The Annals of Probability
1995, Vol. 23, No. 4, 1755–1771

LARGE DEVIATIONS FOR THE THREE-DIMENSIONAL SUPER-BROWNIAN MOTION

BY TZONG-YOW LEE[1] AND BRUNO REMILLARD[2]

University of Maryland and Université du Québec à Trois-Rivières

Let $\mu_t(dx)$ denote a three-dimensional super-Brownian motion with deterministic initial state $\mu_0(dx) = dx$, the Lebesgue measure. Let $V\colon \mathbb{R}^3 \mapsto \mathbb{R}$ be Hölder-continuous with compact support, not identically zero and such that $\int_{\mathbb{R}^3} V(x)\,dx = 0$. We show that

$$\log P\left\{\int_0^t \int_{\mathbb{R}^3} V(x)\mu_s(dx)\,ds > bt^{3/4}\right\}$$

is of order $t^{1/2}$ as $t \to \infty$, for $b > 0$. This should be compared with the known result for the case $\int_{\mathbb{R}^3} V(x)\,dx > 0$. In that case the normalization $bt^{3/4}$, $b > 0$, must be replaced by bt, $b > \int_{\mathbb{R}^3} V(x)\,dx$, in order that the same statement hold true. While this result only captures the logarithmic order, the method of proof enables us to obtain complete results for the corresponding moderate deviations and central limit theorems.

1. Introduction. We consider a measure-valued process known as the Dawson–Watanabe process or super-Brownian motion. Its sample paths $(\mu_t(dx),\ t \geq 0)$ are nonnegative Radon measures on $\mathbb{R}^d$. For $\mu_0(dx) = \sigma(dx)$, we denote by P_σ and E_σ the corresponding probability measure and expectation, respectively. We shall simply write P_x and E_x when the measure σ is the Dirac measure at x, and write P and E when σ is the Lebesgue measure. The process is uniquely characterized by the following Laplace functional of its transition function (see, e.g., [14, Theorem 1.1]):

$$E_\sigma\left\{\exp\left(-\int_{\mathbb{R}^d} \psi(x)\mu_t(dx)\right)\right\} = \exp\left(-\int_{\mathbb{R}^d} u(t,x)\sigma(dx)\right),$$

where ψ denotes a continuous, nonnegative function with compact support and u is the unique solution of

$$\frac{\partial u}{\partial t} = \Delta u - u^2, \qquad 0 < t,\ x \in \mathbb{R}^d,$$
$$u(0,x) = \psi(x), \qquad x \in \mathbb{R}^d.$$

Received April 1994; revised April 1995.
[1]Supported in part by NSF Grant DMS-92-07928.
[2]Supported in part by the Fonds Institutionnel de Recherche, Université du Québec à Trois-Rivières and by the Natural Sciences and Engineering Research Council of Canada, Grant OGP0042137.
AMS 1991 *subject classifications*. 60B12, 60F10, 60F05, 60J15.
Key words and phrases. Large deviations, super-Brownian motion.

For a construction of this process, see [14, Section 1]. Note that the use of the Laplacian Δ, as opposed to $\frac{1}{2}\Delta$, indicates that the underlying Brownian motion is being run at twice the standard speed.

The super-Brownian motion can be constructed (cf. [6]) as the weak limit of a system of many Brownian particles (with generator Δ) of small mass moving independently of each other and dying or duplicating with probability $\frac{1}{2}$, after each small fixed time interval. More precisely, if we have initially one particle of mass $\varepsilon \ll 1$ at each site of the lattice $\{\varepsilon^{1/d}x;\ x \in \mathbb{Z}^d\}$ and if each particle is dying or duplicating independently after time intervals of length ε, then the distribution converges to P as $\varepsilon \to 0$. For this reason, we classify the super-Brownian motion as a branching model throughout the Introduction. Note that the process can also be constructed without passage to the limit (cf. [9]).

Let V: $\mathbb{R}^d \mapsto \mathbb{R}$ be Hölder-continuous with compact support. If $V \geq 0$ and V is not identically equal to zero, it is known (cf. [15] and [19] for the super-Brownian motion and [5] for the critical branching Brownian motions) that

$$\lim_{t\to\infty} A_{t,d}^{-1} \log P\left\{\int_0^t \int_{\mathbb{R}^d} V(x)\,\mu_s(dx)\,ds > ct\right\}$$

exists and is strictly negative for c greater than and sufficiently close to $\int_{\mathbb{R}^d} V(x)\,dx$, where $A_{t,3} = t^{1/2}$, $A_{t,4} = t/\log t$ and $A_{t,d} = t$ for $d \geq 5$. The corresponding complete large-deviation principles are believed to hold true. However, this is only proved in the case $d = 3$ in [15]. Similar problems have been studied for systems of independent random walks and Brownian motions (e.g., [4], [8], [17] and [18]), with large-deviation principles proved for all dimensions. Much less is known for models of interacting particles; see [2] for the voter model and [16] for the simple exclusion random walks.

Interestingly, all the aforementioned models have in common a property of dimensional dependence as follows. The logarithm of probabilities, given by

$$\log P\left\{\int_0^t \int_{\mathbb{R}^d} V(x)\,\mu_s(dx)\,ds > ct\right\},$$

$V \geq 0$, $V \neq 0$ and $c > \int_{\mathbb{R}^d} V(x)\,dx$, has order $t^{1/2}$ for $d = k+1$, order $t/\log t$ for $d = k+2$ and order t for $d \geq k+3$, as $t \to \infty$. Here the number k depends on the specific model and $\int_{\mathbb{R}^d}$ should be replaced by $\sum_{x \in \mathbb{Z}^d}$ in discrete-space models. For example, we have $k = 0$ for systems of independent random walks (or Brownian motions) and for the simple exclusion process; $k = 2$ for the critical branching Brownian motions (or the super-Brownian motions) and for the voter models.

When the function V satisfies $\int_{\mathbb{R}^d} V(x)\,dx = 0$, only the models of independent nonbranching particles have been studied in [3], [22] and [20]. It is known for the model of independent random walks on $\mathbb{Z}^d$, $d = 1, 2$, that the large-deviation behaviors are different from the case of nonzero $\sum_{x \in \mathbb{Z}^d} V(x)$.

Our main theorem in this article is a corresponding result for the three-dimensional super-Brownian motion. The corresponding moderate deviations and central limit theorem are also obtained. Define

$$(\Delta^{-1}V)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}|x-y|^{-1}V(y)\,dy$$
$$= -\int_0^\infty\int_{\mathbb{R}^3}p(t,x-y)V(y)\,dy\,dt,$$

where $p(t,y)=(4\pi t)^{-3/2}\exp(-|y|^2/4t)$.

MAIN THEOREM. *Let B_0 be the set of all Hölder-continuous functions V from $\mathbb{R}^3$ to $\mathbb{R}$ with compact support, $\int_{\mathbb{R}^3}V(x)\,dx=0$, and let B be the set of all $V\in B_0$ such that*

$$\int_{\mathbb{R}^3}(\Delta^{-1}V)^2(x)\,dx = 1.$$

Then, for all $V\in B_0$, there exists $\lambda>0$ such that $\lambda V\in B$. Moreover, for all $V\in B$, the following properties hold true:

(i) *There exists $\alpha>0$ and two positive functions c_1 and c_2 on $(0,\alpha)$ such that, for all $a\in(0,\alpha)$,*

$$-c_1(a) \le \liminf_{t\to\infty} t^{-1/2}\log P\left\{t^{-3/4}\int_0^t\int_{\mathbb{R}^3}V(x)\mu_s(dx)\,ds > a\right\}$$
$$\le \limsup_{t\to\infty} t^{-1/2}\log P\left\{t^{-3/4}\int_0^t\int_{\mathbb{R}^3}V(x)\mu_s(dx)\,ds > a\right\} \le -c_2(a).$$

(ii) *If $0<\delta<\frac{1}{2}$ and $b\ge 0$, then*

$$\lim_{t\to\infty} t^{\delta-1/2}\log P\left\{t^{(\delta/2)-3/4}\int_0^t\int_{\mathbb{R}^3}V(x)\mu_s(dx)\,ds > b\right\} = -\frac{b^2}{4}.$$

(iii) *For any $c\in\mathbb{R}$,*

$$\lim_{t\to\infty}\log E\left\{\exp\left(ct^{-1/2}\int_0^t\int_{\mathbb{R}^3}V(x)\mu_s(dx)\,ds\right)\right\} = c^2,$$

which implies that $(1/\sqrt{2t})\int_0^t\int_{\mathbb{R}^3}V(x)\mu_s(dx)\,ds$ converges in law to a standard normal distribution.

REMARK 1. Applying this theorem to $-V$, we see that statements similar to (i) and (ii) hold for below 0 as well as above 0.

One should compare this result with the corresponding known result for one-dimensional random walks (see [3, Theorems 2 and 3] and also [22, (3.4) and (3.5)]). The comparison reveals the same normalizing function $t^{3/4}$. For more complex models, one often makes predictions based on known counterpart results for simpler models. Our main theorem is one more instance when such a prediction turns out accurate. By plausible reasoning, we think that the main theorem also holds for the three-dimensional voter model and one-dimensional simple exclusion process. It would be interesting to see this worked out.

A crucial technique, commonly used in [3] and [20], is to stop the particles when they first enter the support of the function V. We do not know how to modify that method to prove the main theorem. Our proof method substantially uses the analytic technique of PDE's and can be modified to prove the counterpart results in [3] and [20].

The remainder of this article contains nine lemmas, from which the main theorem follows.

2. Auxiliary results and the proof of the main theorem. Let $\mathscr{M}$ be the set of Radon measures on $\mathbb{R}^3$. A super-Brownian motion $\mu_t(dx)$, with initial $\mu_0(dx) = \delta_x$, the Dirac measure at x, can be looked at as an $\mathscr{M}$-valued diffusion process with a linear drift and a linear diffusivity. More precisely, we mean the following lemma (cf. [23, Theorems 1.3 and 1.6]).

Before stating the next lemma, define

$$\langle g, \nu \rangle = \int_{\mathbb{R}^3} g(x)\nu(dx),$$

for Radon measures ν and continuous functions g.

LEMMA 1. *Let h be in the domain of the Laplacian Δ. Let*

$$M_t = \langle h, \mu_t - \mu_0 \rangle - \int_0^t \langle \Delta h, \mu_s \rangle \, ds,$$

$$[M_t] = 2\int_0^t \langle h^2, \mu_s \rangle \, ds$$

and

$$A_t(\gamma) = \exp\left[\gamma M_t - \gamma^2 \int_0^t \langle h^2, \mu_s \rangle \, ds\right].$$

Then M_t is a P_x-martingale with increasing process $[M_t]$. Moreover $A_t(\gamma)$, $\gamma \in \mathbb{R}$, are P_x-local martingales, and they are P_x-martingales for $t \le T$, provided that $E_x\{\exp \gamma^2 [M_T]/2\} < \infty$.

PROOF. Theorem 1.6 in [23], together with Theorem 1.3(i) in [23], yields that M_t is a P_x-martingale with increasing process $2\int_0^t \langle h^2, \mu_s \rangle \, ds$. Moreover, $t \mapsto M_t$ is a.s. continuous. The exponential martingale result follows from [13, Theorems III-5.2 and III-5.3]. □

A basic analytic technique used in this paper is a comparison principle for semilinear parabolic differential equations.

LEMMA 2. *Suppose that $f \in C^1(\mathbb{R})$ and*

$$\bar{u}, \underline{u} \in C^{1,2}((0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d).$$

Suppose also that $\bar{u}$ and $\underline{u}$ are bounded in $[0, T'] \times \mathbb{R}^d$ for all $T' < T$. If $\bar{u}(0, x) \geq \underline{u}(0, x)$ for all $x \in \mathbb{R}^d$ and

$$(2.1)\qquad \begin{aligned} &\frac{\partial \bar{u}}{\partial t} - \Delta \bar{u} \geq f(\bar{u}), \qquad (t, x) \in (0, T) \times \mathbb{R}^d, \\ &\frac{\partial \underline{u}}{\partial t} - \Delta \underline{u} \leq f(\underline{u}), \qquad (t, x) \in (0, T) \times \mathbb{R}^d, \end{aligned}$$

then $\bar{u}(t, x) \geq \underline{u}(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Furthermore, for a continuous function $\phi(x)$ satisfying

$$\bar{u}(0, x) \geq \phi(x) \geq \underline{u}(0, x), \qquad x \in \mathbb{R}^d,$$

there exists a unique solution $u(t, x) \in C^{1,2}((0, T) \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d)$ of the following problem:

$$(2.2)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + f(u), \qquad (t, x) \in (0, T) \times \mathbb{R}^d, \\ &u(0, x) = \phi(x), \qquad x \in \mathbb{R}^d, \end{aligned}$$

where u is assumed to be bounded in $[0, T'] \times \mathbb{R}^d$ for all $T' < T$. This unique solution u has the additional property that $\bar{u} \geq u \geq \underline{u}$ in $[0, T] \times \mathbb{R}^d$.

REMARK 2. Lemma 2 is well known (cf. [1] for the first half of the lemma). Interested readers are referred to [25], in which Lemma 2 is proved as a special case by using a maximum principle (cf. [11, Theorem 9]) and a monotone iteration method (cf. [24, Theorem 3.1]).

Let $u(t, x; h)$ denote the solution of

$$(2.3)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + u^2, \qquad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\ &u(0, x) = h(x), \qquad x \in \mathbb{R}^3, \\ &\lim_{|x| \to \infty} \sup_{0 \leq t \leq t_0} |u(t, x)| = 0, \qquad \text{for all } t_0 \geq 0. \end{aligned}$$

It follows from Lemma 2 that the solution is unique if it exists. A special result of Haraux and Weissler (cf. [12, Theorem 5(b)]) is important in our approach. It implies the following lemma.

LEMMA 3 (Haraux and Weissler). *There exists a positive radial function F such that the following hold:*

(i) $\lim_{|x| \to \infty} |x|^2 F(x) = L > 0$;
(ii) $u(t, x; F) = (1 + t)^{-1} F((1 + t)^{-1/2} x)$, $(t, x) \in [0, \infty) \times \mathbb{R}^3$.

Let h_T be the function defined by $h_T(x) = T^{-1/4} h(x)$, $x \in \mathbb{R}^3$. An application of Lemmas 2 and 3 yields the following bound on $u(t, x; h_T)$.

LEMMA 4. *Suppose h is continuous and satisfies the following condition:*

$$K = \sup_{x \in \mathbb{R}^3} \frac{|h(x)|}{F(x)} < \infty.$$

Then, for $T \geq T_0 = K^4$, the solution $u(t, x; h_T)$ of (2.3) *exists and*

$$|u(t, x; h_T)| \leq \left(\frac{T_0}{T}\right)^{1/4} (1+t)^{-1} F\big((1+t)^{-1/2} x\big).$$

PROOF. Suppose that $T_0 = 1$. Consider $\overline{w}(t, x; T^{-1/4})$ and $\underline{w}(t, x; T^{-1/4})$, the solutions of the equations

$$\frac{\partial \overline{w}}{\partial t} = \Delta \overline{w} + T^{1/4} \overline{w}^2, \qquad \overline{w}(0, x) = T^{-1/4} F(x), \qquad (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

$$\frac{\partial \underline{w}}{\partial t} = \Delta \underline{w} - T^{1/4} \underline{w}^2, \qquad \underline{w}(0, x) = -T^{-1/4} F(x), \qquad (t, x) \in (0, \infty) \times \mathbb{R}^d.$$

Note that $\overline{w}(0, x; T^{-1/4}) \geq u(0, x; h_T) \geq \underline{w}(0, x; T^{-1/4})$ and that the three functions $\overline{w} \mapsto T^{1/4} \overline{w}^2$, $u \mapsto u^2$ and $\underline{w} \mapsto -T^{1/4} \underline{w}^2$ are also in decreasing relation. Since $T \geq T_0 = 1$, Lemma 2 then implies that

$$\overline{w}(t, x; T^{-1/4}) \geq u(t, x; h_T) \geq \underline{w}(t, x; T^{-1/4}).$$

Simple computations together with Lemma 3 yield

$$\overline{w}(t, x; T^{-1/4}) = T^{-1/4} (1+t)^{-1} F\big((1+t)^{-1/2} x\big)$$

and

$$\underline{w}(t, x; T^{-1/4}) = -T^{-1/4} (1+t)^{-1} F\big((1+t)^{-1/2} x\big).$$

The proof is now completed for the case $T_0 = 1$. For arbitrary T_0, simply replace all the T's by T/T_0. □

Both the probabilistic tool (Lemma 1) and the analytic tools (Lemmas 2 and 3) are useful in our approach. This will become clear in view of the next lemma, which relates the cumulant generating functions to the solutions of (2.3).

LEMMA 5. *Let h be as in Lemma* 4 *and let $u(t, x; h)$ exist for all $t > 0$. Then, for all $(t, x) \in [0, \infty) \times \mathbb{R}^3$,*

$$\log E_x\{\exp\langle h, \mu_t \rangle\} = u(t, x; h).$$

PROOF. Use the identity in the first paragraph of the Introduction and analytic continuation (cf. [15, Lemma 1.7]). □

The next lemma concerns the limiting behavior of the solutions of (2.3). Taking Lemma 5 into consideration, it is also a result for the cumulant generating functions.

LEMMA 6. *Let* $\beta \in \mathbb{R}$, $V \in B_0$ *and* $h = \beta(-\Delta)^{-1}V$. *Then* $h \in L^2$ *and* (i) *and* (ii) *hold*:

(i)
$$\limsup_{T\to\infty}\left|\int_{\mathbb{R}^3}(u(T,x;h_T) - h_T(x))\,dx\right| < \infty;$$

(ii) *If* $a_T \to 0$ *as* $T \to \infty$, *then*
$$\limsup_{T\to\infty}\left|\int_{\mathbb{R}^3}(u(T,x;a_T h_T) - a_T h_T(x))\,dx\right| = 0.$$

PROOF. We first verify that the supremum K in Lemma 4 is finite for h. If $x, y \in \mathbb{R}^3$, $|y| \le c$ and $|x| \ge 2c$, then

$$\begin{aligned}\left||x-y|^{-1} - |x|^{-1}\right| &= \frac{||x| - |x-y||}{|x|\,|x-y|}\\ &= \frac{\left|2x\cdot y - |y|^2\right|}{|x|\,|x-y|(|x| + |x-y|)}\\ &\le \frac{2c|x| + c^2}{|x|^2(|x| - c)}\\ &\le 6c|x|^{-2},\end{aligned}$$

where "·" stands for the inner product. Let the support of V be contained in the ball of radius c centered at the origin. Now, for any $x \ne 0$,

$$\begin{aligned}-h(x) = \Delta^{-1}V(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}|x-y|^{-1}V(y)\,dy\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(|x-y|^{-1} - |x|^{-1}\right)V(y)\,dy.\end{aligned}$$

Combining the last expressions, we finally get

$$|h(x)| \le \frac{6c}{4\pi|x|^2}\int_{\mathbb{R}^3}|V(y)|\,dy, \qquad |x| \ge 2c.$$

This upper bound, together with the fact that h is continuous, ensures that K is finite. Moreover, it also follows that $h \in L^2$.

Let $\bar{u}(t,x;T)$ and $\underline{u}(t,x;T)$ be the solutions of the equations

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} &= \Delta\bar{u} + g_T, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^3,\\ \bar{u}(0,x) &= a_T h_T(x), \qquad x \in \mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \underline{u}}{\partial t} &= \Delta\underline{u}, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^3,\\ \underline{u}(0,x) &= a_T h_T(x), \qquad x \in \mathbb{R}^3,\end{aligned}$$

where a_T is a constant and

$$g_T(t,x) = a_T^2\left(\left(\frac{T_0}{T}\right)^{-1/4}(1+t)^{-1}F\big((1+t)^{-1/2}x\big)\right)^2.$$

It follows from Lemma 4 and from the maximum principle for parabolic equations that

$$\underline{u}(t,x;T) \le u(t,x;a_T h_T) \le \bar{u}(t,x;T),$$

for $T \ge a_T^4 T_0$ and $T \ge t \ge 0$, $x \in \mathbb{R}^3$. From the relation between h and V in the assumption, it is easy to see that

$$\underline{u}(t,x;T) - a_T h_T(x) = -T^{-1/4}a_T\beta\int_0^t\int_{\mathbb{R}^3} p(s,x-y)V(y)\,dy\,ds.$$

Hence $\underline{u} - a_T h_T \in L^1(dx)$ and $\int_{\mathbb{R}^3}(\underline{u}(t,x;T) - a_T h_T(x))\,dx = 0$. Moreover, $v = \bar{u} - \underline{u} \ge 0$, and v satisfies the equations

$$\frac{\partial v}{\partial t} = \Delta v + g_T, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^3,$$
$$v(0,x) = 0, \qquad x \in \mathbb{R}^3.$$

So v has the following representation:

$$v(t,x;T) = \int_0^t\int_{\mathbb{R}^3} p(t-s,x-y)g_T(s,y)\,dy\,ds.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^3} v(T,x;T)\,dx &= \int_0^T\int_{\mathbb{R}^3} g_T(s,y)\,dy\,ds\\ &= a_T^2\left(\frac{T_0}{T}\right)^{1/2}\int_0^T\int_{\mathbb{R}^3}(1+s)^{-2}F^2\big((1+s)^{-1/2}y\big)\,dy\,ds\\ &= a_T^2\left(\frac{T_0}{T}\right)^{1/2}\int_0^T\int_{\mathbb{R}^3}(1+s)^{-1/2}F^2(z)\,dz\,ds\\ &= 2a_T^2\left(\frac{T_0}{T}\right)^{1/2}\big((1+T)^{1/2}-1\big)\int_{\mathbb{R}^3}F^2(z)\,dz.\end{aligned}$$

Since $\underline{u} - a_T h_T \le u - a_T h_T \le v + \underline{u} - a_T h_T$, we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^3}\big(\underline{u}(T,x;T) - a_T h_T(x)\big)\,dx\\ &\le \int_{\mathbb{R}^3}\big(u(T,x;a_T h_T) - a_T h_T(x)\big)\,dx\\ &\le \int_{\mathbb{R}^3} v(T,x;T)\,dx + \int_{\mathbb{R}^3}\big(\underline{u}(T,x;T) - a_T h_T(x)\big)\,dx\\ &= 2a_T^2\left(\frac{T_0}{T}\right)^{1/2}\big((1+T)^{1/2}-1\big)\int_{\mathbb{R}^3}F^2(z)\,dz.\end{aligned}$$

Recall that $|x|^2 F(x)$ is bounded, so $F \in L^2(dx)$. Therefore, to finish the proof of (i) [resp., (ii)], we just have to let $T \to \infty$, observing that if $a_T = 1$ (resp., $a_T \to 0$), then the right-hand side of the last inequality is bounded (resp., goes to 0). □

Let $v(t, x; \theta\delta_0)$ be the mild solution of

$$\frac{\partial v}{\partial t} = \Delta v + v^2 + \theta\delta_0, \qquad (t, x) \in (0, 1] \times \mathbb{R}^3,$$
$$v(0, x) = 0, \qquad x \in \mathbb{R}^3,$$

and let

$$\Lambda(\theta) = \begin{cases} \int_{\mathbb{R}^3} v(1, x; \theta\delta_0)\, dx, & \text{if } v(1, x; \theta\delta_0) \text{ exists}, \\ +\infty, & \text{otherwise}. \end{cases}$$

Further let

$$A = \left\{ \phi: \mathbb{R}^3 \mapsto [0, \infty),\ \phi \text{ is nonnegative, Hölder-continuous with compact support and } \int_{\mathbb{R}^3} \phi(x)\, dx = 1 \right\}.$$

It was proved in [15, Lemma 1.7, (0.4) and (0.5)] that $v(1, x; \theta\delta_0)$ exists when θ is less than a certain positive number θ_0 and that it does not exist when θ is greater than θ_0. It was also proved that the function Λ is smooth, $\Lambda(0) = 0$, $\Lambda'(0) = 1$, $\Lambda'(\theta) > 0$ for $\theta < \theta_0$ and

$$\text{(2.4)} \qquad \lim_{T \to \infty} T^{-1/2} \log E\left\{ \exp\left(\theta T^{-1/2} \int_0^T \langle \phi, \mu_s \rangle\, ds \right) \right\} = \Lambda(\theta),$$

for $\theta < \theta_0$ and $\phi \in A$.

REMARK 3. One can prove that (2.4) also holds if we replace the condition "ϕ has compact support" by the weaker condition "$\phi \in L^1$."

LEMMA 7. *Suppose* $V \in B$ *and* $\theta \in \mathbb{R}$. *Then* (i), (ii) *and* (iii) *hold*:

(i) $$\limsup_{T \to \infty} T^{-1/2} \log E\left\{ \exp\left(\theta T^{-1/4} \int_0^T \langle V, \mu_s \rangle\, ds \right) \right\} \le \tfrac{1}{2}\Lambda(4\theta^2 +);$$

(ii) *for any* $0 < \delta < \frac{1}{2}$,

$$\limsup_{T \to \infty} T^{\delta - 1/2} \log E\left\{ \exp\left(\theta T^{-\delta/2 - 1/4} \int_0^T \langle V, \mu_s \rangle\, ds \right) \right\} \le \theta^2;$$

(iii) $$\limsup_{T \to \infty} \log E\left\{ \exp\left(\theta T^{-1/2} \int_0^T \langle V, \mu_s \rangle\, ds \right) \right\} \le \theta^2.$$

PROOF. It follows from Lemma 1 that

$$\exp\left(\int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right) = \exp(M_T + \langle -h, \mu_T - \mu_0\rangle)$$
$$= \exp(\langle -h, \mu_T - \mu_0\rangle)\exp\left(\gamma\int_0^T \langle h^2, \mu_s\rangle\, ds\right) A_T(\gamma)^{1/\gamma}.$$

Let α, β and γ be positive numbers such that $\beta \geq \gamma$ and $1/\alpha + 1/\beta + 1/\gamma = 1$. The Hölder inequality and Lemma 1 imply that

$$E_x\left\{\exp\int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right\}$$
$$\leq \left[E_x\{\exp\langle -\alpha h, \mu_T - \mu_0\rangle\}\right]^{1/\alpha}\left[E_x\left\{\exp\left(\gamma\beta\int_0^T \langle h^2, \mu_s\rangle\, ds\right)\right\}\right]^{1/\beta}$$
$$\times\left[E_x\{A_T(\gamma)\}\right]^{1/\gamma}.$$

By Lemma 1, the third factor on the right-hand side equals 1 as long as the second factor is finite. If the second factor is infinite, then the inequality is trivial. So we can write

$$\begin{aligned} &E_x\left\{\exp\int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right\} \\ &\quad\leq \left[E_x\{\exp\langle -\alpha h, \mu_T - \mu_0\rangle\}\right]^{1/\alpha}\left[E_x\left\{\exp\left(\gamma\beta\int_0^T \langle h^2, \mu_s\rangle\, ds\right)\right\}\right]^{1/\beta}. \end{aligned} \tag{2.5}$$

Taking logarithms on both sides of (2.5) and integrating with respect to the Lebesgue measure, we get

$$\begin{aligned} \log E\left\{\exp\int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right\} &= \int_{\mathbb{R}^3} \log E_x\left\{\exp\int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right\} dx \\ &\leq \frac{1}{\alpha}\int_{\mathbb{R}^3} \log E_x\{\exp\langle -\alpha h, \mu_T - \mu_0\rangle\}\, dx \\ &\quad + \frac{1}{\beta}\int_{\mathbb{R}^3} \log E_x\left\{\exp\left(\beta\gamma\int_0^T \langle h^2, \mu_s\rangle\, ds\right)\right\} dx \\ &= \frac{1}{\alpha}\int_{\mathbb{R}^3} (u(T, x; -\alpha h) + \alpha h(x))\, dx \\ &\quad + \frac{1}{\beta}\log E\left\{\exp\left(\beta\gamma\int_0^T \langle h^2, \mu_s\rangle\, ds\right)\right\}, \end{aligned} \tag{2.6}$$

where the last step uses Lemma 5.

Let $h = \theta T^{-(\delta/2+1/4)}(-\Delta)^{-1}V$, so $-\Delta h = \theta T^{-(\delta/2+1/4)}V$; also let $\phi = (\Delta^{-1}V)^2$. It now follows from Lemma 6(i) for the case $\delta = 0$ and from Lemma 6(ii) for the case $0 < \delta \le \frac{1}{2}$ that, for any $0 \le \delta \le \frac{1}{2}$,

$$\begin{aligned}(2.7)\qquad &\limsup_{T\to\infty} T^{\delta-1/2} \log E\left\{\exp\left(\theta T^{-(\delta/2+1/4)}\int_0^T \langle V, \mu_s\rangle\, ds\right)\right\}\\ &\quad\le \limsup_{T\to\infty} \frac{1}{\beta} T^{\delta-1/2} \log E\left\{\exp\left(\beta\gamma\theta^2 T^{-(\delta+1/2)}\int_0^T \langle \phi, \mu_s\rangle\, ds\right)\right\}.\end{aligned}$$

To prove (i), set $\delta = 0$ in (2.7). By using (2.4) we obtain that the right-hand side of (2.7) is equal to $(1/\beta)\Lambda(\gamma\beta\theta^2)$. Letting $\alpha \to \infty$ and then $\beta = \gamma \to 2$ from above, we obtain (i), since $\beta\gamma > 4$ whenever $\alpha < \infty$.

Next suppose that $0 < \delta \le \frac{1}{2}$. Set

$$\Lambda_T(a) = T^{-1/2} \log E\left\{\exp\left(aT^{-1/2}\int_0^T \langle \phi, \mu_s\rangle\, ds\right)\right\}.$$

We know that Λ_T is convex, $\Lambda_T(a)$ is finite for small positive a, $\Lambda_T(0) = 0$, $\Lambda'(0) = 1$ and $\Lambda_T \to \Lambda$ as $T \to \infty$. Moreover, there exists $a_0 > 0$ such that $\Lambda_T(a) < \infty$ for $0 \le a < a_0$. It follows that $\Lambda_T(a)/a$ is nondecreasing for $a_0 > a > 0$.

Now, for arbitrary $\varepsilon > 0$, we have $T^{-\delta} < \varepsilon$, if T is large enough. Thus

$$\limsup_{T\to\infty} T^{-\delta}\Lambda_T(\beta\gamma\theta^2 T^{-\delta}) \le \limsup_{T\to\infty} \frac{1}{\varepsilon}\Lambda_T(\varepsilon\beta\gamma\theta^2) = \frac{1}{\varepsilon}\Lambda(\beta\varepsilon\gamma\theta^2).$$

Therefore, the last argument, combined with (2.7), proves that

$$\limsup_{T\to\infty} T^{\delta-1/2} \log E\left\{\exp\left(\theta T^{-(\delta/2+1/4)}\int_0^T \langle V, \mu_s\rangle\, ds\right)\right\} \le \frac{1}{\varepsilon\beta}\Lambda(\beta\varepsilon\gamma\theta^2).$$

The proof of (ii) and (iii) is completed by letting ε go to 0 and by letting γ go to 1 in the last expression. □

LEMMA 8. *Suppose $V \in B$ and $\theta \in \mathbb{R}$. Then the following hold*:

(i)

$$\liminf_{T\to\infty} T^{-1/2} \log E\left\{\exp\left(\theta T^{-1/4}\int_0^T \langle V, \mu_s\rangle\, ds\right)\right\} \ge -\Lambda\left(\frac{-\theta^2}{2}\right),$$

for all $|\theta| \le 2\sqrt{\theta_0}$;

(ii) *for any* $0 < \delta < \frac{1}{2}$,

$$\liminf_{T\to\infty} T^{\delta-1/2} \log E\left\{\exp\left(\theta T^{-(\delta/2+1/4)}\int_0^T \langle V, \mu_s\rangle\, ds\right)\right\} \ge \theta^2;$$

$$\text{(iii)}\qquad \liminf_{T\to\infty} \log E\left\{\exp\left(\theta T^{-1/2}\int_0^T \langle V, \mu_s\rangle\, ds\right)\right\} \ge \theta^2.$$

PROOF. Let $\alpha, \beta, \gamma > 1$ and $1/\alpha + 1/\beta + 1/\gamma = 1$.

By Lemma 1, $A_{t\wedge T}(1/\gamma)$ is a P_x-martingale for all $t \geq 0$, and $E_x\{A_T(1/\gamma)\} = 1$, as long as

$$E_x\left\{\exp \int_0^T \frac{1}{\gamma^2}\langle h^2, \mu_s\rangle\, ds\right\}$$

is finite.

Since

$$A_T\left(\frac{1}{\gamma}\right) = \exp\left\{\left\langle \frac{h}{\gamma}, \mu_T - \mu_0\right\rangle\right\}\exp\left\{\int_0^T \left\langle \frac{-h^2}{\gamma^2}, \mu_s\right\rangle ds\right\} \times \exp\left\{\int_0^T \left\langle \frac{-\Delta h}{\gamma}, \mu_s\right\rangle ds\right\},$$

Hölder's inequality yields

$$1 = E_x\left\{A_T\left(\frac{1}{\gamma}\right)\right\}$$

$$\leq E_x\left\{\exp\left\langle \frac{\alpha h}{\gamma}, \mu_T - \mu_0\right\rangle\right\}^{1/\alpha} E_x\left\{\exp \int_0^T \left\langle \frac{-\beta h^2}{\gamma^2}, \mu_s\right\rangle ds\right\}^{1/\beta} \times E_x\left\{\exp \int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right\}^{1/\gamma}. \tag{2.8}$$

Taking logarithms in (2.8) and rearranging terms then give

$$\log E_x\left\{\exp \int_0^T \langle -\Delta h, \mu_s\rangle\, ds\right\} \geq -\frac{\gamma}{\alpha}\log E_x\left\{\exp\left\langle \frac{\alpha h}{\gamma}, \mu_T - \mu_0\right\rangle\right\} - \frac{\gamma}{\beta}\log E_x\left\{\exp \int_0^T\left\langle \frac{-\beta h^2}{\gamma^2}, \mu_s\right\rangle ds\right\}. \tag{2.9}$$

Recall that $\phi = (\Delta^{-1}V)^2$ is such that $\int_{\mathbb{R}^3}\phi(x)\,dx = 1$. Now using the same h as in the proof of Lemma 7, that is, $h = \theta T^{-(\delta/2+1/4)}(-\Delta)^{-1}V$, and integrating (2.9) with respect to the Lebesgue measure, we get

$$\liminf_{T\to\infty} T^{\delta-1/2}\log E\left\{\exp\left(\theta T^{-(\delta/2+1/4)}\int_0^T\langle V, \mu_s\rangle\, ds\right)\right\} \geq -\frac{\gamma}{\beta}\limsup_{T\to\infty} T^{\delta-1/2}\log E\left\{\exp\int_0^T\left\langle \frac{-\beta\theta^2 T^{-(\delta+1/2)}\phi}{\gamma^2}, \mu_s\right\rangle ds\right\}, \tag{2.10}$$

for any $0 \leq \delta \leq \frac{1}{2}$, using Lemmas 5 and 6.

To complete the proof of (i), we take $\delta = 0$ in (2.10) and use (2.4). Taking $\beta = \gamma = 2\alpha/(\alpha - 1) > 2$, we see that

$$E_x\left\{\exp \theta^2\gamma^{-2}T^{-1/2}\int_0^T\langle \phi, \mu_s\rangle\, ds\right\}$$

is finite whenever $|\theta| < 2\sqrt{\theta_0}$. The proof is completed by letting α tend to infinity.

To complete the proof of (ii), we just use the fact that $\Lambda_T(-a)/a$ is nondecreasing for $a > 0$. The rest of the proof is similar to the one for the (ii) and (iii) in the previous lemma. □

Had the upper bound in Lemma 7 and the lower bound in Lemma 8 agreed, the Gärtner–Ellis theorem (cf. [10]) would have implied a large-deviation principle. Thanks to the following extension of a lemma of Cox and Griffeath (cf. [5, Lemma 7, pages 1130–1131]), Lemmas 7 and 8 do guarantee that the logarithmic order of decay of the probabilities is as asserted in our main theorem.

LEMMA 9. *Let $\{Y_t,\ t > 0\}$ be a sequence of random variables, and let a_t be a normalizing sequence increasing to infinity, with*

$$\psi_t(\lambda) = a_t^{-1} \log E\{\exp \lambda Y_t\}.$$

Let $\bar{\psi}$ and $\underline{\psi}$ be two functions such that, for some $0 < \lambda_0 \le \infty$, we have the following:

(i) *$\bar{\psi}$ and $\underline{\psi}$ are convex on $[0, \lambda_0)$, and $\underline{\psi}(\lambda)/\lambda$ is not constant on $(0, \lambda)$;*

(ii) *$\bar{\psi}(0) = \underline{\psi}(0) = 0$, and $D^+\bar{\psi}(0) = D^+\underline{\psi}(0) = \mu$, where D^+f is the right derivative of f;*

(iii) *$\underline{\psi} \le \liminf_{t\to\infty} \psi_t \le \limsup_{t\to\infty} \psi_t \le \bar{\psi}$, on $[0, \lambda_0)$.*

Set $\bar{c}(\alpha) = \sup_{\lambda\in[0,\lambda_0)} \alpha\lambda - \bar{\psi}(\lambda)$. Then there exist $\bar{\alpha} > \mu$ and a function $\underline{c}$ such that, for all $\alpha \in (\mu, \bar{\alpha})$, $\underline{c}(\alpha) > 0$, $\bar{c}(\alpha) > 0$ and

$$\begin{aligned} -\underline{c}(\alpha) &\le \liminf_{t\to\infty} a_t^{-1} \log P\{a_t^{-1} Y_t > \alpha\} \\ &\le \limsup_{t\to\infty} a_t^{-1} \log P\{a_t^{-1} Y_t > \alpha\} \\ &\le -\bar{c}(\alpha); \end{aligned}$$

$\bar{\alpha}$, $\bar{c}$ and $\underline{c}$ only depend on $\bar{\psi}$ and $\underline{\psi}$, both restricted to $\lambda \in [0, \lambda_0)$.

PROOF. The upper bound is an easy application of Chebyshev's inequality. For any $\lambda \in [0, \lambda_0)$, we have

$$P\{a_t^{-1} Y_t > \alpha\} = P\{\exp(\lambda Y_t) > \exp(a_t \lambda\alpha)\} \le \exp[a_t(\psi_t(\lambda) - \lambda\alpha)].$$

Thus

$$\limsup_{t\to\infty} a_t^{-1} \log P\{a_t^{-1} Y_t > \alpha\} \le -\lambda\alpha + \bar{\psi}(\lambda), \qquad \lambda \in [0, \lambda_0).$$

Hence

$$\limsup_{t\to\infty} a_t^{-1} \log P\{a_t^{-1} Y_t > \alpha\} \le \inf_{\lambda\in[0,\lambda_0)} -\left(\lambda\alpha - \bar{\psi}(\lambda)\right) = -\bar{c}(\alpha).$$

Since $\bar{\psi}$ is convex on $[0, \lambda_0)$, $\bar{\psi}(0) = 0$ and $D^+\bar{\psi}(0) = \mu < \alpha$, $\bar{\psi}(\lambda)/\lambda$ converges to μ as $\lambda \downarrow 0$. Therefore $\bar{c}(\alpha) > 0$.

Next set $\overline{\alpha} = \lim_{\lambda \uparrow \lambda_0} \underline{\psi}(\lambda)/\lambda$. Then $\overline{\alpha} > \mu$. For if this is not the case, then

$$\mu = D^+ \underline{\psi}(0) \le \underline{\psi}(\lambda)/\lambda \le \mu, \qquad \lambda \in (0, \lambda_0),$$

proving that $\underline{\psi}(\lambda)/\lambda$ is constant over $(0, \lambda_0)$, which contradicts assumption (i).

Let $\alpha \in (\mu, \overline{\alpha})$ be given. Then one can find $\lambda \in [0, \lambda_0)$ and $\delta > 0$ such that $\lambda + \delta \in [0, \lambda_0)$ and $\alpha < \underline{\psi}(\lambda)/\lambda$. Finally, let $M > \alpha$ be such that

$$L = \lambda M - \underline{\psi}(\lambda) > 0 \quad \text{and} \quad M > \frac{\overline{\psi}(\lambda + \delta) - \underline{\psi}(\lambda)}{\delta}.$$

Then

$$0 < L < U = \min\big(\lambda(M - \alpha), (\lambda + \delta)M - \overline{\psi}(\lambda + \delta)\big).$$

Now

$$E\big\{\exp(\lambda Y_t) 1_{\{a_t^{-1} Y_t \le \alpha\}}\big\} \le \exp(a_t \lambda \alpha),$$

$$E\big\{\exp(\lambda Y_t) 1_{\{\alpha < a_t^{-1} Y_t \le M\}}\big\} \le \exp(a_t \lambda M) P\big\{a_t^{-1} Y_t > \alpha\big\},$$

and

$$\begin{aligned} E\big\{\exp(\lambda Y_t) 1_{\{a_t^{-1} Y_t > M\}}\big\} &= E\big\{\exp[(\lambda + \delta) Y_t] \exp(-\delta Y_t) 1_{\{a_t^{-1} Y_t > M\}}\big\} \\ &\le \exp(-a_t \delta M) \exp\big[a_t \psi_t(\lambda + \delta)\big]. \end{aligned}$$

Therefore,

$$\begin{aligned} \exp(-a_t \lambda M) \exp\big[a_t \psi_t(\lambda)\big] &\le P\big\{a_t^{-1} Y_t > \alpha\big\} + \exp\big[-a_t \lambda(M - \alpha)\big] \\ &\quad + \exp\big[a_t(\psi_t(\lambda + \delta) - M(\lambda + \delta))\big]. \end{aligned}$$

Next

$$\liminf_{t \to \infty} a_t^{-1} \log \exp(-a_t \lambda M) \exp\big[a_t \psi_t(\lambda)\big] \ge -\lambda M + \underline{\psi}(\lambda) = -L,$$

and

$$\begin{aligned} &\limsup_{t \to \infty} a_t^{-1} \log\big\{\exp\big[-a_t \lambda(M - \alpha)\big] + \exp\big[a_t(\psi_t(\lambda + \delta) - M(\lambda + \delta))\big]\big\} \\ &\quad \le \max\big(-\lambda(M - \alpha), -(\lambda + \delta)M + \overline{\psi}(\lambda + \delta)\big) \\ &\quad = -U. \end{aligned}$$

Since $L < U$, we obtain

$$\liminf_{t \to \infty} a_t^{-1} \log P\big\{a_t^{-1} Y_t > \alpha\big\} \ge -L.$$

The proof is completed by setting $\underline{c}(\alpha) = L$. □

We are now in a position to prove the main theorem.

PROOF OF THE MAIN THEOREM. The first part of the theorem, namely, that for any $V \in B_0$ one can find $\lambda > 0$ such that $\lambda V \in B$, has been proved in Lemma 6. Next the function $\overline{\psi}$: $\theta \mapsto \frac{1}{2}\Lambda(4\theta^2 +)$, appearing in Lemma 7(i), is strictly convex with derivative 0 at $\theta = 0$. Similarly the function $\psi: \theta \mapsto -\Lambda(-\theta^2/2)$, appearing in Lemma 8, is strictly convex on $|\theta| \le 2\sqrt{\theta_0}$ and has derivative 0 at $\theta = 0$. Therefore Lemma 9 applies, completing the proof of (i). Finally, statements (ii) and (iii) follow from the matching upper and lower bounds in (ii) and (iii) of Lemmas 7 and 8. □

In view of the corresponding more complete result for independent particles [3, 22] one is tempted to make the following conjecture. It would be interesting to prove or disprove it.

CONJECTURE 1. *We have*

$$\lim_{t \to \infty} t^{-1/2} \log P\left\{ t^{-3/4} \int_0^t \int_{\mathbb{R}^3} V(x) \mu_s(dx)\, ds > \alpha \right\} = I(\alpha),$$

where $I(\alpha) = \sup_{\theta \in \mathbb{R}}(\alpha\theta - \Lambda(\theta^2))$.

Combining the last theorem and results from [15], we obtain the analogue of Theorem 1.3 in [22] and Corollary 3.1 in [21].

COROLLARY 1. *Consider the occupation time process* $L_T(dx) = T^{-1}\int_0^T \mu_s(dx)\, ds$ *with values in* $\mathscr{M}$*, the space of Radon measures on* $\mathbb{R}^3$ *equipped with the usual projective limit topology of the spaces* $\{\mathscr{M}_K;\ K \subset \mathbb{R}^3,\ K \text{ compact}\}$*, where* $\mathscr{M}_K$ *is the space of finite Radon measures on* K*. Then* $\{\mathscr{M}, L_T, T^{1/2}\}$ *is a large-deviation system with action functional* $\hat{I}$*, where*

$$\hat{I}(\sigma) = \begin{cases} \sup\limits_{\theta \in \mathbb{R}} (c\theta - \Lambda(\theta)), & \text{if } \sigma = c\lambda, \\ +\infty, & \text{otherwise}, \end{cases} \qquad \sigma \in \mathscr{M},$$

and λ *is the Lebesgue measure.*

PROOF. Let ϕ_0 be a fixed continuous density function on $\mathbb{R}^3$ with compact support. Then, for any continuous function V on $\mathbb{R}^3$ with compact support and unit integral, it follows from the proof of Lemmas 7 and 8 that

$$\lim_{T \to \infty} T^{-1/2} \log E\left\{ \exp\left(T^{-1/2}\theta \int_0^T \langle V - \phi_0, \mu_s \rangle\, ds \right) \right\} = 0. \tag{2.11}$$

Using (2.11) and the Hölder inequality, we see that (2.4) holds for all V; that is, the nonnegativity assumption is not needed, and, for any continuous V with compact support, we have

$$\lim_{T \to \infty} T^{-1/2} \log E\left\{ \exp\left(T^{-1/2} \int_0^T \langle V, \mu_s \rangle\, ds \right) \right\} = \Lambda(\overline{V}),$$

whenever $\overline{V} = \int_{\mathbb{R}^3} V(x)\, dx < \theta_0$.

The Hölder-continuity assumption can also be removed; see the second-to-last paragraph of the introduction of [15]. Using Theorem 3.4 of [7], we then obtain that, for any compact subset K of $\mathbb{R}^3$, $\{\mathscr{M}_K, L_T^{(K)}, T^{1/2}\}$ is a large-deviation system with action functional $\hat{I}_K$, where $L_T^{(K)}$ is the restriction of L_T to K and

$$\hat{I}_K(\sigma) = \sup_{V \in C_K} \langle V, \sigma \rangle - \Lambda(\overline{V}), \tag{2.12}$$

where C_K is the set of continuous functions on $\mathbb{R}^3$ with support in K. Although the limit of the cumulant generating function $\Lambda(\overline{V})$ is infinity for $\overline{V} > \theta_0$, formula (2.12) persists because the steepness condition $\Lambda'(\theta_0 -) = \infty$ is verified in [15, Lemma 1.7]. See [15, Theorem 0.4] for more detail. Next we show

$$\hat{I}_K(\sigma) = \begin{cases} \sup_{\theta \in \mathbb{R}} (c\theta - \Lambda(\theta)), & \text{if } \sigma = c\lambda_K, \\ +\infty, & \text{otherwise,} \end{cases} \qquad \sigma \in \mathscr{M}_K, \tag{2.13}$$

and λ_K is the restriction of the Lebesgue measure to K.

If $\sigma = c\lambda_K$ for some $c \geq 0$, then there is nothing to prove. Suppose next that $\sigma \neq c\lambda_K$ for every $c \geq 0$. Then one can find $V \in C_K$ such that $\overline{V} = 0$ and $\langle V, \sigma \rangle = 1$. Therefore,

$$\hat{I}_K(\sigma) \geq \sup_{\theta > 0} \langle \theta V, \sigma \rangle - \Lambda(\theta \overline{V}) = \sup_{\theta > 0} \theta = +\infty.$$

Hence $\hat{I}_K(\sigma) = +\infty$.

In view of (2.13) it is readily seen that

$$\sup_{\substack{K \subset \mathbb{R}^3 \\ K \text{ compact}}} \hat{I}_K(\sigma_K) = \hat{I}(\sigma),$$

where σ_K is the restriction of σ to K.

Since the topology on $\mathscr{M}$ is the projective limit topology of $\{\mathscr{M}_K;\, K \text{ compact}\}$, we can apply Theorem 3.3 of [7] to complete the proof. □

Acknowledgment. We are grateful to the referees for their useful comments.

REFERENCES

[1] ARONSON, D. G. and WEINBERGER, H. F. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30** 33–76.

[2] BRAMSON, M., COX, J. T. and GRIFFEATH, D. (1988). Occupation time large deviation of the voter model. *Probab. Theory Related Fields* **77** 401–413.

[3] COX, J. T. and DURRETT, R. (1990). Large deviations for independent random walks. *Probab. Theory Related Fields* **84** 67–82.

[4] COX, J. T. and GRIFFEATH, D. (1984). Large deviations for Poisson systems of independent random walks. *Z. Wahrsch. Verw. Gebiete* **69** 543–558.

[5] COX, J. T. and GRIFFEATH, D. (1985). Occupation times for critical branching Brownian motions. *Ann. Probab.* **13** 1108–1132.

[6] DAWSON, D. A. (1977). The critical measure diffusion process. *Z. Wahrsch. Verw. Gebiete* **40** 125–145.

[7] DAWSON, D. A. and GÄRTNER, J. (1987). Large deviations for McKean–Vlasov limit of weakly interacting diffusions. *Stochastics* **20** 247–308.

[8] DONSKER, M. D. and VARADHAN, S. R. S. (1987). Large deviations for noninteracting infinite particle systems. *J. Statist. Phys.* **46** 1195–1232.
[9] DYNKIN, E. B. (1989). Superprocesses and their linear additive functionals. *Trans. Amer. Math. Soc.* **314** 255–282.
[10] ELLIS, R. S. (1985). *Entropy, Large Deviations and Statistical Mechanics.* Springer, New York.
[11] FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Englewood Cliffs, NJ.
[12] HARAUX, A. and WEISSLER, F. B. (1982). Nonuniqueness for a semilinear initial value problem. *Indiana Univ. Math. J.* **31** 167–189.
[13] IKEDA N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. North-Holland/Kodansha, Amsterdam.
[14] ISCOE, I. (1986). A weighted occupation time for a class of measure-valued branching processes. *Z. Wahrsch. Verw. Gebiete* **71** 85–116.
[15] ISCOE, I. and LEE, T.-Y. (1993). Large deviations for occupation times of measure-valued branching Brownian motions. *Stochastics Stochastics Rep.* **45** 177–209.
[16] LANDIM, C. (1992). Occupation time large deviations for the symmetric simple exclusion process. *Ann. Probab.* **20** 206–231.
[17] LEE, T.-Y. (1988). Large deviations for noninteracting infinite particle systems. *Ann. Probab.* **16** 1537–1558.
[18] LEE, T.-Y. (1989). Large deviations for systems of noninteracting recurrent particles. *Ann. Probab.* **17** 46–57.
[19] LEE, T.-Y. (1993). Some limit theorems for super-Brownian motion and semilinear differential equations. *Ann. Probab.* **21** 979–995.
[20] LEE, T.-Y. and REMILLARD, B. (1994a). Occupation times in systems of null recurrent Markov processes. *Probab. Theory Related Fields* **98** 245–259.
[21] LEE, T.-Y. and REMILLARD, B. (1994b). Occupation time limit theorems for independent random walks. In *Measure-Valued Processes, Stochastic Partial Differential Equations and Interacting Systems* (D. A. Dawson, ed.) 151–163. Amer. Math. Soc., Providence.
[22] REMILLARD, B. (1990). Asymptotic behaviour of the Laplace transform of weighted occupation times of random walks and applications. In *Diffusion Processes and Related Problems in Analysis, I* (M. A. Pinsky, ed.) 497–519. Birkhäuser, Boston.
[23] ROELLY-COPOLETTA, S. (1986). A criterion of convergence of measure-valued processes: applications to measure branching processes. *Stochastics* **17** 43–65.
[24] SATTINGER, D. (1973). *Topics in Stability and Bifurcation Theory. Lecture Notes in Math.* **309**. Springer, Berlin.
[25] WANG, X. (1993). On Cauchy problems for reaction–diffusion equations. *Trans. Amer. Math. Soc.* **337** 549–590.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742

DÉPARTEMENT DE MATHÉMATIQUES
ET D'INFORMATIQUE
UNIVERSITÉ DU QUÉBEC À TROIS-RIVIÈRES
TROIS-RIVIÈRES, QUÉBEC
CANADA G1K 7P4