain precise upper bounds for $\mathbb{E}(|S_n|\mathbb{1}_{|S_n|\ge a_n})$ for some level of truncation a_n to be defined later. This will be done via an extension of the inequalities of Fuk and Nagaev (1971) to the strongly mixing case, which improves on the previous inequalities of Utev (1985). Since these inequalities are mainly based on Bennett's exponential inequality [see Pollard (1984), page 192] for independent summands, we will start by proving an efficient Bennett-type inequality for strongly mixing random variables.

Maximal inequalities for strongly mixing sequences. Let us first state the main tool for proving these inequalities, which is the following new coupling theorem for real-valued random variables. This result generalizes Berbee's (1979) coupling lemma for β-mixing random variables to strongly mixing real-valued random variables.

THEOREM 4. *Let $\mathscr{A}$ be a σ-field of $(\Omega,\mathscr{T},\mathbb{P})$ and let X be a real-valued random variable taking a.s. its values in $[a,b]$. Suppose furthermore that there exists a random variable δ with uniform distribution over $[0,1]$, independent of $\mathscr{A}\vee\sigma(X)$. Then there exists some random variable X^* independent of $\mathscr{A}$ and with the same distribution as X such that*

$$\mathbb{E}(|X - X^*|) \le 2(b-a)\,\alpha(\mathscr{A},\sigma(X)).$$

Moreover, X^* is a $\mathscr{A}\vee\sigma(X)\vee\sigma(\delta)$-measurable random variable.

In Section 3, we will derive the following Bennett-type inequality from Bennett's exponential inequality for independent random variables and Theorem 4.

THEOREM 5. *Let $(Y_i)_{i>0}$ be a strongly mixing sequence of mean-zero real-valued random variables each a.s. bounded by some positive M, with sequence of strong mixing coefficients $(\alpha_k)_{k>0}$. Suppose now that $Y_i = 0$ a.s. if $i > n$. Let k be any positive integer and let v_k be any positive real such that*

$$v_k \geq \sum_{i>0} \operatorname{Var}\Big(\sum_{ik-k<l\leq ik} Y_l \Big).$$

Then, for any positive t,

$$\mathbb{P}\Big(\sup_{j\leq n} \Big| \sum_{i=1}^{j} Y_i \Big| \geq 2Mk - 2t \Big) \leq 4\exp\Big(-\frac{v_k}{(kM)^2} h\Big(\frac{tkM}{v_k} \Big) \Big) + 4n\frac{\alpha_k}{k}$$
$$\leq 4\exp\Big(-\frac{t}{2kM} \log\Big(1 + \frac{tkM}{v_k} \Big) \Big) + 4n\frac{\alpha_k}{k},$$

where $h(x) = (1+x)\log(1+x) - x$.

In Section 4, we will derive the following Fuk–Nagaev-type inequality for strongly mixing sequences from Theorem 5. The corresponding inequality for independent random variables was done by Petrov (1989), who applied it to moment inequalities. In a recent note [see Rio (1994)], we gave some applications of this new inequality for strongly mixing sequences to maximal moment inequalities for partial sums of a strongly mixing sequence. These moment inequalities improve on the previous inequalities of Yokoyama (1980) and Doukhan and Portal (1983).

THEOREM 6. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strongly mixing sequence of real-valued integrable random variables with mean zero. For any positive integer i, we set $Q_i = Q_{X_i}$. For any u in $[0,1]$, let*

$$v_{n,u} = \sum_{i=1}^{n} \int_0^1 [\alpha^{-1}(t\vee u) \wedge n] Q_i^2(t\vee u)\, dt.$$

Let the quantile function Q_* be defined by

$$tQ_*(t) = \sup_{i>0} \int_0^t Q_i(u)\, du. \tag{1.6}$$

Then there exists some positive constant c such that, for any positive integer n, the following inequality holds: for any $r\geq 1$, any u in $]0,\alpha_0[$ and any $t \geq [\alpha^{-1}(u)\wedge n]Q_*(u)$,

$$\mathbb{P}(S_n^* \geq crt) \leq c\big[1 + \big(rt^2/v_{n,u}\big)\big]^{-r} + cnu\big[\alpha^{-1}(u)\wedge n\big]^{-1}$$

REMARK. Let

$$k = \alpha^{-1}(u) \wedge n \quad \text{and} \quad \overline{U}_i = \sum_{l=ik+1-k}^{ik \wedge n} X_l \mathbb{1}_{|X_e| < Q_l(u)}.$$

It follows from the proof of Theorem 6 [see Section 4] that (a) still holds after replacing $v_{n,u}$ by $\bar{v}_{n,u} = \sum_{i>0} \operatorname{Var} \overline{U}_i$.

2. A coupling result for strongly mixing real-valued random variables. In this section, we prove Theorem 4. The random variable X^* will be defined from the initial random variable X by means of the conditional quantile transformation. The main idea behind the proof is that the quantile transformation minimizes the L^1-distance between X and X^* [see Major (1978)].

Let $F_{\mathscr{A}}$ denote the conditional distribution function of X, which is defined by

$$F_{\mathscr{A}}(t) = \mathbb{P}(X \le t \mid \mathscr{A}).$$

We set $F_{\mathscr{A}}(t-0) = \lim_{s \nearrow t} F_{\mathscr{A}}(s)$. Let F denote the distribution function of X.

Since δ is independent of $\mathscr{A} \vee \sigma(X)$ and has the uniform distribution over $[0,1]$,

$$V = F_{\mathscr{A}}(X-0) + \delta(F_{\mathscr{A}}(X) - F_{\mathscr{A}}(X-0)) \tag{2.1}$$

is independent of $\mathscr{A}$ and has the uniform distribution over $[0,1]$. It follows that

$$X^* = F^{-1}(V) \tag{2.2}$$

is independent of $\mathscr{A}$ and has the same distribution function as X. Now we have to bound $\mathbb{E}(|X - X^*|)$ from above. By (3.1), $X = F_{\mathscr{A}}^{-1}(V)$ almost surely. Hence

$$\mathbb{E}(|X - X^*|) = \mathbb{E}\left(\int_0^1 |F_{\mathscr{A}}^{-1}(v) - F^{-1}(v)|\, dv\right). \tag{2.3}$$

Since X takes its values in $[a, b]$,

$$\int_0^1 |F_{\mathscr{A}}^{-1}(v) - F^{-1}(v)|\, dv = \int_a^b |F_{\mathscr{A}}(t) - F(t)|\, dt. \tag{2.4}$$

So, we have

$$\mathbb{E}(|X - X^*|) = \int_a^b \mathbb{E}(|F_{\mathscr{A}}(t) - F(t)|)\, dt. \tag{2.5}$$

Now, for any real t,

$$\mathbb{E}(|F_{\mathscr{A}}(t) - F(t)|) \le 2\alpha(\mathscr{A}, \sigma(X)),$$

which, together with (2.5), implies Theorem 4.

3. A Bennett-type inequality for strongly mixing sequences. In this section, we prove Theorem 5. The proof will be done using the coupling result stated in Theorem 4.

For any positive integer i, let $U_i = \sum_{ik-k<l\le ik} Y_l$. Since $Y_l = 0$ if $l > n$,

$$U_{2i} = U_{2i-1} = 0 \quad \text{a.s. if } i > \left[\frac{n}{2k}\right] + 1,$$

where the square brackets designate the integer part. Since

$$\sup_{j\le n}\left|\sum_{i=1}^{j} Y_i\right| \le Mk + \sup_{j>0}\left|\sum_{i=1}^{j} U_i\right|,$$

Theorem 5 follows from the inequality below: for any positive t,

$$\mathbb{P}\left(\sup_{j>0}\left|\sum_{i=1}^{j} U_i\right| \ge Mk + 2t\right) \le 4\exp\left(-\frac{v_k}{(kM)^2} h\left(\frac{tkM}{v_k}\right)\right) + 4n\frac{\alpha_k}{k}, \tag{3.1}$$

where $h(x) = (1+x)\log(1+x) - x$. The second part of the inequality is an immediate consequence of the elementary fact that

$$h(x) = x\int_0^1 \log(1+tx)\,dt \ge x\log(1+x)\int_0^1 t\,dt = x\log(1+x)/2.$$

To prove (3.1), let $(\delta_j)_{j>0}$ be a sequence of independent random variables with uniform distribution over $[0,1]$, independent of $(U_j)_{j>0}$. By Theorem 4, for any positive i, there exists a measurable function F_i such that $U_i^* = F_i(U_1,\dots,U_{i-1},U_i,\delta_i)$ satisfies the conditions of Theorem 4 with $\mathscr{A} = \sigma(U_l\colon l < i-1)$. The sequence $(U_i^*)_{i>0}$ so defined has the following properties:

1. For any positive i, the random variable U_i^* has the same distribution as U_i.
2. The random variables $(U_{2i}^*)_{i>0}$ are independent and the random variables $(U_{2i-1}^*)_{i>0}$ are independent.
3. Moreover,
$$\sum_{i>0}\mathbb{E}(|U_i - U_i^*|) \le 4nM\alpha_k.$$

Now

$$\sup_{j>0}\left|\sum_{i=1}^{j} U_i\right| \le \sup_{j>0}\left|\sum_{i=1}^{j} U_{2i}^*\right| + \sup_{j>0}\left|\sum_{i=1}^{j} U_{2i-1}^*\right| + \sum_{i>0}|U_i - U_i^*|. \tag{3.2}$$

By property 3 and Markov's inequality,

$$\mathbb{P}\left(\sum_{i>0}|U_i - U_i^*| \ge Mk\right) \le 4n\frac{\alpha_k}{k}. \tag{3.3}$$

In view of (3.2) and (3.3) it only remains to prove that

$$\mathbb{P}\left(\sup_{j>0}\left|\sum_{i=1}^{j} U_{2i}^*\right| \ge t\right) \le 2\exp\left(-\frac{v_k}{(kM)^2} h\left(\frac{tkM}{v_k}\right)\right) \tag{3.4a}$$

and

$$\mathbb{P}\left(\sup_{j>0}\left|\sum_{i=1}^{j} U_{2i-1}^*\right| \ge t\right) \le 2\exp\left(-\frac{v_k}{(kM)^2} h\left(\frac{tkM}{v_k}\right)\right). \tag{3.4b}$$

The proofs of (3.4a) and (3.4b) being similar, we only prove (3.4a). By property 2, the random variables $(U_{2i}^*)_{i>0}$ are independent. Now, by property 1, for any positive i, $\operatorname{Var} U_i^* \le \operatorname{Var} U_i$ and $\|U_i^*\|_\infty \le Mk$, so we may apply the maximal version of Bennett's inequality [see Pollard (1984), page 192], and (3.4a) follows, therefore completing the proof of Theorem 5.

4. A Fuk–Nagaev type inequality for strongly mixing sequences. In this section, we prove Theorem 6.

Throughout this section $(X_i)_{i\in\mathbb{Z}}$ is a strongly mixing sequence of integrable real-valued random variables with mean zero. For the sake of brevity, let $Q_i = Q_{X_i}$. In order to prove Theorem 6, we need the following upper bound on $\operatorname{Var} S_n$, which is stated in Theorem 1.2 in Rio (1993).

PROPOSITION 1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strongly mixing sequence of real-valued random variables. Then*

$$\operatorname{Var} S_n \le 8 \sum_{i=1}^{n} \int_0^1 [\alpha^{-1}(x) \wedge n] Q_i^2(x)\, dx.$$

PROOF OF THEOREM 6. We set

$$\bar{X}_i = X_i \mathbb{1}_{|X_i| < Q_i(u)} \quad \text{and} \quad \tilde{X}_i = |X_i - \bar{X}_i|. \tag{4.1}$$

Let $k = \alpha^{-1}(u) \wedge n$. When $n < \alpha_0$, $k > 0$. We set $\bar{U}_i = \sum_{l=ik+1-k}^{ik\wedge n} \bar{X}_l$. Clearly,

$$\sup_{j\le n} |S_j| \le \sup_{j\le n} \left| \sum_{i=1}^{j} \bar{X}_i \right| + \sum_{l=1}^{n} \tilde{X}_l, \tag{4.2}$$

which, together with the elementary inequality $|\mathbb{E}\bar{X}_i| \le \mathbb{E}(\tilde{X}_i)$ implies that

$$\sup_{j\le n} |S_j| \le \sup_{j\le n} \left| \sum_{i=1}^{j} \left(\bar{X}_i - \mathbb{E}(\bar{X}_i)\right) \right| + \sum_{l=1}^{n} \left(\mathbb{E}(\tilde{X}_l) + \tilde{X}_l\right). \tag{4.3}$$

Let $t(u) = [\alpha^{-1}(u) \wedge n] Q_*(u)$. Noting that

$$Q_{\tilde{X}_l}(x) \le Q_l(u)\, \mathbb{1}_{x\le u} \tag{4.4}$$

and applying Markov's inequality, we get

$$\mathbb{P}\left(\sum_{l=1}^{n} \left(\mathbb{E}(\tilde{X}_l) + \tilde{X}_l\right) \ge t(u) \right) \le \frac{2nuQ_*(u)}{t(u)} = \frac{2nu}{\alpha^{-1}(u) \wedge n}. \tag{4.5}$$

Combining (4.3) and (4.5), it is not difficult to check that the proof of Theorem 6 will be achieved if we prove that, for any $t \ge t(u)$,

$$\mathbb{P}\left(\sup_{j\le n} \left| \sum_{i=1}^{j} \left(\bar{X}_i - \mathbb{E}(\bar{X}_i)\right) \right| \ge crt \right) \le c\left[1 + \frac{rt^2}{v_{n,u}}\right]^{-r} + \frac{cnu}{\alpha^{-1}(u) \wedge n}. \tag{4.6}$$

To prove (4.6), apply Proposition 1 to the random variables $\overline{X}_l$ and note that

$$Q_{\overline{X}_l}(x) \le Q_l(u + x) \le Q_l(u \vee x).$$

Then we obtain

$$\sum_{i>0} \operatorname{Var} \overline{U}_i \le 8v_{n,u}. \tag{4.7}$$

So, we may apply Theorem 5 to the random variables $(\overline{X}_i - \mathbb{E}(\overline{X}_i))_{i\in[1,n]}$ with $M = 2Q_*(u)$, $k = \alpha^{-1}(u) \wedge n$ and $v_k = 8v_{n,u}$, yielding

$$\begin{aligned}&\mathbb{P}\left(\sup_{j\le n}\left|\sum_{i=1}^{j}\left(\overline{X}_i - \mathbb{E}(\overline{X}_i)\right)\right| \ge 2t(u) + crt\right)\\ &\qquad \le c\exp\left(-\frac{rt}{t(u)}\log\left(1 + \frac{rt(u)t}{v_{n,u}}\right)\right) + \frac{cnu}{\alpha^{-1}(u) \wedge n}\end{aligned} \tag{4.8}$$

for some positive constant c. Both the fact that $t \ge t(u)$ and the concavity of $u \to \log(1 + u)$ ensure that

$$\log\left(1 + \frac{rt(u)t}{v_{n,u}}\right) \ge \frac{t(u)}{t}\log\left(1 + \frac{rt^2}{v_{n,u}}\right),$$

and (4.6) follows; hence, Theorem 6. □

5. A strong invariance principle. In this section, we prove Theorem 2. First we prove that the saturated function Q_* defined by (1.6) still satisfies (1.3). Clearly, $Q_*(t) \le 3\sup_{u\le t}(u/t)^{2/3}Q(u)$, which, together with Claim 1 in Doukhan Massart and Rio (1994), ensures that

$$\int_0^1 \alpha^{-1}(u)Q_*^2(u)\,du < \infty. \tag{5.1}$$

Second, we may assume, increasing the numbers α_n if necessary, that $(\alpha_n)_n$ is a strictly decreasing sequence. As usual in mixing sequences theory, the proof is mainly based on a blocking technique. Let Ψ be the following class of increasing functions:

$$\Psi = \left\{\psi\colon \mathbb{N} \to \mathbb{N},\ \psi \text{ increasing},\ \lim_{n\to\infty}\frac{\psi(n)}{n} = \infty,\ \lim_{n\to\infty}\frac{\psi(n)}{n^{5/4}} = 0\right\}.$$

Let ψ be some element of Ψ, which will be defined later on. Let $M_0 = 0$,

$$M_n = \sum_{k=1}^{n}(\psi(k) + k) \tag{5.2a}$$

and define the random variables $(U_n)_{n>0}$, $(V_n)_{n>0}$ and $(U_n')_{n>0}$ by

$$U_n = \sum_{i=M_n+1}^{M_n+\psi(n)} X_i, \qquad V_n = \sum_{i=M_{n+1}+1-n}^{M_{n+1}} X_i, \qquad U_n' = \sum_{i=M_n+1}^{M_{n+1}} |X_i|. \tag{5.2b}$$

(Note that the blocks U_n and V_n have different lengths.) We also define the truncated random variables $(\bar{U}_n)_{n>0}$ from the initial blocks by

$$\bar{U}_n = \left(U_n \wedge crn/\sqrt{LLn}\right) \vee \left(-crn/\sqrt{LLn}\right), \tag{5.2c}$$

where $Lx = \max(1, \log x)$.

The main step of the proof is the following almost sure approximation result.

PROPOSITION 2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary and strongly mixing sequence of real-valued centered random variables satisfying* (1.3). *Then there exists some function ψ in Ψ and some sequence $(W_i)_{i>0}$ of independent $N(0, \psi(n)\sigma^2)$-distributed random variables such that*

$$\text{(a)} \qquad \sum_{i=1}^{n} \left(W_i - \bar{U}_i\right) = o\left(\sqrt{M_n LLn}\right) \quad a.s.,$$

$$\text{(b)} \qquad \sum_{n=1}^{\infty} \frac{\mathbb{E}\left(|U_n - \bar{U}_n|\right)}{n\sqrt{LLn}} < +\infty$$

and

$$\text{(c)} \qquad U_n' = o\left(n\sqrt{LLn}\right) \quad a.s.,$$

where $(M_n)_n$ and $(\bar{U}_n)_n$ are defined from ψ by (5.2).

PROOF. We start by proving (b). Clearly,

$$|U_n - \bar{U}_n| = \sup\left(0, |U_n| - crn/\sqrt{LLn}\right).$$

Hence

$$\mathbb{E}\left(|U_n - \bar{U}_n|\right) = cr\int_{n/\sqrt{LLn}}^{\infty} \mathbb{P}\left(|U_n| > t\right) dt. \tag{5.3}$$

Let $u_n = \inf\{u \geq 0 \colon \alpha^{-1}(u)Q_*(u) \leq n/\sqrt{LLn}\}$. We set $\sigma_1^2 = 8M_{2,\alpha}$,

$$\begin{aligned} A_{1,n} &= \int_{n/\sqrt{LLn}}^{\infty} \left(1 + \frac{t^2}{\psi(n)\sigma_1^2}\right)^{-r} dt, \\ A_{2,n} &= \int_0^{u_n} -u\, dQ_*(u) + \sum_{\{k \colon \alpha_k \leq u_n\}} Q_*(\alpha_k)\frac{\alpha_k}{k}. \end{aligned} \tag{5.4a}$$

Applying Theorem 6 and noting that $v_{n,u} \leq n\sigma_1^2$, we get

$$\mathbb{E}\left(|U_n - \bar{U}_n|\right) \leq C_r\left(A_{1,n} + \psi(n)A_{2,n}\right) \tag{5.4b}$$

for some positive constant C_r. On the one hand, since $n^{-5/4}\psi(n)$ converges to 0 as $n \to \infty$, $A_{1,n} = O(n^{-2})$ if $r > 4$, which ensures that

$$\sum_{n>0} \frac{A_{1,n}}{n\sqrt{LLn}} < \infty. \tag{5.5}$$

On the other hand, the series

$$\sum_{n>0} \frac{\psi(n) A_{2,n}}{n\sqrt{LLn}}$$

is convergent for some ψ in Ψ if and only if

$$\sum_{n>0} \frac{A_{2,n}}{\sqrt{LLn}} < +\infty. \tag{5.6}$$

Let

$$A_{3,n} = -\int_0^{u_n} u\, dQ_*(u) \quad \text{and} \quad A_{4,n} = \sum_{\{k:\, \alpha_k \le u_n\}} Q_*(\alpha_k)\frac{\alpha_k}{k}.$$

Clearly $A_{2,n} = A_{3,n} + A_{4,n}$. Since $-u\,dQ_*(u) = [Q_*(u) - Q(u)]\,du$, we have

$$A_{3,n} \le \int_0^{u_n} Q_*(u)\, du. \tag{5.7}$$

Let $\chi(u) = \sum_{\{n:\ u_n > u\}}(LLn)^{-1/2}$. It follows from (5.7) that the series $\sum_{n>0}(LLn)^{-1/2}A_{3,n}$ is convergent if

$$\int_0^1 \chi(u)Q_*(u)\, du < \infty. \tag{5.8}$$

Clearly $u_n > u$ if and only if $\alpha^{-1}(u)Q_*(u) > n(LLn)^{-1/2}$, which ensures that $\chi(u) \sim \alpha^{-1}(u)Q_*(u)$ as $u \searrow 0$ [we may w.l.o.g. assume that $\lim_{u\searrow 0} \alpha^{-1}(u)Q_*(u) = \infty$]. Hence (5.8) is equivalent to (5.1). It only remains to prove that $\sum_{n>0} A_{4,n}(LLn)^{-1/2} < \infty$. Now

$$\sum_{n>0} A_{4,n}(LLn)^{-1/2} = \sum_{k>0} \chi(\alpha_k^-)Q_*(\alpha_k)\frac{\alpha_k}{k}, \tag{5.9}$$

which ensures that the series $\sum_{n>0} A_{4,n}(LLn)^{-1/2}$ is convergent whenever

$$\sum_{k>0} \alpha_k[Q_*(\alpha_k)]^2 < \infty. \tag{5.10}$$

Since

$$\sum_{k>0} \alpha_k[Q_*(\alpha_k)]^2 \le \int_0^1 \alpha^{-1}(u)[Q_*(u)]^2\, du,$$

(5.6) follows. Hence (b) of Proposition 2 holds.

Next we prove (c). Let

$$T_n \doteq \sum_{i=M_n+1}^{M_{n+1}} (|X_i| - \mathbb{E}|X_i|). \tag{5.11}$$

Clearly,

$$U_n' = (\psi(n) + n)\mathbb{E}(|X_1|) + T_n. \tag{5.12}$$

We may, modifying ψ if necessary, assume that $\psi(n) = o(n\sqrt{LLn})$. Now,

$$T_n \le \frac{n}{\sqrt{LLn}} + \sup\big(0, T_n - n/\sqrt{LLn}\big). \tag{5.13}$$

Arguing as in the proof of (b) of Proposition 2, we get that

$$\sum_{n>0} \frac{\mathbb{E}\big(\sup\big(0, T_n - n/\sqrt{LLn}\big)\big)}{n\sqrt{LLn}} < \infty,$$

which, together with Kronecker's lemma, implies that

$$\sup\big(0, T_n - n/\sqrt{LLn}\big) = o\big(n\sqrt{LLn}\big) \quad \text{a.s.} \tag{5.14}$$

Then (5.12), (5.13) and (5.14) yield (c) of Proposition 2.

Finally we prove (a). Throughout, $(\delta_n)_{n>0}$ and $(\eta_n)_{n>0}$ denote independent sequences of independent random variables with uniform distribution over $[0,1]$, independent of $(X_i)_{i\in\mathbb{Z}}$. Theorem 4 together with Skorohod's (1976) lemma ensures that there exists a sequence $(\overline{U}_n^*)_{n>0}$ of independent random variables with the same distribution as the random variables $\overline{U}_n$ such that $\overline{U}_n^*$ is a measurable function of $(\overline{U}_l, \delta_l)_{l\le n}$ and

$$\mathbb{E}\big(|\overline{U}_n - \overline{U}_n^*|\big) \le 4cr\frac{n\alpha_n}{\sqrt{LLn}}. \tag{5.15}$$

It follows from (5.15) that

$$\sum_{n>0} \frac{\mathbb{E}\big(|\overline{U}_n - \overline{U}_n^*|\big)}{n} < \infty,$$

which implies that

$$\sum_{i=1}^{n} \big(\overline{U}_i - \overline{U}_i^*\big) = o\Big(\sqrt{M_n LLn}\Big) \quad \text{a.s.} \tag{5.16}$$

via Kronecker's lemma.

By (i) and (ii) of Theorem 1, $(\psi(n))^{-1}\operatorname{Var} U_n$ converges to σ^2 and $(\psi(n))^{-1/2}U_n$ converges weakly to the normal distribution $N(0,\sigma^2)$. It implies the uniform integrability of the sequence $(U_n^2/\psi(n))_{n>0}$, via Theorem 5.4 in Billingsley (1968). Since the random variables $\overline{U}_n^*$ have the same distribution as the random variables $\overline{U}_n$, it follows from both the above facts, Strassen's representation theorem [see Dudley (1968)] and Skorohod's (1976) lemma that one can construct a sequence $(W_n)_{n>0}$ of $\sigma(\overline{U}_n^*, \eta_n)$-measurable random variables with respective distributions $N(0, \psi(n)\sigma^2)$ in such a way that

$$\mathbb{E}\Big(\big(\overline{U}_n^* - W_n\big)^2\Big) = o(\psi(n)) \quad \text{as } n \to +\infty. \tag{5.17}$$

Let

$$\overline{W}_n = \big(W_n \wedge n/\sqrt{LLn}\big) \vee \big(-n/\sqrt{LLn}\big).$$

Clearly

$$\sum_{n>0} \mathbb{E}\left(|W_n - \overline{W}_n|\right) < \infty,$$

which ensures that

$$\sum_{i=1}^{n} \left(W_i - \overline{W}_i\right) = o\left(\sqrt{M_n LLn}\right) \quad \text{a.s.} \tag{5.18}$$

via Kronecker's lemma.

In view of (5.16), (5.18) and Proposition 2(b), it only remains to prove that

$$\sum_{i=1}^{n} \left(\overline{W}_i + \mathbb{E}\left(\overline{U}_i^*\right) - \overline{U}_i^*\right) = o\left(\sqrt{M_n LLn}\right) \quad \text{a.s.} \tag{5.19}$$

Let us prove (5.19). By definition of $\overline{W}_n$,

$$\operatorname{Var}\left(W_n - \overline{W}_n\right) = o(\psi(n)) \quad \text{as } n \to \infty,$$

which shows that

$$\operatorname{Var}\left(\overline{W}_n - \overline{U}_n^*\right) \leq \varepsilon_n \psi(n) \tag{5.20}$$

for some sequence $(\varepsilon_n)_n$ of positive reals decreasing to 0 as n tends to infinity. Since the random variables $(\overline{W}_i - \overline{U}_i^*)_{i \leq n}$ are independent and a.s. bounded by $(cr + 1)n(LLn)^{-1/2}$, the maximal version of Bernstein's inequality for independent random variables [see Pollard (1984), page 190] yields

$$\begin{aligned} &\mathbb{P}\left(\sup_{j \leq n} \left|\sum_{i=1}^{j} \left(\overline{W}_i + \mathbb{E}\left(\overline{U}_i^*\right) - \overline{U}_i^*\right)\right| \geq Ct\right) \\ &\quad \leq 2\exp\left(-\frac{t^2}{\sum_{i=1}^{2} \varepsilon_i \psi(i)}\right) + 2\exp\left(-\frac{t\sqrt{LLn}}{n}\right) \end{aligned} \tag{5.21}$$

for some positive constant C. Now we may apply (5.21) with $n = 2^N$ and

$$t = t_N = 2n\sqrt{LLn} + \left(2LLn \sum_{i=1}^{n} \varepsilon_i \psi(i)\right)^{1/2}.$$

Using (5.20) and the Borel–Cantelli lemma, we then get (5.19). Hence Proposition 2(a) holds. □

PROOF OF THEOREM 2. By Skorohod's (1976) lemma, there exists a sequence $(Y_i)_{i>0}$ of independent $N(0, \sigma^2)$-distributed random variables such that, for any positive n,

$$W_n = \sum_{i=M_n+1}^{M_n+\psi(n)} Y_i.$$

An elementary calculation on Gaussian random variables shows that

$$\sum_{i=1}^{k} Y_i - \sum_{j=1}^{M_{n(k)}} W_j = o\left(\sqrt{kLLk}\right) \quad \text{a.s.,} \tag{5.22}$$

where $n(k) = \sup\{n \geq 0\colon M_n \leq k\}$. Using (5.22) and Proposition 2(c), it is easily seen that it is sufficient to prove that

$$\sum_{i=1}^{n} (U_i + V_i - W_i) = o\left(\sqrt{M_n \log\log n}\right) \quad \text{a.s.} \tag{5.23}$$

Clearly,

$$\sum_{i=1}^{n} (U_i + V_i - W_i) = \sum_{i=1}^{n} \left(\overline{U}_i - W_i\right) + \sum_{i=1}^{n} \left(U_i - \overline{U}_i\right) + \sum_{i=1}^{n} V_i. \tag{5.24}$$

On the one hand, by (a) and (b) of Proposition 2,

$$\sum_{i=1}^{n} \left(\overline{U}_i - W_i\right) + \sum_{i=1}^{n} \left(U_i - \overline{U}_i\right) = o\left(\sqrt{M_n \log\log n}\right) \quad \text{a.s.} \tag{5.25}$$

On the other hand, the sequence $(V_n)_{n>0}$ has the same properties as $(U_n)_{n>0}$. Hence, using the same arguments as in the proof of (a) and (b) of Proposition 2, one can prove that there exists a sequence $(W'_n)_{n>0}$ of independent $N(0, n\sigma^2)$-distributed random variables such that

$$\sum_{i=1}^{n} (V_i - W'_i) = o\left(\sqrt{M_n \log\log n}\right) \quad \text{a.s.}$$

Now, by the law of the iterated logarithm for Gaussian random variables, $\sum_{i=1}^{n} W'_i = O(n\sqrt{LLn})$ a.s. Hence

$$\sum_{i=1}^{n} V_i = o\left(\sqrt{M_n \log\log n}\right) \quad \text{a.s.} \tag{5.26}$$

Equations (5.24), (5.25) and (5.26) then imply (5.23), and Theorem 2 follows. □

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