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HYDRODYNAMIC SCALING LIMITS WITH DETERMINISTIC INITIAL CONFIGURATIONS

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We discuss the hydrodynamic scaling limits starting with deterministic configurations for different models. Certain estimates on the entropy of the system are derived.

1. Introduction. For a large system of interacting particles, the derivation of hydrodynamic scaling limits, besides being important for the understanding of the macroscopic behavior of the system, is of much interest in itself. Recently Guo, Papanicolaou and Varadhan [4] gave a new approach to the hydrodynamic scaling limit problem. The method involved is to control certain aspects of the microscopic evolution by estimates on the entropy. If the initial data have proper entropy bounds, they concluded that the system has the hydrodynamic limit as a weak solution of certain nonlinear parabolic equations. This approach has been applied to derive hydrodynamic limits of various models. Fritz [3] generalized the method to the infinite volume Ginzburg–Landau model, Varadhan to the nongradient Ginzburg–Landau model [12], Quastel to the colored diffusion [7] and Varadhan to interacting diffusion models [13]; see, for example, [1] and [8] for a review.

In this paper, we again consider the hydrodynamic limits, but we start with deterministic configurations. In this situation, the initial entropy is often infinite, so the bound on the initial entropy is not available. Proper understanding of the evolution of the microscopic structure is necessary. To get the hydrodynamic limit, we show that after a finite period of microscopic time the entropy has certain bounds, and at the same time the macroscopic structure has changed very little.

The main idea is based on the fact that since the evolution of the underlying system is elliptic, reasonable conditions on the models as well as the initial configurations imply moderate control on the entropy in a microscopically finite time.

Here we carry out our approach for the Ginzburg–Landau model and the interacting diffusion model. These two models are different because the corresponding systems conserve different quantities: the first one is a discrete lattice model and the second one is continuous. In fact our approach can also be applied to many other models, for example, simple exclusion models, zero

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range attractive models and these models in high dimensions and with infinite volume [6]. Furthermore, the nature of our approach allows us to understand the behavior of dynamics with more than one phase; see [15]. This is because we assume no more than certain macroscopic bounds of initial configurations, which is different from the assumptions given in [4] and [14], where in addition to the macroscopic bounds on the initial configurations, a proper choice of reference states (Gibbs or local Gibbs states) is required for the initial states.

The rest of the paper is organized as follows. In Section 2, we discuss the Ginzburg–Landau models (gradient and nongradient) on the circle. In Section 3, we discuss the interacting diffusion model. The treatments are slightly different because of the nature of the conserved quantities, but the technique is still the same.

2. Ginzburg–Landau model.

2.1. *Model and main results.* For integer $N \geq 1$ let S_N denote the periodic one-dimensional lattice $\{j/N\colon 1 \leq j \leq N\}$, with charges x_j at site j/N. The vector $\mathbf{x} = (x_1, \dots, x_N)$ evolves in time as a diffusion in R^N with generator

$$\begin{aligned}\mathscr{L}_N = {} & \frac{N^2}{2}\sum_{i=1}^{N} a(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 \\ & - \frac{N^2}{2}\sum_{i=1}^{N} W(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right),\end{aligned}$$

where

$$W(x,y) = -a_1(x,y) + a_2(x,y) + a(x,y)(V'(x) - V'(y))$$

for suitable potential function V and strictly positive function $a(x, y)$ with bounded continuous first derivatives satisfying

$$0 < C^{-1} \leq a(x,y) \leq C < \infty.$$

The functions a_1 and a_2 are the partial derivatives

$$a_1 = \frac{\partial a}{\partial x} \quad \text{and} \quad a_2 = \frac{\partial a}{\partial y}.$$

The gradient version of this model has $a(x, y) = 1$, which in many ways is easier to handle since certain cancellations occur when summation is performed.

Assuming

$$\int_{-\infty}^{\infty} e^{-V(x)}\, dz = 1,$$

then the diffusion is stationary and reversible with respect to the probability measure $\Phi_N = \exp(-\sum_{i=1}^{N} V(x_i))\, dx_1 \cdots dx_N$ on R^N.

For simplicity, we also assume

$$\int_{-\infty}^{\infty} \exp(\lambda x - V(x))\, dx = M(\lambda) < \infty \quad \text{for all } \lambda \in R,$$

$$\int_{-\infty}^{\infty} \exp(\sigma |V'(x)| - V(x))\, dx < \infty \quad \text{for all } \sigma \geq 0,$$

and, for some positive constants C_0, C_1 and C_2,

$$-C_0 \leq V''(x) \leq C_1 + C_2 V(x) \quad \text{for all } x \in R.$$

Since the sum $x_1 + \cdots + x_N$ is conserved by the dynamics, the hyperplane $x_1 + \cdots + x_N = N a_N$ of average charge a_N is invariant under the evolution. The diffusion restricted to this hyperplane is elliptic and ergodic with the invariant measure $d\mu_{N, a_N}(x)$ as the conditional distribution of Φ_N given $x_1 + \cdots + x_N/N = a_N$. Therefore,

$$(1) \qquad d\mu_{N, a_N}(x) = \frac{1}{Z_{N, a_N}} \exp\left(-\sum_{i=1}^{N} V(x_i) \right) \delta_{(\sum_{i=1}^N x_i = N a_N)}\, dx_1 \cdots dx_N.$$

The following stochastic differential equation version of the system will be frequently used to discuss the microscopic structure:

$$(2) \qquad \begin{aligned} dx_i(t) &= \frac{N^2}{2}\left[-W(x_i, x_{i+1}) + W(x_{i-1}, x_i) \right] dt \\ &\quad + N\left[\sigma(x_i, x_{i+1})\, d\beta_i - \sigma(x_{i-1}, x_i)\, d\beta_{i-1} \right], \end{aligned}$$

where $\sigma(x_i, x_{i+1}) = \sqrt{a(x_i, x_{i+1})}$.

Let us denote by P_{N, a_N} the diffusion process starting from deterministic configuration $(x_1(0), \dots, x_N(0)) = (\eta_1^N, \dots, \eta_N^N)$ with $(1/N)\sum_{i=1}^N \eta_i^N = a_N$, and by f_N^t the density of the process with respect to $d\mu_{N, a_N}$ at time t.

Here is the main result:

THEOREM 2.1. *Assume there is a constant C_3, such that initial configurations $(\eta_1^N, \dots, \eta_N^N)$ satisfy the following conditions*:

(i) $(1/N)\sum_{i=1}^N V(\eta_i^N) \leq C_3$ *for all* N.

(ii) *There is a function m_0 on S such that, for all smooth functions J on S,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N} \right) \eta_i^N = \int_S J(\theta) m_0(\theta)\, d\theta.$$

Then for every $t > 0$, each $\delta > 0$ and all continuous functions $J(\cdot)$,

$$(3) \qquad \lim_{N \to \infty} P_{N, a_N}\left\{ x: \left| \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N} \right) x_i(t) - \int J(\theta) m(\theta, t)\, d\theta \right| \geq \delta \right\} = 0,$$

where $a_N = (1/N)\sum_{i=1}^N \eta_i^N$ and $m(\theta, t)$ is the unique weak solution of the *nonlinear diffusion equation*

$$(4)\qquad \begin{aligned} \frac{\partial m(\theta,t)}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial\theta}\left[\hat{a}(m(\theta,t))\frac{\partial}{\partial\theta}(h'(m(\theta,t)))\right],\\ m(\theta,0) &= m_0(\theta), \end{aligned}$$

where $h(m)$ and $\hat{a}(m)$ are the quantities to be defined below.

For each $m \in R$, let $h(m) = \sup_\lambda[\lambda m - \log M(\lambda)]$ and let $\lambda = h'(m)$; with this choice of λ, we have a product measure Q_m on $\prod_{-\infty}^{\infty} R = \Omega$ with each coordinate having the distribution $(1/M(\lambda))e^{\lambda x - V(x)}\,dx$. Let $g(x_{-l},\dots,x_l)$ be a smooth function of $(2l+1)$ variables. We view this as a function $g(\omega)$ defined on Ω. If we denote by T the shift operator $(T\omega)(i) = x_{i+1}$ if $\omega(i) = x_i$, we can form the formal infinite sum

$$\xi = \sum_{k=-\infty}^{\infty} g(T^k\omega).$$

Although ξ does not really make sense, the partial derivatives $\partial\xi/\partial x_i$ are all well defined. We can now write down the formula for $\hat{a}(m)$:

$$\hat{a}(m) = \inf_g E^{Q_m}\left\{a(x_1,x_2)\left[1 - \left(\frac{\partial\xi}{\partial x_1} - \frac{\partial\xi}{\partial x_2}\right)\right]^2\right\},$$

where the infimum is taken over all functions g, varying l as well as the function of the $(2l+1)$ variables. The detailed discussion of this characterization can be found in [12]. Especially $\hat{a} = 1$ when $a = 1$, which is the gradient case.

For a density function f_N with respect to μ_{N,a_N} on the hyperplane of average a_N, we define the entropy function as follows:

$$H_{N,a_N}(f_N) = \int f_N \log f_N \, d\mu_{N,a_N}(x).$$

Let $(y_1(t),\dots,y_N(t)) = (x_1(t/N^2),\dots,x_N(t/N^2))$ denote the nonspeeded diffusion process. Then the diffusion generator is

$$\frac{1}{2}\sum_{i=1}^{N} a(y_i,y_{i+1})\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)^2 - \frac{1}{2}\sum_{i=1}^{N} W(y_i,y_{i+1})\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right).$$

We continue to use $\mathscr{L}_N$, P_{N,a_N} and f_N^t to denote the generator, the diffusion process and the density for the time changed process $(y_1(t),\dots,y_N(t))$. We will show the following theorem.

THEOREM 2.2. *Under the same conditions as in Theorem* 2.1, *for all $T > 0$, there exists a constant $C_4 = C_4(T)$ such that, for all $0 < T/2 \le t \le T$,*

$$(5)\qquad H_{N,a_N}(f_N^t) \le C_4 N$$

and, for every smooth function J on S and $\delta > 0$,

$$(6) \qquad \lim_{N\to\infty} P_{N,a_N}\left\{\left|\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)y_i(t) - \int J(\theta)m_0(\theta)\,d\theta\right| \ge \delta\right\} = 0.$$

Theorem 2.2 reveals that although the initial configuration is far away from equilibrium or local equilibrium states, the system demonstrates the property that after a finite amount of time the entropy achieves linear growth rate in the number of sites, whereas the macroscopic behavior is still the same.

The proof of Theorem 2.1 needs a combination of Theorem 2.2 and the following version of results given in [12].

THEOREM 2.3. *If the initial density* f_N^0 *on the hyperplane of average* a_N *satisfies*

$$H_{N,a_N}(f_N^0) \le C_5 N \quad \text{for all } N,$$

and if there is a function m_0 *on S such that, for all smooth functions J on S and every* $\delta > 0$,

$$\lim_{N\to\infty} P_{N,a_N}\left\{\left|\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)x_i(0) - \int J(\theta)m_0\,d\theta\right| \ge \delta\right\} = 0,$$

then for every $t > 0$, *each* $\delta > 0$ *and all continuous functions J on S*,

$$\lim_{N\to\infty} P_{N,a_N}\left\{\left|\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)x_i(t) - \int J(\theta)m(t,\theta)\,d\theta\right| \ge \delta\right\} = 0$$

with $m(t,\theta)$ *being the weak solution of the nonlinear diffusion equation* (4).

The difference between Theorem 2.3 and the result of [12] is that the system considered in this paper occurs on typical hyperplanes instead of on the whole space. Since the same techniques can be applied with slight modifications, we omit the proof here. The interested reader can refer to [12].

2.2. *Entropy bound.* From earlier works (see [4], [11] and [12]), we know that a proper bound on the entropy growth plays a key role in deriving hydrodynamic scaling limits. In this section, we study properties of the entropy growth for the Ginzburg–Landau model with deterministic initial configurations. The main result is the following proposition.

PROPOSITION 2.1. *If the initial configurations satisfy the conditions in Theorem* 2.1, *then there exists a* $C_6 = C_6(T)$ *such that, for all* $0 < T/2 \le t \le T$,

$$\int f_N^t \log f_N^t \, d\mu_{N,a_N}(x) \le C_6 N.$$

We introduce a new diffusion process Q_{N,a_N}, which is described by the following stochastic differential equation:

$$\begin{aligned} dy_i(t) &= \tfrac{1}{2}\big[\big(a_1(y_i, y_{i+1}) - a_2(y_i, y_{i+1})\big) \\ &\qquad - \big(a_1(y_{i-1}, y_i) - a_2(y_{i-1}, y_i)\big)\big]\, dt \\ &\qquad + \sigma(y_i, y_{i+1})\, d\beta_i - \sigma(y_{i-1}, y_i)\, d\beta_{i-1}, \\ y_i(0) &= \eta_i. \end{aligned}$$

The new process has the same initial configuration as that of P_{N,a_N}, and has its infinitesimal generator in divergence form:

$$\hat{\mathscr{L}}_N = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial}{\partial y_{i+1}} - \frac{\partial}{\partial y_i}\right) a(y_i, y_{i+1}) \left(\frac{\partial}{\partial y_{i+1}} - \frac{\partial}{\partial y_i}\right).$$

In addition, the invariant measure on the same hyperplane $\sum_{i=1}^{N} y_i = Na_N$ of average a_N is

$$d\nu_{N,a_N}(x) = \delta_{(\sum_{i=1}^{N} y_i = Na_N)}\, dx_1 \cdots dx_N.$$

Obviously the process Q_{N,a_N} occurs on the hyperplane of average a_N. If we rewrite the generator $\hat{\mathscr{L}}_N$ as

$$\hat{L}_N = \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial}{\partial y_i} a_{ij}(y) \frac{\partial}{\partial y_j}, \tag{7}$$

then

$$A(y) = \big(a_{ij}(y)\big)$$

$$= \begin{pmatrix} a_{N,1} + a_{1,2} & -a_{1,2} & 0 & \cdots & 0 & -a_{N,1} \\ -a_{1,2} & a_{1,2} + a_{2,3} & -a_{2,3} & \cdots & 0 & 0 \\ 0 & -a_{2,3} & a_{2,3} + a_{3,4} & \cdots & 0 & 0 \\ & & & \ddots & & \\ -a_{N,1} & 0 & 0 & \cdots & -a_{N-1,N} & a_{N-1,N} + a_{N,1} \end{pmatrix},$$

with $a_{ij} = a(y_i, y_j)$ and for $\boldsymbol{\alpha} \in R^N$,

$$\boldsymbol{\alpha}' A(y) \boldsymbol{\alpha} = \sum_{i=1}^{N} a_{i,i+1}(y)(\alpha_i - \alpha_{i+1})^2$$

by the assumption on the boundedness of $a(\cdot,\cdot)$,

$$C^{-1} B_N \le A(y) \le C B_N, \tag{8}$$

with

$$B_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & & \\ & & & \ddots & & \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

We denote by g_N^t the density function of the diffusion process Q_{N,a_N} with respect to $d\mu_{N,a_N}$ at time t. Then

$$(9)\qquad H_{N,a_N}(f_N^t) = \int f_N^t \log\frac{f_N^t}{g_N^t}\, d\mu_{N,a_N}(y) + \int f_N^t \log g_N^t\, d\mu_{N,a_N}(y).$$

The first term is simply the relative entropy of density functions on the same hyperplane. To estimate this term, we need the following lemma.

LEMMA 2.1. *For any* $t > 0$,

$$\left.\frac{dQ_{N,a_N}}{dP_{N,a_N}}\right|_{\mathscr{F}_0^t} = \exp\left\{\int_0^t \frac{1}{2}\sum_{i=1}^N \sigma(y_i, y_{i+1})(V'(y_i) - V'(y_{i+1}))\, d\beta_i \right.$$
$$\left. - \frac{1}{8}\int_0^t \sum_{i=1}^N [V'(y_{i+1}) - V'(y_i)]^2 a(y_i, y_{i+1})\, ds\right\}.$$

This is an application of the Cameron–Girsanov theorem (see Stroock and Varadhan [9]). Since both f_N^t and g_N^t are marginal densities of the corresponding diffusion processes, by the property of relative entropy for processes, we have

$$\int f_N^t \log\frac{f_N^t}{g_N^t}\, d\mu_{N,a_N}(y) \le E^{P_{N,a_N}} \log\left.\frac{dP_{N,a_N}}{dQ_{N,a_N}}\right|_{\mathscr{F}_0^t}$$
$$= E^{P_{N,a_N}}\left[\frac{1}{8}\int_0^t \sum_{i=1}^N a(y_i, y_{i+1})[V'(y_{i+1}) - V'(y_i)]^2\, ds\right].$$

Moreover, $g_N^t = G_N^t/(\Psi_{N,a_N}(y))$, where G_N^t is the density of the diffusion Q_{N,a_N} at time t, but with respect to $d\nu_{N,a_N}$ on the hyperplane of average a_N, and

$$\Psi_{N,a_N}(y) = \frac{1}{Z_{N,a_N}}\exp\left(-\sum_{i=1}^N V(y_i)\right).$$

So

$$\int f_N^t \log g_N^t\, d\mu_{N,a_N}(y) = \int f_N^t \log G_N^t\, d\mu_{N,a_N}(y)$$
$$+ \int f_N^t \log \Psi_{N,a_N}(y)^{-1}\, d\mu_{N,a_N}(y).$$

By our assumption on the potential V,

$$Z_{N,a_N} = \int_{S^N} \exp\left(-\sum_{i=1}^{N-1} V(y_i) - V\left(Na_N - \sum_{i=1}^{N-1} y_i\right)\right) dy_1 \cdots dy_{N-1}$$
$$\le \exp(C_7)$$

for some constant $C_7 > 0$ dependent on the lower bound of V. Then

$$\int f_N^t \log \Psi_{N,a_N}(y)^{-1}\, d\mu_{N,a_N}(y) = \log Z_{N,a_N} + E^{f_N^t}\left[\sum_{i=1}^N V(y_i)\right]$$
$$= E^{P_{N,a_N}}\left[\sum_{i=1}^N V(y_i(t))\right] + C_7.$$

So the proof of Proposition 2.1 is reduced to the following lemma.

LEMMA 2.2. *There exist constants $C_8 = C_8(T)$ and $C_9 = C_9(T)$ such that, for all $0 < T/2 \le t \le T$,*

$$E^{f_N^t} \log G_N^t \le C_8 N \tag{10}$$

and, for $0 \le t \le T$,

$$E^P\left[\sum_{i=1}^N V(y_i(t))\right] \le C_9 N, \tag{11}$$

$$E^P\left[\int_0^t \sum_{i=1}^N a(y_i, y_{i+1})(V'(y_{i+1}) - V'(y_i))^2\, ds\right] \le C_9 N. \tag{12}$$

The proof of Lemma 2.2 needs the following upper bound for the heat kernel of a general operator in divergence form.

For an elliptic operator in divergence form,

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}, \tag{13}$$

with symmetric $a(x) = (a_{ij})_{i,j=1}^n$, satisfying

$$\tilde{C}\boldsymbol{\alpha}' B \boldsymbol{\alpha} \le \boldsymbol{\alpha}' a(x) \boldsymbol{\alpha} \le \tilde{C}^{-1} \boldsymbol{\alpha}' B \boldsymbol{\alpha} \tag{14}$$

for some positive constant $\tilde{C}$ and symmetric matrix B with positive eigenvalues, we have following estimate of the upper bound for the heat kernel:

PROPOSITION 2.2. *The heat kernel $\Gamma_t(x, y)$ of the parabolic operator $\mathscr{L} - \partial/\partial t$ has the Gaussian-type upper bound*

$$\Gamma_t(x, y) \le \left(\frac{\tilde{C}_1}{t}\right)^{n/2} \frac{1}{|B|^{1/2}} \exp\left(-\frac{(y-x)' B^{-1} (y-x)}{2\tilde{C}_2 t}\right), \tag{15}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are positive, uniform constants independent of the dimension and the operator, depending only on $\tilde{C}$.

Proposition 2.2 can be proved by using the argument in Fabes and Stroock [2] and a proper change of coordinate scales; see Lu [6].

Now we proceed to the proof of Lemma 2.2.

PROOF OF LEMMA 2.2. [(10).] Since the coefficients of the generator $\tilde{\mathscr{L}}_N$ in (7) satisfy relation (8), by Proposition 2.2, we have the following upper bound for the density function G_N^t:

$$G_N^t(\eta, y) \le \frac{C_{10}^{N-1}}{t^{N-1}|B_{a_N}|^{1/2}} \exp\left(- \frac{(y-\eta)' B_{a_N}^{-1} (y-\eta)}{2tC_{11}} \right).$$

Here B_{a_N} is the restriction of B_N to the subspace defined by the hyperplane of average a_N, $B_{a_N}^{-1}$ is the inverse of B_{a_N} on the hyperplane and C_{10}, C_{11} are functions of C. A simple calculation shows that B_{a_N} has positive eigenvalues

$$\lambda_j = 2 - 2\cos(2j\pi/N), \qquad j = 1, 2, \dots, N-1,$$

so $|B| = \prod_{i=1}^{N-1} \lambda_i$. Since $\log(1/\prod \lambda_i) \le C_{12} N$ for some uniform constant C_{12}, there exists a $C_{13}(T)$ such that, for all $T/2 \le t \le T$,

$$\log\big(G_N^t\big) \le C_{13}(T) N.$$

Note that we need the condition $t \ge T/2 > 0$ to get the correct upper bound.

[(11).] By Itô's formula,

$$\begin{aligned} d \sum_{i=1}^N V(y_i(t)) &= -\tfrac{1}{2} \sum_{i=1}^N [V'(y_i) - V'(y_{i+1})] W(y_i, y_{i+1})\, dt \\ &\quad + \tfrac{1}{2} \sum_{i=1}^N [V''(y_i(t)) + V''(y_{i+1}(t))] a(y_i, y_{i+1})\, dt \\ &\quad + d\text{martingale}. \end{aligned} \tag{16}$$

By the boundedness of a_x, a_y, we have

$$\begin{aligned} &(V'(x) - V'(y)) W(x, y) \\ &\quad = (V'(x) - V'(y)) [a(x,y)(V'(x) - V'(y)) + a_2(x,y) - a_1(x,y)] \\ &\quad \ge -C_{14} + \tfrac{1}{2} a(x,y)(V'(x) - V'(y))^2 \end{aligned}$$

for some fixed constant C_{14} which depends on the bounds on a_x, a_y. By the assumption that $V''(x) \le C_1 + C_2 V(x)$ and (16),

$$\begin{aligned} &E^{P_{N,a_N}} \sum_{i=1}^N V(y_i(t)) \\ &\quad \le C_{15} N - E^{P_{N,a_N}} \int_0^t \tfrac{1}{2} \sum_{i=1}^N (a(y_i, y_{i+1})) [V'(y_{i+1}) - V'(y_i)]^2\, ds \\ &\qquad + C_{15} \int_0^t E^{P_{N,a_N}} \sum_{i=1}^N V(y_i(s))\, ds + \sum_{i=1}^N V(y_i(0)). \end{aligned} \tag{17}$$

Using Gronwall's inequality and the assumption on initial configurations, there exists a constant $C_{16}(T)$ such that, for all $0 \le t \le T$,

$$E^{P_{N,a_N}} \sum_{i=1}^{N} V(y_i(t)) \le C_{16}(T)N.$$

[(12).] It is obvious from (17) that

$$E^{P_{N,a_N}} \int_0^t \sum_{i=1}^{N} a(y_i, y_{i+1})[V'(y_{i+1}) - V'(y_i)]^2 \, ds$$
$$\le C_{17}\left[\sum_{i=1}^{N} V(\eta_i) - E\sum_{i=1}^{N} V(y_i(t))\right] + C_{17}\int_0^t E\left[\sum_{i=1}^{N} V(y_i(s))\, ds\right].$$

By the assumption of the lower bound on the potential V and Lemma 2.2 (11), then

$$E^{P_{N,a_N}} \int_0^t \sum_{i=1}^{N} a(y_i, y_{i+1})[V'(y_{i+1}) - V'(y_i)]^2 \le C_{18}N,$$

for all $0 \le t \le T$ with $C_{18} = C_{18}(T)$.

Thus we have proved Lemma 2.2. Proposition 2.1 is an immediate consequence. □

2.3. *Macroscopic behavior.* In this section, we will prove the following proposition.

PROPOSITION 2.3. *For any finite $t > 0$, $\delta > 0$ and all smooth functions J on S,*

$$\lim_{N\to\infty} P_{N,a_N}\left\{\left|\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) y_i(t) - \int_S J(\theta) m_0(\theta)\, d\theta\right| \ge \delta\right\} = 0.$$

This result tells us that in a finite period of time, the nonspeeded diffusion process has made little change macroscopically.

PROOF OF PROPOSITION 2.3. For a smooth function J, we use Itô's formula:

$$\frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) y_i(t) - \frac{1}{N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)\eta_i^N$$
$$= -\frac{1}{N^2}\int_0^t \frac{1}{2}\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right) W(y_i(s), y_{i+1}(s))\, ds$$
$$+ \frac{1}{N^2}\int_0^t \sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)\sigma(y_i(s), y_{i+1}(s))\, d\beta_i(s).$$

Here $\nabla J(i/N) = N(J((i+1)/N) - J(i/N))$ is the discrete differential of J at i/N. Since $W(x,y) = a_2(x,y) - a_1(x,y) + a(x,y)(V'(x) - V'(y))$, and with the assumption on $a(\cdot,\cdot)$ and inequality (12) in Lemma 2.2, then we know the right-hand side goes to zero in probability. Therefore, by our assumption on initial configurations, in any finite time $t > 0$, the macroscopic structure of the system remains unchanged. □

2.4. *Remarks.*

1. As we see in the proof of Proposition 2.1, we need the bound on the distribution of the configuration $\{x_1(t), \dots, x_N(t)\}$; see Lemma 2.1. So the assumption on the initial configuration in Theorem 2.1 is one of many natural choices. Basically, it requires that initial configurations be distributed reasonably well; that is, satisfy condition (i) in Theorem 2.1.
2. It is obvious that the gradient model is a special case of the model discussed here. In fact, the derivation for the gradient model is straightforward because the heat kernel G_N^t for the new process can be computed explicitly; see [6]. As for the infinite volume Ginzburg–Landau model, certain prior estimates are needed to control the boundary terms; the techniques developed in [3] can be used. See [5] and [6].
3. As seen from the proof of Theorem 2.2, the constructed process with a special choice of potential function can help us to get the entropy bound in a finite microscopic time. In fact, the selection of the potential function is not unique; for example, we can choose a quadratic potential function. We can achieve similar estimations by the same method.

3. Interacting diffusions on the circle. In contrast to the Ginzburg–Landau model, which conserves the total charge, the interacting diffusions considered below conserve the total number of particles.

3.1. *Model and main results.* We denote by S the circle of unit circumference and consider a system of N interacting Brownian motions with S as a state space satisfying the following system of stochastic differential equations:

$$dx_i(t) = -N \sum_{j:\, j \neq i}^{N} V'\big(N\big(x_i(t) - x_j(t)\big)\big)\, dt + d\beta_i(t), \qquad i = 1, 2, \dots, N. \tag{18}$$

Here $(\beta_1, \dots, \beta_N)$ are N independent Brownian motions on the circle and $V(\cdot)$ is an even function on R satisfying the following assumptions:

(A1) V is twice continuously differentiable;
(A2) $V \geq 0$, $V(0) > 0$ and V has compact support;
(A3) $V(\cdot)$ is repulsive in the sense that $\Psi(z) = -zV'(z) \geq 0$.

Then the process $\{x_1(t), \dots, x_N(t)\}$ is a Markov process of diffusion type on S^N, the N-fold copy of S, with an infinitesimal generator given by

$$\mathscr{L}_N = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - N \sum_{j \neq i}^{N} V'\big(N(x_i - x_j)\big) \frac{\partial}{\partial x_i}. \tag{19}$$

It is easily verified that

$$\mathscr{L}_N = \frac{1}{2}\sum_{i=1}^{N} \exp\left(\sum_{j,k}^{N} V(N(x_j - x_k))\right)\frac{\partial}{\partial x_i}\exp\left(-\sum_{j,k}^{N} V(N(x_j - x_k))\right)\frac{\partial}{\partial x_i},$$

so that $\mathscr{N}_N$ is formally symmetric with respect to the measure

$$d\mu_N(x) = \frac{1}{Z_N}\exp\left[-\sum_{i,j}^{N}\big(V(N(x_i - x_j))\big)\right] dx_1 \cdots dx_N,$$

where Z_N is a normalization constant and is chosen to make μ_N into a probability measure.

We start $\mathbf{X}(0) = (x_1(0), \dots, x_N(0))$ at a deterministic configuration. We denote by $f_N^t(x_1, \dots, x_N)$ the density at time t with respect to μ_N, which is a solution of the forward equation

$$\text{(20)} \qquad \begin{aligned} \frac{\partial}{\partial t} f_N^t &= \mathscr{L}_N f_N^t, \\ f_N^t\big|_{t=0} &= \frac{1}{N!}\sum_{\pi \in S_N} \delta_{(a^N_{\pi(1)}, \dots, a^N_{\pi(N)})}. \end{aligned}$$

Here $\mathscr{S}_N$ is the N-permutation group and $(a_1^N, \dots, a_N^N)$ is the initial deterministic configuration. We choose the initial density in the form (3) because the initial data $(x_1(0), \dots, x_N(0))$ can be any arrangement from the initial configuration $(a_1^N, \dots, a_N^N)$ without changing the initial empirical measure as given below. The empirical distribution of the process at time t is defined by

$$\xi_N(t, A) = \frac{1}{N}\sum_{i=1}^{N} \chi_A(x_i(t)) \quad \text{for } A \subset S,$$

and $\xi_N(t)$ is viewed as a random measure on S. If we denote by $\mathscr{M}_1(S)$ the space of probability measures on S, we can view $\xi_N(\cdot)$ as a stochastic process with values in $\mathscr{M}_1(S)$. Let Q_N denote the induced measure by $\xi_N(\cdot)$ starting with the deterministic configuration $(a_1^N, \dots, a_N^N)$ on $C([0, T], \mathscr{M}_1(S))$. Our main result is to show that under suitable assumptions on $(a_1^N, \dots, a_N^N)$ the measure Q_N as $N \to \infty$ will concentrate on a single measure-valued trajectory which is a solution of a certain nonlinear diffusion equation.

In order to describe this nonlinear diffusion equation, we have to introduce some thermodynamic functions of one-dimensional systems with pair interaction given by $V(\cdot)$. The partition function in a finite region $[0, l]$ with activity λ is given by

$$\hat{Z}(l, \lambda) = \sum_{n=0}^{\infty} \frac{e^{n\lambda}}{n!}\int_0^l \cdots \int_0^l \exp\left(-\sum_{i,j} V(x_i - x_j)\right) dx_1 \cdots dx_n;$$

free energy is defined by

$$F(\lambda) = \lim_{l \to \infty} \frac{\log \hat{Z}(l, \lambda)}{l},$$

and it exists and is a convex function of λ for all λ; and

$$\rho(\lambda) = \frac{dF}{d\lambda}$$

is the "density" corresponding to the activity λ and is a continuous strictly monotone function of λ. This function can be inverted to yield $\lambda = \lambda(\rho)$ as a function of ρ. The free energy expressed as a function of ρ, that is,

$$P(\rho) = F(\lambda(\rho)),$$

is called the pressure and is again a continuous strictly montone function of density ρ. For a detailed discussion of these thermodynamic quantities, see [13].

Here is our main result:

THEOREM 3.1. *Assume there is a constant c such that the initial configurations* $(a_1^N, \dots, a_N^N)$ *satisfy*

$$\frac{1}{N} \sum_{i,j} V\big(N\big(a_i^N - a_j^N\big)\big) \le c, \tag{21}$$

for all N, and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} J\big(a_i^N\big) = \int_S J(\theta) \rho_0(\theta) \, d\theta,$$

for all smooth functions $J(\cdot)$ *on S and some density function* $\rho_0(\theta)$ *on S such that* $\rho_0(\theta) \ge 0$ *and* $\int \rho_0(\theta) \, d\theta = 1$.

Then for every $t > 0$ *and any continuous* $J(\cdot)$ *and each* $\delta > 0$,

$$\lim_{N \to \infty} \int_{E_N^t} f_N^t \, d\mu_N = 0,$$

where

$$E_N^t = \left\{ x; \left| \frac{1}{N} \sum_{i=1}^{N} J(x_i) - \int_S J(\theta) \rho(t, \theta) \, d\theta \right| \ge \delta \right\}$$

and $\rho(t, \theta)$ *is defined as the unique solution of*

$$\frac{\partial}{\partial t} \rho(t, \theta) = \frac{1}{2} [P(\rho(t, \theta))]_{\theta\theta}, \qquad \rho(t, \theta)|_{t=0} = \rho_0(\theta). \tag{22}$$

If we look at the behavior of the system on the microscopic scale, that is, the behavior $(y_1(t), \dots, y_N(t)) = (x_1(t/N^2), \dots, x_1(t/N^2))$, then their dynamics can be described as

$$dy_i(t) = -\frac{1}{N}\sum_{j;\, j\neq i}^{N} V'\big(N\big(y_i(t) - y_j(t)\big)\big)\,dt + \frac{1}{N}\,d\beta_i(t).$$

We still use f_N^t to denote the density of $(y_1(t), \dots, y_N(t))$ with the same initial configuration $(a_1^N, \dots, a_N^N)$. Similar to Section 2, we will establish the following theorem.

THEOREM 3.2. *Assume that the initial configuration satisfies the same conditions given in Theorem* 3.1. *Then there exists* $c_1 = c_1(T)$ *such that, for* $T/2 \le t \le T$,

$$\int f_N^t \log f_N^t \, d\mu_N \le c_1 N$$

and, for all smooth functions J *on* S,

$$\lim_{N\to\infty} P\left\{\left|\frac{1}{N}\sum_{i=1}^{N} J(y_i(t)) - \int_S J(\theta)\rho_0(\theta)\,d\theta\right| \ge \delta\right\} = 0.$$

3.2. *Some results involving Gaussian densities.* In this section, we will give some results concerning Gaussian densities and related estimations.

LEMMA 3.1. *If* $W(x)$ *is a function that is supported on* $[-1/(2l), 1/(2l)]$ *and is bounded there by a constant* $\|W\|$, *then for any* $(x_1, \dots, x_N) \in S^N$,

$$\sum_{i,j}^{N} W(x_i - x_j) \le \|W\|\frac{4K}{\eta}\sum_{i,j}^{N} V\big(N(x_i - x_j)\big).$$

Here K *and* η *satisfy the following relations*: $\eta = \frac{1}{2}V(0)$ *and* $N/(2lK) = \delta$, *with* $\delta > 0$ *chosen so that* $V(x) \ge \eta$, *if* $|x| \le \delta$.

For the proof of this lemma, please refer to Varadhan [13].

For later convenience, let $\Psi_t(x) = 1/\sqrt{2\pi t}\sum_{k=-\infty}^{\infty}\exp(-(x-k)^2/2t)$, with $x \in S$, be the Gaussian density on the circle.

LEMMA 3.2. *For given positive constants* α_1 *and* $0 < a < b < 1$, *there is a universal positive constant* C, *depending only on* α_1, *and* $0 < a < b < 1$ *such that, for* $x \ge 0$,

$$x^2\exp(-\alpha_1 x^2) \le C\int_a^b [1 - \cos(xu)]\,du.$$

This can be proved by an elementary analysis.

LEMMA 3.3. *There exists a constant $c_T(V)$, depending on T and V, such that, for $0 < T/2 \le t \le T$ and any $(x_1, \dots, x_N) \in S^N$,*

$$\sum_{i,j}^{N} \Psi_{t/N^2}(x_i - x_j) \le c_T(V) N \sum_{i,j}^{N} V\big(N(x_i - x_j)\big).$$

PROOF. *Step 1.* At first we will find a nice U which is supported on $[-\frac{1}{2}, \frac{1}{2}]$ and $U \ge \eta > 0$ for $|x| \le \frac{1}{4}$ for this lemma to hold. Notice that both $\Psi(x)$ and $U(x)$ are functions on S. Then

$$\begin{aligned}
\Psi_{t/N^2}(x) &= \sum_{k=-\infty}^{\infty} \exp(2\pi i k x) \hat{\Psi}_N^t(k),\\
U(Nx) &= \sum_{k=-\infty}^{\infty} \exp(2\pi i k x) \hat{U}_N(k),\\
\sum_{i,j}^{N} \Psi_{t/N^2}(x_i - x_j) &= \sum_{k=-\infty}^{\infty} \left| \sum_{l=1}^{N} \exp(2\pi i x_l k) \right|^2 \hat{\Psi}_N^t(k),\\
\sum_{i,j}^{N} U\big(N(x_i - x_j)\big) &= \sum_{k=-\infty}^{\infty} \left| \sum_{l=1}^{N} \exp(2\pi i x_l k) \right|^2 \hat{U}_N(k).
\end{aligned}$$

Here $\hat{\Psi}_N^t(k)$ and $\hat{U}_N(k)$ are the Fourier coefficients of $\Psi_{t/N^2}(x)$ and $U(Nx)$.

We see that $\hat{\Psi}_N^t(k) = \exp(-2\pi^2 k^2 t/N^2)$ by a simple computation, so from the relations given above we only need to construct a function U which is even, nonnegative and such that

$$\hat{\Psi}_N^t(k) \le c_2 N \hat{U}_N(k) \tag{23}$$

for a certain constant $c_2 > 0$.

We construct U as follows:

$$U(x) = \int_{1/4}^{1/2} f_\alpha(x)\, d\alpha,$$

with

$$f_\alpha(x) = \begin{cases} 0, & x < -\alpha,\\ 1 + \dfrac{x}{\alpha}, & -\alpha \le x \le 0,\\ 1 - \dfrac{x}{\alpha}, & 0 \le x \le \alpha,\\ 0, & x > \alpha. \end{cases}$$

Then we can see U is positive, even with compact support. Additionally, $U(0) = \frac{1}{4}$, $U(x) = 0$ for $|x| \ge \frac{1}{2}$ and $U(x) \ge \eta > 0$ for $|x| \le \frac{1}{8}$ for proper η, and

$U(Nx)$ has Fourier coefficients as follows:

$$\begin{aligned}\hat{U}_N(k) &= \int_{-1/2}^{1/2} \exp(2\pi ikx) U(Nx)\,dx \\ &= \int_{1/4}^{1/2} d\alpha \int_{-1/2}^{1/2} \exp(2\pi ikx) f_\alpha(Nx)\,dx \\ &= \int_{1/4}^{1/2} d\alpha \left[\frac{2}{N} \int_0^\alpha \cos\left(2\pi \frac{k}{N} x\right)\left(1 - \frac{x}{\alpha}\right) dx \right] \\ &= \int_{1/4}^{1/2} d\alpha \frac{2N(1 - \cos(2\pi(k/N)(\alpha)))}{4\pi^2 k^2 \alpha},\end{aligned}$$

so for $0 < T/2 \le t \le T$, (23) is a consequence of Lemma 3.2 by the proper choice of c_2.

Step 2. Let W be the constructed function U in Step 1. By applying Lemma 3.1, we then get the desired inequality with a proper change of constant. □

3.3. *Entropy estimation.* For scaled interacting Brownian motions, with state space S, the dynamics of the circle of unit circumference, starting with deterministic initial conditions, can be described by the following SDE:

$$dx_i(t) = -N \sum_{j;\, j \ne i}^{N} V'\big(N(x_i(t) - x_j(t))\big)\,dt + d\beta_i(t), \qquad i = 1, 2, \dots, N,$$

$$\sum_{i=1}^{N} \delta_{x_i(0)} = \frac{1}{N!} \sum_{\pi \in \mathscr{S}_N} \left(\sum_{i=1}^{N} \delta_{a_{\pi(i)}} \right).$$

Similar to the Ginzburg–Landau model, we study behavior at a microscopic level, that is, the behavior of $\{y_i(t)\}_{i=1}^N = \{x_i(t/N^2)\}_{i=1}^N$, which can be described by the following dynamics equation:

$$dy_i(t) = -\frac{1}{N} \sum_{j;\, j \ne i} V'\big(N(y_i(t) - y_j(t))\big)\,dt + \frac{1}{N}\,d\beta_i(t), \qquad i = 1, 2, \dots, N,$$

with the same initial condition. We denote by $f^t_{\{a\},N}(y_1, y_2, \dots, y_N)$ the density function at time t with $f^0_{\{a\},N}$ the same as in (20). In this section, we show the following proposition.

PROPOSITION 3.1. *If the initial configuration satisfies the conditions as given in Theorem* 3.1, *then there exists a constant* $c_4 = c_4(T)$ *such that, for all* $T/2 \le t \le T$,

$$\int f^t_{\{a\},N} \log f^t_{\{a\},N}\, d\mu_N \le c_4 N.$$

This gives the estimation of entropy growth for interacting diffusions at a microscopic level. Before the proof, we need some notation. For any $\pi \in \mathscr{S}_N$

(the permutation group) we denote by $f^t_{\pi\vec{a},N}(y_1,\dots,y_N)$ the density function of $(y_1(t),\dots,y_N(t))$ with initial data $(y_1(0),\dots,y_N(0))=(a_{\pi(1)},\dots,a_{\pi(N)})$. Then $f^t_{\{a\},N}(y_1,\dots,y_N)=(1/N!)\Sigma_{\pi\in\mathscr{S}_N}f^t_{\pi\vec{a},t}(y_1,\dots,y_N)$. To apply our argument, we are going to choose a new process Q_N with potential $V=0$ and starting from the same initial data. Denote by $g^t_{\{a\},N}(y_1,\dots,y_N)$ the corresponding density function at time t, and also denote by $g^t_{\pi\vec{a},N}(y_1,\dots,y_N)$ the density function of $(y_1(t),\dots,y_N(t))$ with initial data $(y_1(0),\dots,y_N(0))=(a_{\pi(1)},\dots,a_{\pi(N)})$ corresponding to the dynamics with $V(\cdot)=0$. The following relation is obvious:

$$g^t_{\{a\},N}(y_1,\dots,y_N)=\frac{1}{N!}\sum_{\pi\in\mathscr{S}_N}g^t_{\pi a,N}(y_1,\dots,y_N).$$

In fact,

$$g^t_{\pi\vec{a},N}(y_1,\dots,y_N)=\sum_{\substack{i=1,\dots,N\\k_i=-\infty}}^{\infty}\frac{N^N}{\left(\sqrt{2\pi t}\right)^N}\exp\left(-\frac{N^2}{2t}\sum_{j=1}^{N}\left(y_j-a_{\pi(i)}-k_i\right)^2\right).$$

Then we have

$$\begin{aligned}\int f^t_{\{a\},N}\log f^t_{\{a\},N}\,d\mu_N&=E^{f^t_{\{a\},N}}\log\frac{f^t_{\{a\},N}}{g^t_{\{a\},N}}+E^{f^t_{\{a\},N}}\log\frac{g^t_{\{a\},N}}{\Psi_N}\\ &\le 2E^{f^t_{\{a\},N}}\log\frac{f^t_{\{a\},N}}{g^t_{\{a\},N}}+\log E^{g^t_{\{a\},N}}\left(g^t_{\{a\},N}\right)\\ &\quad-E^{f^t_{\{a\},N}}\log(\Psi_N)\end{aligned}\tag{24}$$

$$=(\mathrm{I})+(\mathrm{II})+(\mathrm{III}).\tag{25}$$

Here we have used the entropy inequality and the fact that Ψ_N is the density of μ_N with respect to Lebesgue measure on S^N.

For part (III), we have

$$\begin{aligned}E^{f^t_{\{a\},N}}\log(\Psi_N)^{-1}&=\log Z_N+E^{f^t_{\{a\},N}}\left[\sum_{i\neq j}^{N}V\left(N(y_i-y_j)\right)\right],\\ \log Z_N&=\log\left(\int_{S^N}\exp\left(-\sum_{i\neq j}^{N}V\left(N(x_i-x_j)\right)\right)dx_1\cdots x_N\right)\le 0.\end{aligned}\tag{26}$$

Proposition 3.1 is a consequence of the following lemmas.

LEMMA 3.4. *Assume the conditions in Theorem* 3.1. *There exists a constant* $c_5(T)>0$ *such that, for* $0<t\le T$,

$$E^{P_N}\left[\sum_{i\neq j}^{N}V\left(N\left(y_i(t)-y_j(t)\right)\right)\right]\le c_5(T)N,$$

$$E^{P_N}\int_0^t\sum_{i=1}^{N}\left[\sum_{j=1}^{N}V'\left(N\left(y_i(s)-y_j(s)\right)\right)\right]^2ds\le c_5(T)N.$$

PROOF. For the process P_N, we have

$$\begin{aligned}
d\sum_{i\neq j} V\big(N\big(y_i(t)-y_j(t)\big)\big)
&= -\sum_{i,j,k} V'\big(N\big(y_i(t)-y_j(t)\big)\big)\big[V'\big(N\big(y_i(t)-y_k(t)\big)\big) \\
&\qquad\qquad -V'\big(N\big(y_j(t)-y_k(t)\big)\big)\big]\,dt \\
&\quad+\sum_{i\neq j} V''\big(N\big(y_i(t)-y_j(t)\big)\big)\,dt + d\text{martingale}.
\end{aligned}$$

By assumption on potential V, we have

$$\begin{aligned}
d\sum_{i\neq j} V\big(N\big(y_i(t)-y_j(t)\big)\big)
&= -2\sum_{i,j,k} V'\big(N\big(y_i(t)-y_j(t)\big)\big)V'\big(N\big(y_i(t)-y_k(t)\big)\big)\,dt \\
&\quad+\sum_{i\neq j} V''\big(N\big(y_i(t)-y_j(t)\big)\big)\,dt + d\text{martingale} \\
&= -2\sum_{i=1}^{N}\left[\sum_{j=1}^{N} V'\big(N\big(y_i(t)-y_j(t)\big)\big)\right]^2 \\
&\quad+\sum_{i\neq j} V''\big(N\big(y_i(t)-y_j(t)\big)\big)\,dt + d\text{martingale}.
\end{aligned}$$

Taking $W(x) = V''(Nx)$ in Lemma 3.1, we have

$$\begin{aligned}
\sum_{i\neq j} V''\big(N\big(y_i(t)-y_j(t)\big)\big) &= \sum_{i,j} V''\big(N\big(y_i(t)-y_j(t)\big)\big) + N(-V'(0)) \\
&\leq c_6\sum_{i,j} V\big(N\big(y_i(t)-y_j(t)\big)\big) + N(-V''(0)) \\
&= c_6\sum_{i\neq j} V\big(N\big(y_i(t)-y_j(t)\big)\big) + N(c_6V(0)-V''(0)).
\end{aligned}$$

Using Gronwall's inequality,

$$\begin{aligned}
E^{P_N}&\left[\sum_{i\neq j} V\big(N\big(y_i(t)-y_j(t)\big)\big)\right] \\
&\leq \left[\sum_{i\neq j} V\big(N\big(a_i^N-a_j^N\big)\big) + N(c_6V(0)-V''(0))t\right](1+c_6te^{c_6t}),
\end{aligned}$$

so Lemma 3.4 follows by the assumption on the initial configurations. □

LEMMA 3.5. *Assume the conditions in Theorem* 3.1. *For all* $T > 0$, *there exists a constant* $c_7(T) > 0$ *such that, for all* $0 < T/2 < t \le T$,

$$\log E^{g^t_{\{a\},N}} g^t_{\{a\},N} \le c_7 N.$$

PROOF. By symmetry, we have

$$\begin{aligned} E^{g^t_{\{a\},N}} g^t_{\{a\},N} &= \frac{1}{N!} \sum_{\pi \in \mathscr{S}_N} E^{g^t_{\pi a,N}} g^t_{\{a\},N} \\ &= E^{g^t_{a,N}} g^t_{\{a\},N} \\ &= \frac{1}{N!} \sum_{\pi \in \mathscr{S}_N} E^{g^t_{a,N}} g^t_{\pi a,N} \end{aligned}$$

and

$$\begin{aligned} & E^{g^t_{a,N}} g^t_{\pi a,N} \\ &\quad = \int_{S^N} \left\{ \frac{N^N}{\left(\sqrt{2\pi t}\right)^N} \sum_{k^i_1 = -\infty}^{\infty} \exp\left(-\frac{N^2}{2t} \sum_{i=1}^{N} \left(x_i - a_{\pi(i)} - k^i_1\right)^2 \right) \right\} \\ &\qquad \times \left\{ \frac{N^N}{\left(\sqrt{2\pi t}\right)^N} \sum_{k^i_2 = -\infty}^{\infty} \exp\left(-\frac{N^2}{2t} \sum_{i=1}^{N} \left(x_i - a_{\pi(i)} - k^i_2\right)^2 \right) \right\} dx_1 \cdots dx_N \qquad (27) \\ &\quad = \int_{S^N} \left(\frac{N}{\sqrt{2\pi t}} \right)^{2N} \prod_{i=1}^{N} \left[\sum_{k^i_1 = -\infty}^{\infty} \exp\left(-\frac{N^2}{2t} \left(x_i - a_{\pi(i)} - k^i_1\right)^2 \right) \right] \\ &\qquad \times \left[\sum_{k^i_2 = -\infty}^{\infty} \exp\left(-\frac{N^2}{2t} \left(x_i - a_{\pi(i)} - k^i_2\right)^2 \right) \right] dx_1 \cdots dx_N, \end{aligned}$$

and for any $x \in S$ a simple calculation shows

$$\begin{aligned} & \left[\sum_{k^i_1 = -\infty}^{\infty} \exp\left(-\frac{\alpha}{4} \left(x - a - k^i_1\right)^2 \right) \right] \left[\sum_{k^i_2 = -\infty}^{\infty} \exp\left(-\frac{\alpha}{2} \left(x - a + b - k^i_2\right)^2 \right) \right] \\ &\qquad \le \left[\sum_{k^i_1 = -\infty}^{\infty} \exp\left(-\frac{\alpha}{4} \left(x - a - k^i_1\right)^2 \right) \right] \left[\sum_{k^i_2 = -\infty}^{\infty} \exp\left(-\frac{\alpha}{6} \left(b - k^i_2\right)^2 \right) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} E^{g^t_{a,N}} g^t_{\pi a,N} &= \int_{S^N} \left(\frac{N}{\sqrt{2\pi t}} \right)^{2N} \prod_{i=1}^{N} \left[\sum_{k^i_1 = -\infty}^{\infty} \exp\left(-\frac{N^2}{2t} \left(x_i - a_{\pi(i)} - k^i_1\right)^2 \right) \right] \\ &\qquad \times \left[\sum_{k^i_2 = -\infty}^{\infty} \exp\left(-\frac{N^2}{2t} \left(x_i - a_i - k^i_2\right)^2 \right) \right] dx_1 \cdots dx_N \end{aligned}$$

$$\le (\sqrt{6})^N \prod_{i=1}^{N} \left(\sum_{k_1^i=-\infty}^{\infty} \frac{N}{\sqrt{6\pi t}} \exp\left(-\frac{N^2}{6t} \left(a_i - a_{\pi(i)} - k_1^i \right)^2 \right) \right)$$

$$= (6)^{N/2} \prod_{i=1}^{N} \Psi_{3t/N^2}(a_i - a_{\pi(i)})$$

and, moreover,

$$\frac{1}{N!} \sum_{\pi \in \mathscr{S}_N} \prod_{i=1}^{N} \Psi_{3t/N^2}(a_i - a_{\pi(i)}) \le \frac{1}{N!} \prod_{i=1}^{N} \left(\sum_{j=1}^{N} \Psi_{3t/N^2}(a_i - a_j) \right)$$

$$\le \frac{1}{N!} \left[\frac{1}{N} \sum_{i,j=1}^{N} \Psi_{3t/N^2}(a_i - a_j) \right]^N.$$

Using Lemma 3.3, there exists a constant $c_T(V) > 0$ such that, for $T/2 \le t \le T$,

$$\sum_{i,j=1}^{N} \Psi_{3t/N^2}(a_i - a_j) \le c_T(V) N \sum_{i,j=1}^{N} V(N(a_i - a_j))$$

and, by our assumptions on initial configurations,

$$\left[\frac{1}{N} \sum_{i,j=1}^{N} \Psi_{3t/N^2}(a_i - a_j) \right]^N \le c_8^N N^N.$$

Therefore, for $T/2 \le t \le T$,

$$\frac{1}{N!} \sum_{\pi \in \mathscr{S}_N} \prod_{i=1}^{N} \Psi_{3t/N^2}(a_i - a_{\pi(i)}) \le \frac{1}{N!} c_8^N N^N$$

and

$$\log E^{g^t_{\{a\},N}} g^t_{\{a\},N} \le c_7 N \quad \text{for } 0 < T/2 \le t \le T$$

for some uniform constant c_7. □

LEMMA 3.6. *Assume the conditions in Theorem* 3.1. *Then there exists a constant* $c_9 = c_9(T) > 0$ *such that, for* $0 < t \le T$,

$$E^{f^t_{\{a\},N}} \log \frac{f^t_{\{a\},N}}{g^t_{\{a\},N}} \le c_9 N.$$

The proof follows the same way as in Section 2. We once again use the entropy property for diffusion processes, the Cameron–Martin formula and also the estimates in Lemma 3.4.

3.4. *Macroscopic behavior.*

PROPOSITION 3.2. *For any finite time $t > 0$ and smooth function J on S,*

$$\lim_{N\to\infty} P\left\{\left|\frac{1}{N}\sum_{i=1}^{N} J(y_i(t)) - \int_S J(\theta)\rho_0(\theta)\,d\theta\right| \geq \delta\right\} = 0.$$

The proof of this proposition is similar to the proof of Proposition 2.3. We use Itô's formula and certain estimates derived in Section 3.3.

Theorem 3.2 is just a combination of Propositions 3.1 and 3.2, and Theorem 3.1 is a consequence of Theorem 3.2 and [13].

3.5. *Remarks.* The condition on the initial configurations is still a requirement on the distribution of the initial configurations. Simply notice that if all configurations $(a_1^N, \dots, a_N^N)$ are concentrated around one point, $(1/N)\Sigma_{i,j}V(N(a_i^N - a_j^N))$ is almost N^2. After a finite microscopic time the particle can only move by distance $1/N$, so the entropy is still very large (about $N \log N$) and the desired bound cannot be attained. The natural condition on the initial configurations is therefore that the configurations should be properly distributed on the unit circle. This property is displayed in our condition (21). We have already seen that at a later time, this property is carried through in the evolution of the system; see Lemma 3.3. Recently, by using different estimations, Uchiyama [10] derived entropy bound similar to that in Theorem 2.2.

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