The Annals of Probability
1995, Vol. 23, No. 3, 1139–1162

POISSON APPROXIMATION FOR THE FINAL STATE OF A GENERALIZED EPIDEMIC PROCESS

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A so-called generalized epidemic model is considered that describes the spread of an infectious disease of the SIR type with any specified distribution for the infectious period. The statistic under study is the number of susceptibles who ultimately survive the disease. In a pioneering paper, Daniels established for a particular case that when the population is large, this variable may have a Poisson-like behavior. This result was discussed later by several authors. In the present work, a necessary and sufficient condition is derived that guarantees the validity of such a Poisson approximation for the generalized epidemic. The proof relies on two key ideas, namely, the building of an equivalent Markovian representation of the model and the use of a suitable coupling via a random walk.

1. Introduction. There is a considerable body of mathematical literature concerned with the spread of an infectious disease of the susceptible–infected–removed (SIR) type. We refer the reader to the book by Bailey (1975) for the relevant work through 1974 and to the paper by Lefèvre (1990) for a short review of the more recent theory.

A rather general standard model can be defined as follows. Consider a population of n initial susceptible individuals and let the epidemic start by introducing m newly infected individuals. It is assumed that any pair of individuals make contact at the points of a homogeneous Poisson process of rate β and that contacts between different pairs are mutually independent. A susceptible if ever contacted by an infective is infected and becomes immediately infectious. Any infective i, initial or subsequent, is infectious for a period of time of random length D_i. All the D_i's are i.i.d. and distributed as the variable D, say. After that period, the infective dies or is immune, and plays no further role in the infection process: it is regarded as removed from the population.

This model is referred to here as the generalized epidemic process. It extends a classical model, named the general epidemic, which corresponds to the particular case where D is exponentially distributed (of parameter μ, say).

Now, for this generalized epidemic, it is clear that the disease process eventually terminates at some finite time A, as soon as there are no more infectives present in the population. Then a statistic of great relevance is $S(A)$, the ultimate number of susceptibles surviving the disease.

We indicate that when interest is focused on the variable $S(A)$, the generalized epidemic covers another standard model, known as the Reed–Frost

Received April 1994; revised September 1994.
[1]Supported in part by a grant from the Fonds National de la Recherche Scientifique Belge.
AMS 1991 *subject classifications*. 60F05, 92A15
Key words and phrases. Poisson convergence, epidemic process, random walk, coupling.

epidemic, which is obtained by supposing that D is equal to some constant [see, e.g., Von Bahr and Martin-Löf (1980)].

Moreover, it is worth mentioning that as far as $S(A)$ is concerned, the model can be reformulated as a special case of the randomized Reed–Frost epidemic introduced by Von Bahr and Martin-Löf (1980). Briefly, the latter model describes a similar SIR infection schema, but it is here supposed that during its infectious period, any infective i fails to transmit the disease to any given susceptible with the random probability Q_i. All the Q_i's are i.i.d. and distributed as the variable Q, say. Thus, we see that the generalized epidemic corresponds to the situation where $Q = \exp(-\beta D)$. Further extensions have been developed by Martin-Löf (1986) and Picard and Lefèvre (1990).

The distribution of $S(A)$ has been studied extensively. There is no simple explicit form for the exact law, but Picard and Lefèvre (1990) derived a compact expression for its probability generating function in terms of a nonstandard family of polynomials. Much of the research deals with the asymptotic behavior of $S(A)$ as the initial susceptible population size is large ($n \to \infty$) and the initial global infection rate per infective is constant ($\beta = \beta_n/n$ with $\beta_n \to \beta$). Very briefly, under these conditions, there exists a threshold phenomenon such that for small outbreaks the final size $n - S(A)$ is finite and distributed as the total progeny in a branching process (possibly conditioned on extinction), while for large outbreaks, the final size is a positive fraction of n and $S(A)$ has a Gaussian limit approximation [see, e.g., Von Bahr and Martin-Löf (1980)]. The alternative asymptotic approximation of $S(A)$ by a Poisson law has been much less investigated. This is the object of the present paper.

In a pioneering work, Daniels (1967) proved for the general epidemic that when roughly "m is finite, μ/β is large and n is much larger," then $S(A)$ has a Poisson-like behavior. The asymptotic expansions used by Daniels (1967), however, are not very enlightening. A formal and direct proof was given by Sellke (1983); in addition, the assumption that m is small was dispensed with. Lefèvre and Picard (1990) examined a similar approximation of $S(A)$ for the Reed–Frost epidemic, but the results obtained are only partial. For the above generalized epidemic, the question was discussed by Ball (1986) in a heuristic way. Recently, Ball and Barbour (1990) considered that model [and the more general process of Martin-Löf (1986)], and they established, for m small, a Poisson limit theorem for $S(A)$ with an order of magnitude of the accuracy.

Our purpose is precisely to provide a thorough treatment of this problem of Poisson approximation for the final state of the generalized epidemic model. For that, we construct a sequence of epidemics indexed by $n \to \infty$, and defined as above from the parameters (n, m_n, β_n, D). Note that the distribution of D is supposed to be independent of n, which is not restrictive in practice. Let A_n be the end of the epidemic and let $S_n(\infty) \equiv S_n(A_n)$ be the ultimate number of susceptibles, with law denoted by $\mathscr{L}(S_n(\infty))$. The main theorem gives a necessary and sufficient condition on D in order that for any sequence $\{m_n\}$ there exists a sequence $\{\beta_n\}$ such that the distributions $\mathscr{L}(S_n(\infty))$ converge weakly to $\mathscr{P}(b)$, $0 < b < \infty$, as $n \to \infty$.

Our method, natural and powerful, exploits directly the probabilistic structure of the epidemic process. It is based on two key ideas, namely, the building of an equivalent Markovian representation of the model and the application of a suitable coupling via a random walk. We indicate that Ball and Barbour (1990) developed a different technique relying on the Stein–Chen method and another coupling via a branching process.

2. The main result. Consider a sequence of generalized epidemic models indexed by $n \to \infty$ and with parameters (n, m_n, β_n, D). Let $S_n(\infty)$ be the final size of the susceptible population.

DEFINITION 2.1. The quantity $S_n(\infty)$ is said to obey a Poisson limit theorem (PLT) if for any sequence $\{m_n\}$, there exists a sequence $\{\beta_n\}$ such that

$$\mathscr{L}(S_n(\infty)) \to_w \mathscr{P}(b), \qquad 0 < b < \infty, \text{ as } n \to \infty. \tag{2.1}$$

It is clear that if $m_n = 1, n \geq 1$, then for any n and β_n,

$$P(S_n(\infty) = n | m_n = 1) \geq P(D = 0). \tag{2.2}$$

Thus, a PLT implies that

$$P(D = 0) = 0. \tag{2.3}$$

From now on, we will suppose that the condition (2.3) holds true.

Put

$$h(x) = \int_0^x P(D > y)\, dy, \tag{2.4}$$

$$g(x) = \int_x^\infty P(D > y)\, dy, \tag{2.5}$$

$$t(x) = 1 - E[e^{-xD}]. \tag{2.6}$$

Hereafter, we will use the concept of slowly varying function (in Karamata's sense). An extensive treatment of this subject can be found in the book by Bingham, Goldie and Teugels (1987).

PROPOSITION 2.2. *The quantity $S_n(\infty)$ obeys a Poisson limit theorem* (*PLT*) *if and only if the two following conditions are satisfied:*

$$h(x) \textit{ is a slowly varying function}, \tag{2.7}$$

$$x \ln(x) P(D > x)/h(x) \to 0 \quad \textit{as } x \to \infty. \tag{2.8}$$

Moreover, under (2.7) *and* (2.8), *a PLT holds with a given sequence of $\{\beta_n\}$ if and only if this sequence satisfies the condition*

$$(n + m_n) t(\beta_n) - \ln(n/b_n) \to 0 \quad \textit{with } b_n \to b, \textit{ as } n \to \infty, \tag{2.9}$$

or equivalently,

$$(n+m_n)\beta_n h(1/\beta_n) - \ln(n/b_n) \to 0 \quad \textit{with } b_n \to b. \tag{2.10}$$

We will prove in Proposition 6.5 that (2.8) implies

$$h(x)/\ln(x) \to 0 \quad \text{and} \quad xP(D > x) \to 0 \quad \text{as } x \to \infty. \tag{2.11}$$

Therefore, Proposition 2.2 might be valid even when $E(D) = \infty$. This is confirmed in the corollary below.

COROLLARY 2.3. *Suppose that there are positive constants* c *and* x_0 *such that*

$$P(D > x) = c/(x\ln(x)) \quad \textit{for all } x > x_0. \tag{2.12}$$

Then $S_n(\infty)$ *obeys a PLT* [*with* $E(D) = \infty$].

PROOF. Under (2.12), we have, for $x > x_0$,

$$h(x) = \int_0^x P(D > y)\, dy = c_1 \ln\ln(x) + c_2,$$

c_1 and c_2 being appropriate constants. Thus, $h(x)$ is a slowly varying function as required by (2.7). Moreover, we directly obtain that (2.8) too is satisfied, which yields the result. □

For many applications, $E(D)$ will be finite. It is easily seen that Proposition 2.2 then becomes the following corollary.

COROLLARY 2.4. *When* $E(D) < \infty$, $S_n(\infty)$ *obeys a PLT if and only if*

$$x\ln(x)P(D > x) \to 0 \quad \textit{as } x \to \infty. \tag{2.13}$$

Under (2.13), *a PLT holds with any given sequence* $\{\beta_n\}$ *such that*

$$(n+m_n)\beta_n[E(D) - g(1/\beta_n)] - \ln(n/b_n) \to 0 \qquad \textit{with } b_n \to b, \textit{ as } n \to \infty. \tag{2.14}$$

In the references indicated in the Introduction, $E(D)$ is supposed to be finite, and Poisson convergence is generally stated under some specific condition involving $E(D)$. This leads us to introduce the following definition.

DEFINITION 2.5. Let $E(D) < \infty$. Then $S_n(\infty)$ is said to obey a strong Poisson limit theorem (SPLT) when it obeys a PLT with a sequence $\{\beta_n\}$ satisfying the condition

$$(n+m_n)\beta_n E(D) - \ln(n/b_n) \to 0 \quad \text{with } b_n \to b, \text{ as } n \to \infty. \tag{2.15}$$

Consider the Reed–Frost epidemic defined from $(n, m_n, \beta_n, E(D))$ and let $\hat{S}_n(\infty)$ denote the corresponding final number of susceptibles. From Proposition 2.2, we see that in fact (2.15) means that $\hat{S}_n(\infty)$ too obeys a PLT.

COROLLARY 2.6. *Let $E(D) < \infty$. Then a SPLT holds if and only if*

$$\ln(x)g(x) \to 0 \quad \text{as } x \to \infty. \tag{2.16}$$

PROOF. Suppose that $S_n(\infty)$ obeys a SPLT. From (2.14) and (2.15), we get

$$(n + m_n)\beta_n g(1/\beta_n) \to 0, \tag{2.17}$$

and using again (2.15),

$$\ln(n)g(1/\beta_n) \to 0 \quad \text{as } n \to \infty. \tag{2.18}$$

Now we may put $m_n = 1$, $n \geq 1$, in (2.15). Then we obtain that $\ln(1/\beta_n) \sim \ln(n)$, and inserting this in (2.18) yields (2.16). Conversely, suppose that (2.16) holds. Clearly, (2.13) then holds too, so that $S_n(\infty)$ obeys a PLT. It remains to verify that any sequence $\{\beta_n\}$ satisfying (2.15) necessarily satisfies (2.14). This means that the sequence should satisfy (2.17), or equivalently (2.18), which is true by (2.16). □

We indicate that if a SPLT holds, conditions (2.14) and (2.15) are equivalent. In other words, under (2.16), any sequence $\{\beta_n\}$ that yields a PLT is of the form given through (2.15). Indeed, (2.16) implies (2.18), which is equivalent to (2.17) by (2.10), so that (2.14) reduces to (2.15). Moreover, note that condition (2.16) does hold true for the general epidemic and the Reed–Frost epidemic.

The proof of Proposition 2.2 is rather long, yet natural. It has been subdivided into four parts that are presented in the next four sections. In the last section, we examine briefly the accuracy of the approximation, applying in particular some well-known results given by the Stein–Chen method.

3. Using a Markov chain representation of the epidemic.

3.1. *An alternative representation.* To begin with, we are going to associate with the epidemic process an "equivalent" Markov chain model. By equivalent, we mean that the variable under study $S_n(\infty)$ has the same distribution as the state of this Markov chain when stopped at a suitable Markov time. We mention that a similar idea was used before in various works [see, e.g., Ball and Barbour (1990)].

Specifically, we make a change of time scale and we define a new artificial time t, $t = 1, 2, \ldots$, as the cumulative number of removals in the course of real time. Put $X_n(0) = n$ and for $t \geq 1$, let $X_n(t)$ denote the number of individuals that escape infectious contacts with the first t infectives removed. For $t \geq 1$, let D_t be the length of the infectious period of the tth infective removed, and denote by $Z_{n,i}(t)$, $1 \leq i \leq n$, the indicator of the event [the susceptible i escapes contacts with that infective (during D_t)]. Clearly, we have

$$X_n(t) = \sum_{i=1}^{X_n(t-1)} Z_{n,i}(t), \qquad t \geq 1. \tag{3.1}$$

Moreover, for $t \geq 1$, each vector $\mathbf{Z}_n(t) = \{Z_{n,i}(t),\ 1 \leq i \leq n\}$ is a family of n exchangeable variables having mixed Bernoulli distributions with random parameter $Q_{n,t} = \exp(-\beta_n D_t)$. Therefore,

$$X_n(t) =_d \mathscr{M}\mathscr{B}(X_n(t-1), Q_{n,t}), \qquad t \geq 1, \tag{3.2}$$

where $\mathscr{M}\mathscr{B}$ denotes a mixed binomial law. All the vectors $\mathbf{Z}_n(t)$ are independent and all the $Q_{n,t}$'s are i.i.d. and distributed as the variable $Q_n = \exp(-\beta_n D)$. Put $I_n(0) = m_n$ and for $t \geq 1$, let $I_n(t)$ denote the number of infected individuals that are still present just after the tth removal. We have

$$t + X_n(t) + I_n(t) = n + m_n, \qquad t \geq 0. \tag{3.3}$$

We now define T_n as the first time when there are no more infectives, that is,

$$T_n = \inf\{t\colon\ t + X_n(t) = n + m_n\}. \tag{3.4}$$

Intuitively, we feel that the corresponding state $X_n(T_n)$ has the same distribution as the ultimate number of susceptibles $S_n(\infty)$. This result is established in the proposition below.

PROPOSITION 3.1. *The process $\{X_n(t),\ t \geq 0\}$ is a decreasing Markov chain, with*

$$X_n(t) =_d \mathscr{M}\mathscr{B}\left(n, \prod_{s=1}^{t} Q_{n,s}\right), \qquad t \geq 1. \tag{3.5}$$

At time T_n, the state $X_n(T_n)$ has the same law as the variable $S_n(\infty)$, which is provided by the n following relations:

$$E\left\{\binom{X_n(T_n)}{k} \Big/ \left[E(Q_n^k)\right]^{n+m_n-X_n(T_n)}\right\} = \binom{n}{k}, \qquad 1 \leq k \leq n. \tag{3.6}$$

PROOF. The first assertion is obvious from (3.1). The law (3.5) for $X_n(t)$ is obtained from (3.2) by induction and using the well-known fact that (in obvious notation)

$$\mathscr{M}\mathscr{B}(\mathscr{B}(l,u),v) =_d \mathscr{B}(l,uv). \tag{3.7}$$

From (3.5), we get for $1 \leq k \leq n$ that

$$E\left[\binom{X_n(t)}{k}\right] = \binom{n}{k}[E(Q_n^k)]^t, \qquad t \geq 0, \tag{3.8}$$

which shows that the process

$$\left\{\binom{X_n(t)}{k}[E(Q_n^k)]^{-t},\ t \geq 0\right\}$$

forms a martingale. Now, by (3.4), T_n is a Markov time, and applying the optional stopping theorem then yields the n relations (3.6). These constitute a triangular set of n linear equations in the n ultimate state probabilities $P[S_n(T_n) = k]$, $1 \le k \le n$. The probability for $k = 0$ follows. Finally, we note that the system (3.6) is identical with the n relations providing the law of $S_n(\infty)$ [see, e.g., Picard and Lefèvre (1990)]. □

3.2. *Poisson convergence.* As a corollary, a PLT for $S_n(\infty)$ may be directly translated in terms of $X_n(T_n)$. We are going to prove that this is equivalent to a PLT for $X_n(n + m_n)$ with the supplementary condition that $\{S_n(\infty)\}$ is bounded in probability. Observe that $X_n(T_n) \ge X_n(n + m_n)$, so that the addition of such a condition is not totally unexpected.

LEMMA 3.2. *For any $a \in \mathbb{N}$,*

$$P[S_n(\infty) \neq X_n(n+m_n)] \le P[S_n(\infty) > a] + a^2[1 - E(Q_n)]. \tag{3.9}$$

PROOF. We have, for $a \in \mathbb{N}$,

$$\begin{aligned} P[S_n(\infty) &\neq X_n(n+m_n)] \\ &\le P[S_n(\infty) > a] + P[S_n(\infty) \le a;\ S_n(\infty) \neq X_n(n+m_n)] \\ &= P[S_n(\infty) > a] \\ &\quad + \sum_{k=1}^{a} P[X_n(T_n) = k] P[X_n(n+m_n) \neq k | X_n(T_n) = k]. \end{aligned} \tag{3.10}$$

Consider the conditional probabilities in (3.10). By construction, $T_n = n + m_n - k$. Now, arguing as for (3.5), we see that, for $t > \zeta \ge 0$,

$$\begin{aligned} P[X_n(t) \neq k | X_n(t-\zeta) = k] &= 1 - P[X_n(t) = k | X_n(t-\zeta) = k] \\ &= 1 - P\left[\mathscr{M}\mathscr{B}\left(k, \textstyle\prod_{s=t-\zeta+1}^{t} Q_{n,s}\right) = k\right] \\ &= 1 - [E(Q_n^k)]^{\zeta} \le k\zeta[1 - E(Q_n)]. \end{aligned} \tag{3.11}$$

Thus, using (3.11), we obtain from (3.10) that

$$P[S_n(\infty) \neq X_n(n+m_n)] \le P[S_n(\infty) > a] + [1 - E(Q_n)] \sum_{k=1}^{a} k^2 P[X_n(T_n) = k];$$

hence inequality (3.9). □

PROPOSITION 3.3. *The variable $S_n(\infty)$ obeys a PLT if and only if the two following conditions hold:*

$$X_n(n+m_n) \text{ obeys a PLT (in identical terms)}, \tag{3.12}$$

$$\text{the sequence } \{S_n(\infty)\} \text{ is bounded in probability.} \tag{3.13}$$

PROOF. Suppose that (3.12) and (3.13) are true. From Lemma 3.2, we then see that $S_n(\infty)$ will obey the same PLT if $E(Q_n) \to 1$. To show that the latter condition is satisfied, we proceed by contradiction and we suppose that

$$E(Q_n) \to q < 1 \quad \text{as } n \to \infty. \tag{3.14}$$

From (3.5) and (3.14), we obtain that

$$E[X_n(n+m_n)] = n[E(Q_n)]^{n+m_n} \to 0 \quad \text{as } n \to \infty, \tag{3.15}$$

but (3.15) is in contradiction with the assumption (3.12). Reciprocally, the weak convergence of $\mathscr{L}(S_n(\infty))$ to a nondegenerate (Poisson) law implies the property (3.13). For (3.12), Lemma 3.2 shows that a sufficient condition is again that $E(Q_n) \to 1$. Proceeding by contradiction, suppose that (3.14) holds. We have, for any $\varepsilon > 0$ and $a \in \mathbb{N}$,

$$P[S_n(\infty) > \varepsilon] \le P[X_n(T_n) > a] + P[a \ge X_n(T_n) > \varepsilon]. \tag{3.16}$$

By construction, the event $[a \ge X_n(T_n)]$ is equivalent to $[T_n \ge n + m_n - a]$. Therefore, we obtain from (3.16) that

$$P[S_n(\infty) > \varepsilon] \le P[X_n(T_n) > a] + P[X_n(n + m_n - a) > \varepsilon]. \tag{3.17}$$

Now, by (3.13), $P[X_n(T_n) > a] < \varepsilon$ for a large enough. Applying Markov's inequality to $P[X_n(n + m_n - a) > \varepsilon]$ and using (3.5) and (3.14), we then deduce from (3.17) that, for any $\varepsilon > 0$,

$$P[S_n(\infty) > \varepsilon] \le 2\varepsilon \quad \text{for } n \ge n_\varepsilon. \tag{3.18}$$

This is in contradiction with the Poisson convergence of $S_n(\infty)$. □

We are now in a position to give a general sketch of the proof of the main result announced.

PROOF OF PROPOSITION 2.2. Suppose that $S_n(\infty)$ obeys a PLT. By definition, we are allowed to choose $m_n = 1$ for all n. From Proposition 3.3, we thus know that $X_n(n+1)$ obeys a PLT. Remember that $X_n(n+1)$ has a mixed binomial law given by (3.5). In Proposition 5.2, we will prove that combining these two facts implies that

$$n\{E[\exp(-\beta_n D)]\}^{n+1} \to b, \tag{3.19}$$

$$(n+1)\mathrm{Var}[\exp(-\beta_n D)] \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

Using the function $t(x)$ defined in (2.6), we see that (3.19) can be reexpressed as (2.9), and (3.20) as

$$n[2t(\beta_n) - t(2\beta_n)] \to 0. \tag{3.21}$$

In Proposition 6.2, we will establish that (2.9) together with (3.21) lead to the properties (2.7) and (2.8). Moreover, we will show in Proposition 6.4 that (2.9) and (2.10) are then equivalent. Reciprocally, now let $\{m_n\}$ be any sequence, and suppose that (2.7) and (2.8) hold true, as well as (2.9) or (2.10). As stated in Proposition 6.4, both (2.9) and (2.10) are then necessarily satisfied. By reversing the above argument, we deduce from Propositions 6.3 and 5.2 that $X_n(n+m_n)$ obeys a PLT. Thus, by Proposition 3.3, a PLT holds for $S_n(\infty)$ if the condition (3.13) is satisfied. The proof of this property is the object of Section 4. □

4. Boundedness of the final epidemic state.

4.1. *A suitable coupling.* For $t \geq 1$, put

$$W_n(t) = 1 + X_n(t) - X_n(t-1) \tag{4.1}$$

$$= 1 - \sum_{i=1}^{X_n(t-1)} [1 - Z_{n,i}(t)]. \tag{4.2}$$

From (3.4) and using (4.1), we observe that

$$\begin{aligned} m_n &= T_n + X_n(T_n) - n \\ &= W_n(1) + \cdots + W_n(T_n). \end{aligned} \tag{4.3}$$

Now, let r be any positive integer. With r, we associate the sequence of random variables $G_{n,r}(t)$'s, $t \geq 1$, defined as

$$G_{n,r}(t) = 1 - \sum_{i=1}^{r} [1 - Z_{n,i}(t)]. \tag{4.4}$$

We point out that the $G_{n,r}(t)$'s are i.i.d., their distribution being given by

$$1 - G_{n,r}(t) =_d \mathscr{M}\mathscr{B}(r, 1 - Q_{n,t}), \qquad t \geq 1. \tag{4.5}$$

Applying a coupling argument, we are going to establish the crucial inequality below.

PROPOSITION 4.1. *For any* $r \in \mathbb{N}$,

$$P[X_n(T_n) \geq r] \leq P\left[\sup_{k \geq T_n} \sum_{t=1}^{k} G_{n,r}(t) \geq m_n\right]. \tag{4.6}$$

PROOF. Since $\{X_n(t),\ t \geq 0\}$ is decreasing, we obtain from (4.2) and (4.4) that on the set $[X_n(T_n) \geq r]$, we have

$$W_n(t) \leq G_{n,r}(t), \qquad 1 \leq t \leq T_n. \tag{4.7}$$

From (4.3) and (4.7), we deduce that

$$m_n \leq G_{n,r}(1) + \cdots + G_{n,r}(T_n)$$

and, therefore,

$$m_n \le \sup_{k \ge T_n} \sum_{t=1}^{k} G_{n,r}(t). \tag{4.8}$$

The inequality (4.6) then follows directly. □

From (4.6), we can write, for any r and $d \in \mathbb{N}$,

$$P[X_n(T_n) \ge r] \le P(T_n \le d) + P\left[\sup_{k \ge d} \sum_{t=1}^{k} G_{n,r}(t) \ge m_n\right]. \tag{4.9}$$

Using (4.9), we propose to establish that if a PLT holds, then the sequence $\{X_n(T_n)\}$ is bounded in probability.

LEMMA 4.2. *Under* (2.7), (2.8) *and* (2.9), *for any* $a \in (0,1)$,

$$P(T_n \le an) \to 0 \quad \text{as } n \to \infty. \tag{4.10}$$

PROOF. Fix $a \in (0,1)$. Since $T_n \ge m_n$, $P(T_n \le an) = 0$ when $m_n > an$. Thus, to establish (4.10), it suffices to discuss the case where $m_n \le an$ for all $n \ge 1$. Put $r_{1,n} = [(1-a)n] \in \mathbb{N}$ and consider the variables $G_{n,r_{1,n}}(t)$'s, $t \ge 1$, introduced in (4.4). With these r.v.'s, we associate a new sequence of random variables $H_{n,r_{1,n}}(t)$'s, $t \ge 1$, defined by

$$H_{n,r_{1,n}}(t) = G_{n,r_{1,n}}(t) I[G_{n,r_{1,n}}(t) \ge 0] - I[G_{n,r_{1,n}}(t) < 0], \tag{4.11}$$

where $I[\cdot]$ denotes an indicator variable. From (4.4), (4.5) and (4.11), we see that the $H_{n,r_{1,n}}(t)$'s are i.i.d., their distribution being concentrated on three points -1,0,1. We will show in (4.19) that their mean is negative for n large. It is also clear that

$$H_{n,r_{1,n}}(t) \ge G_{n,r_{1,n}}(t), \qquad t \ge 1. \tag{4.12}$$

Now, by construction, we have

$$P(T_n \le an) \le P\{X_n(T_n) \ge [(1-a)n]\},$$

and using successively (4.6) and (4.12),

$$\begin{aligned} P[T_n \le an] &\le P\left[\sup_{k \ge 1} \sum_{t=1}^{k} G_{n,r_{1,n}}(t) \ge m_n\right] \\ &\le P\left[\sup_{k \ge 1} \sum_{t=1}^{k} H_{n,r_{1,n}}(t) \ge m_n\right]. \end{aligned} \tag{4.13}$$

To bound the latter probability, we apply Lemma 4.4 below (see Appendix 4.2). We then obtain from (4.13) and (4.36) that, for n large,

$$P(T_n \le an) \le 8 \operatorname{Var}[H_{n,r_{1,n}}(t)] / \{E[H_{n,r_{1,n}}(t)]\}^2. \tag{4.14}$$

From (4.5) and (4.11), the law of $H_{n,r_{1,n}}(t)$ is given by

$$P[H_{n,r_{1,n}}(t)=1] = P[G_{n,r_{1,n}}(t)=1] = E(Q_n^{r_{1,n}}) = E[\exp(-r_{1,n}\beta_n D)], \tag{4.15}$$

$$P[H_{n,r_{1,n}}(t)=0] = P[G_{n,r_{1,n}}(t)=0] = r_{1,n}E[Q_n^{r_{1,n}-1}(1-Q_n)] = r_{1,n}E\{[\exp(-(r_{1,n}-1)\beta_n D)][1-\exp(-\beta_n D)]\}, \tag{4.16}$$

the point -1 carrying the remaining probability mass. Note that, for n large, $r_{1,n}-1 > (1-a)n-2 > (1-a)n/2$, so that (4.16) then yields

$$P[H_{n,r_{1,n}}(t)=0] \le r_{1,n}E\{[\exp(-((1-a)/2)n\beta_n D)][1-\exp(-\beta_n D)]\}. \tag{4.17}$$

Now, we will prove in Proposition 6.6 that if a PLT holds, any sequence $\{\beta_n\}$ satisfying (2.9) with $m_n \le an,\ n\ge 1$, is such that

$$n\beta_n \to \infty \quad \text{as } n\to\infty. \tag{4.18}$$

Moreover, we will show in Lemma 4.5 below that under (2.3), the condition (4.18) implies that the right-hand sides of (4.15) and (4.17) tend to 0. Therefore, we deduce that

$$E[H_{n,r_{1,n}}(t)] \to -1, \tag{4.19}$$

$$\operatorname{Var}[H_{n,r_{1,n}}(t)] \to 0 \quad \text{as } n\to\infty. \tag{4.20}$$

Inserting (4.19) and (4.20) in (4.14) then leads to (4.10). □

PROPOSITION 4.3. *Under* (2.7), (2.8) *and* (2.9),

$$P[X_n(T_n)\ge l] \to 0 \quad \textit{as } l\to\infty, \textit{ uniformly in } n. \tag{4.21}$$

PROOF. Fix $a\in(0,1)$ and $\varepsilon>0$ very small. By Lemma 4.2 and since $T_n \ge m_n$, we have, for n large enough,

$$P[T_n \le a(n+m_n)/2] \le \varepsilon. \tag{4.22}$$

Thus, choosing $d=[a(n+m_n)/2]\in\mathbb{N}$ in (4.9) and using (4.22), we obtain, for any $r\in\mathbb{N}$,

$$P[X_n(T_n)\ge r] \le \varepsilon + P\left[\sup_{k\ge[a(n+m_n)/2]} \sum_{t=1}^{k} G_{n,r}(t) \ge m_n\right]. \tag{4.23}$$

Applying Lemma 4.4 below to (4.23) would then give

$$P[X_n(T_n)\ge r] \le \varepsilon + 8\operatorname{Var}[G_{n,r}(t)]/(\{E[G_{n,r}(t)]\}^2[a(n+m_n)/2]). \tag{4.24}$$

From (4.5), we easily get

$$E[G_{n,r}(t)] = 1 - r[1 - E(Q_n)], \tag{4.25}$$

$$\operatorname{Var}[G_{n,r}(t)] = r(r-1)\operatorname{Var}(Q_n) + rE(Q_n)[1 - E(Q_n)]. \tag{4.26}$$

Therefore, we have

$$\begin{aligned}\frac{\operatorname{Var}[G_{n,r}(t)]}{\{E[G_{n,r}(t)]\}^2} &\le \left\{\frac{r[1-E(Q_n)]}{1-r[1-E(Q_n)]}\right\}^2 \\ &\quad\times\left\{\frac{\operatorname{Var}(Q_n)}{[1-E(Q_n)]^2} + \frac{1}{r[1-E(Q_n)]}\right\}.\end{aligned} \tag{4.27}$$

Put $r_{2,n} = 1 + [2/(1 - E(Q_n))] \in \mathbb{N}$. We note that by (4.25), $E[G_{n,r_{2,n}}(t)] < 0$ and by (2.9), $r_{2,n} \sim 2(n + m_n)/\ln(n)$. From (4.27), we obtain that, for some constant c,

$$\begin{aligned}\frac{\operatorname{Var}[G_{n,r_{2,n}}(t)]}{\{E[G_{n,r_{2,n}}(t)]\}^2} &\le c\left\{\frac{\operatorname{Var}(1-Q_n)}{[1-E(Q_n)]^2} + 1\right\} \\ &\le c + \frac{c}{[1-E(Q_n)]}.\end{aligned} \tag{4.28}$$

Combining (4.24) and (4.28) yields, for some constant $\tilde{c}$,

$$P[X_n(T_n) \ge r_{2,n}] \le \varepsilon + \tilde{c}/(n + m_n) + \tilde{c}/\{(n + m_n)[1 - E(Q_n)]\}. \tag{4.29}$$

By (2.9), $(n + m_n)[1 - E(Q_n)] \sim \ln(n)$. Thus, we deduce from (4.29) that

$$P[X_n(T_n) \ge r_{2,n}] \le 2\varepsilon \quad \text{for } n \ge n_\varepsilon. \tag{4.30}$$

Now, let $l > 0$. For any r and $n \in \mathbb{N}$, we have

$$P[X_n(T_n) \ge l] \le P[X_n(T_n) \ge r] + P[r > X_n(T_n) \ge l]. \tag{4.31}$$

In (4.31), choose $r = r_{2,n}$ defined above. From (4.30) and the construction of the model, we get, for $n \ge n_\varepsilon$,

$$\begin{aligned}P[X_n(T_n) \ge l] &\le 2\varepsilon + P[T_n > n + m_n - r_{2,n} \text{ and } X_n(T_n) \ge l] \\ &\le 2\varepsilon + P[X_n(n + m_n - r_{2,n}) \ge l].\end{aligned} \tag{4.32}$$

Using Markov inequality and (3.5), we obtain from (4.32) that

$$\begin{aligned}P[X_n(T_n) \ge l] &\le 2\varepsilon + E[X_n(n + m_n - r_{2,n})]/l \\ &= 2\varepsilon + n[E(Q_n)]^{n+m_n-r_{2,n}}/l.\end{aligned} \tag{4.33}$$

From Propositions 3.1, 3.3 and 5.2, we know that

$$n[E(Q_n)]^{n+m_n} \to b \quad \text{as } n \to \infty. \tag{4.34}$$

Furthermore, by definition of $r_{2,n}$,

$$[E(Q_n)]^{-r_{2,n}} \sim \exp\{[1 - E(Q_n)]r_{2,n}\} \le \exp(3). \tag{4.35}$$

Inserting (4.34) and (4.35) in (4.33) then leads to (4.21). □

4.2. *Appendix.* We establish here two preliminary results used above. The former provides an upper bound for the probability that a random walk with a negative drift reaches a given positive level. This bound is probably not new but seems to be little known.

LEMMA 4.4. *Let $U(t)$, $t \geq 1$, be a sequence of i.i.d. r.v.'s (distributed as the variable U, say), with $E(U) < 0$. Put $S(k) = U(1) + \cdots + U(k)$, $k \geq 1$. Then, for any $r \in \mathbb{N}$,*

$$P[\sup_{k\geq r} S(k) \geq 1] \leq 8 \operatorname{Var}(U)/\{r[E(U)]^2\}. \tag{4.36}$$

PROOF. Suppose that $r \in [2^j, 2^{j+1})$ for some j and let $\bar{S}(k) = S(k) - kE(Y)$, $k \geq 1$. By a standard argument, we have

$$\begin{aligned} P\Big[\sup_{k\geq r} S(k) \geq 1\Big] &= P\Big\{\sup_{k\geq r}[\bar{S}_k + kE(U)] \geq 1\Big\} \\ &\leq \sum_{i=j}^{\infty} P\Big\{\max_{2^i\leq k<2^{i+1}}[\bar{S}_k + kE(U)] \geq 1\Big\} \\ &\leq \sum_{i=j}^{\infty} P\Big[\max_{1\leq k<2^{i+1}} \bar{S}_k \geq 1 - 2^i E(U)\Big]. \end{aligned} \tag{4.37}$$

Applying the maximal inequality to (4.37), we obtain

$$\begin{aligned} P\Big[\sup_{k\geq r} S(k) \geq 1\Big] &\leq \sum_{i=j}^{\infty} \operatorname{Var}(\bar{S}_{2^{i+1}})/[1 - 2^i E(U)]^2 \\ &= 2 \operatorname{Var}(U) \sum_{i=j}^{\infty} 2^i/[1 - 2^i E(U)]^2. \end{aligned} \tag{4.38}$$

From (4.38), we then deduce that

$$P[\sup_{k\geq r} S(k) \geq 1] \leq \{2 \operatorname{Var}(U)/[E(U)]^2\} \sum_{i=j}^{\infty}(1/2^i),$$

which leads to (4.36). □

The latter result is technical.

LEMMA 4.5. *Let $a > 0$, let X be a positive r.v. with $P(X = 0) = 0$, and let $\{j_n\}$ be a sequence of positive numbers such that $nj_n \to \infty$. Then,*

$$E[\exp(-nj_n X)] \to 0, \tag{4.39}$$

$$nE\{\exp(-anj_n X)[1 - \exp(-j_n X)]\} \to 0 \quad \text{as } n \to \infty. \tag{4.40}$$

PROOF. We have, for any $y > 0$,

$$E[\exp(-nj_nX)] = E[\exp(-nj_nX)I(nj_nX > y)] + E[\exp(-nj_nX)I(nj_nX \le y)] \le \exp(-y) + P(nj_nX \le y). \tag{4.41}$$

Choosing in (4.41), $y = \sqrt{nj_n}$ yields

$$E[\exp(-nj_nX)] \le \exp(-\sqrt{nj_n}) + P(X \le 1/\sqrt{nj_n}). \tag{4.42}$$

However, by hypothesis, $nj_n \to \infty$ and $P(X \le x) \to 0$ as $x \to 0+$. Thus, from (4.42) we then deduce (4.39). Now, since $1 - \exp(-x) \le x$ and $x\exp(-x) \le 2\exp(-x/2)$ for $x \ge 0$, we have

$$nE\{\exp(-anj_nX)[1 - \exp(-j_nX)]\} \le nE[j_nX \ \exp(-anj_nX)] \le (2/a)E[\exp(-anj_nX/2)]. \tag{4.43}$$

By (4.39), the right-hand side of (4.43) tends to 0, implying (4.40). □

5. Poisson convergence of mixed binomial laws. In the proof of Proposition 2.2 (see Section 3), we exploited a characterization of the Poisson convergence of a certain class of mixed binomial distributions. Because of its own interest, this matter is investigated here separately. We begin with a very simple result, most probably known but apparently rarely referred to in the literature.

PROPOSITION 5.1. *Let Y_n, $n \ge 1$, be a sequence of r.v.'s defined on $[0,1]$. For $n \ge 1$, let Z_n be a r.v. having the mixed binomial law $\mathscr{MB}(n, Y_n)$. Then, as $n \to \infty$, $\mathscr{L}(Z_n) \to_w \mathscr{P}(b)$, $0 < b < \infty$, if and only if*

$$nY_n \to_P b. \tag{5.1}$$

PROOF. Let us argue in terms of probability generating functions. Fix x in $[0,1]$. We have, for any $c > 0$,

$$\begin{aligned} E(x^{Z_n}) &= E\{[1 - Y_n(1-x)]^n\} \\ &= E\{I(nY_n < cb)\exp[n\ln(1 - Y_n(1-x))]\} \\ &\quad + E\{I(nY_n \ge cb)[1 - Y_n(1-x)]^n\} \\ &= E\{I(nY_n < cb)\exp[n\ln(1 - Y_n(1-x))]\} + \theta_{c,n}\exp(-cb), \end{aligned} \tag{5.2}$$

where $\theta_{c,n}$ is some constant in $[0,1]$. Choose c large enough in (5.2). We then see that as $n \to \infty$,

$$E(x^{Z_n}) \sim E\{\exp[nY_n(x-1)]\},$$

which tends to $\exp[b(x-1)]$ if and only if (5.1) holds true. □

We now deduce the result used earlier.

PROPOSITION 5.2. *Let $Q_{n,s}$, $1 \le s \le n + m_n$, $m_n \ge 0$ and $n \ge 1$, be a double sequence of r.v.'s such that within each row, that is, for each n, the $Q_{n,s}$'s are i.i.d. and distributed on* $[0,1]$ *as the variable Q_n, say. For $n \ge 1$, let $X_n(n + m_n)$ be a r.v. having the law*

$$X_n(n+m_n) =_d \mathscr{M}\mathscr{B}\left(n, \prod_{s=1}^{n+m_n} Q_{n,s}\right). \tag{5.3}$$

Then, as $n \to \infty$, $\mathscr{L}(X_n(n+m_n)) \to_w \mathscr{P}(b)$, $0 < b < \infty$, if and only if the following two conditions are satisfied:

$$n[E(Q_n)]^{n+m_n} \to b, \tag{5.4}$$

$$(n+m_n)\mathrm{Var}(Q_n) \to 0. \tag{5.5}$$

PROOF. By Proposition 5.1, it is equivalent to show that as $n \to \infty$,

$$nY_n \equiv n \prod_{s=1}^{n+m_n} Q_{n,s} \to_P b \tag{5.6}$$

if and only if (5.4) and (5.5) hold true. First, suppose that these two conditions are satisfied. A direct calculation yields

$$E(nY_n) = n[E(Q_n)]^{n+m_n} \to b, \tag{5.7}$$

$$\mathrm{Var}(nY_n) = n^2\left\{[E(Q_n^2)]^{n+m_n} - [E(Q_n)]^{2(n+m_n)}\right\}$$

$$= \{n[E(Q_n)]^{n+m_n}\}^2 \left\{\left[1 + \frac{\mathrm{Var}(Q_n)}{(E(Q_n))^2}\right]^{n+m_n} - 1\right\} \tag{5.8}$$

$$\sim b^2(n+m_n)\mathrm{Var}(Q_n) \to 0. \tag{5.9}$$

By Tchebychev inequality, we then deduce (5.6). Let us prove the converse. We start by noting that $E(Q_n) \to 1$ and thus $Q_n \to_P 1$. Indeed, if $E(Q_n) \to q < 1$, then $E(nY_n) \to 0$, which is in contradiction with (5.6). Now, let us express (5.6) as

$$\ln(n) + \sum_{s=1}^{n+m_n} \ln(Q_{n,s}) \to_P \ln(b). \tag{5.10}$$

For $1 \le s \le n + m_n$ and $n \ge 1$, put

$$X_{n,s} = [1/(n+m_n)]\ln(n/b) + \ln(Q_{n,s}). \tag{5.11}$$

Clearly, the variables $X_{n,s}$ are i.i.d. within each row and $X_{n,s} \to_P 0$ as $n \to \infty$. Thus they obey the condition of infinite smallness [see, e.g., Petrov (1975), page 63]. Moreover, (5.10) states that

$$\sum_{s=1}^{n+m_n} X_{n,s} \to_P 0. \tag{5.12}$$

By a standard result [Petrov (1975), Theorem 3], (5.12) together with the previous property are equivalent to the three following conditions:

$$(n+m_n)P[\mid X_{n,1}\mid \geq t] \to 0, \tag{5.13}$$

$$(n+m_n)E\left\{[\mid X_{n,1}\mid]I[\mid X_{n,1}\mid < t]\right\} \to 0, \tag{5.14}$$

$$(n+m_n)\mathrm{Var}\left\{[\mid X_{n,1}\mid]I[\mid X_{n,1}\mid < t]\right\} \to 0, \tag{5.15}$$

for every $t > 0$. Using (5.11), these become

$$(n+m_n)P(Q_n \leq q) \to 0, \tag{5.16}$$

$$(n+m_n)E\{[(1/(n+m_n))\ln(n/b) + \ln(Q_n)]I(Q_n > q)\} \to 0, \tag{5.17}$$

$$(n+m_n)\mathrm{Var}\{[(1/(n+m_n))\ln(n/b) + \ln(Q_n)]I(Q_n > q)\} \to 0, \tag{5.18}$$

for every q in $(0,1)$. Furthermore, inserting (5.16) in (5.17) leads to

$$\ln(n/b) + (n+m_n)E[\ln(Q_n)I(Q_n > q)] \to 0. \tag{5.19}$$

After multiplication by $[1/(n+m_n)]\ln(n/b)$ $(\to 0)$, (5.19) implies that

$$\ln(n/b)E[\ln(Q_n)I(Q_n > q)] \to 0. \tag{5.20}$$

By (5.17) and (5.20), (5.18) reduces to

$$(n+m_n)E\{[\ln(Q_n)]^2 I(Q_n > q)\} \to 0. \tag{5.21}$$

Since $\mid \ln(1-x)\mid \geq x$ for $x \geq 0$, (5.21) implies that

$$(n+m_n)E[(1-Q_n)^2 I(Q_n > q)] \to 0. \tag{5.22}$$

We are now ready to derive (5.4) and (5.5). From (5.16), (5.22) and the fact that $E(Q_n) \to 1$, we directly obtain (5.5). By (5.22), (5.19) yields

$$\ln(n/b) + (n+m_n)E[(Q_n - 1)I(Q_n > q)] \to 0, \tag{5.23}$$

and applying (5.16) then leads to (5.4). □

In Section 7, we will estimate the accuracy of the Poisson approximation in a PLT. To this end, we need to examine that question within Propositions 2.1 and 2.2 above. Let $d(\cdot,\cdot)$ denote the total variation distance between probability distributions over $\mathbb{N}$; for a concise presentation of this notion, see, for example, Appendix A.1 of the book by Barbour, Holst and Janson (1992).

PROPOSITION 5.3. *Let Z_n, $n \geq 1$, be a sequence of r.v.'s distributed as in Proposition 5.1. Then for any $b > 0$,*

$$\begin{aligned} &d[\mathscr{L}(Z_n), \mathscr{P}(b)] \\ &\quad \leq \min(1, 1/b)\{n^2\,\mathrm{Var}(Y_n) + n[E(Y_n)]^2 + n[E(Y_n)] - b\}. \end{aligned} \tag{5.24}$$

In particular, if $Z_n \equiv X_n(n+m_n)$, $n \geq 1$, is distributed as in (5.3), *then* (5.24) *holds with $E(nY_n)$ and $\mathrm{Var}(nY_n)$ given by* (5.7) *and* (5.8), *respectively.*

PROOF. Obviously, $Z_n =_d \mathscr{M}\mathscr{B}(n, Y_n)$ is the sum of n indicators that are increasing functions of Y_n. By Theorem 2.E in Barbour, Holst and Janson (1992), these indicators are positively related. Thus, applying their Corollary 2.E.1 yields

$$d[\mathscr{L}(Z_n), \mathscr{P}(b)] \leq \min(1, 1/b)\{\operatorname{Var}(Z_n) - b + 2n[E(Y_n)]^2\},$$

which reduces to (5.24). The special case is straightforward. □

6. Application to the embedded Markov model.

6.1. *Convergence condition.* Let us come back to the epidemic context (and to the proof of Proposition 2.2). To obtain the characterization (2.7) and (2.8) on D ensuring a PLT holds, we used Proposition 5.2 through the three results that are derived below. We start by stating a simple technical lemma from real analysis.

LEMMA 6.1. *Let $g_i(x)$, $i = 1, \dots, k + l$, with k, $l \geq 1$, be monotone continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. If there exists a sequence $\{j_n\}$ such that*

$$j_n \to \infty, \qquad j_n/j_{n+1} \to 1,$$

$$g_i(j_n)/g_i(j_{n+1}) \to 1, \qquad i = 1, \dots, k + l,$$

$$\prod_{i=1}^{k} g_i(j_n) \Big/ \prod_{i=k+1}^{k+l} g_i(j_n) \to \lambda \quad \text{as } n \to \infty,$$

then

$$\prod_{i=1}^{k} g_i(x) \Big/ \prod_{i=k+1}^{k+l} g_i(x) \to \lambda \quad \text{as } x \to \infty.$$

PROPOSITION 6.2. *Let $\{\beta_n\}$ be a sequence satisfying* (2.9) *with $m_n = 1$, $n \geq 1$. If, in addition, it satisfies* (3.21), *that is,*

$$n[2t(\beta_n) - t(2\beta_n)] \to 0 \quad \textit{as } n \to \infty, \tag{6.1}$$

then (2.7) *and* (2.8) *hold true.*

PROOF. First, let us show that under (6.1), $\{\beta_n\}$ is such that, for any $\lambda \in [1, 2]$,

$$t(\lambda\beta_n)/t(\beta_n) \to \lambda \quad \text{as } n \to \infty. \tag{6.2}$$

Note that (6.1) implies (6.2) for $\lambda = 2$. Since $t(x)$ is a continuous increasing concave function, with $t(0) = 0$ by (2.3), we can write, for $\lambda \in (1, 2)$,

$$(1/\lambda)t(\lambda x) \leq t[(1/\lambda)\lambda x + (1 - 1/\lambda)0] = t(x), \tag{6.3}$$

$$t(\lambda x) = t[(2 - \lambda)x + (\lambda - 1)2x] \geq (2 - \lambda)t(x) + (\lambda - 1)t(2x). \tag{6.4}$$

From (6.3), (6.4) and (6.2) with $\lambda = 2$, we get

$$\limsup[t(\lambda\beta_n)/t(\beta_n)] \leq \lambda,$$

$$\liminf[t(\lambda\beta_n)/t(\beta_n)] \geq 2 - \lambda + (\lambda - 1)2 = \lambda,$$

which yields (6.2). Now, it is clear that, for any sequence $\{\beta_n\}$ satisfying (2.9),

$$t(\beta_n)/t(\beta_{n+1}) \to 1 \quad \text{as } n \to \infty. \tag{6.5}$$

Furthermore, under (6.1), we have

$$\beta_n/\beta_{n+1} \to 1 \quad \text{as } n \to \infty. \tag{6.6}$$

Indeed, suppose that there exists a subsequence $\{n'\}$ such that, for instance, $\beta_{n'} \geq \tilde{\lambda}\beta_{n'+1}$ for some $\tilde{\lambda} > 1$. Put $\lambda = \tilde{\lambda} \wedge 2$. Since $t(x)$ is increasing and using (6.2), we obtain that

$$t(\beta_{n'})/t(\beta_{n'+1}) \geq t(\lambda\beta_{n'+1})/t(\beta_{n'+1}) \to \lambda > 1,$$

which is in contradiction with (6.5). At this point, we are ready to establish that $t(1/x)$ is a regularly varying function of index -1, that is [see Bingham, Goldie and Teugels (1987)], for any $\lambda > 0$,

$$t(\lambda/x)/t(1/x) \to \lambda \quad \text{as } x \to \infty. \tag{6.7}$$

In fact, combining (6.2), (6.5) and (6.6) and applying Lemma 6.1, we find that (6.7) holds true for $\lambda \in [1, 2]$. From this, we directly see that (6.7) is also valid for $\lambda \in [1/2, 1]$. Now note that any $\lambda > 0$ can be expressed as $\lambda = \theta^r$ for some $r \in \mathbb{N}$ and $\theta \in [1/2, 2]$. Therefore, writing

$$t(\theta^r/x)/t(1/x) = [t(\theta^r/x)/t(\theta^{r-1}/x)] \cdots [t(\theta/x)/t(1/x)],$$

we deduce that (6.7) holds for any $\lambda > 0$. Moreover, we observe that the Laplace–Stieltjes transform of $h(x)$ is given by

$$\int_0^\infty e^{-sx}\, dh(x) = t(s)/s, \qquad \mathbb{R}e(s) > 0. \tag{6.8}$$

From (6.7) and (6.8), we then obtain [see Bingham, Goldie and Teugels (1987), Theorem 1.7.1] that, as stated by (2.7), $h(x)$ is a slowly varying function. We mention that, as a consequence,

$$xt(1/x)/h(x) \to 1 \quad \text{as } x \to \infty. \tag{6.9}$$

Let us prove (2.8). We have

$$\begin{aligned} 2t(\beta_n) - t(2\beta_n) &= 2\int_0^\infty P(D > t/\beta_n)e^{-t}(1 - e^{-t})\, dt \\ &\geq 2P(D > 1/\beta_n)\int_0^1 e^{-t}(1 - e^{-t})\, dt, \end{aligned} \tag{6.10}$$

so that (6.1) implies that

$$nP(D > 1/\beta_n) \to 0 \quad \text{as } n \to \infty. \tag{6.11}$$

However, (2.9) with $m_n = 1,\ n \geq 1$, and (6.9) yield successively

$$n \sim \ln[1/t(\beta_n)]/t(\beta_n) \sim \ln[1/t(\beta_n)]/[\beta_n h(1/\beta_n)]. \tag{6.12}$$

Furthermore, from (6.3), (6.9) and (2.7), we get that, for every $\varepsilon > 0$,

$$\beta_n t(1) \leq t(\beta_n) \sim \beta_n h(1/\beta_n) \leq \beta_n^{1-\varepsilon} c(n, \varepsilon), \tag{6.13}$$

where $c(n, \varepsilon) \to 0$ as $n \to \infty$ [see Bingham, Goldie and Teugels (1987), Theorem 1.5.4]. Thus, from (6.12) and (6.13), we obtain that

$$n \sim \ln(1/\beta_n)/[\beta_n h(1/\beta_n)]. \tag{6.14}$$

Inserting (6.14) in (6.11) and applying Lemma 6.1 then leads to (2.8). □

PROPOSITION 6.3. *Under* (2.7), (2.8) *and* (2.9),

$$(n + m_n)[2t(\beta_n) - t(2\beta_n)] \to 0 \quad \textit{as } n \to \infty. \tag{6.15}$$

PROOF. From (6.10), we have

$$\begin{aligned} 0 &\leq 2t(\beta_n) - t(2\beta_n) \\ &\leq 2\int_0^1 P(D > t/\beta_n)t\, dt + 2P(D > 1/\beta_n)\int_1^\infty e^{-t}\, dt \\ &= 2\beta_n^2 \int_0^{1/\beta_n} P(D > t)t\, dt + (2/e)P(D > 1/\beta_n). \end{aligned} \tag{6.16}$$

Thus, a sufficient condition for (6.15) is that when multiplied by $n + m_n$, each term in the right-hand side of (6.16) tends to 0. Arguing as for (6.12) and (6.14), we obtain from (2.9) and (2.7) that, for n large,

$$(n + m_n) \leq c\ \ln(1/\beta_n)/[\beta_n h(1/\beta_n)], \tag{6.17}$$

for some constant c. Put $x = 1/\beta_n \to \infty$. From (2.8) and (6.17) we then deduce that

$$\begin{aligned} &(n + m_n)P(D > 1/\beta_n) \\ &\qquad \leq cx\ \ln(x)P(D > x)/h(x) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{6.18}$$

Moreover, from (6.25) below, we get

$$\begin{aligned} &(n + m_n)\beta_n^2 \int_0^{1/\beta_n} P(D > t)t\, dt \\ &\qquad \leq c\ \ln(x)\int_0^x P(D > t)t\, dt/xh(x) \to 0, \end{aligned} \tag{6.19}$$

which completes the proof. □

PROPOSITION 6.4. *Under* (2.7) *and* (2.8), *the two conditions* (2.9) *and* (2.10) *for* $\{\beta_n\}$ *are equivalent.*

PROOF. We have

$$
\begin{aligned}
&| t(\beta_n) - \beta_n h(1/\beta_n) | \\
&= \left| \int_0^1 P(D > t/\beta_n)(1-e^{-t})\, dt - \int_1^\infty P(D > t/\beta_n)e^{-t}\, dt \right| \qquad (6.20)\\
&\le \beta_n^2 \int_0^{1/\beta_n} P(D > t)t\, dt + (1/e)P(D > 1/\beta_n).
\end{aligned}
$$

Suppose that $\{\beta_n\}$ satisfies (2.9). We have shown in Proposition 6.3 that (2.7) and (2.8) imply (6.18) and (6.19). Thus we obtain from (6.20) that

$$(n + m_n)[t(\beta_n) - \beta_n h(1/\beta_n)] \to 0 \quad \text{as } n \to \infty. \tag{6.21}$$

Combining (2.9) with (6.21) then leads to (2.10). Conversely, suppose that $\{\beta_n\}$ satisfies (2.10). By (6.20), a sufficient condition for (2.9) is that (6.18) and (6.19) hold true. From (2.7) and (2.10), we find that $1/\beta_n \ge \sqrt{n}$ for n large. Therefore, we get from (2.10) that, for n large,

$$(n + m_n) \le 2\ \ln(n/b_n)/[\beta_n h(1/\beta_n)] \le c\ \ln(1/\beta_n)/[\beta_n h(1/\beta_n)], \tag{6.22}$$

for some constant c. Now, as above, (6.22) together with (2.7) and (2.8) imply (6.18) and (6.19). □

6.2. *Complementary results.* We derive hereafter two related propositions that have been applied earlier [see (2.11), (6.19) and (4.18)].

PROPOSITION 6.5. *Under* (2.7) *and* (2.8),

$$xP(D > x) \to 0, \tag{6.23}$$

$$h(x)/\ln(x) \to 0, \tag{6.24}$$

$$\ln(x) \int_0^x P(D > t)t\, dt/[xh(x)] \to 0 \quad \textit{as } x \to \infty. \tag{6.25}$$

PROOF. We start by showing that (2.8) alone implies (6.23) and (6.24). Put $\varepsilon_x = x\ \ln(x)P(D > x)/h(x)$, $x > 0$. By (2.8), there exists x_0 such that $\varepsilon_{x_0} \le \ln(2)$ for all $x \ge x_0$. We have, for $x \ge x_0$,

$$
\begin{aligned}
h(2x) &= h(x) + \int_x^{2x} P(D > t)\, dt \le h(x) + xP(D > x)\\
&= h(x)[1 + \varepsilon_x/\ln(x)] \le \ln(2x)[h(x)/\ln(x)].
\end{aligned}
$$

By iterating, we obtain, for $x \ge 2^{k-1}x_0$, $k \ge 1$, that

$$h(x)/\ln(x) \ge h(2x)/\ln(2x) \ge \cdots \ge h(2^k x)/\ln(2^k x). \tag{6.26}$$

Since $h(x)$ is increasing, we deduce from (6.26) that, for $x \in [2^{k-1}x_0, 2^k x_0]$, $k \geq 1$,

$$\begin{aligned} h(x) &\leq h(2^k x_0) \leq \ln(2^k x_0) h(x_0)/\ln(x_0) \\ &\leq \ln(2x) h(x_0)/\ln(x_0) \leq c(x_0)\ln(x), \end{aligned} \tag{6.27}$$

for some constant $c(x_0)$. Inserting (6.27) in (2.8) then yields (6.23). Moreover, fix $u > e$. We have, for $x \geq u$,

$$\begin{aligned} h(x) &\leq u + \int_u^x P(D > t)\, dt \\ &\leq u + \{\sup_{t \geq u}[P(D > t)t]\}\ln(x). \end{aligned} \tag{6.28}$$

Choosing $u = \sqrt{\ln(x)}$ in (6.28) and using (6.23) then leads to (6.24). Now, let us prove that (2.7) and (2.8) imply (6.25). By (2.7), $h(x)/\ln(x)$ is a slowly varying function [see Bingham, Goldie and Teugels (1987), Proposition 1.3.6]. Thus, we obtain by their Proposition 1.5.8 that, for $x > e$,

$$\int_e^x [h(t)/\ln(t)]\, dt \leq \tilde{c}(h) x h(x)/\ln(x), \tag{6.29}$$

for some constant $\tilde{c}(h)$. From (6.29), we find, for $x > e$,

$$\begin{aligned} \int_0^x P(D > t)t\, dt &= \int_0^{x^{1/3}} P(D > t)t\, dt + \int_{x^{1/3}}^x \varepsilon_t [h(t)/\ln(t)]\, dt \\ &\leq x^{2/3} + \varepsilon_{x^{1/3}} \int_{x^{1/3}}^x [h(t)/\ln(t)]\, dt \\ &\leq x^{2/3} + \varepsilon_{x^{1/3}} \tilde{c}(h) x h(x)/\ln(x), \end{aligned}$$

which yields

$$\ln(x) \int_0^x P(D > t)t\, dt / x h(x) \leq \ln(x)/x^{1/3} h(x) + \varepsilon_{x^{1/3}} \tilde{c}(h). \tag{6.30}$$

Since by (2.8), $\varepsilon_{x^{1/3}} \to 0$ as $x \to \infty$, (6.25) follows directly from (6.30). □

PROPOSITION 6.6. *Under* (2.7), (2.8) *and* (2.9), *and if* $m_n \leq an$, $n \geq 1$, *for some* $a > 0$, *then* $n\beta_n \to \infty$ *as* $n \to \infty$.

PROOF. From (2.7) and using (6.9), we can write that for n large,

$$t(\beta_n) \leq c_1 \beta_n h(1/\beta_n) \leq c_2 \sqrt{\beta_n}, \tag{6.31}$$

for some constants c_1, c_2. As $m_n \leq an$, we obtain from (6.31) that, for n large,

$$(n + m_n)t(\beta_n) - \ln(n/b_n) \leq (1 + a)c_2 n\sqrt{\beta_n} - \ln(n/b_n). \tag{6.32}$$

By (2.9), the left-hand side of (6.32) tends to 0. Thus, we may say that there exists a constant c_3 such that $c_3 \ln(1/\beta_n) \leq \ln(n/b_n)$ for n large. From this together with $m_n \le an$, we get, for n large,

$$(6.33)\qquad \begin{aligned} &(n+m_n)\beta_n h(1/\beta_n) - \ln(n/b_n) \\ &\quad \le h(1/\beta_n)[(1+a)n\beta_n - c_3 \ln(1/\beta_n)/h(1/\beta_n)]. \end{aligned}$$

By Proposition 6.4, the left-hand side of (6.33) tends to 0. Since by (6.24), $\ln(1/\beta_n)/h(1/\beta_n) \to \infty$, we then deduce that $n\beta_n \to \infty$ as announced. □

7. On the accuracy of the Poisson approximation. Let us examine the case, generally true in practice, where $E(D^2) < \infty$. We directly see that the condition (2.16) is then satisfied. Therefore, by Corollary 2.6, a SPLT holds and $\{\beta_n\}$ is necessarily of the form (2.15). Now, a further problem is concerned with quantifying the accuracy of the Poisson approximation. The technique developed in this work enables one to obtain an order of magnitude for the error involved.

A main point is that when m_n is small, the rate of convergence is rather slow, typically of order $1/\ln(n)$. The reason for that comes from the nature of the model itself. For illustration, consider the general epidemic process (D having an exponential law of parameter μ), with $m_n = 1$, $n \ge 1$. From (2.15), we have

$$P[S_n(\infty) = n \mid \text{ this model }] = \mu/(\mu + \beta_n n) = O(1/\ln(n)).$$

Thus, the total variation distance between $\mathscr{L}(S_n(\infty))$ and $\mathscr{P}(b)$ is such that

$$(7.1)\qquad \begin{aligned} &d[\mathscr{L}(S_n(\infty) \mid \text{ this model }),\ \mathscr{P}(b)] \\ &\quad \ge \mu/(\mu + \beta_n n) - P[\mathscr{P}(b) = n] = O(1/\ln(n)). \end{aligned}$$

In fact, it can be shown that the upper bound for $d[\cdot,\cdot]$ is also of the same order. That result is in agreement with Corollary 2.6 in Ball and Barbour (1990).

We are going to point out that when m_n is sufficiently large, the approximation becomes significantly better.

PROPOSITION 7.1. *Suppose that $E(D^2) < \infty$ and $m_n \ge an$, $n \ge 1$, for some $a > 0$. Then, for any $b > 0$, when n is large,*

$$(7.2)\qquad d[S_n(\infty),\ \mathscr{P}(b)] \le c(1+b)^2[\ln(n)]^3/n,$$

where c is a constant function of a, $E(D)$ and $E(D^2)$, but independent of b.

PROOF. To begin with, we indicate that all the constants $c_1, c_2, \dots$ introduced hereafter have the above property. We also note that since $1 - \exp(-x) \le x$ for $x \ge 0$,

$$(7.3)\qquad E[(1-Q_n)^2] \le \beta_n^2 E(D^2).$$

By well-known properties of $d[\cdot,\cdot]$, we can write

$$\begin{aligned} d[\mathscr{L}(S_n(\infty)),\ \mathscr{P}(b)] &\le d[\mathscr{L}(S_n(\infty)),\ \mathscr{L}(X_n(n+m_n))] \\ &\quad + d[\mathscr{L}(X_n(n+m_n)),\ \mathscr{P}(b)] \\ &\le P[S_n(\infty) \ne X_n(n+m_n)] \\ &\quad + d[\mathscr{L}(X_n(n+m_n)),\ \mathscr{P}(n[E(Q_n)]^{n+m_n})] \\ &\quad + d[\mathscr{P}(n[E(Q_n)]^{n+m_n}),\ \mathscr{P}(b)] \\ &\equiv p_1(\cdot) + d_2[\cdot,\cdot] + d_3[\cdot,\cdot], \quad \text{say.} \end{aligned} \tag{7.4}$$

By Theorem 1.C in Barbour, Holst and Janson (1992), we have

$$d_3[\cdot,\cdot] \le \mid n[E(Q_n)]^{n+m_n} - b \mid .$$

From (2.15) and since $m_n \ge an$, we then obtain, for n large,

$$d_3[\cdot,\cdot] \le c_1 b[\ln(n)]^2/n. \tag{7.5}$$

From (3.5) and Proposition 5.3, we get

$$d_2[\cdot,\cdot] \le n^2 \operatorname{Var}\left(\prod_{s=1}^{n+m_n} Q_{n,s}\right) + [E(Q_n)]^{n+m_n}.$$

Combining (2.15), (5.8), (7.3) and $m_n \ge an$ then leads, for n large, to

$$d_2[\cdot,\cdot] \le c_2 b^2[\ln(n)]^2/n. \tag{7.6}$$

To bound $p_1(\cdot)$, we will use Lemma 3.2 and an argument similar to that followed in Proposition 4.3. First, note that $T_n \ge m_n \ge an$. With $r_{2,n}$ defined as before, we deduce from (2.15), (4.9), (4.24), (4.27) and (7.3) that, for n large,

$$P[S_n(\infty) \ge r_{2,n}] \le c_3/n. \tag{7.7}$$

Let us choose in (4.31), $r = r_{2,n}$ and $l = \tilde{c}b\ln(n)$, for any $\tilde{c} > 0$. From (7.7), we then obtain

$$P[S_n(\infty) \ge \tilde{c}b\ \ln(n)] \le c_3/n + P[X_n(n+m_n-r_{2,n}) \ge \tilde{c}b\ \ln(n)]. \tag{7.8}$$

However, it is clear that

$$\begin{aligned} &P[X_n(n+m_n-r_{2,n}) \ge \tilde{c}b\ln(n)] \\ &\quad \le d[\mathscr{L}(X_n(n+m_n-r_{2,n})),\ \mathscr{P}(n[E(Q_n)]^{n+m_n-r_{2,n}})] \\ &\qquad + P[\mathscr{P}(n[E(Q_n)]^{n+m_n-r_{2,n}}) \ge \tilde{c}b\ln(n)] \\ &\quad \equiv d_4[\cdot,\cdot] + \mu_5(\cdot), \quad \text{say.} \end{aligned} \tag{7.9}$$

Arguing as for $d_2[\cdot,\cdot]$, it can be shown that $d_4[\cdot,\cdot]$ has an upper bound of the same order [given in (7.6)]. In $\mu_5(\cdot)$ choose $\tilde{c}$ sufficiently large. Since for any λ, k and $h \ge 0$,

$$P[\mathscr{P}(\lambda) \ge k] \le \exp\{-hk - \lambda[1-\exp(h)]\},$$

we get from (2.15) and $m_n \geq an$ that, for n large, an upper bound for $\mu_5(\cdot)$ is of order $1/n$. Inserting these bounds in (7.9) and (7.8) yields

$$P[S_n(\infty) \geq \tilde{c}b \ln(n)] \leq c_4(1+b^2)[\ln(n)]^2/n. \tag{7.10}$$

Now, let us apply (3.9) with $a = \tilde{c}b\ \ln(n)$. From (7.10) and (2.15) with $m_n \geq an$, we obtain that, for n large,

$$\begin{aligned} p_1(\cdot) &\leq c_4(1+b^2)[\ln(n)]^2/n + c_5 b^2[\ln(n)]^3/n \\ &\leq c_6(1+b^2)[\ln(n)]^3/n. \end{aligned} \tag{7.11}$$

Combining (7.4), (7.5), (7.6) and (7.11) then implies (7.2). □

Acknowledgments. We would like to thank the referee and an Associate Editor for constructive comments.

REFERENCES

[1] BAILEY, N. T. J. (1975). *The Mathematical Theory of Infectious Diseases and Its Applications*. Griffin, London.

[2] BALL, F. G. (1986). A unified approach to the distribution of total size and total area under the trajectory of infectives in epidemic models. *Adv. in Appl. Probab.* **18** 289–310.

[3] BALL, F. G. and BARBOUR, A. D. (1990). Poisson approximation for some epidemic models. *J. Appl. Probab.* **27** 479–490.

[4] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation*. Oxford Univ. Press.

[5] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation*. Cambridge Univ. Press.

[6] DANIELS, H. E. (1967). The distribution of the total size of an epidemic. In *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **4** 281–293. Univ. California Press, Berkeley.

[7] LEFÈVRE, C. (1990). Stochastic epidemic models for S-I-R infectious diseases: A brief survey of the recent general theory. *Stochastic Processes in Epidemic Theory. Lecture Notes in Biomath.* **86** 1–12. Springer, Berlin.

[8] LEFÈVRE, C. and PICARD, P. (1990). A non-standard family of polynomials and the final size distribution of Reed–Frost epidemic processes. *Adv. in Appl. Probab.* **22** 25–48.

[9] MARTIN-LÖF, A. (1986). Symmetric sampling procedures, general epidemic processes and their threshold limit theorems. *J. Appl. Probab.* **23** 265–282.

[10] PETROV, V.V. (1975). *Sums of Independent Random Variables*. Springer, Berlin.

[11] PICARD, P. and LEFÈVRE, C. (1990). A unified analysis of the final size and severity distribution in collective Reed–Frost epidemic processes. *Adv. in Appl. Probab.* **22** 269–294.

[12] SELLKE, T. (1983). On the asymptotic distribution of the size of a stochastic epidemic. *J. Appl. Probab.* **20** 390–394.

[13] VON BAHR, B. and MARTIN-LÖF, A. (1980). Threshold limit theorems for some epidemic processes. *Adv. in Appl. Probab.* **12** 319–349.

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