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# CROSSING VELOCITIES AND RANDOM LATTICE ANIMALS

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We consider a Brownian motion in a Poissonian potential conditioned to reach a remote location. We show that for typical configurations the expectation of the time $H$ to reach this goal grows at most linearly in the distance from the goal to the origin. In spite of the fact that $H$ has no finite exponential moment, we derive three exponential estimates, one of which concerns the size of a natural lattice animal attached to the trajectory of the process up to the goal.

**Introduction.** We consider in this paper a Poissonian potential on $\mathbb{R}^d$, $d \geq 2$, obtained by translating a given fixed shape function $W(\cdot)$, which is bounded, nonnegative and measurable with compact support, at the points $x_i$ of a Poissonian cloud configuration $\omega = \sum_i \delta_{x_i}$. We assume that $W$ is not a.s. equal to zero and denote by $\mathbb{P}$ the law of the Poissonian cloud with fixed intensity $\nu > 0$, on the space $\Omega$ of simple pure point measures $\omega$ on $\mathbb{R}^d$.

We study in this article a Brownian motion in the Poissonian potential conditioned to reach a remote location. Our aim is to derive controls on the time it takes our conditioned process to "reach its goal."

We let $P_x$ stand for the Wiener measure starting from $x$, and we let $Z_.$ stand for the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. For $y \in \mathbb{R}^d$, a "remote location," the conditioned process is described by the measure

$$\hat{P}_0(dw) = \frac{1}{u(0)} 1\{H < \infty\} \exp\left\{ -\int_0^H V(Z_s(w), \omega)\, ds \right\} P_0(dw), \tag{I.1}$$

where $V(x, \omega) = \sum_i W(x - x_i) = \int W(x - x')\omega(dx')$ is the Poissonian potential, $H$ is the entrance time of $Z_.$ into the "goal," the closed ball of radius 1 around $y$, and $u(0)$ is the normalizing constant. In other words, $H$ is the "time to reach the goal," our main object of interest in the present work. Let us mention that the dependence on $y, \omega$ is dropped from the notation. In a slightly formal way, the conditioned process is a Brownian motion feeling up to time $H$ a drift which depends on the cloud structure

$$\begin{aligned} dZ_s &= d\beta_s + \frac{\nabla u}{u}(Z_s)\, ds, \qquad 0 \leq s \leq H, \\ Z_0 &= 0, \end{aligned} \tag{I.2}$$

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provided for $z \in \mathbb{R}^d$, in a consistent way with (I.1),

$$u(z) = e_0(z, y, \omega), \tag{I.3}$$

and with the notation of [6],

$$e_\lambda(x, y, \omega) = E_x\left[\exp\left\{-\int_0^H (\lambda + V)(Z_s, \omega)\, ds\right\}, H < \infty\right], \tag{I.4}$$

for $\lambda \geq 0$, $x, y \in \mathbb{R}^d$, $\omega \in \Omega$, if $H$ is as above. That is, $0 \leq u(\cdot) \leq 1$ is the $V(\cdot, \omega)$ equilibrium potential of $\overline{B}(y, 1)$ which formally satisfies

$$\begin{aligned} \tfrac{1}{2}\Delta u - Vu &= 0 \quad \text{on } \overline{B}(y,1)^c, \\ u &= 1 \quad \text{on } \overline{B}(y,1), \\ u &= 0 \quad \text{at infinity} \quad \text{(for typical configurations)}. \end{aligned} \tag{I.5}$$

We show here that the conditioned process reaches its goal with a nondegenerate velocity, in the sense that (see Theorem 2.2)

$$\begin{aligned} &\text{when } d \geq 2, \text{there exist } \kappa(d, \nu, W) \in (0, \infty) \text{ such that} \\ &\mathbb{P}\text{-a.s.} \quad \limsup_{y \to \infty} \frac{1}{|y|} \hat{E}_0[H] < \kappa(d, \nu, W). \end{aligned} \tag{I.6}$$

The difficulty in proving (I.6) comes from the occurrence within the cloud of big empty pockets where the conditioned Brownian motion "almost looks like a usual Brownian motion." For instance, in dimension 2, using scaling arguments, the expected time to exit a circular pocket is comparable to the surface of the pocket and in fact, for arbitrarily shaped pockets, no more than a universal constant times the surface of the pocket, by a general result of Cranston and McConnell [1]. Of course in such large pockets, the conditioned process tends to have a vanishing velocity, which endangers our claim (I.6).

Moreover, a companion difficulty when working with the crossing time $H$ is the fact that any exponential moment of $H$ under $\hat{P}_0$ is infinite for typical cloud configurations. As we shall explain further below, we palliate this lack of integrability with an exponential estimate of a different kind concerning the size of a certain random lattice animal attached to the process. Let us mention that a similar estimate to (I.6) holds in the one-dimensional case and follows from the ergodic theorem together with the controls derived in Theorem 2.6 of [6].

The estimate (I.6) has a natural application to the study of certain Lyapounov exponents we introduced in [6]. These Lyapounov coefficients $\alpha_\lambda(x)$, $\lambda \geq 0$, $x \in \mathbb{R}^d$, satisfy

$$\mathbb{P}\text{-a.s., for } M > 0, \qquad \lim_{y \to \infty} \sup_{0 \leq \lambda \leq M} \frac{1}{|y|} |-\log e_\lambda(0, y, \omega) - \alpha_\lambda(y)| = 0. \tag{I.7}$$

We know from [6] that the nonnegative function $\alpha_\lambda(x)$ is jointly continuous, defines a norm on $\mathbb{R}^d$ for fixed $\lambda$ and is concave increasing for fixed $x$. From (I.6) follows the differentiability at the origin of $\lambda \to \alpha_\lambda(x)$, for $x \in \mathbb{R}^d$ (see Corollary 2.3).

This point has an interesting consequence on the study of the asymptotic behavior of "quenched Brownian motion with a constant drift $h$ in the Poissonian potential $V(\cdot, \omega)$"; see Section 3 for precise definitions. As follows from our results in [6] and [7], this model exhibits a transition between the small drift and large drift regimes. There is a certain critical threshold for $|h|$ which is in general direction dependent and can be expressed in terms of the norm $\alpha_0(\cdot)$. Below this threshold, the process has "zero velocity," and above this threshold the process has "positive velocity." In fact, much more is known and we refer to Section 3 for precise statements. As we show here the transition occurs with a "jump in the velocity," which above the critical threshold is no less than $1/\kappa$, where $\kappa$ is the constant from (I.6).

Let us now explain how the paper is organized. In Section 1 we derive our key exponential estimates. We chop $\mathbb{R}^d$ into large cubes of size $l$, $l$ large but independent of $|y|$. To simplify things there are essentially two types of cubes: cubes which receive points of the cloud and cubes which do not (the true story is somewhat more complicated). We derive, on one hand, in Theorem 1.1 and Proposition 1.2, exponential estimates on the time spent up to time $H$ by the process in cubes receiving a point of the cloud. We show that a suitably small exponential moment of this time does not grow faster than geometrically in $|y|$. Intuitively, the cubes which receive a point of the cloud are unpleasant for the process which is "near obstacles" in such cubes.

On the other hand, we introduce the random lattice animal $\mathscr{A}$ on $\mathbb{Z}^d$, made of the labels of cubes visited by the process up to time $H$. We show in Theorem 1.3 that a suitably small exponential moment of the size of $\mathscr{A}$ does not grow faster than geometrically in $N$. Ideas there are in the spirit of the work of Cox, Gandolfi, Griffin and Kesten [2] and Fontes and Newman [3]. For typical cloud configurations, a large $|\mathscr{A}|$ implies that $\mathscr{A}$ meets a nonvanishing fraction of cubes receiving a point of the cloud. This is an "unpleasant experience" for the process and the extent to which it is "unpleasant" is quantified by a suitable supermartingale introduced in the proof of Theorem 1.3.

In Section 2, we prove (I.6) in Theorem 2.2. The main task is to control the time spent in empty cubes. We use Harnack's inequality together with our exponential controls on the size of $\mathscr{A}$. The two-dimensional situation presents additional difficulty and in fact we explicitly use a result of Fontes and Newman [3] in this case.

In Section 3 we discuss the applications to quenched Brownian motion with a constant drift in the Poissonian potential.

**1. Exponential estimates.** Our objective in this section is to prove the exponential estimates mentioned in the Introduction. These exponential estimates are of two kinds. Roughly speaking, one type of estimate is concerned with the time spent by the process in the vicinity of obstacles, and another type of estimate with the size of the random lattice animal made of blocks (to be defined below) visited by the process.

We keep here the notation from the Introduction. When $A$ is a closed subset of $\mathbb{R}^d$, $d \geq 2$, $H_A$ denotes the entrance time of $Z_.$ in $A$, whereas for $U$ an open subset of $\mathbb{R}^d$, $T_U$ is the exit time of $Z_.$ from $U$. We let $a = a(W) > 0$ stand for the smallest number such that $W(\cdot) = 0$ outside $\overline{B}(0, a)$. When $z \in \mathbb{R}^d$, we write

$$B(z) = \overline{B}(z, 1), \tag{1.1}$$

for the closed ball of radius 1 around $z$.

Let us now introduce the paving of $\mathbb{R}^d$ which comes in the definition of the random lattice animal and of the notion of vicinity of obstacles. For $q \in \mathbb{Z}^d$, we consider the cube of size $l$ and center $lq$:

$$C(q) = \left\{ z \in \mathbb{R}^d \colon -\frac{l}{2} \leq z^i - lq^i < \frac{l}{2}, i = 1, \dots, d \right\}. \tag{1.2}$$

Here $l(d, \nu, a) > 8a$ is a large enough number such that

$$\sum_{n=1}^{\infty} 9^{dn} p_n(l, \nu) < \infty, \tag{1.3}$$

where $p_n(l, \nu)$ stands for the probability that a binomial variable with parameters $n$ and $p = 1 - \exp\{-\nu(l^d/4^d)\}$ takes a value smaller than $n/2$. That such a choice of $l$ is possible follows from standard exponential estimates on the binomial distribution with success probability $p$ close to 1. Let us now explain the meaning of (1.3).

The factor $9^{dn} = (3^d)^{2n}$ represents a (rough) upper bound on the number of animals $\Gamma$ (i.e., finite connected sets) on $\mathbb{Z}^d$, of size $n$, containing 0, when the adjacency relation of two sites $q, q'$ in $\mathbb{Z}^d$ is defined via

$$\|q - q'\| = \sup_{i=1,\dots,d} |q^i - q'^i| \leq 1.$$

To see this, one uses a spanning tree of $\Gamma$ with $n$ vertices and $n - 1$ nearest neighbor edges, and one constructs a nearest neighbor path starting at 0 of length at most $2n$ "walking around the spanning tree"; see Lemma 1 of Cox, Gandolfi, Griffin and Kesten [2] and Harris [4].

The quantity $9^{dn} p_n(l, \nu)$ is then an upper bound on the $\mathbb{P}$-probability that there exists an animal $\Gamma$ containing 0, of size $n$ and such that

$$\sum_{q \in \Gamma} 1\{\text{the open cube of side length } l/4 \text{ centered at } lq \text{ receives a point of } \omega\} \leq n/2. \tag{1.4}$$

Let us incidentally mention that the random lattice animal, which we later on associate with $Z_.$, corresponds in contrast to (1.4) to the usual adjacency relation on $\mathbb{Z}^d$, where $q, q'$ are adjacent if $\sum_{i=1}^d |q^i - q'^i| \leq 1$.

Let us introduce some more notation. For $y \in \mathbb{R}^d$, we partition the collection of boxes $C(q)$, $q \in \mathbb{Z}^d$, into three disjoint classes indexed by $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$. The dependence on $y$ is dropped from the notation. Define

$$(1.5)\quad \mathscr{C}_3 = \{q \in \mathbb{Z}^d \colon \exists\, q' \in \mathbb{Z}^d, \|q' - q\| \leq 3 \text{ and } C(q') \cap B(y) \neq \phi\}.$$

Hence $\mathscr{C}_3$ is the class of boxes "neighboring the goal $B(y)$";

$$(1.6)\qquad \mathscr{C}_2 = \left\{q \in \mathbb{Z}^d \setminus \mathscr{C}_3 \colon \omega\left(\bigcup_{q' \colon \|q' - q\| \leq 1} C(q')\right) = 0\right\}.$$

That is, $q \in \mathbb{Z}^d \setminus \mathscr{C}_3$ is in $\mathscr{C}_2$ if no neighboring box of $C(q)$, $C(q)$ included, receives a point of the cloud $\omega$. Finally the remaining boxes, not in $\mathscr{C}_3$ and "neighboring the obstacles" are indexed by the class

$$(1.7)\qquad \mathscr{C}_1 = \left\{q \in \mathbb{Z}^d \setminus \mathscr{C}_3 \colon \omega\left(\bigcup_{q' \colon \|q - q'\| \leq 1} C(q')\right) \geq 1\right\}.$$

Accordingly, we define for $i = 1, 2, 3$,

$$(1.8)\qquad H_i = \sum_{q \in \mathscr{C}_i} \int_0^H 1(Z_s \in C(q))\, ds.$$

That is, $H_i$ is the time spent by the process in boxes indexed by the class $\mathscr{C}_i$ until it reaches $B(y)$. We are now ready to state the first exponential estimate as the following theorem.

THEOREM 1.1. *There exist* $\gamma_1(d, \nu, W) > 0$ *such that, for* $y \in \mathbb{R}^d$ *and* $\omega \in \Omega$,

$$(1.9)\qquad \hat{E}_0[\exp\{\gamma_1 H_1\}] \leq \frac{1}{u(0)}$$

[*see* (I.1) *and* (I.3) *for the definition of* $u(\cdot)$].

PROOF. We express $\mathbb{Z}^d$ as the disjoint union

$$\mathbb{Z}^d = \bigcup_{I \in \{0, \dots, 4\}^d} 5\mathbb{Z}^d + I,$$

and for $I \in \{0, \dots, 4\}^d$, we define

$$(1.10)\qquad \mathscr{C}_{1,I} = \mathscr{C}_1 \cap (5\mathbb{Z}^d + I) \quad \text{and} \quad H_{1,I} = \sum_{q \in \mathscr{C}_{1,I}} \int_0^H 1(Z_s \in C(q))\, ds.$$

To prove (1.9) it clearly suffices to prove an analogous statement for each $H_{1,I}$, $I \in \{0, \dots, 4\}^d$. We now keep such an $I$ fixed. We define the open set $O$

which is the pairwise disjoint union of open cubes of side length $5l$ centered at points in $5l\mathbb{Z}^d + lI$,

$$(1.11)\qquad O = \bigcup_{q \in 5\mathbb{Z}^d + I} \Bigl( \bigcup_{q' : \|q' - q\| \le 2} C(q') \Bigr)^0,$$

and the closed set $A$ contained in $O$,

$$(1.12)\qquad A = \bigcup_{q \in \mathscr{C}_{1,I}} \overline{C(q)}.$$

We now define two increasing sequences of stopping times of the natural right continuous filtration $\mathscr{F}_t^+$ on $C(\mathbb{R}_+, \mathbb{R}^d)$. These stopping times describe the successive returns to $A$ and departures from $O$ of the process:

$$(1.13)\qquad \begin{aligned} R_1 &= \inf\{s \ge 0, Z_s \in A\} \le \infty, \\ D_1 &= \inf\{s \ge R_1, Z_s \in O^c\} \le \infty, \end{aligned}$$

and by induction, letting $\theta_t$, $t \ge 0$, stand for the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$,

$$\begin{aligned} R_{n+1} &= R_1 \circ \theta_{D_n} + D_n \le \infty, \qquad n \ge 1 \\ D_{n+1} &= D_1 \circ \theta_{D_n} + D_n \le \infty, \end{aligned}$$

so that

$$0 \le R_1 \le D_1 \le \cdots \le R_n \le D_n \le \cdots,$$

these inequalities, except for the first one, being strict as soon as the left member is finite. We now have, on $\{H < \infty\}$,

$$(1.14)\qquad H_{1,I} = \sum_{i=1}^{\infty} \int_{R_i \wedge H}^{D_i \wedge H} 1(Z_s \in A)\, ds = \sum_{i=1}^{\infty} \biggl( \int_0^{D_1 \wedge H} 1(Z_s \in A)\, ds \biggr) \circ \theta_{R_i \wedge H}.$$

We now define

$$(1.15)\qquad c_1(d, \nu, W) = \sup_{\substack{\|z\| \le l/2 \\ \|x\| \le 3l/2}} E_z \exp\biggl\{ -\int_0^{T_U} W(Z_s - x)\, ds \biggr\} < 1,$$

where

$$(1.16)\qquad U = \Bigl( \bigcup_{\|q\| \le 2} C(q) \Bigr)^0.$$

At this point we use the fact that $a < l/2$ (in fact, $a < l/8$ for later use) and therefore the obstacle attached to $x$ does meet $U$ and is, in fact, strictly within $U$. We now pick $\gamma(d, \nu, W) > 0$ small enough that

$$(1.17)\qquad E_z[\exp\{\gamma T_U\}] - 1 + c_1 \le 1 \quad \text{for } z \in \mathbb{R}^d.$$

It now follows from (1.14) that

$$
\begin{aligned}
(1.18)\qquad & u(0)\hat{E}_0[\exp\{\gamma H_{1,I}\}] \\
& \le E_0\left[\exp\left\{\gamma \sum_{i\ge 1} (D_1 \wedge H)\circ\theta_{R_i\wedge H} - \int_0^H V(Z_s,\omega)\,ds\right\}, H<\infty\right].
\end{aligned}
$$

Now for $k \ge 0$,

$$
\begin{aligned}
& E_0\left[\exp\left\{\gamma \sum_{i=1}^{k+1} (D_1 \wedge H)\circ\theta_{R_i\wedge H} - \int_0^H V(Z_s,\omega)\,ds\right\}, H<\infty\right] \\
& \quad \le E_0\left[\exp\left\{\gamma \sum_{i=1}^{k+1} (D_1 \wedge H)\circ\theta_{R_i\wedge H} - \int_0^{D_{k+1}\wedge H} V(Z_s,\omega)\,ds\right\},\right. \\
(1.19)\qquad & \qquad\qquad \left. D_{k+1}\wedge H<\infty\right] \\
& \quad \le E_0\left[\exp\left\{\gamma \sum_{i=1}^{k} (D_1 \wedge H)\circ\theta_{R_i\wedge H} - \int_0^{R_{k+1}\wedge H} V(Z_s,\omega)\,ds\right\},\right. \\
& \qquad R_{k+1}\wedge H<\infty, \\
& \qquad \left. E_{Z_{R_{k+1}\wedge H}}\left[\exp\left\{\gamma D_1\wedge H - \int_0^{D_1\wedge H} V(Z_s,\omega)\,ds\right\}\right]\right].
\end{aligned}
$$

Now when $H<\infty$ and $H \le R_{k+1}$,

$$
(1.20)\qquad E_{Z_{R_{k+1}\wedge H}}\left[\exp\left\{\gamma D_1\wedge H - \int_0^{D_1\wedge H} V(Z_s,\omega)\,ds\right\}\right] = 1.
$$

On the other hand, when $R_{k+1} < H$, then $Z_{R_{k+1}\wedge H} \in A$, and therefore there exists $q \in \mathscr{C}_{1,I} \subset \mathscr{C}_3^c$ such that $Z_{R_{k+1}\wedge H} \in \overline{C}(q)$. It follows that whenever $C(q') \cap B(y) \ne \phi$, $\|q'-q\| \ge 4$; see (1.5). Consequently, when $R_{k+1} < H$, $P_{Z_{R_{k+1}\wedge H}}$-a.s. $D_1 < H$ and the left member of (1.20) now equals

$$
\begin{aligned}
& E_{Z_{R_{k+1}}}\left[\exp\left\{\gamma D_1 - \int_0^{D_1} V(Z_s,\omega)\,ds\right\}\right] \\
& \quad \le E_{Z_{R_{k+1}}}[e^{\gamma D_1}] - 1 + E_{Z_{R_{k+1}}}\left[\exp\left\{-\int_0^{D_1} V(Z_s,\omega)\,ds\right\}\right] \\
& \quad \le E_{Z_{R_{k+1}}}[e^{\gamma D_1}] - 1 + c_1 \le 1,
\end{aligned}
$$

combining (1.15) and (1.17) since one of the cubes neighboring $C(q)$ receives a point of the cloud. Therefore, the left member of (1.19) is smaller than

$$
E_0\left[\exp\left\{\gamma \sum_{i=1}^{k} (D_1 \wedge H)\circ\theta_{R_i\wedge H} - \int_0^{D_k\wedge H} V(Z_s,\omega)\,ds\right\}, D_k\wedge H<\infty\right]
$$

and by induction is smaller than 1.

Using Fatou's lemma, the left member of (1.18) is smaller than 1, and this proves (1.9) as explained above. □

The second exponential estimate is more straightforward:

PROPOSITION 1.2. *There exist $\gamma_3(d, \nu, W) > 0$ such that for large enough $|y|$, for any $\omega \in \Omega$,*

$$\hat{E}_0[\exp\{\gamma_3 H_3\}] \le \frac{2}{u(0)}. \tag{1.21}$$

PROOF. Denote by $A$ the closed ball $\overline{B}(y, L)$, where $L = 4\sqrt{d}\,l + 1$. Then from (1.5), $A$ contains $\bigcup_{q \in \mathscr{C}_3} \overline{C(q)}$. Moreover, for $|y| \ge L$, $\gamma > 0$,

$$u(0)\hat{E}_0[\exp\{\gamma H_3\}] \le \sup_{|z-y|=L} E_z\left[\exp\left\{\gamma \int_0^H 1(Z_s \in A)\, ds\right\}\right].$$

Now when $d \ge 3$, using scaling and translation invariance, the last expression is smaller than

$$\sup_{|z|=1} E_z\left[\exp\left\{\gamma L^2 \int_0^\infty 1\big(Z_s \in \overline{B}(0,1)\big)\, ds\right\}\right].$$

It is classical that there exists $\gamma_0(d)$ such that when $\gamma L^2 \le \gamma_0(d)$ the last expression is finite and, in fact, smaller than 2 if $\gamma_0(d)$ is chosen small enough. This proves (1.21) when $d \ge 3$, with $\gamma_3 = \gamma_0/L^2$. When $d = 2$, one picks an open ball $O$, centered in $y$, containing $A$, large enough so that

$$P_z[H < T_O] \ge \tfrac{3}{4} \quad \text{when } z \in A. \tag{1.22}$$

Then one introduces the successive returns to $A$ and departures from $O$, as in (1.13). For $|y| \ge L$, $z$ with $|z - y| = L$ and $\gamma > 0$, we have

$$\begin{aligned} &E_z\left[\exp\left\{\gamma \int_0^H 1(Z_s \in A)\, ds\right\}\right] \\ &\quad \le \sum_{k=1}^{\infty} E_z\left[R_k < H \le R_{k+1}, \exp\left\{\gamma \sum_{j=1}^{k} D_1 \circ \theta_{R_j}\right\}\right]. \end{aligned} \tag{1.23}$$

Pick $\gamma$ small enough so that

$$\sup_{z \in O} E_z[\exp\{\gamma D_1\}] - 1 + \tfrac{1}{4} \le \tfrac{1}{3}. \tag{1.24}$$

Then for $k \ge 1$,

$$\begin{aligned} &E_z\left[R_k < H, \exp\left\{\gamma \sum_{j=1}^{k} D_1 \circ \theta_{R_j}\right\}\right] \\ &\quad = E_z\left[R_k < H, \exp\left\{\gamma \sum_{j=1}^{k-1} D_1 \circ \theta_{R_j}\right\} E_{Z_{R_k}}[\exp\{\gamma D_1\}]\right] \\ &\quad \le \tfrac{4}{3} E_z\left[R_k < H, \exp\left\{\gamma \sum_{j=1}^{k-1} D_1 \circ \theta_{R_j}\right\}\right]. \end{aligned}$$

Now when $k \geq 2$,

$$E_z\left[R_k < H, \exp\left\{\gamma \sum_{j=1}^{k-1} D_1 \circ \theta_{R_j}\right\}\right]$$

$$= E_z\left[R_{k-1} < H, \exp\left\{\gamma \sum_{j=1}^{k-2} D_1 \circ \theta_{R_j}\right\} E_{Z_{R_{k-1}}}[\exp\{\gamma D_1\}, D_1 < H]\right].$$

Using (1.24) and (1.22), on $R_{k-1} < H$,

$$E_{Z_{R_{k-1}}}[\exp\{\gamma D_1\}, D_1 < H] \leq \tfrac{1}{3}.$$

Therefore the above expression is smaller than

$$\tfrac{1}{3} E_z\left[R_{k-1} < H, \exp\left\{\gamma \sum_{j=1}^{k-2} D_1 \circ \theta_{R_j}\right\}\right] \leq \left(\tfrac{1}{3}\right)^{k-1}$$

by induction. We now find that

$$E_z\left[\exp\left\{\gamma \int_0^H 1(Z_s \in A)\, ds\right\}\right] \leq \tfrac{4}{3} \sum_1^{\infty} \left(\tfrac{1}{3}\right)^{k-1} = 2.$$

This proves (1.21) when $d = 2$. □

As already mentioned in the Introduction, we cannot hope for an exponential estimate in the spirit of Theorem 1.1 or Proposition 1.2 in the case of $H_2$ defined in (1.8). We shall instead derive our exponential estimate on the total number of cubes visited by the process $Z_{.}$ up to time $H$. To this end, we define

$$(1.25) \qquad \mathscr{A}(w) = \left\{q = \mathbb{Z}^d \colon H_{\overline{C}(q)} < H\right\},$$

where again we dropped the dependence on $y$ in the notation. Observe that the path $Z_{.}$ goes from one box to another box through "faces" and not "corners" because the set of points in $\mathbb{R}^d$ with at least two coordinates of the form $l/2 + \mathbb{Z}l$ is polar. Now $\hat{P}_0$-a.s. $H$ is finite and, therefore,

$$(1.26) \qquad P_0\text{-a.s. } \mathscr{A}(w) \text{ is a lattice animal of } \mathbb{Z}^d \text{ containing } 0,$$

where we use the standard adjacency relation for which $q, q'$ are adjacent if $\sum_1^d |q^i - q'^i| \leq 1$.

Let us briefly explain the strategy underlying the proof of an exponential estimate under $\hat{P}_0$ on the size of $\mathscr{A}$. The idea is to exploit (1.3) and (1.4), so that for typical configurations $\omega$ and large $|\mathscr{A}|$, the number of occupied sites in $\mathscr{A}$ represents a nonvanishing fraction ($\geq \frac{1}{2}$) of $|\mathscr{A}|$. On the other hand, using exponential controls in the spirit of the proof of Theorem 1.1, one has exponential bounds under $\hat{P}_0$ on the number of occupied cubes visited by $Z_{.}$ and, therefore, on $|\mathscr{A}|$. Let us also mention that our exponential estimate do not hold when $P_0$ replaces $\hat{P}_0$, even when $d = 2$.

THEOREM 1.3. *There exists a set* $\tilde{\Omega}$ *of full* $\mathbb{P}$*-measure and* $\gamma_2(d, \nu, W) > 0$ *such that, for* $\omega \in \tilde{\Omega}$,

$$\sup_y \left( u(0) \hat{E}_0 \left[ \exp\{\gamma_2 |\mathscr{A}|\} \right] \right) < \infty \tag{1.27}$$

($|\mathscr{A}|$ *denotes the cardinality of* $\mathscr{A}$).

PROOF. We define the successive times of travel of $Z_.$ at $\| \ \|$ distance $3l/4$:

$$\begin{aligned} &S_0 = 0, \qquad S_1 = \inf\left\{ s \geq 0, \|Z_s - Z_0\| \geq \frac{3l}{4} \right\}, \\ &S_{i+1} = S_1 \circ \theta_{S_i} + S_i \quad \text{for } i \geq 1. \end{aligned} \tag{1.28}$$

Observe that $Z_{S_i}$ and $Z_{S_{i+1}}$ lie in neighboring boxes $C(q)$ for the neighboring relation $\|q - q'\| \leq 1$. Moreover during the time interval $[Z_{S_i}, Z_{S_{i+1}}]$, $i \geq 0$, $Z_s$ cannot visit more than $3^d$ distinct boxes $\overline{C(q)}$. Therefore, if we define

$$\tilde{A}(w) = \left\{ q \in \mathbb{Z}^d, \exists\, i \geq 0, Z_{S_i \wedge H} \in C(q) \right\}, \tag{1.29}$$

then $\tilde{\mathscr{A}}$ is $\hat{P}_0$-a.s. a $\| \ \|$ lattice animal on $\mathbb{Z}^d$ (with obvious meaning) and

$$\hat{P}_0\text{-a.s.} \qquad |\mathscr{A}| \leq 3^d |\tilde{\mathscr{A}}|. \tag{1.30}$$

It therefore suffices to prove an analogous estimate to (1.27) with $\mathscr{A}$ replaced by $\tilde{\mathscr{A}}$. From the discussion preceding (1.4), we can pick a set $\tilde{\Omega}$ of full $\mathbb{P}$-measure such that

(1.31) for $\omega \in \tilde{\Omega}$, there is $n_0(\omega)$ so that for $n \geq n_0(\omega)$ and $\Gamma$ a $\| \ \|$ lattice animal containing $O$, with $|\Gamma| = n$, $\sum_{q \in \Gamma} 1\{\omega(lq + (-l/8, l/8)^d) \geq 1\} \geq n/2$

[in other words, for $\Gamma$ as above, (1.4) does not hold].

From now on we fix a given $\omega \in \tilde{\Omega}$ and consider $y \in \mathbb{R}^d$. To derive an exponential estimate on $|\mathscr{A}|$, we shall use a suitable supermartingale. However, we still need to introduce some more notations. We define the function $Oc(\cdot)$ on $\mathbb{R}^d$ via

$$\begin{aligned} Oc(z) &= 1, \quad \text{if the unique } q \in \mathbb{Z}^d \text{ such that } z \in C(q) \text{ is not in } \mathscr{C}_3 \\ &\qquad\quad \text{and is “occupied,” that is, } \omega\left(lq + (-l/8, l/8)^d\right) \geq 1, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{1.32}$$

We also introduce $\chi(d, \nu, W) \in (0, 1)$ via

$$\chi(d, \nu, W) = \sup_{\substack{\|z\| \leq l/2 \\ \|x\| \leq l/8}} E_z \left[ \exp\left\{ -\int_0^{S_1} W(Z_s - x)\, ds \right\} \right] < 1, \tag{1.33}$$

where we now use $l/2 + l/8 + a < 3l/4$ since $8a < l$ so that the obstacle is strictly within $\| \ \|$ distance $3l/4$ from $z$.

If $\mathscr{F}^+_{S_m}$, $m \geq 0$, denotes the $\sigma$-field associated to the $\mathscr{F}^+_t$ stopping time $S_m$, we define the sequence of $\mathscr{F}^+_{S_m}$ variables as

$$M_0 = 1, \qquad (1.34)$$
$$M_m = \prod_{i=0}^{m-1} \chi^{-Oc(Z_{S_i})1(S_{i+1} < H)} \exp\left\{ -\int_0^{S_m \wedge H} V(Z_s, \omega)\, ds \right\} \quad \text{for } m \geq 1.$$

Our next step is to show that

$$(M_m)_{m \geq 0} \text{ is an } \mathscr{F}^+_{S_m} \text{ supermartingale under } P_0. \qquad (1.35)$$

Indeed, for $n \geq 0$,

$$\begin{aligned} E_0\left[ M_{m+1} | \mathscr{F}^+_{S_m} \right] &= E_0\left[ M_m \left( 1(S_m \geq H) + 1(S_m < H) \chi^{-Oc(Z_{S_m})1(S_{m+1} < H)} \right.\right. \\ &\qquad \left.\left. \times \exp\left\{ -\int_{S_m}^{(S_1 \wedge H) \circ \theta_{S_m} + S_m} V(Z, \omega)\, ds \right\} \right) | \mathscr{F}^+_{S_m} \right] \\ &= M_m \left( 1(S_m \geq H) + 1(S_m < H) \right. \\ &\qquad \left. \times E_{Z_{S_m}} \left[ \chi^{-Oc(Z_0)1(S_1 < H)} \exp\left\{ -\int_0^{S_1 \wedge H} V(Z_s, \omega)\, ds \right\} \right] \right). \end{aligned}$$

Observe that when $Z_{S_m} \in \bigcup_{q \in \mathscr{C}_3} C(q)$, $P_{Z_{S_m}}$-a.s. $Oc(Z_{S_0}) = 0$ and

$$E_{Z_{S_m}} \left[ \chi^{-Oc(Z_0)1(S_1 < H)} \exp\left\{ -\int_0^{S_1 \wedge H} V(Z_s, \omega)\, ds \right\} \right] \leq 1. \qquad (1.36)$$

On the other hand, when $Z_{S_m} \notin \bigcup_{q \in \mathscr{C}_3} C(q)$, $P_{Z_{S_m}}$-a.s., $S_1 < H$, so that (1.36) holds again by (1.33). This proves that $E_0[M_{m+1} | \mathscr{F}^+_{S_m}] \leq M_m$ and therefore (1.35). Using Fatou's lemma, we see that

$$\begin{aligned} 1 \leq E[M_\infty] \geq E_0 &\left[ \prod_{i \geq 0} \chi^{-Oc(Z_{S_i})1(S_{i+1} < H)} \right. \\ &\left. \times \exp\left\{ -\int_0^H V(Z_s, \omega)\, ds \right\}, H < \infty \right]. \end{aligned} \qquad (1.37)$$

If $n_0$ denotes the integer from (1.31) corresponding to $\omega$, then $\hat{P}_0$-a.s.,

$$\begin{aligned} &\text{either} \quad |\tilde{\mathscr{A}}| \leq n_0 \\ &\text{or} \quad |\tilde{\mathscr{A}}| \leq 2 \sum_{q \in \tilde{\mathscr{A}}} 1\left\{ \omega\left( lq + \left( -\frac{l}{8}, \frac{l}{8} \right)^d \right) \geq 1 \right\} \\ &\qquad \leq 2\left( \sum_{i \geq 0} Oc(Z_{S_i}) 1(S_{i+1} < H) + |\mathscr{C}_3| \right), \end{aligned} \qquad (1.38)$$

Therefore, from (1.30),

$$\hat{E}_0\left[\exp\left\{\frac{1}{2\cdot 3^d}\log\left(\frac{1}{\chi}\right)|\mathscr{A}|\right\}\right] \le \hat{E}_0\left[\exp\left\{\frac{1}{2}\log\left(\frac{1}{\chi}\right)|\tilde{\mathscr{A}}|\right\}\right]$$

and using (1.37) and (1.38),

$$\le \frac{1}{u(0)}\left(\chi^{-n_0} + \chi^{-|\mathscr{C}_3|}\right),$$

from which our claim (1.27) follows. □

**2. Nonvanishing of asymptotic crossing velocities.** The main object of this section is to show that for typical cloud configurations, $\hat{E}_0[H]$ grows asymptotically linearly with $|y|$. The proof of this result will rely on the exponential estimates derived in the previous section. Our main task in this section is the control of the time spent by the process in boxes $C(q)$ with $q \in \mathscr{C}_2$, that is, essentially in the "holes" within the cloud. As we shall explain below, we shall derive such controls with the help of the exponential estimate on the size of the random lattice animal $\mathscr{A}$ in Theorem 1.3, and Harnack's inequality.

First we need some notation. We define for $y \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^d$, $s > 0$, $\omega \in \Omega$,

$$r_y(s, z, z', \omega) = p_s(z, z') E^s_{z,z'}\left[\exp\left\{-\int_0^s V(Z_u, \omega)\,du\right\}, H > s\right], \tag{2.1}$$

where $p_s(z, z')$ is the Brownian transition density and $E^s_{z,z'}$ is the Brownian bridge in time $s$ from $z$ to $z'$. When $z$ or $z'$ belongs to $B(y)$, then $r_y(s, z, z', \omega) = 0$. So $r_y$ is the transition subdensity of Brownian motion in the Poissonian potential killed when entering $B(y)$. We also define $r(s, z, z', \omega)$ analogously, except that we omit the condition $H > s$ in the Brownian bridge expectation. In other words, $r$ is the transition subdensity of Brownian motion in the Poissonian potential.

The corresponding Green's functions are

$$\begin{aligned} g_y(z, z', \omega) &= \int_0^\infty r_y(s, z, z', \omega)\,ds,\\ g(z, z', \omega) &= \int_0^\infty r(s, z, z', \omega)\,ds. \end{aligned} \tag{2.2}$$

Of course, when $d \ge 3$,

$$\begin{aligned} g_y(z, z', \omega) &\le g(z, z', \omega) \le g^0(z, z', \omega)\\ &= \int_0^\infty p_s(z, z')\,ds \quad \text{as defined.} \end{aligned} \tag{2.3}$$

In dimension 2, we know from [6], (1.36) and (1.37) that on a set of full $\mathbb{P}$ probability, $g(z, z', \omega) < \infty$, for $z \ne z'$.

We omit the proof of the following lemma which is classical and follows straightforwardly from the strong Markov property of Wiener measure.

LEMMA 2.1. *Let $C$ be a closed subset of* $\mathbb{R}^d$,

$$\hat{E}_0\left[\int_0^H 1(Z_s \in C)\,ds\right] = \hat{E}_0\left[H_C < H,\, \hat{E}_{Z_{H_C}}\left[\int_0^H 1(Z_s \in C)\,ds\right]\right], \tag{2.4}$$

*and for* $z \in \mathbb{R}^d$,

$$\hat{E}_z\left[\int_0^H 1(Z_s \in C)\,ds\right] = \frac{1}{u(z)}\int_C g_y(z, z')u(z')\,dz', \tag{2.5}$$

*where* $u(\cdot)$ *is defined in* (I.3).

We are now going to state the main result of this section.

THEOREM 2.2 ($d \geq 2$). *There is a set* $\overline{\Omega}$ *of full* $\mathbb{P}$ *probability and* $\kappa(d, \nu, W) \in (0, \infty)$ *such that, for* $\omega \in \overline{\Omega}$,

$$\limsup_{y \to \infty} \frac{1}{|y|}\hat{E}_0[H] < \kappa. \tag{2.6}$$

PROOF. With the notations of (1.8), we know that $H = H_1 + H_2 + H_3$. Moreover, from (1.9), (1.27) and the estimate (I.7) in the case $\lambda = 0$, on a set of full $\mathbb{P}$-probability,

$$\limsup_{y \to \infty} \hat{E}_0\left[\exp\left\{\gamma_1 \frac{H_1}{|y|}\right\}\right] \leq \limsup_{y \to \infty}\left(\frac{1}{u(0)}\right)^{1/|y|} = \exp\left\{\sup_{|e|=1} \alpha_0(e)\right\}, \tag{2.7}$$

$$\limsup_{y \to \infty} \hat{E}_0\left[\exp\left\{\gamma_3 \frac{H_3}{|y|}\right\}\right] \leq \limsup_{y \to \infty}\left(\frac{2}{u(0)}\right)^{1/|y|} = \exp\left\{\sup_{|e|=1} \alpha_0(e)\right\}, \tag{2.8}$$

Therefore our claim will follow once we prove an estimate analogous to (2.6) with $H$ replaced by $H_2$. Observe that, as follows from Lemma 2.1,

$$\hat{E}_0[H_2] = \sum_{q \in \mathscr{C}_2} \hat{E}_0\Big[H_{\overline{C}(q)} < H,\, \frac{1}{u\big(Z_{H_{\overline{C}(q)}}\big)}\int_{\overline{C}(q)} g_y(Z_{H_{\overline{C}(q)}}, z)u(z)\,dz\Big]. \tag{2.9}$$

Now when $q \in \mathscr{C}_2$, it follows from (1.6) that no point of the cloud $\omega$ falls in the box $C(q)$ and its neighboring boxes. Moreover, neither $C(q)$ nor its neighboring boxes intersect $B(y)$. Since $a < l/8$, $V(\cdot, \omega) = 0$ on the homothetic of $C(q)$ with ratio $3/2$ and center $ql$. Consequently, $u(z) = E_z[\exp\{-\int_0^H V(Z_s, \omega)\,ds\},\, H < \infty]$ is a positive harmonic function on the interior of this set which contain $\overline{C(q)}$. From Harnack's inequality we can find a constant $K(d)$ such that

$$\text{when } y \in \mathbb{R}^d,\ \omega \in \Omega,\ q \in \mathscr{C}_2,\ z, z' \in \overline{C}(q),\ u(z)/u(z') \le K(d). \tag{2.10}$$

Inserting in (2.9), we obtain

$$\hat{E}_0[H_2] \le \sum_{q \in \mathscr{C}_2} K(d) \hat{P}_0\big[H_{\overline{C}(q)} < H\big] \sup_{\overline{C}(q)} \int_{\overline{C}(q)} g_y(z, z', \omega)\, dz'. \tag{2.11}$$

In dimension $d \ge 3$, from (2.3) together with $\sup_{\overline{C}(q)} \int_{\overline{C}(q)} g^0(z, z')\, dz' = \text{const}(d, l) < \infty$, we see that

$$\hat{E}_0[H_2] \le C(d, \nu, W) \cdot \hat{E}_0[|\mathscr{A}|] \tag{2.12}$$

and our claim (2.6), with $H$ replaced by $H_2$, now follows from (1.27) and (I.7) in the case $\lambda = 0$. This finishes the proof of (2.6) when $d \ge 3$.

Let us now discuss the two-dimensional situations. We introduce in this case for $q \in \mathbb{Z}^d$, $W_q$ the possibly empty "cluster of unoccupied sites in $\mathbb{Z}^d$" which contain $q$. That is, $W_q$ is the connected component attached to $q$ of sites $q' \in \mathbb{Z}^d$ with $Oc(q'l) = 0$; see (1.32) for the definition, with the usual adjacency relation on $\mathbb{Z}^d$: $\sum_{i=1}^d |q_1^i - q_2^i| \le 1$. When $Oc(ql) = 1$ ($q$ is occupied), then $W_q$ is empty.

In fact, from (1.3) and the discussion following (1.3) the probability that a site $q \in \mathbb{Z}^d$ is empty, that is, $\exp\{-\nu l^d / 4^d\}$, is nonpercolating for the standard site percolation problem on $\mathbb{Z}^d$. Therefore, on a set $\Omega_1$ of full $\mathbb{P}$ probability the sets $W_q$, $q \in \mathbb{Z}^d$, are all finite.

From the description of $\mathscr{C}_2$, all $q'$ with $\|q' - q\| \le 1$ belong to $W_q$ when $q \in \mathscr{C}_2$. We also define, for $q \in \mathbb{Z}^d$, the open set

$$O_q = \Big( \bigcup_{q' \in W_q} C(q') \Big)^{\circ}. \tag{2.13}$$

Consider now some fixed $\omega \in \Omega_1$ and $q \in \mathscr{C}_2$. Introduce as in Section 1 the successive returns to $\overline{C}(q)$ and departures from $O_q$, so that $0 \le R_1 < D_1 < R_2 < \cdots < R_n < D_n < \cdots < \infty$, $P_z$-a.s., for $z \in \mathbb{R}^2$, since $O_q$ is bounded when $\omega \in \Omega_1$. Then for $z \in \overline{C}(q)$,

$$\begin{aligned} &\int_{\overline{C}(q)} g(z, z', \omega)\, dz' \\ &\quad = \int_0^\infty E_z\Big[ Z_s \in \overline{C}(q), \exp\Big\{ -\int_0^s V(Z_u, \omega)\, du \Big\} \Big]\, ds \\ &\quad = \sum_{i=1}^\infty E_z\Big[ \int_{R_i}^{R_i + D_1 \circ \theta_{R_i}} 1(Z_s \in \overline{C}(q)) \exp\Big\{ -\int_0^s V(Z_u, \omega)\, du \Big\}\, ds \Big] \\ &\quad \le \sum_{i=1}^\infty E_z\Big[ \exp\Big\{ -\int_0^{R_i} V(Z_u, \omega)\, du \Big\} E_{Z_{R_i}}\Big[ \int_0^{D_1} 1(Z_s \in \overline{C}(q)\, ds) \Big] \Big]. \end{aligned} \tag{2.14}$$

However, when $z \in \overline{C}(q)$,

$$E_z[D_1] = E_z\big[T_{O_q}\big] \le \big(2/\lambda_{-\Delta/2}(O_q)\big)\log\Big(E_x\Big[\exp\Big\{\tfrac{1}{2}\lambda_{-\Delta/2}(O_q)T_{O_q}\Big\}\Big]\Big),$$

provided $\lambda_{-\Delta/2}(O_q)$ denotes the principal Dirichlet eigenvalue of $-\Delta/2$ in $O_q$. Now by (1.18) of [5] (see also Cranston and McConnell [1] Lemma 2.1), the above quantity is smaller than $\text{const}(d = 2)/\lambda_{-\Delta/2}(O_q) \le \text{const}(d = 2)|O_q|$, using Faber Krahn's inequality in the last step. Therefore, for $z \in \overline{C}(q)$,

$$\int_{\overline{C}(q)} g(z, z', \omega)\, dz' \le \text{const} \cdot \sum_{i=1}^{\infty} E_z\left[\exp\left\{-\int_0^{R_i} V(Z_s, \omega)\, ds\right\}\right]|O_q|, \tag{2.15}$$

where the constant is "numerical" and is independent of $\nu, W, y, \omega, q$.

Now for $i \ge 1$,

$$\begin{aligned} &E_z\left[\exp\left\{-\int_0^{R_{i+1}} V(Z_u, \omega)\, du\right\}\right] \\ &\quad = E_z\left[\exp\left\{-\int_0^{D_i} V(Z_u, \omega)\, du\right\} E_{Z_{D_i}}\left[\exp\left\{-\int_0^{R_1} V(Z_u, \omega)\, du\right\}\right]\right]. \end{aligned} \tag{2.16}$$

Now $P_z$-a.s. we can find $q' \in \mathbb{Z}^d$ with $O_c(q'l) = 1$ and $Z_{D_i} \in \overline{C}(q')$. As already pointed out, we necessarily have $\|q - q'\| \ge 2$. Therefore,

$$E_{Z_{D_i}}\left[\exp\left\{-\int_0^{R_1} V(Z_u, \omega)\, du\right\}\right] \le E_{Z_{D_i}}\left[\exp\left\{-\int_0^{S_1} V(Z_u, \omega)\, du\right\}\right] \le \chi,$$

with the notations of (1.28) and (1.33). Using induction,

$$E_z\left[\exp\left\{-\int_0^{R_i} V(Z_u, \omega)\, du\right\}\right] \le \chi^{i-1} \quad \text{for } i \ge 1 \text{ and } z \in \overline{C}(q).$$

Therefore, when $q \in \mathscr{C}_2$ and $z \in \overline{C}(q)$,

$$\int_{\overline{C}(q)} g(z, z', \omega)\, dz' \le C(d = 2, \nu, W)|O_q| = C'(d = 2, \nu, W)|W_q|.$$

Inserting in (2.11), we find

$$\begin{aligned} \hat{E}_0[H_2] &\le C(d = 2, \nu, W)\hat{E}_0\left[\sum_{q \in \mathscr{A}} |W_q|\right] \\ &= C(d = 2, \nu, W)\hat{E}_0\left[\frac{1}{|\mathscr{A}|}\sum_{q \in \mathscr{A}} |W_q| \cdot |\mathscr{A}|\right]. \end{aligned} \tag{2.17}$$

Since unoccupied sites are nonpercolating, from Theorem 4 of Fontes and Newman [3], we have

$$\mathbb{P}\text{-a.s.} \qquad \limsup_{n \to \infty} \sup_{\substack{|\mathscr{B}| = n \\ 0 \in \mathscr{B}}} \frac{1}{n} \sum_{q \in \mathscr{B}} |W_q| = \text{const}(\nu, l) < \infty, \tag{2.18}$$

where $\mathscr{B}$ runs over lattice animals containing 0.

It is easy to argue that when $y$ tends to infinity,

$$\inf\{|\mathscr{A}(w)|;\ w \in C_0(\mathbb{R}_+, \mathbb{R}^d) \text{ with } H(w) < \infty\}$$

tends to infinity (in fact linearly with $|y|$). Therefore, on a set of full $\mathbb{P}$-measure,

$$\text{for large } |y|,\ \hat{E}_0[H_2] \le \text{const}(d = 2, \nu, W)\hat{E}_0[|\mathscr{A}|]. \tag{2.19}$$

We then conclude as in (2.12). This finishes the proof of Theorem 2.2. □

We now give an application of Theorem 2.2 to the control of the derivative at 0 of $\lambda \in [0, \infty) \to \alpha_\lambda(x)$, for $x \in \mathbb{R}^d$.

COROLLARY 2.3 ($d \ge 2$). *For $x \in \mathbb{R}^d$, the function $\lambda \in [0, \infty) \to \alpha_\lambda(x) \in \mathbb{R}_+$ is differentiable at* 0. *If $\kappa(d, \nu, W)$ denotes the constant in* (2.6),

$$\sup_{|e|=1} \alpha_0'(e) \le \kappa \tag{2.20}$$

(*for the one-dimensional version of this result*, *see Theorem* 2.6 *of* [6]).

PROOF. Recall that $\alpha_\lambda(\cdot)$ is a norm on $\mathbb{R}^d$. Therefore, it suffices to consider $e \in \mathbb{R}^d$, with $|e| = 1$. Then since $\lambda \to \alpha_\lambda(e)$ is concave and increasing, $\alpha_0'(e) = \lim_{n\to\infty} \uparrow n(\alpha_{1/n}(e) - \alpha_0(e))$.

Now for fixed $n$, by (I.7),

$$\begin{aligned} n\big(\alpha_{1/n}(e) - \alpha_0(e)\big) &= \lim_{N\to\infty} \frac{n}{N}\big(-\log e_{1/n}(0, Ne, \omega) - \log e_0(0, Ne, \omega)\big) \quad (\mathbb{P}\text{-a.s.}) \\ &= \lim_{N\to\infty} \frac{n}{N} \int_0^{1/n} E_0\left[H \exp\left\{-\int_0^H (\lambda + V)(Z_s, \omega)\, ds\right\},\right. \\ &\qquad\qquad \left. H < \infty\right] \Big/ e_\lambda(0, Ne, \omega)\, d\lambda. \end{aligned}$$

The function $\lambda \to -\log e_\lambda(0, Ne, \omega)$ is concave on $[0, \infty)$, and the expression inside the integral decreases in $\lambda$. Consequently, the last term in smaller than

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} E_0\left[H \exp\left\{-\int_0^H V(Z_s, \omega)\, ds\right\}, H < \infty\right] \Big/ e_0(0, Ne, \omega) \\ &\quad = \limsup_{N\to\infty} \hat{E}_0\left[\frac{H}{N}\right] \end{aligned}$$

and claim (2.20) now follows from Theorem 2.2. □

**3. Application to quenched Brownian motion with a constant drift in a Poissonian potential.** We shall now discuss the consequences of Corollary 2.3 on the asymptotic behavior of quenched Brownian motion with a constant drift $h$ among Poissonian obstacles. By this we mean the path measures on $C(\mathbb{R}_+, \mathbb{R}^d)$ defined for $t > 0$ and $\omega \in \Omega$ through

$$(3.1)\qquad \begin{aligned} Q^h_{t,\omega} &= \frac{1}{S^h_{t,\omega}} \exp\left\{-\int_0^t V(Z_s,\omega)\,ds\right\} P^h_0(dw) \\ &= \frac{1}{\tilde{S}^h_{t,\omega}} \exp\left\{h\cdot Z_t - \int_0^t V(Z_s,\omega)\,ds\right\} P_0(dw), \end{aligned}$$

provided $P^h_0$ is Wiener measure with a constant drift $h$ and $S^h_{t,\omega}, \tilde{S}^h_{t,\omega}$ are the respective normalizing constants. In the absence of drift ($h = 0$), we simply write $Q_{t,\omega}$. We shall now recall results from [6] and [7] which show that a transition of regime takes place as one goes from small values of $h$ to large values of $h$.

We introduced in [6] the rate function

$$(3.2)\qquad I(x) = \sup_{\lambda \ge 0}\,(\alpha_\lambda(x) - \lambda)$$

and the direction dependent critical threshold

$$(3.3)\qquad c(e,d,\nu,W) = \inf\{\alpha_0(x),\ x\cdot e \ge 1\}, \quad \text{when } e \in \mathbb{R}^d \text{ with } |e| = 1.$$

When $W$ is rotationally invariant, $\alpha_0(\cdot)$ is rotationally invariant as well and $c(\cdot, d, \nu, W)$ is independent of the direction. We know from Theorem 3.1 of [6] and Theorem 3.1 of [7] that on a set $\overline{\Omega}$ of full $\mathbb{P}$-measure, for $h \in \mathbb{R}^d$,

(3.4) $Z_t/t$ satisfies a large deviation principle with rate function $I^h(x) = I(x) - h\cdot x + \sup_y(h\cdot y - I(y))$, under $Q^h_{t,\omega}$ as $t \to \infty$, (when $h = 0$, $I^h = I$);

(3.5) if $h = |h|e$ and $|h| < c(e,d,\nu,W)$, $Z_t/(\log t)^{-2/d}$ satisfies a large deviation principle with rate function $\alpha_0(\cdot)$ under $Q^h_{t,\omega}$ as $t \to \infty$.

In particular, $Z_t/(t(\log t)^{-2/d}) \to 0$ in $Q^h_{t,\omega}$ probability as $t \to \infty$.

(3.6) When $h = |h|e$ with $|h| > c(e,d,\nu,W)$, if $F$ is a small enough neighborhood of 0 in $\mathbb{R}^d$, $\lim_{t\to\infty}(1/t)\log Q^h_{t,\omega}(Z_t \in tF) < 0$.

The statements (3.4)–(3.6) describe a transition from a "subballistic regime" to a "ballistic regime," as $|h|$ crosses the critical value $c(e,d,\nu,W)$. We also introduced in (2.30) of [6] the star-shaped (at the origin) compact set $K$,

$$(3.7)\qquad K = \{v \in \mathbb{R}^d,\ \alpha_0'(v) \le 1\},$$

and showed in (2.31) and (3.12) of [6] that

$$(3.8)\qquad K = \{x \in \mathbb{R}^d,\ I(x) = \alpha_0(x)\}$$

and

(3.9) $\mathbb{P}$-a.s. for $|h| > c(e,d,\nu,W)$, for any compact subset $F$ of $K^0$ $\lim_{t\to\infty}(1/t)\log Q^h_{t,\omega}(Z_t \in tF) < 0$.

Of course (3.8) and (3.9) become truly interesting when $K$ is nondegenerate and, for instance, has nonempty interior. This was shown to be the case in the one-dimensional situation in Theorem 2.6 of [6]. When $K$ has nonempty interior, (3.9) and (3.5) predict a "jump in the velocity" at the crossing of the critical value. Our main contribution here is to be able to treat the $d \geq 2$ situation.

THEOREM 3.1. *Let $d \geq 2$ and $\kappa$ denote the constant from* (2.6). *Then*

$$K \supset \overline{B}(0, 1/\kappa) \tag{3.10}$$

*and if* $v^* = \inf\{|v|, v \in K^c\} \geq 1/\kappa$,

$$\mathbb{P}\text{-a.s., for } |h| > c(e, d, \nu, W) \text{ and } v < v^*, \quad \lim_{t \to \infty} \frac{1}{t} \log Q^h_{t,\omega}(|Z_t| \leq vt) < 0. \tag{3.11}$$

PROOF. Now (3.10) is an immediate application of (2.20) and (3.7), and (3.11) follows from (3.9). □

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## REFERENCES

[1] CRANSTON, M. and MCCONNELL, T. R. (1983). The lifetime of conditioned Brownian motion. *Z. Wahrsch. Verw. Gebiete* **65** 1–11.

[2] COX, J. T., GANDOLFI, A., GRIFFIN, P. S. and KESTEN, H. (1993). Greedy lattice animals I: Upper bounds. *Ann. Appl. Probab.* **3** 1151–1169.

[3] FONTES, L. and NEWMAN, C. (1993). First passage percolation for random colorings of $\mathbb{Z}^d$. *Ann. Appl. Probab.* **3** 746–762.

[4] HARRIS, T. E. (1956). First passage and recurrence distributions. *Trans. Amer. Math. Soc.* **73** 471–486.

[5] SZNITMAN, A. S. (1993). Brownian motion in a Poissonian potential. *Probab. Theory Related Fields* **97** 447–477.

[6] SZNITMAN, A. S. (1994). Shape theorems Lyapounov exponents and large deviations for Brownian motion in a Poissonian potential. *Comm. Pure Appl. Math.* **47** 1655–1688.

[7] SZNITMAN, A. S. (1995). Quenched critical large deviations for Brownian motion in a Poissonian potential. *J. Funct. Anal.* To appear.

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