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DIVERGENCE OF SHAPE FLUCTUATIONS IN TWO DIMENSIONS

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We consider stochastic growth models, such as standard first-passage percolation on $\mathbb{Z}^d$, where to leading order there is a linearly growing deterministic shape. Under natural hypotheses, we prove that for $d = 2$, the shape fluctuations grow at least logarithmically in all directions. Although this bound is far from the expected power law behavior with exponent $\chi = 1/3$, it does prove divergence. With additional hypotheses, we obtain inequalities involving χ and the related exponent ξ (which is expected to equal 2/3 for $d = 2$). Combining these inequalities with previously known results, we obtain for standard first-passage percolation the bounds $\chi \geq 1/8$ for $d = 2$ and $\xi \leq 3/4$ for all d.

1. Introduction. A subject that has attracted considerable attention in recent years is the nature of the fluctuations of growing interfaces [see Krug and Spohn (1991) for a review]. In this paper we consider several models (including standard first-passage percolation) of a stochastically growing subset $\tilde{B}(t)$, of the lattice $\mathbb{Z}^d$ $(d \geq 2)$, at time t. The interface between $\tilde{B}(t)$ and its complement will (under natural hypotheses) grow linearly in t with a deterministic shape. The magnitude of the fluctuations of this interface about its mean shape is believed (under further hypotheses, as discussed below) to be typically of the order of t^χ with χ depending on the dimension d. The exponent χ is predicted to equal 1/3 for $d = 2$ [see Huse and Henley (1985), Kardar (1985), Huse, Henley and Fisher (1985) and Kardar, Parisi and Zhang (1986)]. There have been varying discussions about the nature of χ for higher dimensions ranging from the possible independence of χ on d [Kardar and Zhang (1987)] through the picture of χ decreasing with d while always remaining strictly positive [see Wolf and Kertész (1987) and Kim and Kosterlitz (1989)] to the possibility that for d above some d_c, $\chi = 0$ and perhaps the fluctuations do not even diverge [see Natterman and Renz (1988), Halpin-Healy (1989) and Cook and Derrida (1990)].

Relatively few rigorous results have been obtained about the shape fluctuations. For some models, Kesten (1993) has proved that $\chi \leq 1/2$ [see also Alexander (1995) for related bounds]. For other models, Wehr and Aizenman (1990) have derived a rigorous lower bound, $\chi \geq (1-(d-1)\xi)/2$, in terms of another exponent ξ about which little was known rigorously. The exponent ξ is such that n^ξ is the order of the fluctuations about the mean for the location

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where the growing $\tilde{B}(t)$ first touches a hyperplane at distance n from the origin. It is conjectured that χ and ξ should satisfy the relation $\chi = 2\xi - 1$ for all d [see Krug and Spohn (1991)]. There had been no proof that the shape fluctuations diverge for any model in any dimension $d > 1$. In this paper, by building on the methods and results of Wehr and Aizenman (1990), Kesten (1993) and Alexander (1995), we obtain two types of results. The first type is a proof of divergence (at least logarithmically fast) valid for several different models when $d = 2$. We note that for an important special case of first-passage percolation, a different proof was obtained by Pemantle and Peres (1994). The second type of result is an upper bound on ξ. For standard first-passage percolation, this bound, together with the results of Kesten (1993) and Alexander (1995), yields $\xi \le 3/4$ for all d which, when combined with the Wehr–Aizenman inequality, gives $\chi \ge 1/8$ for $d = 2$.

The models we consider are defined in terms of a family $\{\tau(e)\}$ of (nonnegative) random variables indexed by the nearest neighbor edges e of $\mathbb{Z}^d$. We will generally think of the $\tau(e)$'s as passage times through the edges (although in some contexts it is natural to regard them as energies associated with the edges). These models, known in the probabilistic literature as first-passage percolation models, were invented by Hammersley and Welsh (1965) to model the spread of a fluid through a porous material. Early results are surveyed in the book by Smythe and Wierman (1978) and more recent results by Kesten (1986, 1987).

The set $\tilde{B}(t)$ is defined as the collection of all sites in $\mathbb{Z}^d$ reachable from the origin by a (nearest neighbor, self-avoiding) path along which the sum of the $\tau(e)$'s does not exceed t. In one type of model, paths are restricted to have no coordinate ever decrease. We will call these directed first-passage percolation models; in the physics literature they are often regarded as directed polymer models (at zero temperature) with the $\tau(e)$'s regarded as edge energies. Note that in the directed case $\tilde{B}(t)$ is a subset of the positive orthant $\mathbb{Z}^d_+$, and for $d = 2$ the directed paths are equivalent (by a 45° rotation) to the space–time paths of a (discrete time) simple random walk on $\mathbb{Z}^1$. In the second type of model, there is no restriction on the paths. We will call these undirected first-passage percolation models (or simply first-passage percolation models).

The rate at which $\tilde{B}(t)$ grows with t depends on the percolation properties of the edges with $\tau(e) = 0$. Let us consider, for example, (undirected) first-passage percolation with independent identically distributed (i.i.d.) $\tau(e)$'s. (This will be the first of three models we treat in this paper.) Let $p_c = p_c(d)$ denote the critical value for standard independent nearest neighbor bond percolation on $\mathbb{Z}^d$ [see Grimmett (1989)]. It is clear that when $P(\tau(e) = 0) > p_c$, then $\tilde{B}(t)$ becomes infinite after a finite time. One of the basic results of the subject (which we review in more detail in the next section) is that when $P(\tau(e) = 0) < p_c$, $\tilde{B}(t)$ is (for large t) approximately the intersection of $\mathbb{Z}^d$ with tB_0, where B_0 is a deterministic bounded convex subset of $\mathbb{R}^d$. A similar result is valid for directed first-passage percolation with i.i.d. $\tau(e)$'s (the second of the three models we treat). In this case the appropriate hypothesis is that $P(\tau(e) = 0) < p_c^{\text{dir}}$, where p_c^{dir} is the critical value for independent nearest neighbor *directed* bond percolation on $\mathbb{Z}_+^d$.

The directed percolation critical value plays a different, but related, role when for some $\lambda > 0$, $P(\tau(e) < \lambda) = 0$ while $P(\tau(e) = \lambda) > p_c^{\text{dir}}$. In this case, $\tilde{B}(t)$ grows linearly but B_0 is not strictly convex [see Durrett and Liggett (1981)]; this is so for *both* directed and undirected first-passage percolation. For example, in the directed case when $d = 2$, the boundary of B_0 includes a straight line segment

$$\{(x_1, x_2)\colon x_1 + x_2 = 1/\lambda,\ |x_2 - x_1| \le a_0/\lambda\}, \tag{1.1}$$

where a_0 in $(0, 1)$ is a function of $P(\tau(e) = \lambda) - p_c^{\text{dir}}$. In the $d = 2$ undirected case, the boundary of B_0 includes the line segment (1.1) together with the three segments obtained upon rotating (1.1) by multiples of $\pi/2$.

To analyze the fluctuations of the boundary of $\tilde{B}(t)$, one should consider separately the parts of the boundary growing in different directions. Thus we define $T_n(\hat{x})$, for a unit vector $\hat{x}$ in $\mathbb{R}^d$, to be the time t at which $\tilde{B}(t)$ first reaches the lattice site closest to $n\hat{x}$ (with some deterministic rule for breaking ties). The time constant in direction $\hat{x}$ is

$$\mu(\hat{x}) = \lim_{n\to\infty} \frac{T_n(\hat{x})}{n} = \lim_{n\to\infty} \frac{E(T_n(\hat{x}))}{n} \tag{1.2}$$

and the asymptotic shape B_0 (as a subset of $\mathbb{R}^d$ or of $\mathbb{R}_+^d$) is

$$B_0 = \{r\hat{x}\colon \hat{x} \text{ is a unit vector and } r \le 1/\mu(\hat{x})\}. \tag{1.3}$$

Our investigation of the fluctuations of $\tilde{B}(t)$ is in terms of the asymptotic growth of the variance of $T_n(\hat{x})$,

$$\operatorname{var}(T_n(\hat{x})) = E(T_n(\hat{x})^2) - [E(T_n(\hat{x}))]^2. \tag{1.4}$$

We remark that the results we obtain for "point-to-point" passage times like $T_n(\hat{x})$ easily extend (by the same methods) to other passage times, like the "point-to-plane" passage time from the origin to a plane (perpendicular to $\hat{x}$) at distance n from the origin or to other "point-to-region" passage times.

Under appropriate hypotheses, the predicted asymptotic behavior of $\operatorname{var}(T_n(\hat{x}))$ is given by a power law usually denoted by $n^{2\chi}$ so that χ designates the growth exponent of the standard deviation of $T_n(\hat{x})$ as $n \to \infty$. For $d = 2$, χ is believed to be 1/3 and the appropriate hypotheses should presumably be whatever is needed to guarantee that $\mu(\hat{x}) > 0$ and that the boundary of B_0 have a finite radius of curvature at the point $\hat{x}/\mu(\hat{x})$ where it intersects the ray, $R_{\hat{x}} = \{r\hat{x}\colon r > 0\}$. We will call such an $\hat{x}$ a direction of curvature for B_0; a precise definition will be given in Section 3 below. For a discussion of the relation between curvature and the exponents χ and ξ, see Section 7 of Krug and Spohn (1991).

As remarked at the beginning of this introduction, in high dimensions the question of divergence of fluctuations is somewhat controversial and a finite radius of curvature of B_0 in a given direction might not imply divergent fluctuations. For $d = 2$, however (and for other low dimensions), $\operatorname{var}(T_n(\hat{x}))$ is expected to diverge as $n \to \infty$ *except* in the special cases associated with either vanishing time constants or asymptotic shapes which are not strictly convex. These special cases, where $\mathrm{var}(T_n(\hat{x}))$ does not diverge for all or some $\hat{x}$'s, are as follows:

CASE 1. i.i.d. undirected first-passage percolation with $P(\tau(e) = 0) > p_c$.

CASE 2. i.i.d. undirected first-passage percolation with $P(\tau(e) < \lambda) = 0$, $P(\tau(e) = \lambda) > p_c^{\mathrm{dir}}$, $\lambda > 0$.

CASE 3. i.i.d. directed first-passage percolation with $P(\tau(e) = 0) > p_c^{\mathrm{dir}}$.

CASE 4. i.i.d. directed first-passage percolation with $P(\tau(e) < \lambda) = 0$, $P(\tau(e) = \lambda) > p_c^{\mathrm{dir}}$, $\lambda > 0$.

In Case 1, $T_n(\hat{x})$ stays bounded as $n \to \infty$, for any $\hat{x}$, and in Case 3 it does so if $\hat{x} = (\hat{x}_1, \hat{x}_2)$ in $\mathbb{R}_+^2$ is such that $|\hat{x}_2 - \hat{x}_1| < a_0(\hat{x}_1 + \hat{x}_2)$. Here a_0 is the same as in (1.1). In Cases 2 and 4, $T_n(\hat{x}) - \lambda n(\hat{x}_1 + \hat{x}_2)$ stays bounded as $n \to \infty$ if $\hat{x}$ in $\mathbb{R}_+^2$ is such that $|\hat{x}_2 - \hat{x}_1| < a_0(\hat{x}_1 + \hat{x}_2)$.

The first set of results in this paper for i.i.d. directed or undirected first-passage percolation with $d = 2$ are proofs that the hypotheses believed necessary to guarantee $n^{2\chi}$ behavior for $\mathrm{var}(T_n(\hat{x}))$ for *every* $\hat{x}$ are in fact sufficient to yield divergence of $\mathrm{var}(T_n(\hat{x}))$ for every $\hat{x}$ (at least logarithmically fast). Theorem 2 of Section 2 covers the undirected case away from Cases 1 or 2 (or the limits of those cases), while Theorem 3 of Section 2 covers the directed case away from Cases 3 or 4 (or their limits). We do not treat in this paper the cases where $\mathrm{var}(T_n(\hat{x}))$ should behave like $n^{2\chi}$ only for certain $\hat{x}$'s. We note, however, that for *directed* first-passage percolation our techniques can be readily adapted to prove divergence of $\mathrm{var}(T_n(\hat{x}))$ for precisely those $\hat{x}$'s. This is possible both when $P(\tau(e) = 0) \geq p_c^{\mathrm{dir}}$ and when $P(\tau(e) = \lambda) \geq p_c^{\mathrm{dir}}$ for $\lambda > 0$ [while $P(\tau(e) < \lambda) = 0$]. On the other hand, either additional hypotheses or improved techniques seem to be needed to obtain such a result for *undirected* first-passage percolation [i.e., in the $P(\tau(e) = \lambda) \geq p_c^{\mathrm{dir}}$ situation].

The third and last model we treat in this paper is undirected first-passage percolation, but with dependent $\tau(e)$'s related to the $d = 2$ standard Ising ferromagnet $\{\sigma_x\colon x \in \mathbb{Z}^2\}$ according to

$$\tau((x, y)) = -\tfrac{1}{2}(\sigma_x \sigma_y - 1) = \begin{cases} +1, & \text{if } \sigma_x \neq \sigma_y, \\ 0, & \text{if } \sigma_x = \sigma_y. \end{cases} \tag{1.5}$$

This model is closely related to a model of random surfaces in three dimensions [Abraham and Newman (1988, 1989, 1991)] and our fluctuation results are applied elsewhere [Abraham, Fontes, Newman and Piza (1994)] to study the roughness of those surfaces. Our fluctuation result for the Ising first-passage model (Theorem 4 of Section 2) is completely analogous to our result for the undirected i.i.d. case with $\tau(e)$ taking only the values 0 and 1; that is, away from the parameter region where the edges with $\tau(e) = 0$ percolate (and off the boundary of that region), $\text{var}(T_n(\hat{x}))$ diverges as $n \to \infty$ at least logarithmically for every $\hat{x}$.

The proofs of all our logarithmic results have the same overall structure: First, a general lower bound on variances and a nonprobabilistic inequality about sequences (see Theorem 8 and Lemma 1 in Section 3) together show that for $d = 2$, $\text{var}(T_n(\hat{x}))$ will diverge at least logarithmically in n, if the expected number of certain "local defects" for $T_n(\hat{x})$ within distance k of the origin diverges at least linearly in k. Then, arguments related to the shape theorem will yield this linear divergence. Roughly speaking, a local defect for $T_n(\hat{x})$ is a local region in $\mathbb{Z}^2$, where a change of configuration will reduce $T_n(\hat{x})$. The exact meaning of local defect will depend on the exact model being considered, as will the exact proof of the linear divergence of their mean number.

The second set of results in this paper concern power law lower bounds for $\text{var}(T_n(\hat{x}))$. For i.i.d. undirected percolation, our result (roughly speaking) is that $\chi \geq 1/8$ for $d = 2$. More precisely, away from Cases 1 or 2 above (or their limits), we prove that, for any $\varepsilon > 0$, $\text{var}(T_n(\hat{x}))$ diverges faster than $n^{1/8-\varepsilon}$, provided two additional hypotheses are valid: (i) $E(\exp[\beta\tau(e)]) < \infty$ for some $\beta > 0$, and (ii) the unit vector $\hat{x}$ is a direction of curvature for B_0. The first additional hypothesis ought not to be necessary. The second additional hypothesis ought to be extraneous since it should automatically be valid for all $\hat{x}$ away from Cases 1 and 2 (or their limits). Unfortunately, all we know for sure is that there are at least some $\hat{x}$'s which are directions of curvature. This follows from Lemma 5 of Section 6, which asserts the existence of at least one such $\hat{x}$. The symmetry of $\mathbb{Z}^2$ under rotations by multiples of $\pi/2$ then implies that there are at least four such directions. Because of the nonconstructive nature of Lemma 5 (and of the shape theorem), we cannot assert that any particular $\hat{x}$ (such as a coordinate direction) is a direction of curvature. Clearly the subject of first-passage percolation is in need of some good qualitative results on the nature of the asymptotic shape B_0.

The power law results are based on arguments which restrict the local defects for $T_n(\hat{x})$ to lie within a strip parallel to $\hat{x}$ of width n^{ξ}. The key point here is to get an upper bound on ξ. This is done by deriving the exponent inequality $\xi \leq (1 + \chi')/2$ (see Theorem 6; here χ' is an exponent related to χ) and using the bound, $\chi' \leq 1/2$ for all d, of Kesten (1993) and Alexander (1995).

The remainder of the paper is organized as follows. In Section 2, we define more precisely the various first-passage percolation models under consideration, review the asymptotic shape theorems known for them and state our main results about divergence of fluctuations. Theorems 2, 3 and 4 of Section 2 are our logarithmic bound results for the three types of models we consider. Theorems 5, 6 and 7 of Section 2 are power law bound results; although the inequality $\chi \geq 1/8$ of Theorem 7 is only proved for i.i.d. undirected first-passage percolation with $d = 2$, versions of the exponent inequalities of Theorems 5 and 6 (and parts of Theorem 7) are valid for other models and for $d > 2$ (see the remarks following Theorem 7). In Section 3, we first prove Theorem 2 in the special case of 0 or 1 valued $\tau(e)$'s. We then give (in Theorem 8) a farreaching generalization of the first part of that proof in the form of a lower bound on the variance of a random variable [such as $T_n(\hat{x})$] in terms of probabilities of local defect events. This lower bound, which is very similar to lower bounds of Wehr and Aizenman (1990), is applied in Section 4 to independent first-passage percolation (leading to the proofs of Theorems 2 and 3) and is applied in Section 5 to prove the Ising first-passage percolation result, Theorem 4. It is applied in Section 6 along with geometric arguments related to the curvature of B_0 to prove the power law results, Theorems 5, 6 and 7.

2. Definitions and main results.

In this section we define more precisely the various first-passage percolation models referred to in the Introduction and state our main results of both types—logarithmic and power law.

A first-passage percolation model on $\mathbb{Z}^d$ is defined as follows. To each (nearest neighbor) edge of $\mathbb{Z}^d$ we attach a nonnegative random variable $\tau(e)$ (the "passage time" for the edge e). In the simplest setup we take these variables as i.i.d. random variables, but we will also consider below (some) cases where the $\tau(e)$'s are dependent.

We define then for a path r consisting of edges $e_1, e_2, \dots, e_n$ the passage time of r as

$$T(r) = \sum_{i=1}^{n} \tau(e_i). \tag{2.1}$$

Different models may be obtained by considering restrictions on the allowed paths. The *directed* first-passage percolation model is obtained if we restrict ourselves to directed paths; that is, we are allowed to travel only along paths where coordinates never decrease. The case where the paths are unrestricted will be referred to as the undirected first-passage percolation model. Although we will not do so, it is of course possible to also consider intermediate situations where coordinates are allowed to decrease in some directions but not in others.

We then define for two vertices $u, v \in \mathbb{Z}^d$ the passage time from u to v as

$$T(u,v) = \inf\{T(r)\colon\ r \text{ is an allowed path from } u \text{ to } v\}. \tag{2.2}$$

For a unit vector $\hat{x}$ in $\mathbb{R}^d$ we define $v(n,\hat{x})$ to be the point in $\mathbb{Z}^d$ closest (in Euclidean distance) to $n\hat{x}$ (with some deterministic rule for breaking ties) and

$$T_n(\hat{x}) = T(0, v(n,\hat{x})). \tag{2.3}$$

This is equivalent to the definition of $T_n(\hat{x})$ given in Section 1.

The time constant in the $\hat{x}$ direction, $\mu(\hat{x})$, is defined as

$$\mu(\hat{x}) = \lim_{n\to\infty} \frac{T_n(\hat{x})}{n}. \tag{2.4}$$

The existence of this limit (almost surely and in L^1) for the case $\hat{x} = \hat{e}_1 = (1, 0, \dots, 0)$ immediately follows [under mild conditions on the $\tau(e)$'s] from the subadditivity property, $T(u,v) \le T(u,w) + T(w,v)$. The existence of $\mu(\hat{x})$ for general $\hat{x}$ may be rephrased in terms of the properties of the asymptotic behavior of the set $\tilde{B}(t)$ of vertices which can be reached from the origin by time t. For technical convenience, $\tilde{B}(t)$ is usually replaced by the subset of $\mathbb{R}^d$,

$$B(t) = \{v + \overline{U}\colon\ v \in \tilde{B}(t)\}, \tag{2.5}$$

where $\overline{U}$ is the unit cube:

$$\overline{U} = \{(x_1, \dots, x_d)\colon\ |x_i| \le 1/2 \text{ for each } i\}. \tag{2.6}$$

Under various natural conditions on the $\tau(e)$'s, it is known that the behavior of $B(t)$ for large t is governed by a "shape theorem," according to which $B(t)$ grows linearly with t and has an asymptotic shape which is nonrandom. In the undirected first-passage percolation model, with i.i.d. $\tau(e)$'s, a version of the result [with a stricter hypothesis on the $\tau(e)$'s than necessary] may be stated as follows [Richardson (1973), Cox and Durrett (1981) and Kesten (1986)]:

THEOREM 1. *Consider the undirected first-passage percolation model with i.i.d. $\tau(e)$'s such that $E(\tau(e)^2) < \infty$. If $\mu(\hat{e}_1) > 0$, there exists a nonrandom, compact, convex subset B_0 in $\mathbb{R}^d$ (with nonempty interior) such that, almost surely,*

$$\forall\, \varepsilon > 0, \qquad (1-\varepsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\varepsilon)B_0 \quad \text{for all large } t. \tag{2.7}$$

If $\mu(\hat{e}_1) = 0$, then, almost surely,

$$\forall \text{ bounded } K \text{ in } R^d, \qquad K \subset \tfrac{1}{t}B(t) \quad \text{for all large } t. \tag{2.8}$$

Furthermore $\mu(\hat{e}_1) > 0$ if and only if

$$P(\tau(e) = 0) < p_c, \tag{2.9}$$

where $p_c = p_c(d)$ is the critical value for independent nearest neighbor bond percolation on $\mathbb{Z}^d$.

The above result establishes a strong law of large numbers for the sequences $T_n(\hat{x})$. A natural question one may ask concerns the fluctuations of these quantities. We will consider the asymptotic behavior of the variance of $T_n(\hat{x})$:

$$\operatorname{var}(T_n(\hat{x})) = E(T_n(\hat{x})^2) - [E(T_n(\hat{x}))]^2. \tag{2.10}$$

Logarithmic lower bounds. Our first result is a logarithmic lower bound for $\operatorname{var}(T_n(\hat{x}))$ for $d = 2$. The hypotheses are related to nonoccurrence of percolation of edges with $\tau(e)$ assuming the lowest possible value. To make this precise we define

$$\lambda \equiv \inf\{x\colon\ P(\tau(e) \le x) > 0\} \tag{2.11}$$

and $p_c^{\text{dir}} = p_c^{\text{dir}}(d)$ to be the critical value for independent nearest neighbor *directed* bond percolation on $\mathbb{Z}^d$. Our first result may now be stated as follows:

THEOREM 2. *Consider undirected first-passage percolation on $\mathbb{Z}^d$ with i.i.d. passage times such that $E(\tau(e)^2) < \infty$ and $\text{var}(\tau(e)) > 0$. Assume in addition that one of the following two conditions is satisfied:*

$$\lambda = 0 \quad \textit{and} \quad P(\tau(e) = 0) < p_c(d), \tag{2.12}$$

$$\lambda > 0 \quad \textit{and} \quad P(\tau(e) = \lambda) < p_c^{\text{dir}}(d). \tag{2.13}$$

If $d = 2$, then there exists a constant $B > 0$ such that

$$\text{var}(T_n(\hat{x})) \geq B \log(n) \tag{2.14}$$

for $n \geq 1$ and all unit vectors $\hat{x}$ in $\mathbb{R}^2$.

We remark that the special case of Theorem 2 with exponentially distributed $\tau(e)$'s was proved, using different methods, by Pemantle and Peres (1994).

In the case of directed first-passage percolation, one can mimic the proof of Theorem 1 and obtain a shape theorem, but with a weakened version of (2.7). In particular, for $d = 2$, with i.i.d. $\tau(e)$'s such that $E(\tau(e)^2) < \infty$ and $P(\tau(e) = 0) < p_c^{\text{dir}}(2)$, there exists a nonrandom, convex subset B_0 of $\mathbb{R}_+^2$ (with nonempty interior) such that, almost surely, for any $\delta > 0$, a modified version of (2.7) is valid in which B_0 and $B(t)$ are replaced by their intersection with the cone in $\mathbb{R}_+^2$, $\{(x_1, x_2)\colon\ \delta < x_1/x_2 < 1/\delta\}$. The difficulty with the unmodified version of (2.7) is that $\mu(\hat{x})$ may not be continuous as $\hat{x}$ aproaches a coordinate direction. Nevertheless, we obtain the following result on fluctuations.

THEOREM 3. *Consider directed first-passage percolation on $\mathbb{Z}^d$ with i.i.d. passage times such that $E(\tau(e)^2) < \infty$ and $\text{var}(\tau(e)) > 0$. Assume that*

$$P(\tau(e) = \lambda) < p_c^{\text{dir}}(d), \tag{2.15}$$

If $d = 2$, then there exists a strictly positive constant C such that

$$\text{var}(T_n(\hat{x})) \geq C \log(n)$$

for $n \geq 1$ and all unit vectors $\hat{x}$ in $\mathbb{R}_+^2$.

Next we consider a first-passage percolation model with dependent $\tau(e)$'s related to the standard two-dimensional ferromagnetic Ising model in an external magnetic field h. [For more information on Ising models, see, e.g., Georgii (1988)]. The Hamiltonian is given by

$$H = -\frac{J}{2} \sum_{(x,y)} \sigma_x \sigma_y - h \sum_x \sigma_x \tag{2.16}$$

with $J > 0$, the first sum over nearest neighbor edges and the Ising spin variables σ_x taking values ± 1. The passage time for the edge between nearest neighbor vertices x and y is defined as

$$\tau((x, y)) = -\tfrac{1}{2}(\sigma_x \sigma_y - 1) = \begin{cases} +1, & \text{if } \sigma_x \neq \sigma_y, \\ 0, & \text{if } \sigma_x = \sigma_y. \end{cases} \tag{2.17}$$

Here, we take $\{\sigma_x\colon x \in \mathbb{Z}^2\}$ to have as its joint distribution the Gibbs distribution [with formal density proportional to $\exp(-H)$] obtained by taking the standard infinite volume limit with free boundary condition.

The shape theorem holds in exactly the same form as for the undirected model with i.i.d. $\tau(e)$'s except that the criterion for the positivity of the time constant $\mu(\hat{e}_1)$ is different here. Let J_c denote the critical value of the coupling constant J, above which there are multiple infinite volume Gibbs distributions (when $h = 0$). It was proved recently by Higuchi (1993a, b) that percolation of like spins [which of course is equivalent to percolation of $\tau(e) = 0$ edges] occurs only when $J > J_c$ or when $J \le J_c$ and $|h| > h_c(J)$, where $h_c(J) > 0$ for $J < J_c$ and $h_c(J_c) = 0$; in the complement of this region there is no percolation. This extended earlier results of Coniglio, Nappi, Peruggi and Russo (1976). In a recent paper, Fontes and Newman (1993) have proved [see also Chayes (1993)] that $\mu(\hat{e}_1) > 0$ in the interior of the nonpercolating regime, that is, when $J < J_c$ and $|h| < h_c(J)$. We have then the following result on fluctuations.

THEOREM 4. *For the $d = 2$ Ising first-passage percolation model defined above, there exist constants $D(J, h) > 0$ such that*

$$\operatorname{var}(T_n(\hat{x})) \ge D(J, h)\log(n) \tag{2.18}$$

for $n \ge 1$ and all J, h in the interior of the nonpercolating regime.

Power law results. Except in some remarks, we restrict attention now to i.i.d. undirected first-passage percolation. Before stating the first power law result, we need to define two critical exponents. For any direction $\hat{x}$, define

$$\chi_{\hat{x}} \equiv \sup\{\gamma \ge 0\colon \text{ for some } C > 0,\ \operatorname{var}(T_n(\hat{x})) \ge Cn^{2\gamma} \text{ for all } n\}. \tag{2.19}$$

Although $\chi_{\hat{x}}$ is not believed to depend on $\hat{x}$ (under the assumptions of our theorems), there is no proof that this is so. A similar situation occurs for the next critical exponent $\xi_{\hat{x}}$, which concerns the transverse fluctuations of minimizing paths for $T_n(\hat{x})$.

Let $M_n(\hat{x})$ denote the (random) set of all sites in $\mathbb{Z}^2$ belonging to some time-minimizing path from the origin to $v(n, \hat{x})$. Let $L_{\hat{x}}$ denote the line $\{\alpha\hat{x}\colon \alpha \in \mathbb{R}\}$ and, for $\gamma > 0$, let $\Lambda_n^\gamma(\hat{x})$ denote the cylinder of radius n^γ parallel to $\hat{x}$; that is,

$$\Lambda_n^\gamma(\hat{x}) = \{z \in \mathbb{R}^d\colon d(z, L_{\hat{x}}) \le n^\gamma\}. \tag{2.20}$$

Here $d(z, A)$ for $A \subset \mathbb{R}^d$ denotes the Euclidean distance $\inf_{y \in A} \|z - y\|$. The exponent $\xi_{\hat{x}}$ is then defined as

$$\begin{aligned} \xi_{\hat{x}} \equiv \inf\{\gamma > 0\colon &\text{ for some } C > 0, \\ &P(M_n(\hat{x}) \subset \Lambda_n^\gamma(\hat{x})) \ge C \text{ for all large } n\}. \end{aligned} \tag{2.21}$$

The inequality in the next theorem was originally obtained by Wehr and Aizenman (1990) for directed first-passage percolation, and under the assumption that $\tau(e)$ has an absolutely continuous distribution with bounded density.

THEOREM 5. *Assume the hypotheses of Theorem 2, but for general d. Then, for any unit vector $\hat{x}$ in $\mathbb{R}^d$,*

$$\chi_{\hat{x}} \geq \frac{1-(d-1)\xi_{\hat{x}}}{2}. \tag{2.22}$$

We will refer to (2.22) as the Wehr–Aizenman inequality. It is not expected to help yield a strictly positive lower bound on $\chi_{\hat{x}}$ for any $d \geq 3$ since it is believed that $\xi_{\hat{x}} \geq 1/2$ for any $d \geq 2$. However for $d = 2$, one only needs an upper bound on $\xi_{\hat{x}}$ strictly below 1. The next theorem gives an upper bound on $\xi_{\hat{x}}$ in terms of an exponent χ'; when combined with previous results of Kesten (1993) and Alexander (1995) on χ', it will yield $\xi_{\hat{x}} \leq 3/4$ for $d = 2$ (at least for some $\hat{x}$'s). The exponent χ' takes into account not only the fluctuations of $T_n(\hat{x})$ about its mean (for all $\hat{x}$), but also the deviation of the mean from the asymptotic expression $n\mu(\hat{x})$. Our precise definition is

$$\chi' \equiv \inf\{\kappa\colon (t-t^{\kappa})B_0 \subset B(t) \subset (t+t^{\kappa})B_0 \text{ for large } t, \text{ almost surely}\}. \tag{2.23}$$

Before stating Theorem 6, we need to give a precise definition for $\hat{x}$ to be a direction of curvature for B_0. Basically we require that at the point $z = \hat{x}/\mu(\hat{x})$ on the boundary ∂B_0, a sphere S of finite radius can be inserted (locally) between a tangent plane to B_0 and $\operatorname{int}(B_0)$, the interior of B_0. More precisely we require the existence of some closed Euclidean ball D in $\mathbb{R}^d$ of finite positive radius (and any center) such that the following two conditions hold:

CONDITION 1. There is a subset S' of the sphere boundary ∂D which is open (as a subset of ∂D) and which contains z [$= \hat{x}/\mu(\hat{x})$] but no point of $\operatorname{int}(B_0)$.

CONDITION 2. $D \cap \operatorname{int}(B_0) \neq \varnothing$.

Condition 2 guarantees that D is on the correct side of a tangent plane. For future use we note that because B_0 is convex and compact, the overall definition is equivalent to one in which Conditions 1 and 2 are replaced by the following:

CONDITION 1′. ∂D contains z.

CONDITION 2′. $D \supset B_0$.

THEOREM 6. *Assume the hypotheses of Theorem 1 with $d \geq 2$ and $P(\tau(e) = 0) < p_c(d)$. Then, for any $\hat{x}$ which is a direction of curvature for B_0,*

$$\xi_{\hat{x}} \leq \frac{1+\chi'}{2}. \tag{2.24}$$

REMARKS. Let us define

$$\chi = \sup_{\hat{x}} \chi_{\hat{x}}. \tag{2.25}$$

On a heuristic level, one has $\chi \leq \chi'$ because χ' includes deviations of $E(T_n(\hat{x}))$ from $n\mu(\hat{x})$ as well as fluctuations of $T_n(\hat{x})$ about $E(T_n(\hat{x}))$, but this comparison between χ and χ' is not rigorous since $\chi_{\hat{x}}$ was defined in terms of variance and χ' in terms of almost sure behavior. Nevertheless, (2.24) should be thought of as a rigorous partial affirmation of the nonrigorous identity in the physics literature, $\chi = 2\xi - 1$, which is believed valid for all $d \geq 2$ (with ξ the common value of all the $\xi_{\hat{x}}$'s). Indeed, the proof of (2.24) given below in Section 6 is essentially a rigorized version of the derivation of $\chi = 2\xi - 1$ given by Krug and Spohn (1991). We note that, as in its heuristic version, the argument is mainly geometrical (with probabilistic considerations playing only a peripheral role) and is applicable in rather wide generality to first-passage type models with an asymptotic shape B_0. In particular (2.24) is valid for the $d = 2$ Ising model of Theorem 4.

The last theorem combines Theorems 5 and 6 with previously known results to obtain exponent bounds for $d = 2$:

THEOREM 7. *Assume the hypotheses of Theorem* 2 *with* $d = 2$. *Then*

$$\max(\chi', \chi) \geq 1/5. \tag{2.26}$$

Assume in addition that $E(\exp[\beta\tau(e)]) < \infty$ *for some* $\beta > 0$. *Then, for any direction of curvature* $\hat{x}$,

$$\xi_{\hat{x}} \leq 3/4, \qquad \chi_{\hat{x}} \geq 1/8; \tag{2.27}$$

thus

$$\chi \geq 1/8. \tag{2.28}$$

REMARKS. Inequality (2.26) is also valid for the $d = 2$ Ising model of Theorem 4. The upper bound for $\xi_{\hat{x}}$ of (2.27) comes from (2.24) and the upper bound, $\chi' \leq 1/2$ for all d, of Kesten (1993) and Alexander (1995). Their bound on χ' (see the proof of Theorem 7 in Section 6 below) yields

$$\xi_{\hat{x}} \leq 3/4 \quad \text{for all } d. \tag{2.29}$$

It is believed that $\xi_{\hat{x}} = \xi = 2/3$ for $d = 2$ and that ξ is decreasing in d. We remark that versions of Theorems 5 and 6 and the bound $\chi' \leq 1/2$ remain valid for i.i.d. directed first-passage percolation. In particular, (2.27) is valid for $d = 2$ and (2.29) for all d as long as $\hat{x}$ has no vanishing coordinates. (See the discussion preceding Theorem 3 above.) In order to conclude (2.28), one would simply need to know that there is some direction of curvature, but the soft argument we use for this in the undirected case (see Lemma 5 of Section 6) is not applicable in the directed case.

3. Special case and general approach. We begin this section with a proof of Theorem 2 for the case of 0 or 1 valued $\tau(e)$'s. After that, we present in Theorem 8 below an abstract generalization of one of the basic inequalities used in the 0 or 1 valued case; that generalization will be used in subsequent sections to prove Theorem 2 in its entirety as well as Theorems 3 and 4.

So we now restrict attention to undirected first-passage percolation on $\mathbb{Z}^2$ with i.i.d. $\tau(e)$'s taking the values 0 and 1 with probabilities p and $q = 1 - p$. We will assume that the underlying probability space $(\Omega, \mathscr{F}, P)$ is the canonical one: $\Omega = \{0,1\}^{\mathbb{E}^2}$ (where $\mathbb{E}^2$ is the set of all nearest neighbor edges in $\mathbb{Z}^2$), $\mathscr{F}$ is the usual σ-field generated by cylinder sets, $P = P_p$ is the Bernoulli product measure and $\tau(e)$ is the coordinate variable $\omega(e)$ for each e. Our object is to derive the logarithmic lower bound (2.14) for the variance of $T_n(\hat{x})$ with constant $B > 0$, providing only that $p < p_c = p_c(2)$. The constant B may depend on the value of p but not on the unit vector $\hat{x}$ in $\mathbb{R}^2$.

Our first step is to express $T \equiv T_n(\hat{x})$ as an infinite series of martingale differences. This is in the spirit of Aizenman and Wehr (1990) and of Wehr and Aizenman (1990). To do this, we begin by deterministically ordering all the nearest neighbor edges: $e_1, e_2, \dots$. For future purposes, we will choose a "spiral" ordering, that is, one in which for each L, all the edges within each box $\Lambda_L = \{-L, -L+1, \dots, L\}^d$ come before those in Λ_{L+1} but not in Λ_L. We then define $\mathscr{F}_k$ to be the σ-field generated by $\tau(e_1), \dots, \tau(e_k)$ (and $\mathscr{F}_0$ to be the trivial σ-field). The passage time T is then the almost sure L^1 and L^2 limit of the martingale $E(T \mid \mathscr{F}_k)$ as $k \to \infty$ and so, since martingale differences are uncorrelated, we have the standard identity,

$$\text{var}(T) = \sum_{k=1}^{\infty} \text{var}[E(T \mid \mathscr{F}_k) - E(T \mid \mathscr{F}_{k-1})]. \tag{3.1}$$

Next let us define, for each k, T_k^0 and T_k^1 to be the random variables obtained from T by setting $\tau(e_k)$, respectively, to 0 and 1. That is, we may express ω for each k as $(\omega(e_k), \hat{\omega}^k)$, where $\hat{\omega}^k$ is the restriction of ω to $\mathbb{E}^2 \setminus \{e_k\}$ and then define for $\delta = 0$ or 1, $T_k^\delta(\omega) = T_k^\delta(\omega(e_k), \hat{\omega}^k) = T(\delta, \hat{\omega}^k)$.

A moment's thought on the definition of T in the 0 or 1 valued context shows that the random variable $H_k \equiv T_k^1 - T_k^0$ only takes the values 0 and 1 and should be thought of as the indicator variable of the event that e_k [or more accurately, the value of $\tau(e_k)$] "matters" for T. When $H_k(\omega) = 1$, this means precisely that in the configuration $(1, \hat{\omega}^k)$, *some* minimizing path for T passes through the edge e_k or equivalently it means that in $(0, \hat{\omega}^k)$ *every* minimizing path for T passes through e_k; $H_k(\omega) = 0$ corresponds precisely to the complement. Thus we may write $T = T_k^0 + H_k \tau(e_k)$. Because T_k^0 and H_k only depend on $\hat{\omega}^k$ and $\hat{\omega}^k$ is independent of $\omega(e_k) = \tau(e_k)$, it follows that $E(T_k^0 \mid \mathscr{F}_k) = E(T_k^0 \mid \mathscr{F}_{k-1})$, $E(H_k \tau(e_k) \mid \mathscr{F}_k) = \tau(e_k)\, E(H_k \mid \mathscr{F}_{k-1})$ and

$$E(T \mid \mathscr{F}_k) - E(T \mid \mathscr{F}_{k-1}) = [\tau(e_k) - E(\tau(e_k))]\, E(H_k \mid \mathscr{F}_{k-1}).$$

This latter random variable has zero mean, so its variance may be calculated by squaring, then evaluating the $\mathscr{F}_{k-1}$ conditional expectation using the independence of $\tau(e_k)$ and $\mathscr{F}_{k-1}$ and finally taking an overall expectation. Substituting into (3.1) yields the identity

$$\begin{aligned} \operatorname{var}(T) &= pq\sum_{k=1}^{\infty} E(E(H_k \mid \mathscr{F}_{k-1})^2) \\ &= pq\sum_{k=1}^{\infty} E(P(H_k = 1 \mid \mathscr{F}_{k-1})^2). \end{aligned} \tag{3.2}$$

Let us define F_k to be the event $\{H_k = 1,\ \tau(e_k) = 1\}$ or equivalently the event that $\tau(e_k) = 1$ and *some* minimizing path for T passes through e_k. We will describe this as the event that e_k is a *minimizing* 1-*edge*. Then we have, using independence and the Cauchy–Schwarz inequality, that

$$E(P(H_k = 1 \mid \mathscr{F}_{k-1})^2) = E\left(\frac{1}{q^2}P(F_k \mid \mathscr{F}_{k-1})^2\right) \geq \frac{1}{q^2}P(F_k)^2 \geq P(F_k)^2. \tag{3.3}$$

Thus we obtain the key inequality

$$\operatorname{var}(T) \geq pq\sum_{k=1}^{\infty} P(F_k)^2. \tag{3.4}$$

Let us denote by a_k the probability $P(F_k)$. The next step of the argument is to apply the following lemma [which is a variant of a lemma used by Pemantle and Peres (1994)] to replace the right-hand side of (3.4) by an expression involving the partial sums:

$$\sum_{j=1}^{k} P(F_j) = E(\text{number of minimizing 1-edges among the first } k \text{ edges}). \tag{3.5}$$

Our logarithmic lower bound for $\operatorname{var}(T)$ will then follow after we show that (3.5) is bounded below by $B_1\sqrt{k}$ for $k \leq Cn^2$. We remark that the constant $1/12$ appearing in the lemma is not optimal.

LEMMA 1. *For any positive a_k's and $m \geq 1$,*

$$\sum_{k=1}^{m} a_k^2 \geq \frac{1}{12}\left(\sum_{k=1}^{m} k^{-1}\right)^{-1}\left(\sum_{k=1}^{m-1} k^{-1}\left[k^{-1/2}\sum_{j=1}^{k} a_j\right]\right)^2. \tag{3.6}$$

PROOF. Let us define the following three positive sequences:

$$b_m = \sum_{k=1}^{m} a_k, \qquad c_m = m^{-1/2}, \qquad d_m = m(c_m - c_{m+1}). \tag{3.7}$$

Then, by elementary manipulation of series and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
0 \le b_m c_m &= b_1 c_1 + \sum_{k=2}^{m} [(b_k - b_{k-1})c_k + b_{k-1}(c_k - c_{k-1})] \\
&= \sum_{k=1}^{m} a_k c_k + \sum_{j=1}^{m-1} b_j (c_{j+1} - c_j) \qquad (3.8) \\
&\le \left(\sum_{k=1}^{m} a_k^2\right)^{1/2} \left(\sum_{k=1}^{m} c_k^2\right)^{1/2} - \sum_{k=1}^{m-1} k^{-1} d_k b_k.
\end{aligned}
$$

Our claimed inequality (3.6) is an immediate consequence of (3.8) and the fact that

$$
d_k = k(1 - [k/(1+k)]^{1/2})k^{-1/2} \ge (1/\sqrt{12})k^{-1/2} \quad \text{for } k \ge 1. \tag{3.9}
$$

This can be obtained by noting that

$$
\frac{d}{dx}\left\{x\left(1 - \left[\frac{x}{1+x}\right]^{1/2}\right)\right\} = 1 - \frac{2x+3}{2x+4}\left[\frac{x}{x+1}\right]^{1/2} > 0 \quad \text{for } x > 0, \tag{3.10}
$$

so that

$$
x(1 - [x/(1+x)]^{1/2}) \ge 1 - [1/2]^{1/2} > 1/\sqrt{12} \quad \text{for } x \ge 1. \tag{3.11}
$$

□

Combining this lemma with the key inequality (3.4), we see that

$$
\operatorname{var}(T) \ge \frac{pqB_1^2}{12}\left(\sum_{k \le Cn^2+1} k^{-1}\right)^{-1}\left(\sum_{k \le Cn^2} k^{-1}\right)^2 \ge B\log(n) \tag{3.12}
$$

provided that the mean number of edges in (3.5) is at least $B_1\sqrt{k}$ for $k \le Cn^2$. Now, let us define the passage time $\hat{T}_L$ to the boundary of the box Λ_L as

$$
\hat{T}_L = \min\{T(0, x)\colon \max(|x_1|, |x_2|) = L\} \tag{3.13}
$$

and let us denote by C_L the number of edges totally within Λ_L. If $n\hat{x}$ is outside of $(-L, L)^2$ (which will be so if $L \le n/\sqrt{2}$), then any minimizing path for $T_n(\hat{x})$ must pass through the boundary of the box Λ_L and hence must pass through at least $\hat{T}_L$ edges e within Λ_L which have $\tau(e) = 1$. Thus the number of minimizing 1-edges among the first C_L edges is at least $\hat{T}_L$ and

$$
\sum_{j=1}^{k} P(F_j) \ge E(\hat{T}_{L(k)}) \quad \text{provided } L(k) \le n/\sqrt{2}, \tag{3.14}
$$

where $L(k)$ is the largest L so that $C_L \le k$. There are positive constants B_2 and B_3 such that, for all k, $[B_2\sqrt{k}] \le L(k) \le B_3\sqrt{k}$ and so

$$
\sum_{j=1}^{k} P(F_j) \ge E\big(\hat{T}_{[B_2\sqrt{k}]}\big) \quad \text{provided } k \le (B_3)^2 n^2/2. \tag{3.15}
$$

For small k, where $[B_2\sqrt{k}] = 0$, one has $P(F_1) \geq P(\tau(e_j) = 1$ for $1 \leq j \leq 4) = q^4$ and thus the desired lower bound of the form $B_1\sqrt{k}$ for $\sum_{j=1}^{k} P(F_j)$ with $k \leq Cn^2$ would follow from (3.15) and the fact that

$$\lim_{L\to\infty} E(\hat{T}_L)/L = \mu(p) > 0 \quad \text{for } p < p_c. \tag{3.16}$$

This fact is a consequence of the shape theorem [see (2.7) and (2.9)], which implies that for $p < p_c$, $\hat{T}_L/L$ has the almost sure positive limit μ, equal to the smallest ρ such that the set ρB_0 intersects the boundary of $[-1, 1]^2$. Since $\hat{T}_L/L \leq 1$, the desired limit (3.16) of the mean follows. We remark that μ is easily seen to be $\mu(\hat{e}_1)$ because of the convexity and symmetry properties of the set B_0.

We have now completed the proof of Theorem 2 for 0 or 1 valued $\tau(e)$'s. The remainder of this section will be devoted to generalizing the key inequality (3.4) in a sufficiently abstract setting so that it can be applied to all the cases treated in Theorems 2, 3 and 4 as well as to Theorem 5.

The abstract setting involves a random variable T with $E(T^2) < \infty$ on the probability space $(\Omega, \mathscr{F}, P)$ where $\Omega = \mathbb{R}^I = \{\omega = (\omega_i\colon i \in I)\}$ is the space of real valued sequences indexed by a countable index set I, $\mathscr{F} = \mathscr{B}^I$ is the usual σ-field generated by Borel cylinder sets and P is (for the time being) any probability measure on $(\Omega, \mathscr{F})$. For U any subset of I, $\mathscr{F}(U)$ will denote the σ-field generated by $\{\omega_i\colon i \in U\}$. In the undirected first-passage context of Theorems 2 and 5 with i.i.d. $\tau(e)$'s, I will be $\mathbb{E}^2$, the set of nearest neighbor edges in $\mathbb{Z}^2$; in the directed i.i.d. context of Theorem 3, I will be the set of nearest neighbor edges in $\mathbb{Z}_+^2$. In both of these cases P will be the product measure which is the joint distribution of the $\tau(e)$'s. Finally, in the Ising context of Theorem 4, I will be $\mathbb{Z}^2$ and P will be the infinite volume Gibbs distribution.

In our general setting, we have a sequence $U_1, U_2, \ldots$ of disjoint subsets of I. Sometimes, as in the case of i.i.d. 0 or 1 valued $\tau(e)$'s, the U_i's will be singletons corresponding to some ordering of I. Mimicking the notation used earlier, we will express ω for each k as $(\omega^k, \hat{\omega}^k)$, where ω^k (resp. $\hat{\omega}^k$) is the restriction of ω to U_k (resp. to $I \setminus U_k$). We also have, for each k, disjoint events D_k^0 and D_k^1 in $\mathscr{B}^{U_k}$. These are generalizations of the events $\omega(e_k) = 0$ and $\omega(e_k) = 1$ from the 0 or 1 valued case. Accordingly, we again mimic our earlier notation and define

$$H_k(\omega) = T_k^1(\hat{\omega}^k) - T_k^0(\hat{\omega}^k), \tag{3.17}$$

where

$$T_k^0(\hat{\omega}^k) = \sup_{\omega^k \in D_k^0} T((\omega^k, \hat{\omega}^k)), \qquad T_k^1(\hat{\omega}^k) = \inf_{\omega^k \in D_k^1} T((\omega^k, \hat{\omega}^k)). \tag{3.18}$$

A positive H_k represents a minimum amount that T is reduced by moving ω^k from D_k^1 to D_k^0 while keeping $\hat{\omega}^k$ fixed.

THEOREM 8. *Assume the general setting just described and the following three hypotheses about P, the U_k's, the D_k^δ's and T:*

(i) *Conditional on $\mathscr{F}(I \setminus \bigcup_k U_k)$, the $\mathscr{F}(U_k)$'s are mutually independent.*
(ii) *There exist $p, q > 0$ such that, for any k,*

$$P(\omega^k \in D_k^0 \mid \mathscr{F}(U_k^c)) \geq p, \quad P(\omega^k \in D_k^1 \mid \mathscr{F}(U_k^c)) \geq q \quad a.s. \tag{3.19}$$

(iii) *For every k, $H_k \geq 0$ a.s.*

Suppose that, for some $\varepsilon > 0$ and each k, $F_k \in \mathscr{F}$ is a subset of the event $\{H_k \geq \varepsilon\}$. Then

$$\operatorname{var}(T) \geq pq\varepsilon^2 \sum_k P(F_k)^2. \tag{3.20}$$

Preparatory to proving Theorem 8, we present the following general lemma, where the conditional variance of a random variable with respect to a sub-σ-field is defined, as usual, by

$$\operatorname{var}_{\mathscr{G}}(X) \equiv E(X^2 \mid \mathscr{G}) - [E(X \mid \mathscr{G})]^2 \tag{3.21}$$

and where $\mathscr{G} \vee \mathscr{G}'$ denotes the smallest σ-field containing $\mathscr{G}$ and $\mathscr{G}'$.

LEMMA 2. *Let $T \in L^2(\Omega, \mathscr{F}, P)$ with $(\Omega, \mathscr{F}) = (\mathbb{R}^I, \mathscr{B}^I)$ as above and let $\mathscr{G}, \mathscr{G}_1, \mathscr{G}_2, \dots$ be sub-σ-fields of $\mathscr{F}$. If $\mathscr{G}_1, \mathscr{G}_2, \dots$ are mutually independent, then*

$$\operatorname{var}(T) \geq \sum_k \operatorname{var}(E(T \mid \mathscr{G}_k)). \tag{3.22}$$

If, conditional on $\mathscr{G}$, the $\mathscr{G}_k$'s are mutually independent, then

$$\operatorname{var}(T) \geq \sum_k E \operatorname{var}_{\mathscr{G}}(E(T \mid \mathscr{G} \vee \mathscr{G}_k)). \tag{3.23}$$

PROOF. We recall the standard fact that for any sub-σ-field $\mathscr{H}$,

$$\operatorname{var}(T) = E \operatorname{var}_{\mathscr{H}}(T) + \operatorname{var}(E(T \mid \mathscr{H})) \geq \begin{cases} \operatorname{var}(E(T \mid \mathscr{H})), \\ E \operatorname{var}_{\mathscr{H}}(T). \end{cases} \tag{3.24}$$

For the first part of the lemma, we denote $\mathscr{G}_1 \vee \cdots \vee \mathscr{G}_k$ by $\mathscr{F}_k$, the trivial σ-field by $\mathscr{F}_0$ and $\mathscr{G}_1 \vee \mathscr{G}_2 \vee \cdots$ by $\mathscr{F}_\infty$. Consider the martingale $E(T \mid \mathscr{F}_k)$, which converges (in L^2) to $E(T \mid \mathscr{F}_\infty)$. It follows from (3.24) and standard martingale reasoning that

$$\begin{aligned} \operatorname{var}(T) \geq \operatorname{var} E(T \mid \mathscr{F}_\infty) &= \sum_k \operatorname{var}[E(T \mid \mathscr{F}_k) - E(T \mid \mathscr{F}_{k-1})] \\ &\geq \sum_k \operatorname{var}[E(E(T \mid \mathscr{F}_k) - E(T \mid \mathscr{F}_{k-1}) \mid \mathscr{G}_k)]. \end{aligned} \tag{3.25}$$

Since $\mathscr{G}_k \subset \mathscr{F}_k$, we have $E(E(T \mid \mathscr{F}_k) \mid \mathscr{G}_k) = E(T \mid \mathscr{G}_k)$ and since, for the first part of the lemma, $\mathscr{G}_k$ is independent of $\mathscr{F}_{k-1}$, we have

$E(E(T \mid \mathscr{F}_{k-1}) \mid \mathscr{G}_k) = E(T)$. Hence the right-hand side of (3.25) equals that of (3.22). For the second part of the lemma, we begin with $\mathrm{var}(T) \geq E\ \mathrm{var}_{\mathscr{G}}(T)$. We now apply the first part of the lemma to a regular conditional probability given $\mathscr{G}$, $\mu(\omega, d\omega')$. [See, e.g., Durrett (1991), pages 199–200; here we are using the countability of the index set I so that $\mathbb{R}^I$ can be regarded as a complete separable metric space with $\mathscr{B}^I$ its Borel σ-field, which guarantees the existence of regular conditional probabilities.] The conditional independence of the $\mathscr{G}_k$'s with respect to $\mathscr{G}$ implies that for P-almost all ω, the σ-fields $\mathscr{G} \wedge \mathscr{G}_k$ are independent sub-σ-fields of $\mathscr{F}$ in the probability space $(\Omega, \mathscr{F}, \mu(\omega, \cdot))$. The first part of the lemma then yields

$$\mathrm{var}_{\mathscr{G}}(T) \geq \sum_k \mathrm{var}_{\mathscr{G}}(E(T \mid \mathscr{G} \vee \mathscr{G}_k)). \tag{3.26}$$

Taking the expectation yields (3.23). □

Another lemma we will need is the following.

LEMMA 3. *Suppose* $X \in L^2(\Omega, \mathscr{F}, P)$, D^0 *and* D^1 *are disjoint events in* $\mathscr{F}$ *and* $\mathscr{G}$ *is a sub-σ-field of* $\mathscr{F}$ *such that* $P(D^\delta \mid \mathscr{G}) > 0$ *a.s., for* $\delta = 0$ *and* 1. *Then*

$$\mathrm{var}(X) \geq \frac{P(D^0)\,P(D^1)}{P(D^0) + P(D^1)}\,(x_1 - x_0)^2, \tag{3.27}$$

where $x_\delta = E(X\,1_{D^\delta})/P(D^\delta)$, *and*

$$\mathrm{var}_{\mathscr{G}}(X) \geq \frac{P(D^0 \mid \mathscr{G})\,P(D^1 \mid \mathscr{G})}{P(D^0 \mid \mathscr{G}) + P(D^1 \mid \mathscr{G})}(X_1 - X_0)^2 \quad a.s., \tag{3.28}$$

where

$$X_\delta = E(X\,1_{D^\delta} \mid \mathscr{G})/P(D^\delta \mid \mathscr{G}) \quad \text{for } \delta = 0 \text{ and } 1. \tag{3.29}$$

PROOF. We begin with the top inequality of (3.24) with T replaced by X and $\mathscr{H}$ taken as the σ-algebra generated by D^0 and D^1. Now $E(X \mid \mathscr{H})$ may be represented as a random variable on a three-element probability space. The probabilities of the three elements are $p = P(D^0)$, $q = P(D^1)$ and $1 - p - q = P(D^2)$ with $D^2 = (D^0 \cup D^1)^c$, and the random variable takes values x_0, x_1 and x_2. Since, for any Y, $\mathrm{var}(Y)$ is the infimum over α of $E((Y - \alpha)^2)$, we have

$$\begin{aligned} \mathrm{var}(E(X \mid \mathscr{H})) &= \inf_\alpha [p(x_0 - \alpha)^2 + q(x_1 - \alpha)^2 + (1 - p - q)(x_2 - \alpha)^2] \\ &\geq \inf_\alpha [p(x_0 - \alpha)^2 + q(x_1 - \alpha)^2] = \frac{pq}{p + q}(x_1 - x_0)^2, \end{aligned} \tag{3.30}$$

which yields (3.27). The conditional version (3.28) is obtained by applying (3.27) to a regular conditional distribution for X given $\mathscr{G}$ [see, e.g., Durrett (1991), pages 198–199]. □

PROOF OF THEOREM 8. We let $\mathscr{G}_k = \mathscr{F}(U_k)$ and $\mathscr{G} = \mathscr{F}(I \setminus \bigcup_k U_k)$ so that (3.23) of Lemma 2 is applicable. We then use (3.28) of Lemma 3 with $X = E(T \mid \mathscr{G} \vee \mathscr{G}_k)$ and $D^\delta = D_k^\delta \times \mathbb{R}^{I \setminus U_k}$. Since $T\,1_{D^0} \leq T_k^0\,1_{D^0}$ and $D^0 \in \mathscr{G}_k$, we have

$$\begin{aligned} X\,1_{D^0} &= E(T \mid \mathscr{G} \vee \mathscr{G}_k)1_{D^0} = E(T\,1_{D^0} \mid \mathscr{G} \vee \mathscr{G}_k) \leq E(T_k^0\,1_{D^0} \mid \mathscr{G} \vee \mathscr{G}_k) \\ &= E(T_k^0 \mid \mathscr{G} \vee \mathscr{G}_k)\,1_{D^0} = E(T_k^0 \mid \mathscr{G})\,1_{D^0}, \end{aligned} \tag{3.31}$$

where the last equality is a consequence of the $\mathscr{G} \vee (\bigvee_{j \neq k} \mathscr{G}_j)$-measurability of T_k^0 and of hypothesis (i) of the theorem. This upper bound for $X\,1_{D^0}$ and an analogous lower bound for $X\,1_{D^1}$ show that X_0 and X_1 of (3.29) satisfy

$$X_0 \leq E(T_k^0 \mid \mathscr{G}), \qquad X_1 \geq E(T_k^1 \mid \mathscr{G}), \tag{3.32}$$

so that

$$X_1 - X_0 \geq E(T_k^1 - T_k^0 \mid \mathscr{G}) = E(H_k \mid \mathscr{G}) \geq 0 \quad \text{a.s.}, \tag{3.33}$$

where we have used hypothesis (iii) of the theorem. This and hypothesis (ii), when inserted into (3.28), imply

$$\begin{aligned} \operatorname{var}_{\mathscr{G}}(X) &\geq \frac{pq}{p+q}[E(H_k \mid \mathscr{G})]^2 \geq pq[E(H_k \mid \mathscr{G})]^2 \\ &\geq pq\varepsilon^2[P(H_k \geq \varepsilon \mid \mathscr{G})]^2. \end{aligned} \tag{3.34}$$

Thus, from (3.23) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \operatorname{var}(T) &\geq pq\varepsilon^2 \sum_k E[P(H_k \geq \varepsilon \mid \mathscr{G})]^2 \geq pq\varepsilon^2 \sum_k P(H_k \geq \varepsilon)^2 \\ &\geq pq\varepsilon^2 \sum_k P(F_k)^2. \end{aligned} \tag{3.35}$$

□

4. Proofs of logarithmic bounds for independent first-passage percolation. In this section we apply the methods developed in the last section to undirected and directed first-passage percolation with i.i.d. $\tau(e)$'s to prove Theorems 2 and 3. We begin with Theorem 2, but rearrange the two cases, (2.12) and (2.13):

$$P(\tau(e) = \lambda) < p_c(2), \tag{4.1}$$

$$\lambda > 0 \quad \text{and} \quad p_c(2) \leq P(\tau(e) = \lambda) < p_c^{\text{dir}}(2). \tag{4.2}$$

PROOF OF THEOREM 2, CASE (4.1). We apply Theorem 8 of the last section in the following context: $I = \mathbb{E}^2$ is the set of nearest neighbor edges in $\mathbb{Z}^2$, P is the joint distribution of the i.i.d. $\tau(e)$'s, $T = T_n(\hat{x})$ for some unit vector $\hat{x}$ and $U_k = \{e_k\}$, where $e_1, e_2, \ldots$ is the same "spiral" ordering used at the beginning of the last section to prove the 0 or 1 valued case of Theorem 2. We define

$$D_k^0 = (-\infty, a], \qquad D_k^1 = [b, \infty), \tag{4.3}$$

where a and b are constants [depending on the common distribution of the $\tau(e)$'s] chosen so that $0 \le a < b$ and such that

$$P(\tau(e) \le a) > 0, \qquad P(\tau(e) < b) < p_c(2). \tag{4.4}$$

If $P(\tau(e) = \lambda) > 0$, then a may be chosen as λ. The three numbered hypotheses of Theorem 8 are easily seen to be valid: (i) because the $\mathscr{F}(U_k)$'s are independent even without conditioning; (ii) because (by independence) the inequalities of (3.19) are equalities with $p = P(\tau(e) \le a)$ and $q = P(\tau(e) \ge b)$; (iii) because T is a coordinatewise nondecreasing function of ω. Finally we let $\varepsilon = b - a$ and

$$F_k = \{\tau(e_k) \ge b \text{ and } \textit{some} \text{ minimizing path for } T \text{ passes through } e_k\}. \tag{4.5}$$

Mimicking the language used in the proof in the 0 or 1 valued case, we describe this as the event that e_k is a *minimizing b-edge*.

We may now apply Lemma 1, as in the 0 or 1 valued case, which reduces the proof to showing that $\sum_{j=1}^{k} P(F_j)$, the expected number of minimizing b-edges among the first k edges, is bounded below by $B_1\sqrt{k}$ for $k \le Cn^2$. To do this we consider the undirected first-passage percolation model with i.i.d. passage times

$$\hat{\tau}(e) = \begin{cases} 1, & \text{if } \tau(e) \ge b, \\ 0, & \text{if } \tau(e) < b. \end{cases} \tag{4.6}$$

Note that $\hat{p} \equiv P(\hat{\tau}(e) = 0) < p_c(2)$ by our choice of b [see (4.4)]. We define $\hat{T}_L$ to be the passage time to the boundary of the box Λ_L for this first-passage model. The remainder of the proof is exactly as in the 0 or 1 valued proof given in Section 3 with (3.14)–(3.16) all valid [except that p in (3.16) should be replaced by $\hat{p}$]. □

PROOF OF THEOREM 2, CASE (4.2). Again we apply Theorem 8 with I, P and T as in case (4.1), but with different choices of U_k, D_k^δ and F_k than before. We remark that the arguments we will use in this case are very similar to the block construction arguments used by van den Berg and Kesten (1993). Indeed, it has been pointed out to us by those authors that a modest strengthening of a certain property of minimizing paths they derived [see their (2.16)] would allow us to prove case (4.2) of Theorem 2 with essentially the same choices of U_k, D_k^δ and F_k as we used in case (4.1). They further note that such a strengthening can be derived by their methods. In order to make our proof more self-contained, we will not pursue this approach to case (4.2).

To define the U_k's, we begin by partitioning $\mathbb{Z}^2$ into boxes Λ_x, indexed by x in $\mathbb{Z}^2$, which are translates of the basic box $\Lambda_L = \{-L, -L+1, \dots, L\}^2$ centered at the origin: $\Lambda_x = (2L+1)x + \Lambda_L$. The scale size L will be chosen later. We then partition $\mathbb{E}^2$ into related U_x's with U_x containing all the edges totally within Λ_x and all edges partly within Λ_x which touch, say, the western and southern boundaries of Λ_x. We choose a spiral ordering $x_1, x_2, \dots$ of the sites of $\mathbb{Z}^2$ and take $U_k = U_{x_k}$. We choose

$$D_k^0 = (-\infty, a]^{U_k}, \qquad D_k^1 = (b, \infty)^{U_k}, \tag{4.7}$$

with $a = \lambda$ and $b > \lambda$ a constant such that

$$P(\tau(e) \le b) < p_c^{\mathrm{dir}}(2). \tag{4.8}$$

The three numbered hypotheses of Theorem 8 are again immediately seen to be verified with

$$p = P(\tau(e) = \lambda)^{|U_1|}, \qquad q = P(\tau(e) > b)^{|U_1|}, \tag{4.9}$$

where $|U_1| =$ number of edges in U_1.

Finally we let $\varepsilon = \min(b - \lambda, 2\lambda)$ and choose

$$\begin{aligned} F_k = \{&\text{there exist two sites in } \Lambda_{x_k} \text{ which} \\ &(1)\ \text{both belong to a minimizing path for } T, \text{ and} \\ &(2)\ \text{are separated by Euclidean distance at least } L \text{ and} \\ &(3)\ \text{are not connected by any directed path of edges} \\ &\text{with } \tau(e) \le b\}. \end{aligned} \tag{4.10}$$

In the definition of F_k, a directed path means one which either goes only north and east or else only north and west or else only south and east or else only south and west. To see that $F_k \subset \{H_k \ge \varepsilon\}$, note that if $\omega = (\omega^k, \hat{\omega}^k) \in F_k$ and we first replace all $\tau(e)$'s with e in U_k by λ [i.e., ω^k is replaced by $(\lambda, \lambda, \dots, \lambda)$], then either there was a *directed* segment of a minimizing path between two sites in the box for which the replacement of a minimizing edge with $\tau(e) > b$ by a $\tau(e) = \lambda$ edge reduces the passage time T by at least $b - \lambda$, or else there was a *nondirected* segment of a minimizing path between two sites in the box whose replacement by a directed segment (with at least two fewer edges than the nondirected segment) of $\tau(e) = \lambda$ edges reduces the passage time by at least 2λ. If we next replace all $\tau(e)$'s with e in U_k by values greater than b, then the T [for the $\tau(e) = \lambda$ in U_k situation] is increased by at least $b - \lambda$ (if a new minimizing path still passes through U_k) or else by at least the amount ($\ge \varepsilon$) that it was reduced previously (if the new minimizing path bypasses U_k). We remark that the second condition in the definition of F_k was not used yet, but will be needed below.

As in case (4.1), an application of Lemma 1 reduces the proof to showing that $\sum_{j=1}^{k} P(F_j) \ge B_1\sqrt{k}$ for $k \le Cn^2$. We will show that this is true for sufficiently large L. To do this we will define a *site* first-passage percolation model related to the F_k's. For each x in $\mathbb{Z}^2$, we define τ_x^* to be the indicator variable of the event F_x^* whose complement is

$$\begin{aligned} F_x^{*c} = \{&\text{there exist two sites in } \Lambda_x, \text{ separated by Euclidean} \\ &\text{distance at least } L \text{ and connected by some directed} \\ &\text{path of edges with } \tau(e) \le b\}. \end{aligned} \tag{4.11}$$

Note that $F_k \supset [F^*_{x_k} \cap \hat{F}_{x_k}]$, where

$$\hat{F}_x = \{\text{a minimizing path for } T \text{ touches at least two points in } \Lambda_x \text{ which are at least distance } L \text{ apart}\}. \tag{4.12}$$

The site first-passage percolation model we wish to consider is the one with (i.i.d.) site passage times τ^*_x defined on the graph with vertex set $\mathbb{Z}^2$ and edge set $\mathbb{E}^{2*} = \{\text{pairs of } *\text{-neighbors in } \mathbb{Z}^2\}$. Two distinct sites, (x_1, x_2) and (y_1, y_2) in $\mathbb{Z}^2$, are $*$-neighbors if $|x_1 - y_1| \leq 1$ and $|x_2 - y_2| \leq 1$; that is, $*$-neighbors are either ordinary nearest neighbors or else diagonal nearest neighbors. We wish to compare $\sum_{j=1}^{k} P(F_j)$ to the mean of T^*_M, the passage time to the boundary of the box Λ_M:

$$T^*_M = \min\Bigl(\sum_{x\in\gamma} \tau^*_x\colon\ \gamma \text{ is a } *\text{-connected path of sites in } \mathbb{Z}^2 \text{ from the origin to the boundary of } \Lambda_M\Bigr). \tag{4.13}$$

The point of our choice of $*$-connection is that the set of x's in $\mathbb{Z}^2$ for which $\hat{F}_x$ occurs is a $*$-connected set containing the origin and at least one site on the boundary of Λ_M, providing $n\hat{x}$ [recall $T = T_n(\hat{x})$] is outside of the square $[-L-M(2L+1), L+M(2L+1)]^2$ [which will be so if $M+1 \leq (2L+1)^{-1}n/\sqrt{2}$]. Under the same conditions on M, L and n, the number of k's with x_k in the box Λ_M for which F_k occurs is bounded below by T^*_M and so we have as an analogue of (3.14):

$$\sum_{j=1}^{k} P(F_j) \geq E\bigl(T^*_{[\sqrt{k}/2-1/2]}\bigr) \quad \text{providing } \sqrt{k}/2 + 1/2 \leq (2L+1)^{-1}n/\sqrt{2}. \tag{4.14}$$

If we can show that (for large enough L)

$$\lim_{M\to\infty} E(T^*_M)/M = \mu^* > 0, \tag{4.15}$$

then it will follow that $\sum_{j=1}^{k} P(F_j) \geq B_1\sqrt{k}$ for $k_0 \leq k \leq Cn^2$. This together with an ad hoc bound for $k \leq k_0$ will complete the proof.

The positive limit in (4.15) would follow from the $(\mathbb{Z}^2, \mathbb{E}^{2*})$ site analogue of the shape theorem, providing

$$p^* = p^*(L) \equiv P(\tau^*_x = 0) = P(F^{*c}_x) < p^*_c(2), \tag{4.16}$$

where $p^*_c(2) > 0$ is the critical value for independent site percolation on $(\mathbb{Z}^2, \mathbb{E}^{2*})$. A proof can be obtained by essentially the same arguments used in the "standard" version of Theorem 1 [see Kesten (1986)].

We claim that the inequality of (4.16) is valid for sufficiently large L because the requirement (4.8) on b implies that $p^*(L) \to 0$ as $L \to \infty$. This last claim is a direct consequence of the directed percolation result [see Aizenman and Barsky (1987) for one main ingredient] that since $P(\tau(e) \leq b) < p_c^{\text{dir}}$, there exist $c > 0$ and $c' < \infty$ such that the event $A(x, y)$, that x and y are connected by some directed path of edges with $\tau(e) \leq b$, satisfies

$$P(A(x, y)) \leq c' e^{-c\|x-y\|}. \tag{4.17}$$

Here $\|x\|$ is the Euclidean distance. Thus

$$\begin{aligned} p^*(L) = P(F^{*c}_{(0,0)}) &\leq \sum_{\substack{x,y \text{ in } \Lambda_L \\ \|x-y\| \geq L}} P(A(x, y)) \\ &\leq c'' L^2 e^{-cL} \to 0 \quad \text{as } L \to \infty. \end{aligned} \tag{4.18}$$

This completes the proof of case (4.2) of Theorem 2. □

PROOF OF THEOREM 3. The proof here is mostly like that of case (4.1) of Theorem 2. Again we apply Theorem 8. This time $I = \mathbb{E}^2_+$, the set of nearest neighbor edges in $\mathbb{Z}^2_+$, P is the joint distribution of the i.i.d. $\tau(e)$'s, $T = T_n(\hat{x})$ for some unit vector $\hat{x}$ and $U_k = \{e_k\}$, where $e_1, e_2, \ldots$ is some spiral ordering of $\mathbb{E}^2_+$. The D^δ_k's are defined as in (4.3) with $0 \leq a < b$ chosen so that

$$P(\tau(e) \leq a) > 0, \qquad P(\tau(e) < b) < p_c^{\text{dir}}(2). \tag{4.19}$$

We again define F_k by (4.5) and let $\varepsilon = b - a$. Again Lemma 1 reduces the proof to obtaining the appropriate $B_1\sqrt{k}$ lower bound on the expected number of minimizing b-edges among the first $\sqrt{k}$ edges in $\mathbb{E}^2_+$. To derive this bound, let $\tilde{T}_L$ denote the minimum number of $\tau(e) \geq b$ edges on directed paths from the origin to the boundary of $\Lambda_L \cap \mathbb{Z}^2_+$. Then, even in the absence of a complete shape theorem in the directed case (see the discussion preceding Theorem 3), it can be shown [e.g., by arguments like those used to obtain (4.15) in the proof of case (4.2) of Theorem 2] that $\liminf E(\tilde{T}_L)/L > 0$. We leave further details to the reader. □

5. Proof of logarithmic bound for Ising first-passage percolation. This section is devoted to the proof of Theorem 4. As usual, we begin by applying Theorem 8, but this time we will need to apply it many times with different choices for the U_k's, D^δ_k's and F_k's and then average the resulting lower bounds for var(T). In the Ising context, $I = \mathbb{Z}^2$, P is the infinite volume (free boundary condition) Ising model Gibbs distribution for the given parameters J and h [which is supported on the subset $\{-1, 1\}^I$ of $\mathbb{R}^I$], and $T = T_n(\hat{x})$ for some unit vector $\hat{x}$, defined in terms of the *dependent* edge passage times

$$\tau((x, y)) = \begin{cases} 1, & \text{if } \omega_x \neq \omega_y, \\ 0, & \text{if } \omega_x = \omega_y, \quad \text{for } (x, y) \in \mathbb{E}^2 \end{cases} \tag{5.1}$$

[see (2.17) and note that in the notation we have inherited from Section 3, ω_x is essentially the Ising spin variable σ_x].

Let S denote a (site) lattice animal (at the origin); that is, a nonempty finite subset of $\mathbb{Z}^2$ containing the origin which is connected (by the edges of $\mathbb{E}^2$). Its boundary, denoted ∂S, is the set of sites not in S which are nearest neighbors of some site in S and its closure, denoted $\overline{S}$, is the union of S and ∂S. For each lattice animal S, we will choose the U_k's and so forth as follows. Let R_S be the smallest positive integer such that all the translates, $\overline{S} + R_S x$ for $x \in \mathbb{Z}^2$, are not only mutually disjoint, but also are not nearest neighbors of each other (i.e., $\overline{\overline{S} + R_S x}$ is disjoint from $\overline{S} + R_S y$ for every $x, y \in \mathbb{Z}^2$ with $x \neq y$). We then take $U_k = \overline{S} + R_S x_k$, where $x_1, x_2, \ldots$ is a spiral ordering of $\mathbb{Z}^2$ and note that hypothesis (i) of Theorem 8 follows from the nearest neighbor spatial Markov property of nearest neighbor Ising models. [We remark that for a finite range Ising model, hypothesis (i) would be valid with $U_k = \overline{S} + (R_S + r)x_k$ and r large enough.]

Let us denote $S + R_S x_k$ by U_k^0 so that $\partial U_k^0 = \partial S + R_S x_k$ and $U_k = U_k^0 \cup \partial U_k^0$. We then choose

$$D_k^0 = \{\omega_x = +1 \text{ for } x \in U_k\}, \tag{5.2}$$

$$D_k^1 = \{\omega_x = +1 \text{ for } x \in \partial U_k^0,\ \omega_y = -1 \text{ for } y \in U_k^0\}. \tag{5.3}$$

Hypothesis (ii) of Theorem 8 follows from the usual formula for the conditional probabilities of a Gibbs distribution, with p and q depending on J and h as well as S. Hypothesis (iii) is valid because when $\omega|_{\partial U_k^0} \equiv +1$, the change from $\omega|_{U_k} \equiv +1$ to $\equiv -1$ either reduces T by 2 (if a minimizing path entered and exited U_k^0) or by 1 (if a minimizing path enters but does not exit U_k^0 or vice versa) or else leaves T unchanged. Finally we take $\varepsilon = 1$ and

$$F_k = \{\omega|_{U_k} \in D_k^1\} \cap \{R_S x_k \text{ lies on a minimizing path for } T\}. \tag{5.4}$$

It should be clear from our discussion of hypothesis (iii) that $F_k \subset \{H_k \geq \varepsilon\}$. We will describe the second event on the right-hand side of (5.4) as the event that $R_S x_k$ is a minimizing site.

Theorem 8 gives us the inequality (3.20) with the above definitions. Let us also consider alternate definitions in which the $+1$'s and -1's are interchanged in (5.2) and (5.3) with the resulting F_k^-, p^- and q^- appearing in (3.20). If we average these two inequalities and use the fact that $(a^2 + b^2)/2 \geq (a+b)^2/4$, we obtain

$$\operatorname{var}(T) \geq \tfrac{1}{4} p_S q_S \sum_k P(G_k)^2, \tag{5.5}$$

where $p_S = \min(p, p^-)$, $q_S = \min(q, q^-)$ and

$$G_k = F_k \cup F_k^- = \{R_S x_k \text{ is a minimizing site}\} \cap \{\mathscr{C}(R_S x_k) = R_S x_k + S\}. \tag{5.6}$$

Here $\mathscr{C}(x)$ denotes the "like-spin cluster" at site x, that is, the maximal connected set of sites y containing x and with each $\omega_y = \omega_x$. [To be overly precise, $\mathscr{C}(x)$ should be defined as the empty set if $\omega_x \neq \pm 1$.]

Let us denote by G_x the event given by the right-hand side of (5.6) but with $R_S x_k$ replaced by x. If we were to replace our original choice of U_k's by the translates $\overline{S} + R_S x_k + y$ for a fixed $y \in \mathbb{Z}^2$, we would obtain an analogue of

(5.5) but with the G_x's originally appearing replaced by G_{x+y}. Averaging these inequalities over y in the box $\{0, 1, \dots, R_S - 1\}^2$ yields the bound

$$\operatorname{var}(T) \geq r_S \sum_{x \in \mathbb{Z}^d} P(G_x)^2 \tag{5.7}$$

with $r_S = p_S q_S/(2R_S)^2$. Next we average (5.7) over all lattice animals S with cardinality $|S| \leq M$ and use the fact that the average of squares is greater than the square of the average to conclude that

$$\operatorname{var}(T) \geq r'_M \sum_{x \in \mathbb{Z}^d} P(G_x^M)^2, \tag{5.8}$$

where $r'_M > 0$ depends on M, h and J, and where

$$G_x^M = \{x \text{ is a minimizing site and } |\mathscr{C}(x)| \leq M\}. \tag{5.9}$$

This completes our use of Theorem 8.

To apply Lemma 1, we reorganize (5.8) by using a spiral ordering $x_1, x_2, \dots$ of the sites in $\mathbb{Z}^2$. By the usual reasoning, it then suffices to show that, for $L \leq n/\sqrt{2}$, the mean of

$$Y_L^M = \{\text{number of } x \text{ in } \Lambda_L \text{ which are minimizing sites with } |\mathscr{C}(x)| \leq M\} \tag{5.10}$$

grows at least linearly as $L \to \infty$. However, for $L \leq n/\sqrt{2}$,

$$Y_L^M \geq \min_{|\gamma|=L} \left(\sum_{x \in \gamma} 1_{|\mathscr{C}(x)| \leq M} \right), \tag{5.11}$$

where the minimum is over (site self-avoiding) connected paths starting at the origin containing $|\gamma| = L$ sites. Now the elementary inequalities

$$\begin{aligned} \sum_{x \in \gamma} |\mathscr{C}(x)| &\geq (M+1) \sum_{x \in \gamma} 1_{|\mathscr{C}(x)| > M} + \sum_{x \in \gamma} 1_{|\mathscr{C}(x)| \leq M} \\ &\geq M|\gamma| - M \sum_{x \in \gamma} 1_{|\mathscr{C}(x)| > M} \end{aligned} \tag{5.12}$$

together with (5.11) imply that

$$Y_L^M \geq L \left(1 - M^{-1} \max_{|\gamma|=L} L^{-1} \sum_{x \in \gamma} |\mathscr{C}(x)| \right). \tag{5.13}$$

The desired linearly growing lower bound on Y_L^M (and hence on its mean) is then an immediate consequence (by choosing M large enough) of a result of Fontes and Newman (1993): For (J, h) in the interior of the nonpercolating regime for the standard $\mathbb{Z}^2$ Ising model,

$$\limsup_{L \to \infty} \max_{|\gamma|=L} L^{-1} \sum_{x \in \gamma} |\mathscr{C}(x)| < \infty \quad \text{a.s.} \tag{5.14}$$

We remark that this result was used by Fontes and Newman (1993) to prove the Ising model shape theorem in the nonpercolating regime.

6. Proof of power law inequalities.

PROOF OF THEOREM 5. This proof is along the same lines as the proof of Theorem 2 given in Section 4, except that Lemma 1 will be replaced by a simple Schwarz inequality. We will consider here the general d version of case (4.1), that is, $P(\tau(e) = \lambda) < p_c(d)$; the more complicated case where $\lambda > 0$ and $p_c(d) \leq P(\tau(e) = \lambda) < p_c^{\mathrm{dir}}(d)$ may be handled by the same type of argument used for it in Section 4.

Using the same notation as in the proof of this case of Theorem 2, we again let F_k be the event that e_k is a minimizing b-edge for $T_n(\hat{x})$ so that Theorem 8 yields

$$\mathrm{var}(T_n(\hat{x})) \geq pq(b-a)^2 \sum_{k=1}^{\infty} P(F_k)^2. \tag{6.1}$$

Let us denote by F_e the event that e is a minimizing b-edge for $T_n(\hat{x})$. For any $\gamma > \xi_{\hat{x}}$, we define

$$V_n = [(1+\delta)n\mu(\hat{x})B_0] \cap \Lambda_n^{\gamma}(\hat{x}), \tag{6.2}$$

$$\mathscr{E}_n = \{e \in \mathbb{E}^d\text{: each endpoint of } e \text{ is in } V_n\}, \tag{6.3}$$

$$A_n = \text{the event that } M_n(\hat{x}) \subset V_n. \tag{6.4}$$

Then from (6.1) and the Schwarz inequality,

$$\mathrm{var}(T_n(\hat{x})) \geq pq(b-a)^2 \sum_{e \in \mathscr{E}_n} P(F_e)^2 \geq \frac{pq(b-a)^2}{|\,\mathscr{E}_n\,|}\left[\sum_{e \in \mathscr{E}_n} P(F_e)\right]^2. \tag{6.5}$$

Now $|\,\mathscr{E}_n\,|$ grows like $n^{1+(d-1)\gamma}$ while

$$\sum_{e \in \mathscr{E}_n} P(F_e) \geq P(A_n)\, E\left(\sum_{e \in \mathscr{E}_n} 1_{F_e} \mid A_n\right) = P(A_n)\, E\left(\sum_{k=1}^{\infty} 1_{F_k} \mid A_n\right). \tag{6.6}$$

Now since $\gamma > \xi_{\hat{x}}$, $\mathscr{P}(A_n)$ is bounded away from zero by the definition (2.21) of $\xi_{\hat{x}}$ and by the shape theorem. The shape theorem for the $\hat{\tau}(e)$ variables of (4.6) also implies that $\sum_{k=1}^{\infty} 1_{F_k} \geq Dn$ for large n a.s., which implies at least linear growth for the second factor on the far right of (6.6). Thus $\mathrm{var}(T_n(\hat{x}))$ grows at least like $n^{1-(d-1)\gamma}$. Letting γ approach $\xi_{\hat{x}}$ yields the desired inequality (2.22). □

PROOF OF THEOREM 6. We assume $\chi' < 1$ since in any case $\xi_{\hat{x}} \leq 1$ follows from the shape theorem. We will show that if $\hat{x}$ is a direction of curvature for B_0, then for every $\gamma > (1+\chi')/2$, $P(M_n(\hat{x}) \subset \Lambda_n^{\gamma}(\hat{x})) \to 1$ as $n \to \infty$. We begin by noting that

$$M_n(\hat{x}) = \bigcup_{0 \leq t \leq T_n(\hat{x})} [B_t(0) \cap B_{(T_n(\hat{x})-t)}(v(n,\hat{x}))], \tag{6.7}$$

where

$$B_t(y) = \{z \in \mathbb{Z}^d:\ T(y,z) \le t\}. \tag{6.8}$$

The expression (6.7) for $M_n(\hat{x})$ is just the statement that any site touched by a time-minimizing path from the origin to $v(n,\hat{x})$ must be reached from the origin by some time t [$\le T_n(\hat{x})$] and that it must be possible to reach from there to $v(n,\hat{x})$ in the remaining time. From the definition of χ', we have for any $\kappa > \chi'$ that

$$T_n(\hat{x}) \le n\mu(\hat{x}) + n^\kappa \quad \text{for large } n\text{, a.s.,} \tag{6.9}$$

and for any *fixed* y that

$$B_t(y) \subset y + (t + t^\kappa)B_0 \quad \text{for large } t\text{, a.s.} \tag{6.10}$$

These imply that for any $\kappa > \chi'$ the following is valid with probability approaching 1 as $n \to \infty$:

$$M_n(\hat{x}) \subset \bigcup_{0 \le t \le n\mu(\hat{x}) + n^\kappa} \{(t + n^\kappa)B_0 \cap [n\hat{x} + (n\mu(\hat{x}) - t + 2n^\kappa)B_0]\}. \tag{6.11}$$

The remainder of the proof is a purely geometric (nonprobabilistic) argument that the RHS of (6.11) is contained in the cylinder $\Lambda_n^\gamma(\hat{x})$ of radius n^γ for all large n. The desired result follows immediately from the next lemma by letting κ approach χ'. Recall that $L_{\hat{x}}$ is the straight line passing through the origin and $n\hat{x}$. □

LEMMA 4. *Let $\kappa \in (0,1)$. If $\hat{x}$ is a direction of curvature for B_0, then there is some constant $c \in (0,\infty)$ so that, for any $t_1, t_2 \ge 0$ with $t_1 + t_2 \le n\mu(\hat{x}) + 3n^\kappa$ and any $y \in t_1B_0 \cap [n\hat{x} + t_2B_0]$,*

$$d(y, L_{\hat{x}}) \le cn^{(1+\kappa)/2} \quad \textit{for large } n. \tag{6.12}$$

PROOF. Without loss of generality we may suppose $t_1 + t_2 = n\mu(\hat{x}) + 3n^\kappa$. Choose $t_0 \in [0, n\mu(\hat{x})]$ such that $t_1 - 3n^\kappa \le t_0 \le t_1$ and $n\mu(\hat{x}) - t_2 \le t_0 \le n\mu(\hat{x}) - t_2 + 3n^\kappa$. Denote by H the $(d-1)$-dimensional hyperplane through the origin which is parallel to the tangent hyperplane to B_0 at the point $\hat{x}/\mu(\hat{x})$. If there is more than one tangent hyperplane, we choose one which is also tangent at $\hat{x}/\mu(\hat{x})$ to a ball D satisfying Conditions 1′ and 2′ just above Theorem 6. Let H_i for $i = 1, 2$ denote the closed half-spaces on the two sides of the hyperplane $H + t_0\hat{x}/\mu(\hat{x})$ so that the origin is in H_1 and $n\hat{x}$ is in H_2. Then we have the inclusion

$$\{t_1B_0 \cap [n\hat{x} + t_2B_0]\} \subset \{(t_1B_0 \cap H_2) \cup ([n\hat{x} + t_2B_0] \cap H_1)\}. \tag{6.13}$$

We will show that (6.12) is valid for $y \in t_1B_0 \cap H_2$. A change of coordinates then gives (6.12) for $y \in [n\hat{x} + t_2B_0] \cap H_1$. By Condition 2′ for the ball D, $t_1B_0 \subset t_1D$. We now have a ball D containing the origin which is tangent at

$\hat{x}/\mu(\hat{x})$ to $H+\hat{x}/\mu(\hat{x})$. For $t_0 \le n\mu(\hat{x})$ and $t_1 \in [t_0, t_0+3n^\kappa]$, we need to show (6.12) for y in the intersection of $t_1 D$ and the "far side" of $t_0(H+\hat{x}/\mu(\hat{x}))$. This intersection is contained between $H+t_0\hat{x}/\mu(\hat{x})$ and $H+t_1\hat{x}/\mu(\hat{x})$ and is in fact the "polar cap" of the sphere $t_1 D$ (a sphere of radius $t_1\rho$ for some fixed ρ) sliced off by $H+t_0\hat{x}/\mu(\hat{x})$. The thickness of the cap (along the direction perpendicular to H) is $t_1 - t_0 \le 3n^\kappa$ and thus its transverse diameter is

$$\begin{aligned} &2([\rho t_1]^2 - [\rho t_1 - (t_1 - t_0)]^2)^{1/2} \\ &\quad \le 2[2\rho t_1(t_1-t_0)]^{1/2} \le 2[2\rho(n\mu(\hat{x})+3n^\kappa)3n^\kappa]^{1/2} \\ &\quad = O(n^{(1+\kappa)/2}). \end{aligned} \tag{6.14}$$

Since the line $L_{\hat{x}}$ passes through the cap, (6.12) follows. □

PROOF OF THEOREM 7. Let $\hat{x}$ be a direction of curvature for B_0. We have from Theorems 5 and 6 for $d=2$ that

$$\chi_{\hat{x}} \ge (1-\chi')/4, \tag{6.15}$$

which implies that

$$\max(\chi', \chi_{\hat{x}}) \ge 1/5. \tag{6.16}$$

When $P(\tau(e)=0) < p_c(d)$ and the moment generating function condition on $\tau(e)$ is valid, Kesten (1993) and Alexander (1995) have proved that

$$\chi' \le 1/2 \quad \text{for all } d. \tag{6.17}$$

Inserting this bound into (2.24) and (6.15) yields (2.27) for $d=2$ and (2.29) for $d \ge 2$. Finally (2.26) and (2.28) follow by taking the supremum over $\hat{x}$ and using the next lemma. □

LEMMA 5. *Let B be a nonempty bounded subset of $\mathbb{R}^d$. Denote its interior, closure and boundary, all defined in the usual way by* $\operatorname{int}(B)$, $\overline{B}$ *and* $\partial B = \overline{B}\setminus \operatorname{int}(B)$. *There exists at least one point $z \in \partial B$ such that, for some closed ball D with radius $\rho \in (0,\infty)$, $z \in \partial D$ and $D \supset \overline{B}$.*

PROOF. Let D_r denote the closed ball of radius r centered at the origin. Except for the single special case where B consists of only the origin, we take $\rho = \inf\{r > 0\colon D_r \supset \overline{B}\}$, $D = D_\rho$ and z any point in $\partial D_\rho \cap \overline{B}$. We leave details to the reader. In the special case, take D to be any sphere with positive radius whose boundary passes through the origin. □

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