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THE HAUSDORFF MEASURE OF THE SUPPORT OF TWO-DIMENSIONAL SUPER-BROWNIAN MOTION

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We show that two-dimensional super-Brownian motion is a multiple of the h-Hausdorff measure on its closed support, where $h(r) = r^2 \log^+(1/r)\log^+ \log^+ \log^+(1/r)$. This complements known results in dimensions greater than 2.

1. Introduction and statement of result. The goal of this work is to find an exact Hausdorff measure function for the support of two-dimensional super-Brownian motion at a fixed time. The corresponding result in higher dimensions was proved by Perkins (1989) and refined in Dawson and Perkins (1991). Before stating our main result, we introduce the relevant notation.

We denote by $M_F(E)$ the space of finite measures on a measurable space $(E, \mathscr{E})$. The integral of a function $f\colon E \to \mathbb{R}$ with respect to a measure μ is written as $\langle \mu, f\rangle$, or in the case of a probability, $\mu(f)$. For $\gamma > 0$, let $\mathbb{Q}_\mu^\gamma$ denote the law on $(\Omega_Y, \mathscr{F}_Y) = (C([0,\infty), M_F(\mathbb{R}^d)), \mathscr{B}(\Omega_Y))$ [where $\mathscr{B}(\Omega_Y)$ denotes the Borel subsets of Ω_Y] of d-dimensional super-Brownian motion with branching rate γ. That is, if $Y_t(\omega) = \omega(t)$ on Ω_Y, then under $\mathbb{Q}_\mu^\gamma$, Y is an $M_F(\mathbb{R}^d)$-valued diffusion such that, for all bounded measurable $\phi\colon \mathbb{R}^d \to [0,\infty)$,

$$\mathbb{Q}_\mu^\gamma(\exp(-\langle Y_t, \phi\rangle)) = \exp(-\langle \mu, U_t^\gamma\phi\rangle), \tag{1.1}$$

where $U_t^\gamma = U_t^\gamma\phi$ is the unique solution of (the weak form of)

$$\frac{\partial U_t^\gamma}{\partial t}(x) = \frac{\Delta}{2}U_t^\gamma(x) - \frac{\gamma}{2}(U_t^\gamma(x))^2, \qquad U_0^\gamma = \varphi \tag{1.2}$$

[see Dawson (1993), Chapter 4].

We denote by $h - m(A)$ the Hausdorff h-measure of a set A in $\mathbb{R}^d$ and by $S(Y_t)$ the closed support of Y_t. If $\phi(r) = r^2 \log^+ \log^+(1/r)$, then Dawson and Perkins [(1991), Theorem 5.2] state that for $d \ge 3$ there is a constant $c_0(d) \in (0,\infty)$ such that

$$\begin{aligned} &Y_t(A) = c_0(d)\gamma\phi - m(A \cap S(Y_t)),\\ &\qquad \forall A \in \mathscr{B}(\mathbb{R}^d),\ \mathbb{Q}_\mu^\gamma\text{-a.s.},\quad \forall t > 0,\ \gamma > 0 \text{ and } \mu \in M_F(\mathbb{R}^d). \end{aligned} \tag{1.3}$$

The extension to general $\gamma > 0$ is trivial because the scaling properties of (1.2) show that

$$\mathbb{Q}_\mu^\gamma(Y \in A) = \mathbb{Q}_{\mu/\gamma}^1(\gamma Y \in A). \tag{1.4}$$

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The two-dimensional case is more delicate because the recurrence of planar Brownian motion leads to a longer range dependence in the local structure of Y_t. A less precise result for $d = 2$, which implies $S(Y_t)$ is a Lebesgue null set of dimension 2, or empty, for all $t > 0$ a.s., may be found in Perkins [(1989), Theorem 2]. Our main result is the analogue of (1.3) for $d = 2$.

For the rest of this work we will assume $d = 2$.

NOTATION. Let $h(r) = r^2 \log^+(1/r)\log^+ \log^+ \log^+(1/r)$.

THEOREM 1.1. *There is a universal constant c_0 in $(0,\infty)$ such that*

$$Y_t(A) = c_0 \gamma h - m(A \cap S(Y_t)), \quad \forall A \in \mathscr{B}(\mathbb{R}^2), \mathbb{Q}^{\gamma}_{\mu}\text{-a.s.}, \quad \forall t > 0, \gamma > 0, \mu \in M_F(\mathbb{R}^2). \tag{1.5}$$

This result was conjectured in Perkins (1988). It is interesting to observe that the function h is the same as the one that gives an exact Hausdorff measure for a planar Brownian path [Taylor (1964)].

In view of (1.4) it suffices to prove the theorem for $\gamma = 4$, a value which is well suited to the path-valued process of Le Gall (1993) and so we set $\mathbb{Q}_\mu = \mathbb{Q}^4_\mu$. This path-valued process W is our main tool, and in Section 2 we describe the main features of this process that we will need, including its associated exit measures and the special Markov property for the excursions of W outside of a set. Exit measures and the special Markov property were introduced and studied by Dynkin (1991) for general superprocesses. The upper bound on the Hausdorff measure of the support is established in Section 4. The key technical ingredient is a precise lower bound for the pth moment of the mass in a small disk, under the excursion measure for W (which corresponds to the canonical measure for Y). This result is proved in Section 3 (Corollary 3.3) along with several other bounds for the excursion measures.

The lower bound for the Hausdorff measure of the support is more involved. An outline of the proof is presented in Section 5 along with some technical preliminaries, including a well-known representation of the associated Palm measure and a version of the special Markov property under the Palm measure from Le Gall (1995). The proof of the lower bound is given in Section 6, and a simple zero–one law is used to show that the upper and lower bounds coincide in Section 7.

We do not know if (1.3) (for $d \geq 3$) or (1.5) (for $d = 2$) is valid for all $t > 0$ outside a single null set. Equation (1.3) is valid for all $t > 0$ (and $d \geq 3$) a.s. if different constants are used in the upper and lower bounds, but for $d = 2$ such a global (in t) result is only known with different Hausdorff measure functions [ϕ and $h_1(r) = r^2(\log^+(1/r))^2$] in the lower and upper bounds for Y_t. These results are given in Perkins (1989).

We denote by $P^{u,y}$ the law on $C([u,\infty),\mathbb{R}^2)$ of planar Brownian motion starting at y at time $u \geq 0$, and we write P^y for $P^{0,y}$. We also denote by P_t and $p_t(x)$ the semigroup and transition density, respectively, of planar Brownian motion.

Let $bp\mathscr{E}$ (respectively, $p\mathscr{E}$) denote the space of bounded nonnegative (respectively, nonnegative) $\mathscr{E}$-measurable functions on a measurable space $(E, \mathscr{E})$. The symbols $c_1, c_2, \dots$ represent fixed positive constants, and c is a positive constant whose value may change from line to line.

2. The path-valued process and its special Markov property. In this section we give a rather rapid introduction to the path-valued process of Le Gall (1993, 1994a) in the time-inhomogeneous setting of Le Gall (1995). Our goal here is only to introduce the results needed to resolve the problem at hand.

The state spaces for this process are the spaces

$$\mathscr{W}^{(u)} = \{(w, \zeta) \in C([u, \infty), \mathbb{R}^2) \times [u, \infty): w(s) = w(\zeta)\ \forall\, s \geq \zeta\}, \qquad u \geq 0,$$

of stopped two-dimensional paths w on $[u, \infty)$ with "lifetime" ζ. We will systematically write w in place of (w, ζ), as ζ will be clear from the context, and then write $\zeta(w)$ for ζ and $u(w)$ for u, the starting time of the path. Let $\hat{w} = w(\zeta)$ denote the terminal point of w. The space of all stopped paths $\mathscr{W} = \bigcup_{u \geq 0} \mathscr{W}^{(u)}$ is a Polish space when equipped with the metric

$$d(w, w') = |u(w) - u(w')| + \sup_{r \geq 0} |w(r \vee u(w)) - w'(r \vee u(w'))| + |\zeta(w) - \zeta(w')|.$$

We denote by $(\underline{x}, u)$ (or $\underline{x}$ if $u = 0$) the trivial path in $\mathscr{W}^{(u)}$ which is constant at x and has lifetime u.

Let $\Omega^{(u)} = C(\mathbb{R}_+, \mathscr{W}^{(u)})$ be the space of continuous functions from $\mathbb{R}_+$ to $\mathscr{W}^{(u)}$ with the topology of uniform convergence on compact sets. The canonical process on all of these spaces is denoted by $(W_s, s \geq 0)$, and ζ_s is the lifetime of W_s. We denote by $\Omega_0^{(u)}$ the subspace of $\Omega^{(u)}$ of those W for which $\zeta_s = u$ for large enough s. Let $\Omega_0 = \bigcup_{u \geq 0} \Omega_0^{(u)}$ and equip Ω_0 with the topology of uniform convergence with respect to the metric d and its Borel σ-field $\mathscr{F}^0$.

For $w \in \mathscr{W}^{(0)}$, $\mathbb{P}_w$ denotes the law on $(\Omega, \mathscr{F}) \equiv (\Omega^{(0)}, \mathscr{B}(\Omega^{(0)}))$ of the path-valued process associated with planar Brownian motion. Under $\mathbb{P}_w$, W is a $\mathscr{W}^{(0)}$-valued diffusion and $(\zeta_s, s \geq 0)$ is a one-dimensional reflecting Brownian motion [see Le Gall (1993), Theorem 1.1]. Intuitively, the path W_s grows like a planar Brownian motion when ζ_s "increases" and is erased when ζ_s "decreases." We denote by $(L_s^t, s, t \geq 0)$ the continuous local time of ζ at level t and "time" s, normalized to be a sojourn density for ζ.

If $w(0) = x \in \mathbb{R}^2$, then $W_s(0) = x$ for every $s \geq 0$, $\mathbb{P}_w$-a.s. As 0 is regular for ζ, clearly $\underline{x}$ is a regular point for the diffusion W and so we may introduce $\mathbb{N}_x$, the Itô excursion measure for excursions from $\underline{x}$. The measure $\mathbb{N}_x$ is a σ-finite measure on Ω_0 which is supported by $\Omega_0^{(0)}$. Normalize $\mathbb{N}_x$ so that it is the intensity of the Poisson measure, Π^x, of excursions from $\underline{x}$ completed up to time

$$\tau_0 = \inf\{t: L_t^0 > 1\}.$$

For $u \geq 0$ define $\Theta_u: \mathscr{W}^{(0)} \to \mathscr{W}^{(u)}$ by

$$\Theta_u(w, \zeta) = ((w(t-u), t \geq u), \zeta + u)$$

and define $\mathbb{N}_{u,x}$ on $(\Omega_0, \mathscr{F}^0)$ by

$$\mathbb{N}_{u,x}(W \in A) = \mathbb{N}_x(\Theta_u(W_\cdot) \in A). \tag{2.1}$$

The measure $\mathbb{N}_{u,x}$ is clearly supported by $\Omega_0^{(u)}$. If $W \in \Omega^{(u)}$, let $\sigma(W) = \inf\{s \geq 0: \zeta_s = u\}$.

Proposition 2.2 of Le Gall (1994a) shows that $\mathbb{N}_{u,x}(\zeta \in \cdot)$ is the Itô measure of excursions of linear Brownian motion above u and so the local time of ζ, $(L_s^t,\ t \geq u,\ s \geq 0)$, is also well defined and jointly continuous under $\mathbb{N}_{u,x}$. Define a continuous $M_F(\mathbb{R}^2)$-valued process on $(\Omega, \mathscr{F}, \mathbb{P}_w)$ by

$$Y_t(W) = \int_0^{\tau_0} 1\big(\hat{W}_s \in \cdot\big)\, d_s L_s^t, \qquad t \geq 0.$$

If $W \in \Omega_0$ and ζ has a jointly continuous time $(L_s^t: t \geq u(W),\ s \geq 0)$, define a continuous $M_F(\mathbb{R}^2)$-valued path by

$$X_t(W) = \int_0^{\sigma(W)} 1\big(\hat{W}_s \in \cdot\big)\, d_s L_s^t, \qquad t \geq u(W).$$

In particular, $(X_t,\ t \geq u)$ is a continuous $M_F(\mathbb{R}^2)$-valued process under each $\mathbb{N}_{u,x}$. Theorem 2.1 of Le Gall (1993) (with $\rho = \frac{1}{4}$) and (1.4) show that if δ_x denotes point mass at x, then

$$\mathbb{P}_{\underline{x}}(Y \in \cdot) = \mathbb{Q}_{\delta_x}(\cdot), \qquad \forall\, x \in \mathbb{R}^2. \tag{2.2}$$

Decompose Y_t according to the contributions from the individual excursions of W from $\underline{x}$ to see that

$$Y_t = \int_{\Omega_0} X_t(W)\, d\Pi^x(W), \qquad \forall\, t > 0,\ \mathbb{P}_{\underline{x}}\text{-a.s.} \tag{2.3}$$

This and (2.2) show that, for $\phi \in p\mathscr{B}(\mathbb{R}^2)$,

$$\begin{aligned} \mathbb{Q}_{\delta_x}(\exp(-\langle Y_t, \phi\rangle)) &= \mathbb{P}_{\underline{x}}\left(\exp\left(-\int \langle X_t, \phi\rangle\, d\Pi^x\right)\right) \\ &= \exp\left(-\int (1 - \exp(-\langle X_t, \phi\rangle))\mathbb{N}_x(dW)\right). \end{aligned} \tag{2.4}$$

REMARK 2.1. Equation (2.4) shows that $R_t(x, \cdot) = \mathbb{N}_x(X_t \in \cdot)$ are the canonical measures associated with the infinitely divisible random measures Y_t under $\mathbb{Q}_{\delta_x}$ [see Dawson (1992), Section 3.4 and Chapter 6]. The definitions of $\mathbb{N}_{u,x}$ and X_t imply

$$\mathbb{N}_{u,x}(X_t \in A) = \mathbb{N}_x(X_{t-u} \in A) = R_{t-u}(x, A), \qquad \forall\, t \geq u \geq 0,\ x \in \mathbb{R}^2. \tag{2.5}$$

Take means in (2.3) to conclude

$$\mathbb{N}_{u,x}(\langle X_t, \phi\rangle) = \mathbb{N}_x(\langle X_{t-u}, \phi\rangle) = \mathbb{Q}_{\delta_x}(\langle Y_{t-u}, \phi\rangle) = P_{t-u}\phi(x), \tag{2.6}$$

where we used the superprocess property (the fact that the mean measure of a superprocess is given by the expected value of the underlying Markov process) in the last equality.

We will also use the laws $\mathbb{P}^*_w(\cdot) = \mathbb{P}_w(W_{\cdot \wedge \sigma} \in \cdot)$, $w \in \mathscr{W}^{(0)}$, on Ω (supported by $\Omega_0^{(0)}$) to formulate the strong Markov property of W under $\mathbb{N}_x$. Let $\mathscr{F}_t = \bigcap_{r>t} \sigma(W_s\colon s < r)$ and let θ_t denote the usual shift operators on Ω. If T is an $(\mathscr{F}_t)$-stopping time such that $T > 0$, $\mathbb{N}_y$-a.e., $F \in p\mathscr{F}_T$ and $G \in p\mathscr{F}$, then [see Le Gall (1995)]

$$(2.7) \qquad \mathbb{N}_y(1(T<\infty)F \cdot (G \circ \theta_t)) = \mathbb{N}_y\big(1(T<\infty)F\mathbb{P}^*_{W_T}(G)\big).$$

We now follow Section 3 of Le Gall (1994a) and introduce the exit measures X^D from a fixed open subset D of $\mathbb{R}_+ \times \mathbb{R}^2$ [see also Le Gall (1995)]. If $w \in C([u,\infty), \mathbb{R}^2)$, let

$$\tau(w) = \tau_D(w) = \inf\{t \ge u\colon (t, w(t)) \notin D\}, \qquad \inf \varnothing = \infty,$$

and we abuse the notation slightly by also writing $\tau(w)$ for the "same" exit time if $w \in \mathscr{W}^{(u)}$. Assume $(0,x) \in D$ and $P^x(\tau < \infty) > 0$. If $\gamma_s = (\zeta_s - \tau(W_s))^+$, $A_t = \int_0^t 1(\gamma_s > 0)\, ds$ and $\alpha(s) = \inf\{r\colon A(r) > s\}$, then under $\mathbb{P}_{\underline{x}}$, $(\gamma_{\alpha(s)}, s \ge 0)$ is a reflecting Brownian motion with local time at level zero $\tilde{L}^0_s$ [Proposition 3.1 of Le Gall (1994a)]. Let $L^D_s = \tilde{L}^0_{A(s)}$ and call L^D_s the exit local time of D. As in Section 3 of Le Gall (1994a), we may in the same way define L^D_s on $(\Omega_0, \mathbb{N}_{u,x})$ for $(u,x) \in D$ such that $P^{u,x}(\tau < \infty) > 0$. To see this, argue as in Le Gall (1994a) with space–time planar Brownian motion as the underlying process and note that $\mathbb{N}_{u,x}$ differs from the associated excursion measure in a trivial manner. For (u,x) as above, define the exit measure X^D on $(\Omega_0, \mathbb{N}_{u,x})$ by

$$\langle X^D, \phi\rangle = \int_0^\sigma \phi\big(s, \hat{W}_s\big)\, dL^D_s.$$

If $(u,x) \notin D$ or $P^{u,x}(\tau < \infty) = 0$, set $X^D \equiv 0$ under $\mathbb{N}_{u,x}$. Then $X^D \in M_F(\mathbb{R}_+ \times \mathbb{R}^2)$ is supported on ∂D and

$$(2.8) \qquad \mathbb{N}_{u,x}(\langle X^D, \phi\rangle) = P^{u,x}(\phi(\tau, B_\tau)1(\tau<\infty)), \qquad \forall\, (u,x) \in D$$

[more details on the construction of X^D in this setting can be found in Le Gall (1995)]. It is easy to check that if

$$(2.9) \qquad D^{(u)} = \{(r-u, x)\colon r \ge u, (r,x) \in D\},$$

then

$$(2.10) \qquad \mathbb{N}_{u,y}(X^D \in \cdot) = \mathbb{N}_y\big(\sigma_u(X^{D^{(u)}}) \in \cdot\big),$$

where $\langle \sigma_u(\nu), \varphi\rangle = \langle \nu, \varphi(u + \cdot, \cdot)\rangle$ for any nonnegative function φ on $\mathbb{R}_+ \times \mathbb{R}^2$. [The reader may prefer to use (2.10) as the definition of $\mathbb{N}_{u,y}(X^D \in \cdot)$.]

The special Markov property of W from Le Gall (1995) will play an essential role in our lower bound on the Hausdorff measure of the support. To describe it, fix $(u,x) \in D$ such that $P^{u,x}(\tau_D < \infty) > 0$ and let

$$(2.11) \quad \eta(s) = \inf\left\{t\colon \int_0^t 1(\zeta_r \le \tau(W_r))\, dr > s\right\}, \qquad W'_s = W_{\eta(s)} \text{ for } W \in \Omega_0.$$

Let $\mathscr{E}^D$ be the σ-field of subsets of Ω_0 generated by W' and the $\mathbb{N}_{u,x}$-null sets in Ω_0. The random open set $\{s \in [0,\sigma)\colon \tau(W_s) < \zeta_s\}$ is the countable union of disjoint open intervals $\{(a_i, b_i)\colon i \in I\}$. For each i in I, $\tau(W_s) \equiv \tau^i$ is constant for s in $[a_i, b_i]$ and so is $W_s(t)$ for all $t \le \tau^i$ [see the proof of Proposition 3.1 in Le Gall (1994a)]. Define the ith excursion "outside D" by

$$W_s^i(t) = W_{(a_i+s)\wedge b_i}(t), \qquad t \ge \tau^i, \tag{2.12}$$

so that $W^i \in \Omega^{(\tau_i)}$ has lifetime $\zeta_s^i = \zeta_{(a_i+s)\wedge b_i}$. Let $\mathscr{N}(dW) = \sum_{i\in I}\delta_{W^i}(dW)$.

THEOREM 2.2 [Special Markov property, Le Gall (1995)]. *If $\Phi \in p\mathscr{F}^0$, then*

$$\mathbb{N}_{u,x}\left(\exp\left(-\sum_{i\in I}\Phi(W^i)\right)\middle|\mathscr{E}^D\right) = \exp\left(-\int \mathbb{N}_{t,y}(1-\exp(-\Phi))\,X^D(dt,dy)\right).$$

That is, conditional on $\mathscr{E}^D$, $\mathscr{N}$ is a Poisson measure on Ω_0 with characteristic measure $\int \mathbb{N}_{t,y}(\cdot)X^D(dt,dy)$.

3. Some bounds for $\mathbb{N}$. Let $w_0 \in \mathscr{W}$ have lifetime t_0. By Proposition 2.5 of Le Gall (1994a) there is a Poisson random measure Λ on $[0,t_0)\times\Omega$, defined on $(\Omega,\mathscr{F},\mathbb{P}^*_{w_0})$, such that Λ has characteristic measure $2\,dt\,\mathbb{N}_{w_0(t)}(dW)$ and

$$X_{t_0} = \int_0^{t_0}\int_\Omega X_{t_0-t}(W)\,d\Lambda(t,W), \qquad \mathbb{P}^*_{w_0}\text{-a.s.} \tag{3.1}$$

This expresses $X_{t_0}(W)$ as the sum of the contributions from excursions of ζ above its minimum-to-date.

NOTATION. If $\phi \in p\mathscr{B}(\mathbb{R}^2)$ and $t \ge 0$, set

$$G(\phi,t) = \int_0^t \sup_{y\in\mathbb{R}^2} P_s\phi(y)\,ds.$$

The next result is, in view of Remark 2.1, essentially Lemma 6.5.4 of Dawson (1992). We give a proof using W.

LEMMA 3.1. *If $\phi \in p\mathscr{B}(\mathbb{R}^2)$, $t > 0$ and $y \in \mathbb{R}^2$, then, for every λ in $[0,(4G(\phi,t))^{-1}]$,*

$$\mathbb{N}_y(\exp(\lambda\langle X_t,\phi\rangle)-1) \le 2\lambda P_t\phi(y).$$

PROOF. The Markov property (2.7) shows that

$$\begin{aligned}
&(n!)^{-1}\mathbb{N}_y(\langle X_t,\phi\rangle^n)\\
&\quad = \mathbb{N}_y\left(\int_{0<u_1<\cdots<u_n<\sigma}\prod_1^n \phi(\hat{W}_{u_i})\,dL^t_{u_1}\cdots dL^t_{u_n}\right)\\
&\quad = \mathbb{N}_y\left(\int_{0<u_1<\cdots<u_{n-1}<\sigma}\prod_1^{n-1}\phi(\hat{W}_{u_i})\right.\\
&\qquad\qquad \left.\times\,\mathbb{P}^*_{W_{u_{n-1}}}\left(\int_0^\sigma \phi(\hat{W}_u)\,dL^t_u\right)dL^t_{u_{n-1}}\cdots dL^t_{u_1}\right).
\end{aligned}$$

Since $W_{u_{n-1}}$ has lifetime t, $dL^t_{u_{n-1}}$-a.e., we may set $w_0 = W_{u_{n-1}}$, assume w_0 has lifetime t and use (3.1) to see that

$$\begin{aligned}\mathbb{P}^*_{w_0}\left(\int_0^\sigma \phi(\hat{W}_u)\,dL^t_u\right) &= 2\int_0^t \mathbb{N}_{w_0(r)}(\langle X_{t-r},\phi\rangle)\,dr\\ &= 2\int_0^t P_{t-r}\phi(w_0(r))\,dr \quad [\text{by } (2.6)]\\ &\le 2G(\phi,t).\end{aligned}$$

The obvious induction gives

$$\begin{aligned}(n!)^{-1}\mathbb{N}_y(\langle X_t,\phi\rangle^n) &\le (2G(\phi,t))^{n-1}\mathbb{N}_y(\langle X_t,\phi\rangle)\\ &= (2G(\phi,t))^{n-1}P_t\phi(y).\end{aligned}$$

The desired result follows by multiplying by λ^n and summing over n. □

The above result shows the moments

$$\psi(t,x,\phi,p) = \mathbb{N}_x(\langle X_t,\phi\rangle^p), \qquad \phi\in bp\mathscr{B}(\mathbb{R}^2),\ p\in\mathbb{Z}_+,\ x\in\mathbb{R}^2,$$

are finite. The following recursion relation between these moments is well known.

PROPOSITION 3.2. *We have* $\psi(t,x,\phi,1) = P_t\phi(x)$ *and, for* $p \ge 2$,

$$\psi(t,x,\phi,p) = 2\sum_{j=1}^{p-1}\binom{p}{j}\int_0^t P_{t-s}(\psi(s,\cdot,\phi,j)\psi(s,\cdot,\phi,p-j))(x)\,ds.$$

PROOF. If $u_t(\lambda,x) = U_t(\lambda\phi)(x)$ is as in (1.2) with $\gamma = 4$, then

$$u_t(\lambda,x) = \lambda P_t\phi(x) - 2\int_0^t P_{t-s}\big(u_s(\lambda,\cdot)^2\big)(x)\,ds. \tag{3.2}$$

Equations (1.1) and (2.4) show that

$$\begin{aligned}u_t(\lambda,x) &= \mathbb{N}_x(1-\exp(-\lambda\langle X_t,\phi\rangle))\\ &= \sum_{n=1}^\infty (-1)^{n-1}\frac{\lambda^n}{n!}\psi(t,x,\phi,n),\end{aligned}$$

where the series is absolutely convergent if $|\lambda| \le \lambda_0$, for some $\lambda_0 > 0$, by Lemma 3.1. If $a_n(t,x) = (-1)^{n-1}(n!)^{-1}\psi(t,x,\phi,n)$, then the absolute convergence gives

$$u_t(\lambda,x) = \sum_{n=2}^\infty \lambda^n\left(\sum_{j=1}^{n-1} a_j(t,x)a_{n-j}(t,x)\right),$$

where the series is again absolutely convergent for $|\lambda| \le \lambda_0$. Substitute this into (3.2) and use Fubini's theorem (thanks again to Lemma 3.1) on the right-hand side to conclude that, for $0 \le \lambda \le \lambda_0$,

$$\sum_{n=1}^{\infty} (-1)^{n-1}(n!)^{-1}\psi(t,x,\phi,n)\lambda^n$$
$$= P_t\phi(x)\lambda - 2\sum_{n=2}^{\infty}\sum_{j=0}^{n-1}\int_0^t P_{t-s}\big(a_j(s,\cdot)a_{n-j}(s,\cdot)\big)(x)\,ds\,\lambda^n.$$

Equate coefficients of λ^p to obtain the desired recurrence relation. □

Let $D(x,r)$ denote the open disk in the plane centered at x and with radius r. We slightly abuse the notation and write $\psi(t,x,r,p)$ for $\psi(t,x,\phi,p)$ when $\phi = 1(D(0,r))$.

Fix $c_1 > 1/2$ and let $c_2 = (4c_1 - 2)/(4c_1 - 1) \in (0,1)$. A routine calculation shows there is a constant $c_3 > 0$ such that

$$P^x(B_t \in D(0,r)) \ge c_3\exp(-|x|^2/2t)r^2t^{-1}, \qquad \forall\, r \le \sqrt{t},\ x \in \mathbb{R}^2.$$

Finally let $c_4 = c_3(2c_1)^{-1}$. By convention, $0^0 \equiv 1$.

Corollary 3.3. *We have*

$$\psi(t,x,r,p) \ge 2c_1c_4^p p!\,r^{2p}\exp\big(-c_1|x|^2/t\big)t^{-1}\log^+\big(tc_2^p/r^2\big)^{p-1},$$
$$\forall\, t \ge r^2,\ p \in \mathbb{N},\ x \in \mathbb{R}^2.$$

Proof. We will use induction on p. If $p = 1$, this follows from the definition of c_3 (even with $c_1 = 1/2$). Let $p > 1$ and assume the result for all $p' < p$. By Proposition 3.2 and the induction hypothesis,

$$\begin{aligned}
&\psi(t,x,r,p)\\
&\ge 2p!c_4^pr^{2p}(2c_1)^2\sum_{j=1}^{p-1}\int_{r^2}^t\int p_{t-s}(y-x)\exp\left(\frac{-2c_1|y|^2}{s}\right)s^{-1}\\
&\qquad\times\log^+\big(sc_2^jr^{-2}\big)^{j-1}\log^+\big(sc_2^{p-j}r^{-2}\big)^{p-j-1}\,dy\,s^{-1}\,ds\\
&= 2p!c_4^pr^{2p}(2c_1)^2\sum_{j=1}^{p-1}\int_{r^2}^t\int p_{t-s}(y-x)p_{s/4c_1}(y)\,dy\\
&\qquad\times\frac{2\pi}{4c_1}\log^+\big(sc_2^jr^{-2}\big)^{j-1}\log^+\big(sc_2^{p-j}r^{-2}\big)^{p-j-1}s^{-1}\,ds\\
&= p!c_4^pr^{2p}2c_1\sum_{j=1}^{p-1}\int_{r^2}^t\Big[\exp\Big(-|x|^2\Big(2\Big(t-s\big(1-(4c_1)^{-1}\big)\Big)\Big)^{-1}\Big)\\
&\qquad\qquad\times\Big(t-s\big(1-(4c_1)^{-1}\big)\Big)^{-1}\\
&\qquad\qquad\times\log^+\big(sc_2^jr^{-2}\big)^{j-1}\log^+\big(sc_2^{p-j}r^{-2}\big)^{p-j-1}s^{-1}\Big]\,ds.
\end{aligned}$$

Note that $s \le c_2 t$ implies $2(t - s(1-(4c_1)^{-1})) \ge t/c_1$ and that

$$\log^+\left(sc_2^j r^{-2}\right) \ge \log^+\left(sc_2^{p-1}r^{-2}\right) \quad \text{for } 1 \le j \le p-1.$$

Therefore,

$$\begin{aligned}\psi(t,x,r,p) &\ge c_4^p p! r^{2p}(2c_1)\exp\left(-|x|^2 c_1/t\right)t^{-1}(p-1)\\ &\quad \times \int_{c_2^{1-p}r^2}^{c_2 t} \log\left(sc_2^{p-1}r^{-2}\right)^{p-2} s^{-1}\,ds\, 1\left(c_2^p t \ge r^2\right).\end{aligned} \tag{3.3}$$

Let $u = \log(sc_2^{p-1}r^{-2})$ to see that the integral on the right-hand side of (3.3) equals

$$\int_0^{\log^+(c_2^p t r^{-2})} u^{p-2}\,du = \log^+\left(tc_2^p r^{-2}\right)^{p-1}(p-1)^{-1}.$$

Substitute this into (3.3) to complete the proof. □

Fix $w_0 \in \mathscr{W}$ with lifetime 1 and let $\tilde{w}_0(t) = w_0(1-t) - w_0(1)$. If Λ is as in (3.1) and $r \in (0,1)$, let

$$Z(r) = \int_{1-r}^{1-r^2}\int X_{1-t}(D(w_0(1),r))\Lambda(dt,dW)\left(r^2\log\frac{1}{r}\right)^{-1}$$

and

$$I(r,p) = \int_{r^2}^{r}\exp\left(\frac{-c_1|\tilde{w}_0(s)|^2}{s}\right)\left(\log^+\left(sc_2^p r^{-2}\right)\right)^{p-1}s^{-1}\,ds\left(\log\frac{1}{r}\right)^{-p}p,$$
$$p \in \mathbb{N},$$

where c_1 and c_2 are as in Corollary 3.3. Intuitively, $r^2\log(1/r)Z(r)$ is the contribution to $X_1(D(w_0(1),r))$ from particles which split off from w_0 in $[1-r, 1-r^2]$.

LEMMA 3.4. *There exist constants $c_5 \ge c_6$ such that, for every $r \in (0,1/2)$ and $p \in \mathbb{N}$,*

$$c_5^p p^p \ge \mathbb{P}^*_{w_0}\left(Z(r)^p\right) \ge c_6^p p^p I(r,p).$$

PROOF. An easy calculation shows that

$$G(1(D(0,r)),1) \le 3r^2\log(1/r) \quad \text{for } r \in (0,1/2). \tag{3.4}$$

Therefore if $r \in (0,1/2)$ and $\lambda = 1/12$, then $\lambda(r^2\log(1/r))^{-1} \le$

$(4G(1(D(0,r))\,t))^{-1}$ for all $t \le 1$ and so

$$\mathbb{P}^*_{w_0}(\exp \lambda Z(r)) = \exp\left(\int_{1-r}^{1-r^2}\int \left(\exp\left(\lambda\left(r^2 \log\frac{1}{r}\right)^{-1} X_{1-t}(D(w_0(1),r))\right) - 1\right) \times 2\mathbb{N}_{w_0(t)}(dW)\, dt\right)$$

$$\le \exp\left(\int_0^1 4\lambda\left(r^2 \log\frac{1}{r}\right)^{-1} P_{1-t}(1(D(w_0(1),r)))(w_0(t))\, dt\right) \qquad \text{(Lemma 3.1)}$$

$$\le e \quad [\text{by } (3.4)].$$

The required upper bound is now clear because $\mathbb{P}^*_{w_0}(Z(r)^p) \le p!\lambda^{-p}\mathbb{P}^*_{w_0}(\exp \lambda Z(r))$.

For the lower bounds, use Corollary 3.3 to see that

$$\mathbb{P}^*_{w_0}\left(Z(r)^p\right) \ge \mathbb{P}^*_{w_0}\left(\int_{1-r}^{1-r^2}\int X_{1-t}(D(w_0(1),r))^p \Lambda(dt,dW)\right) r^{-2p}\left(\log\frac{1}{r}\right)^{-p}$$

$$= 2\int_{1-r}^{1-r^2}\int X_{1-t}(D(w_0(1),r))^p \mathbb{N}_{w_0(t)}(dW)\, dt\, r^{-2p}\left(\log\frac{1}{r}\right)^{-p}$$

$$\ge 4c_1c_4^p p!\int_{r^2}^{r}\exp\left(\frac{-c_1|\hat{w}_0(t)|^2}{t}\right)t^{-1}\log^+\left(\frac{tc_2^p}{r^2}\right)^{p-1} dt\left(\log\frac{1}{r}\right)^{-p}$$

$$\ge c_6^p p^p I(r,p). \qquad \square$$

LEMMA 3.5. *There exists a constant c_7 such that if $p = [2A/c_6] + 1$ ($[x]$ is the integer part of x), then*

$$\mathbb{P}^*_{w_0}(Z(r) \ge A) \ge \left((2^p I(r,p) - 1)^+\right)^2 \exp(-c_7 A), \qquad \forall A \ge \frac{c_6}{2},\ r \in \left(0, \frac{1}{2}\right).$$

PROOF. Let $A \ge c_6/2$ and $r \in (0, 1/2)$, so that $2A/c_6 \le p \le 4A/c_6$. Use

$$\mathbb{P}^*_{w_0}\left(Z(r)^p\right) \le A^p + \mathbb{P}^*_{w_0}\left(Z(r)^{2p}\right)^{1/2}\mathbb{P}^*_{w_0}(Z(r) > A)^{1/2}$$

to see that

$$\mathbb{P}^*_{w_0}(Z(r) > A)^{1/2} \ge \left(\mathbb{P}^*_{w_0}\left(Z(r)^p\right) - A^p\right)^+ \mathbb{P}^*_{w_0}\left(Z(r)^{2p}\right)^{-1/2}$$

$$\ge \left((c_6 p)^p I(r,p) - A^p\right)^+ (c_5 2p)^{-p}$$

$$\ge (A/c_5 2p)^p (2^p I(r,p) - 1)^+$$

$$\ge (c_6/8c_5)^p (2^p I(r,p) - 1)^+$$

$$\ge (c_6/8c_5)^{4A/c_6} (2^p I(r,p) - 1)^+$$

$$= \exp(-c_7 A/2)(2^p I(r,p) - 1)^+. \qquad \square$$

4. Upper bound for the Hausdorff measure of the support. Let $r_n = r(n) = 2^{-2^n}$, and if $C \subset \mathbb{R}^2$ and $r > 0$, set $C^r = \{x: d(x, C) \le r\}$, where $d(x, C)$ is the usual distance from x to C. Let

$$\Lambda_n = \Big\{C: C \text{ an open square of side length } r_n \text{ centered at } (\mathbf{j} + \mathbf{e})r_n \text{ for some } \mathbf{j} \in \mathbb{Z}^2, \mathbf{e} \in \{0, 1/2\}^2, C \subset (-n, n)^2\Big\}.$$

We will use a well-known theorem of Rogers and Taylor which reduces the upper bound on $h - m(S(Y_1))$ to showing that, for a sufficiently small $c > 0$,

$$h - m\left(\left\{x \in S(Y_1): \limsup_{r \downarrow 0} Y_1(D(x, r))h(r)^{-1} \le c\right\}\right) = 0, \qquad \mathbb{Q}_{\delta_0}\text{-a.s.}$$

The key to finding a good cover for the above set will be an estimate on

$$\mathbb{N}_0\big(X_1(C) > 0, X_1(C^{r_j}) \le ch(r_j) \text{ for } j = 2^n, \dots, 2^{n+1} - 1\big) \tag{4.1}$$

for C in $\Lambda_{2^{n+1}}$. The strong Markov property for W, (2.7), allows us to estimate the above set by analyzing the two conditions separately. This is of course what is done when bounding the Hausdorff measure of the Brownian path [see Taylor (1964), page 256]. Note, however, that if one works only with the super-Brownian motion Y_1, it is not all clear how to get such an estimate on (4.1). The process W introduces time dynamics to the analysis of Y_1 which are critical for our arguments to work.

For $B > 0$ and $w \in \mathscr{W}$ with $\zeta \ge 1$, let

$$F_{n,B}(w) = \frac{1}{n} \sum_{j=0}^{n-1} 1\big(\sup(|w(\zeta - t) - w(\zeta)|: 0 \le t \le 2^{-j}) > B2^{-j/2}\big).$$

Lemma 1 of Le Gall (1994b) shows that for any $\delta > 0$ there are a $B > 0$ and a $c_8 > 0$ such that

$$\mathbb{N}_0(\exists\, t \ge 0: \zeta_t = 1, F_{n,B}(W_t) > \delta) \le c_8 e^{-n}. \tag{4.2}$$

LEMMA 4.1. *For each B, $c_0 > 0$, there exists an integer $n_0(B, c_0) \in \mathbb{N}$ such that whenever $n \ge n_0$, $w_0 \in \mathscr{W}$ has lifetime 1 and $F_{m,B}(w_0) \le 1/6$ for all $m \ge 2^{2^n}$, then for any $C \in \Lambda_{2^{n+1}}$ such that $w_0(1) \in C$,*

$$\mathbb{P}^*_{w_0}\big(X_1(C^{r_j}) \le c_0 h(r_j) \text{ for } j = 2^n, \dots, 2^{n+1} - 1\big) \le \exp(-2^{n(1-c_7c_0)}).$$

PROOF. Let $\theta(r) = \log^+ \log^+ \log^+(1/r)$. Assume $w_0 \in \mathscr{W}$ and $C \in \Lambda_{2^{n+1}}$ satisfy the hypotheses of the lemma for a given $B > 0$ and $n \in \mathbb{N}$, and let $c_0 > 0$. In the course of the proof we will need to assume $n \ge n_0$ for some $n_0(B, c_0)$. Define Λ and $Z(r)$ as in Section 3. Then

$$\begin{aligned} q_n &\equiv \mathbb{P}^*_{w_0}\big(X_1(C^{r_j}) \le c_0 h(r_j) \text{ for } j = 2^n, \dots, 2^{n+1} - 1\big) \\ &\le \mathbb{P}^*_{w_0}\big(Z(r_j) \le c_0 \theta(r_j) \text{ for } j = 2^n, \dots, 2^{n+1} - 1\big) \\ &= \prod_{j=2^n}^{2^{n+1}-1} \mathbb{P}^*_{w_0}\big(Z(r_j) \le c_0 \theta(r_j)\big) \end{aligned}$$

by the independence property of Λ and the fact that $r_j^2 = r_{j+1}$. Let $p_j = [2c_0\theta(r_j)/c_6] + 1$ and note that $p_j \sim (2c_0/c_6)\log j$ as $j \to \infty$ (the ratio approaches 1). If $n \geq n_0(c_0)$, then Lemma 3.5 shows the above is bounded by

$$\prod_{j=2^n}^{2^{n+1}-1}\Big(1 - \exp\big(-c_7c_0\theta(r_j)\big)\big((2^{p_j}I(r_j,p_j) - 1)^+\big)^2\Big)$$
$$\leq \exp\Bigg(-\sum_{j=2^n}^{2^{n+1}-1} cj^{-c_7c_0}\Bigg(\Bigg(2^{p_j}\int_{r_j^2c_2^{-p_j}}^{r_j}\exp\bigg(\frac{-c_1|\tilde{w}_0(s)|^2}{s}\bigg)$$
$$\times\bigg(\log s + 2^{j+1}\log 2 - p_j\log\frac{1}{c_2}\bigg)^{p_j-1}\frac{ds}{s}$$
$$\times p_j2^{-jp_j}(\log 2)^{-p_j} - 1\Bigg)^+\Bigg)^2\Bigg).$$

If $s \geq 2^j r_j^2$ and $j \geq 2^{n_0}$, then for n_0 sufficiently large it is easy to see

$$\left(\frac{\log s + 2^{j+1}\log 2 - p_j\log(1/c_2)}{\log s + 2^{j+1}\log 2}\right)^{p_j-1} \geq \frac{1}{2}.$$

Use this in the above to see that if

$$I_m = \int_{2^{-(m+1)}}^{2^{-m}}\exp\big(-c_1|\tilde{w}_0(s)|^2/s\big)(\log s + 2^{j+1}\log 2)^{p_j-1}s^{-1}\,ds,$$

then, for $n \geq n_0$,

$$q_n \leq \exp\Bigg(-c\sum_{j=2^n}^{2^{n+1}-1} j^{-c_7c_0}\Bigg(\Bigg(2^{-(j-1)p_j-1}(\log 2)^{-p_j}p_j \times\Bigg(\sum_{m=2^j}^{2^{j+1}-j-1} I_m\Bigg) - 1\Bigg)^+\Bigg)^2\Bigg). \tag{4.3}$$

Suppose

$$\tilde{w}_0 \in A_m = \Big\{w \in \mathscr{W}\colon \zeta = 1,\ \sup_{s\leq 2^{-m}}|w(s)| \leq B2^{-m/2}\Big\}.$$

Then for $2^j \leq m \leq 2^{j+1} - j - 1$,

$$I_m \geq \exp\big(-2c_1B^2\big)\int 1\big(\log 2(2^{j+1} - m - 1) \leq u \leq \log 2(2^{j+1} - m)\big)u^{p_j-1}\,du$$
$$\geq \exp\big(-2c_1B^2\big)(\log 2)^{p_j}(2^{j+1} - (m+1))^{p_j-1},$$

and therefore if $2^j \le m < 3(2^{j-1})$,

$$\begin{aligned} &2^{-(j-1)p_j-1}(\log 2)^{-p_j} p_j I_m \\ &\quad \ge c(B) p_j (4-(m+1)2^{-(j-1)})^{p_j}(2^{j+1}-(m+1))^{-1} \\ &\quad \ge c(B) p_j 2^{-j}. \end{aligned}$$

This, together with (4.3), gives us (for $n \ge n_0$)

$$\begin{aligned} q_n &\le \exp\left(-c\sum_{j=2^n}^{2^{n+1}-1} j^{-c_7c_0}\left(\left(\left(\sum_{m=2^j}^{3(2^{j-1})-1} 1_{A_m}(\tilde{w}_0)\right)c(B)p_j 2^{-j}-1\right)^+\right)^2\right) \\ &\le \exp\left(-c\sum_{j=2^n}^{2^{n+1}-1} j^{-c_7c_0}\left(\left(\left(2^{j-1}-3(2^{j-1})F_{3(2^{j-1}),B}(w_0)\right)c(B)p_j 2^{-j}-1\right)^+\right)^2\right) \\ &\le \exp\left(-c\sum_{j=2^n}^{2^{n+1}-1} j^{-c_7c_0}\left(\left(c(B)p_j/4-1\right)^+\right)^2\right) \\ &\qquad\qquad \left[\text{since } F_{m,B}(w_0) \le 1/6 \text{ for } m \ge 2^{2^n}\right] \\ &\le \exp(-2^{n(1-c_7c_0)}) \end{aligned}$$

provided $n \ge n_0(B, c_0)$. □

THEOREM 4.2. *There is a constant $c_9 \in (0,\infty)$ such that*

$$c_9 h - m(A \cap S(Y_1)) \le Y_1(A), \qquad \forall\, A \in \mathscr{B}(\mathbb{R}^2),\ \mathbb{Q}_{\delta_0}\text{-}a.s.$$

PROOF. Let $c_0 = (2c_7)^{-1}$ and $c_0' = (1+\sqrt{2})^{-2}c_0/2$. Consider the random sets

$$B = \left\{x \in S(Y_1): \limsup_{r\downarrow 0} Y_1(D(x,r))h(r)^{-1} \le c_0'\right\},$$

$$B_N = \left\{x \in S(Y_1): x \in [-N,N]^2, Y_1\left(D\left(x,(1+\sqrt{2})r_j\right)\right)h(r_j)^{-1} \le c_0\ \forall\, j \ge N\right\}$$

and note that $B \subset \bigcup_{N=1}^{\infty} B_N$. By a slight variant of a theorem of Rogers and Taylor [see, e.g., Perkins (1988), Theorem 1.4], it suffices to show that

$$h - m(B_N) = 0, \qquad \forall\, N,\ \mathbb{Q}_{\delta_0}\text{-}a.s. \tag{4.4}$$

We say $C \in \Lambda_{2^{n+1}}$ is bad for $\nu \in M_F(\mathbb{R}^2)$ if and only if $\nu(C) > 0$ and $\nu(C^{r_j}) \le c_0 h(r_j)$ for $j = 2^n, \dots, 2^{n+1}-1$. Assume $C \in \Lambda_{2^{n+1}}$, $N \le 2^n$ and $C \cap B_N \ne \phi$. If $x \in C \cap B_N$, then for $j \le 2^{n+1}$, $C^{r_j} \subset D(x,(1+\sqrt{2})r_j)$ and so the definition of B_N shows that C is bad for Y_1 [note that C open and $C \cap B_N \neq \phi$ imply $Y_1(C) > 0$]. Therefore,

(4.5) $\{C \in \Lambda_{2^{n+1}}: C \text{ is bad for } Y_1\}$ is a cover for B_N whenever $2^n \geq N$.

Assume for the moment we can show there is an $n_0 \in \mathbb{N}$ and $c > 0$ such that, for every $C \in \Lambda_{2^{n+1}}$ and $n \geq n_0$,

$$\begin{aligned} \mathbb{N}_0(C \text{ bad for } X_1) &\leq c\big(\exp(-2^{2^n}) + |\log r(2^{n+1})|^{-1}\exp(-2^{n/2})\big) \\ &= c\big(\exp(-2^{2^n}) + 2^{-2^{n+1}}(\log 2)^{-1}\exp(-2^{n/2})\big) \qquad (4.6) \\ &\equiv \delta(n). \end{aligned}$$

If C is bad for Y_1, it clearly is bad for $X_1(W)$ for some point W in the support of the point process Π^0 in (2.3), and so for C and n as in (4.6),

$$\begin{aligned} \mathbb{Q}_{\delta_0}(C \text{ bad for } Y_1) &\leq 1 - \exp(-\mathbb{N}_0(C \text{ bad for } X_1)) \\ &\leq \delta(n). \end{aligned}$$

Therefore, for $n \geq n_0$,

$$\begin{aligned} &\mathbb{Q}_{\delta_0}\bigg(\sum_{C \in \Lambda_{2^{n+1}}} 1(C \text{ bad for } Y_1) h\big(r(2^{n+1})\big)\bigg) \\ &\quad \leq \operatorname{card}(\Lambda_{2^{n+1}})\,\delta(n) r(2^{n+1})^2 |\log r(2^{n+1})| \log\log|\log r(2^{n+1})| \\ &\quad \leq c 2^{2n}\delta(n) 2^{2^{n+1}}(n+1), \end{aligned}$$

which is summable over n. Off the $\mathbb{Q}_{\delta_0}$-null set

$$\Gamma = \bigg\{\limsup_{n\to\infty} \sum_{C \in \Lambda_{2^{n+1}}} 1(C \text{ bad for } Y_1) h\big(r(2^{n+1})\big) > 0\bigg\},$$

(4.5) shows that (4.4) holds and we are done.

It remains to prove (4.6). Let $C \in \Lambda_{2^{n+1}}$ and define

$$T = \inf\big\{t: \hat{W}_t \in C \text{ and } \zeta_t = 1\big\}.$$

On $\{T < \sigma\}$,

$$X_1(\cdot) = \int_0^T 1\big(\hat{W}_t \in \cdot\big)\, d_t L_t^1 + X_1 \circ \theta_T(\cdot) \geq X_1 \circ \theta_T,$$

and so by the strong Markov property (2.7),

$$\begin{aligned} &\mathbb{N}_0(C \text{ bad for } X_1) \\ &\quad \leq \mathbb{N}_0\big(T < \sigma, X_1 \circ \theta_T(C^{r_j}) \leq c_0 h(r_j), j = 2^n, \dots, 2^{n+1} - 1\big) \\ &\quad = \mathbb{N}_0\big(1(T < \sigma)\mathbb{P}^*_{W_T}\big(X_1(C^{r_j}) \leq c_0 h(r_j), j = 2^n, \dots, 2^{n+1} - 1\big)\big) \\ &\quad \leq \mathbb{N}_0\big(T < \sigma, F_{m,B}(W_T) > 1/6 \text{ for some } m \geq 2^{2^n}\big) \\ &\qquad + \mathbb{N}_0\Big(1\big(T < \sigma, F_{m,B}(W_T) \leq 1/6\ \forall\, m \geq 2^{2^n}\big) \\ &\qquad\qquad \times \mathbb{P}^*_{W_T}\big(X_1(C^{r_j}) \leq c_0 h(r_j), j = 2^n, \dots, 2^{n+1} - 1\big)\Big). \end{aligned}$$

By (4.2) we may choose $B > 0$ and $c_8 > 0$ so that the first term is bounded by $c_8 \exp(1 - 2^{2^n})$. Apply Lemma 4.1 to see there is an n_0 such that, for $n \geq n_0$, the second term is bounded by

$$\mathbb{N}_0(T < \sigma)\exp(-2^{n/2}).$$

Theorem 2 of Le Gall (1994b) states that there is a $c > 0$ such that

$$\mathbb{N}_0(T < \sigma) = \mathbb{N}_0(S(X_1) \cap C \neq \varnothing) \leq c|\log r(2^{n+1})|^{-1}.$$

Combine the above bounds to obtain (4.6) and hence complete the proof. □

5. Lower bound for the Hausdorff measure of the support: outline of the proof and preliminaries. The required lower bound on the h-Hausdorff measure of $S(Y_1) \cap A$ follows from the density theorem of Rogers and Taylor [see, e.g., Perkins (1988), Theorem 1.4] and the following result.

THEOREM 5.1. *There is a constant $c_{10} \in (0, \infty)$ such that*

$$\limsup_{r \downarrow 0} \frac{Y_1(D(x,r))}{h(r)} \leq c_{10}, \qquad Y_1\text{-a.a. } x, \mathbb{Q}_{\delta_0}\text{-a.s.}$$

We know that $S(Y_1)$ is Lebesgue null $\mathbb{Q}_{\delta_0}$-a.s. [e.g., by Perkins (1989)]. Therefore, if Y, Y' are independent, each with law $\mathbb{Q}_{\delta_0}$, we may take a conditional expectation given Y_1 and use the superprocess property to see that

$$Y_1'(S(Y_1)) = 0 \quad \text{a.s.} \tag{5.1}$$

Let $W^1, \ldots, W^N$ be the excursions of W from $\underline{0}$ (under $\mathbb{P}_{\underline{0}}$) completed by time τ_0, for which the lifetime ζ hits 1. Then N is Poisson with mean 1/2, and given $N = n$, $(X_1(W^i), i \leq n)$ are i.i.d. with law $\mathbb{N}_0(X_1 \in \cdot \,|X_1 \neq 0)$. Equation (2.3) becomes

$$Y_1 = \sum_{i=1}^{N} X_1(W^i). \tag{5.2}$$

This and (5.1) clearly imply that

$$X_1(W^i)\big(S\big(X_1(W^j)\big)\big) = 0, \qquad \forall\, i \neq j \leq N, \mathbb{P}_{\underline{0}}\text{-a.s.}$$

and, therefore,

$$\limsup_{r \downarrow 0} X_1(W^j)(D(x,r))h(r)^{-1} = 0, \qquad X_1(W^i)\text{-a.s. } \forall\, i \neq j \leq N, \mathbb{P}_{\underline{0}}\text{-a.s.}$$

It follows easily from the above observations that Theorem 5.1 would be a consequence of

$$\limsup_{r \downarrow 0} X_1(D(x,r))h(r)^{-1} \leq c_{10}, \tag{5.3}$$
$$X_1(dx)\text{-a.e., } \mathbb{N}_0\text{-a.s. for some } c_{10} < \infty.$$

Let $P^{y\to x}$ denote the law of planar Brownian motion starting at y at $t=0$ and conditioned to be x at $t=1$. We may construct on some measurable space a family of probability laws $\{\mathbb{P}^{(w)}: w \in C([0,1],\mathbb{R}^2)\}$, and two random measures $\mathscr{M}_0(d\nu)$ and $\mathscr{M}(d\nu)$ such that, under $\mathbb{P}^{(w)}$, $\mathscr{M}_0(d\nu)$ is a Poisson random measure on $M_F(\mathbb{R}^2)-\{0\}$ with characteristic measure $4\int_0^1 R_{1-t}(w(t),\cdot)\,dt$ and $\mathscr{M}(dW)$ is a Poisson random measure on $(\Omega_0,\mathscr{F}^0)$ with characteristic measure $4\int_0^1 \mathbb{N}_{t,w(t)}(\cdot)\,dt$. Theorem 6.4.1 of Dawson (1992) gives a representation for the Palm measure associated with the canonical measure $R_1(x,\cdot)$, which implies that for each measurable $\Phi: \mathbb{R}^2\times M_F(\mathbb{R}^2)\to [0,\infty)$,

$$\int\int \Phi(x,\nu)\nu(dx)R_1(y,d\nu)$$
$$= \int\int \mathbb{P}^{(w)}\Big(\Phi\Big(x,\int \nu\mathscr{M}_0(d\nu)\Big)\Big)P^{y\to x}(dw)\,p_1(x-y)\,dx.$$

Use (2.5) to see that this gives

$$\mathbb{N}_0\Big(\int \Phi(x,X_1)X_1(dx)\Big)$$
$$= \int\int \mathbb{P}^{(w)}\Big(\Phi\Big(x,\int X_1(W)\mathscr{M}(dW)\Big)\Big)P^{0\to x}(dw)\,p_1(x)\,dx \tag{5.4}$$

(the necessary measurability of $w\mapsto \mathbb{P}^{(w)}$ is trivial to obtain). Define a random measure Z_1 under $\mathbb{P}^{(w)}$ by

$$Z_1 = \int X_1(W)\mathscr{M}(dW). \tag{5.5}$$

In view of the Palm measure representation (5.4), to prove (5.3) it suffices to fix $w\in C([0,1],\mathbb{R}^2)$ and $x=w(1)$ and show there is a $c_{11}<\infty$ such that

$$\limsup_{k\to\infty} Z_1\big(D(x,2^{-k})\big)h(2^{-k})^{-1} \le c_{11}, \qquad \mathbb{P}^{(w)}\text{-a.s.} \tag{5.6}$$

Then (5.3) would follow with $c_{10}=4c_{11}$.

To prove (5.6) for a fixed w in $C([0,1],\mathbb{R}^2)$ and $x=w(1)$, we require a version of the special Markov property (Theorem 2.2) under $\mathbb{P}^{(w)}$. For $W\in\Omega_0$ we abuse our notation slightly and let $u(W)=u(W_s)$ denote the common starting time of the paths traversed by W. Let $D\subset \mathbb{R}_+\times\mathbb{R}^2$ be open, let $\tau(w)=\tau_D(w)=\inf\{t: (t,w(t))\notin D\}\wedge 1$ and define the exit measure from D under $\mathbb{P}^{(w)}$ by

$$Z^D = \int 1(u(W)<\tau(w))X^D(W)\mathscr{M}(dW).$$

Then Z^D is $\mathbb{P}^{(w)}$-a.s. a finite measure supported on ∂D because X^D is supported on ∂D under each $\mathbb{N}_{t,w(t)}$ [see (2.8)] and

$$\mathbb{P}^{(w)}(\langle Z^D,1\rangle) = 4\int_0^{\tau(w)}\mathbb{N}_{t,w(t)}(\langle X^D,1\rangle)\,dt \le 4.$$

To define the analogue of $\mathscr{E}^D$, recall the definition of the map $W \to W'$ on Ω_0 from (2.11) and let $\mathscr{F}^D$ be the σ-field generated by the random measure

$$\int 1(u(W) < \tau(w))\,\delta_{W'}\mathscr{M}(dW)$$

and the collection of $\mathbb{P}^{(w)}$-null sets. If $\{W^{(j)}: j \in J\}$ are the points W of $\mathscr{M}$ such that $u(W) < \tau(w)$, define the excursions of $W^{(j)}$ "outside D," $\{W^{(j),i}: i \in I_j\}$ as in (2.12) and set

$$\mathscr{N}_p(dW) = \sum_{j \in J} \sum_{i \in I_j} \delta_{W^{(j),i}}(dW).$$

The following result is an easy consequence of Theorem 2.2.

PROPOSITION 5.2. *If $\Phi \in p\mathscr{F}_0$, then*

$$\mathbb{P}^{(w)}\left(\exp\left(-\int \Phi(W)\mathscr{N}_p(dW)\right)\middle|\mathscr{F}^D\right) = \exp\left(-\int \mathbb{N}_{t,y}(1-\exp(-\Phi))Z^D(dt,dy)\right).$$

That is, conditional on $\mathscr{F}^D$, $\mathscr{N}_p(dW)$ is a Poisson measure on Ω_0 with characteristic measure $\int \mathbb{N}_{t,y}(\cdot)Z^D(dt,dy)$.

We now outline the proof of (5.6). It is not hard to use a Borel–Cantelli argument to prove

$$\limsup_{n\to\infty} Z_1(D(x,2^{-2^n}))h(2^{-2^n})^{-1} \le c, \qquad \mathbb{P}^{(w)}\text{-a.s. for some } c < \infty.$$

[See Proposition 6.1 below for a closely related result.] The Borel–Cantelli argument unfortunately fails if 2^{-2^n} is replaced by 2^{-k}. Inequality (5.6) does not follow immediately from the above result because $h(2^{-2^{n+1}})/h(2^{-2^n})$ is not bounded away from zero. To prove (5.6), we develop a method to interpolate for k between 2^{n-1} and 2^n. To this end, it is easier to replace $Z_1(D(x,2^{-k}))$ by another quantity, whose asymptotic behavior will be the same and is defined in terms of exit measures. For each k in $\mathbb{N}$, we set

$$H_k = \{(t,y): t \le 1 \text{ and } 1 - t + |y-x|^2 \le 2^{-2k}\} \subset \mathbb{R}_+ \times \mathbb{R}^2$$

and $D_k = H_k^c$ (see Figure 1).

We will show the limiting behavior of the exit measure total mass $\langle Z^{D_k}, 1\rangle$ is essentially the same as that of $Z_1(D(x,2^{-k}))$. This will follow essentially from Lemma 5.4 below.

Our interpolation argument consists of verifying that $\langle Z^{D_k}, 1\rangle$ large for some $2^{n-1} \le k < 2^n$ implies $\langle Z^{D_{2^n}}, 1\rangle$ is large with a probability bounded from below.

First step. For $a > 0$ we introduce the subset $U_k = U_k(a)$ of ∂D_k defined by

$$U_k = \{(t,y): t < 1, (1-t) + |y-x|^2 = 2^{-2k}, |y-x| \le a\sqrt{1-t}\}$$

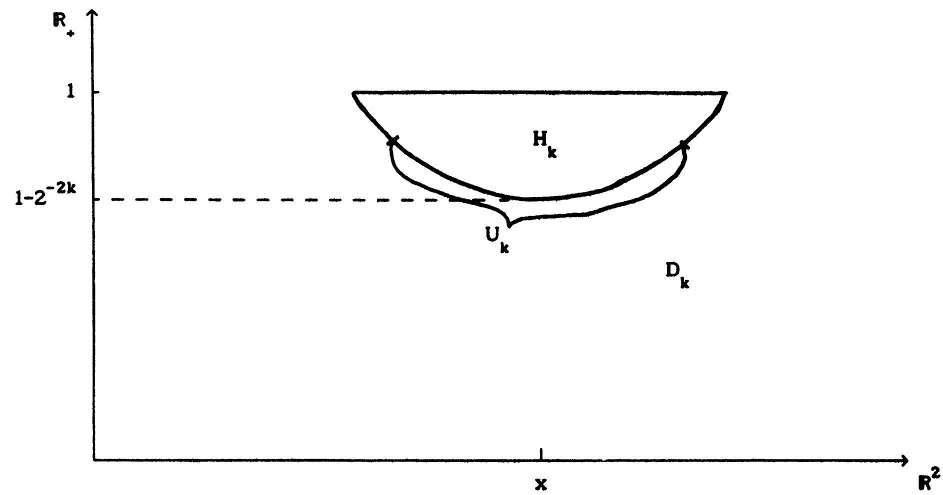

FIG. 1.

(see Figure 1). We can then prove (Proposition 6.2) that if $Z^{D_k}(U_k)$ is large, then so is $\langle Z^{D_{2^n}}, 1\rangle$ with a probability bounded from below.

Second step. It may happen that $\langle Z^{D_k}, 1\rangle$ is large but $Z^{D_k}(U_k)$ is not and hence we cannot use the first step. Proposition 6.4 essentially tells us, however, that if $\langle Z^{D_k}, 1\rangle$ is large, then $Z^{D_{k-1}}(U_{k-1})$ must also be large and so the first step may be used with $k-1$ in place of k.

We start by proving some preliminary estimates.

LEMMA 5.3. *If $(t, y) \in D_k$ and $t < 1$, then*

$$\mathbb{N}_{t,y}(\langle X^{D_k}, 1\rangle^2) \le 80 \cdot 2^{-2k}\big(1 + \log^+((1-t)2^{2k})\big)\mathbb{N}_{t,y}(\langle X^{D_k}, 1\rangle).$$

PROOF. Recall from (2.10) that $\mathbb{N}_{t,y}(\langle X^{D_k}, 1\rangle^2) = \mathbb{N}_y(\langle X^{D_k^{(t)}}, 1\rangle^2)$. Let $(L_s, s \ge 0)$ denote the exit local time of $D_k^{(t)}$. Then $\langle X^{D_k^{(t)}}, 1\rangle = L_\sigma$ and, therefore,

$$\begin{aligned}\mathbb{N}_y(\langle X^{D_k^{(t)}}, 1\rangle^2) &= 2\mathbb{N}_y\left(\int\int 1(0 \le u < v \le \sigma)\, dL_u\, dL_v\right)\\ &= 2\mathbb{N}_y\left(\int_0^\sigma \mathbb{P}^*_{W_u}(L_\sigma)\, dL_u\right)\end{aligned} \tag{5.7}$$

by the Markov property (2.7). Let $\tau_t(w) = \inf\{u: (u, w(u)) \notin D_k^{(t)}\}$. Recall that L_u increases only when $\tau_t(W_u) = \zeta_u$. Let w be a stopped path such that $\tau_t(w) = \zeta$. This implies $\zeta \le 1 - t$ and $(r, w(r)) \in D_k^{(t)}$ for $r < \zeta$. Now use Proposition 2.5 of Le Gall (1994a) as in (3.1) to see that if Λ is as in (3.1) (with w in place of w_0), then

$$X^{D_k^{(t)}} = \int_0^\zeta \int_\Omega X^{D_k^{(t)}}(\Theta_r(W))\, d\Lambda(r, W), \qquad \mathbb{P}^*_w\text{-a.s.}$$

Take means and use (2.1) to see that

$$\begin{aligned}\mathbb{P}^*_w(L_\sigma) &= 2\int_0^\zeta \mathbb{N}_{r,w(r)}(L_\sigma)\,dr\\ &= 2\int_0^\zeta \mathbb{N}_{w(r)}(\langle X^{D_k^{(t+r)}},1\rangle)\,dr \quad [\text{by (2.10)}]\\ &= 2\int_0^\zeta P^{w(r)}(\tau_{t+r}<\infty)\,dr \quad [\text{by (2.8)}].\end{aligned} \tag{5.8}$$

If $(u,z)\in\partial D_k^{(t+r)}$ and $u<1-t-r$, then

$$\begin{aligned}&P^{u,z}(B_{1-t-r}\in D(x,2^{-k}))\\ &\quad= P^0\Big(B_1\in D\big((x-z)(1-t-r-u)^{-1/2},2^{-k}(1-t-r-u)^{-1/2}\big)\Big)\\ &\quad= P^0\Big(B_1\in D\big(w,(1+|w|^2)^{1/2}\big)\Big),\end{aligned} \tag{5.9}$$

where $w=(x-z)(1-t-r-u)^{-1/2}$. The disk $D(w,(1+|w|^2)^{1/2})$ will contain $D(b,1)$ for some $|b|\le 1$ and so the above probability is greater than $\inf_{|x|\le 2} p_1(x)\pi \ge 1/20$. Therefore, the strong Markov property implies

$$P^{w(r)}(\tau_{t+r}<\infty)\le 20P^{w(r)}\big(B_{1-t-r}\in D(x,2^{-k})\big).$$

Returning to (5.8), we have

$$\begin{aligned}\mathbb{P}^*_w(L_\sigma) &\le 40\int_0^\zeta P^{w(r)}\big(B_{1-t-r}\in D(x,2^{-k})\big)\,dr\\ &\le 40\int_0^{1-t}\min\big(2^{-2k}(1-t-r)^{-1},1\big)\,dr\\ &\le 40\cdot 2^{-2k}\big(1+\log^+((1-t)2^{2k})\big).\end{aligned}$$

Use this in (5.7) to complete the proof. □

LEMMA 5.4. *If $(t,y)\in D_k$, then:*

(a) $\mathbb{N}_{t,y}(\exp(\lambda X_1(D(x,2^{-k})))-1|\mathscr{E}^{D_k})\ge \exp((\lambda/20)\langle X^{D_k},1\rangle)-1$, for all $\lambda\ge 0$;

(b) $\mathbb{N}_{t,y}(\exp(\lambda X_1(D(x,2^{-k})))-1|\mathscr{E}^{D_k})\le \exp(2\lambda\langle X^{D_k},1\rangle)$, for all $\lambda\in[0,2^{2k}/4]$.

PROOF. Under $\mathbb{N}_{t,y}$, $X_1(W)(D(x,2^{-k}))$ is the sum of the contributions to this mass from the excursions of W outside D_k. Therefore, if $\mathscr{N}$ is as in Theorem 2.2 with $D=D_k$, then

$$X_1(D(x,2^{-k}))=\int X_1(W)(D(x,2^{-k}))\mathscr{N}(dW)$$

and the special Markov property shows that

$$\begin{aligned}
&\mathbb{N}_{t,y}\big(\exp(\lambda X_1(D(x,2^{-k}))) - 1 \mid \mathscr{E}^{D_k}\big)\\
(5.10)\qquad &= \exp\Big(\int \mathbb{N}_{u,z}\big(\exp(\lambda X_1(D(x,2^{-k}))) - 1\big) X^{D_k}(du,dz)\Big) - 1\\
&\geq \exp\Big(\lambda \int \mathbb{N}_{u,z}\big(X_1(D(x,2^{-k}))\big) X^{D_k}(du,dz)\Big) - 1.
\end{aligned}$$

It is easy to see that X^{D_k} is a.e. supported on $\partial' D_k = \partial D_k \cap \{(u,z)\colon u < 1\}$ [use (2.8)]. For $(u,z) \in \partial' D_k$, (2.6) gives us

$$\mathbb{N}_{u,z}\big(X_1(D(x,2^{-k}))\big) = P^{u,z}\big(B_1 \in D(x,2^{-k})\big) \geq \frac{1}{20},$$

where in the last part we argue as in (5.9) (with $t = r = 0$). Use this in the above to prove (a).

For (b) we again use the equality in (5.10). Lemma 3.1 and (2.5) imply

$$\mathbb{N}_{u,z}\big(\exp(\lambda X_1(D(x,2^{-k}))) - 1\big) \leq 2\lambda P^z\big(B_{1-u} \in D(x,2^{-k})\big) \leq 2\lambda$$

provided that $0 \leq \lambda \leq \frac{1}{4} G(1(D(x,2^{-k})), 1-u)^{-1}$. If $(u,z) \in \partial' D_k$, then $G(1(D(x,2^{-k})), 1-u) \leq 1-u \leq 2^{-2k}$ and (b) follows upon using this bound in the equality of (5.10). □

As an immediate consequence of Lemmas 3.1 and 5.4(a), we see that, for $(t,y) \in D_k$,

$$\begin{aligned}
\mathbb{N}_{t,y}\big(\exp((\lambda/20)\langle X^{D_k},1\rangle) - 1\big) &\leq \mathbb{N}_{t,y}\big(\exp(\lambda X_1(D(x,2^{-k}))) - 1\big)\\
(5.11)\qquad &\leq 2\lambda P^y\big(B_{1-t} \in D(x,2^{-k})\big),\\
&\qquad \forall\, \lambda \in \big[0, \tfrac{1}{4}(1-t)^{-1}\big].
\end{aligned}$$

6. Lower bound for the Hausdorff measure of the support: proof. In this section we prove Theorem 5.1. Fix w in $C([0,1],\mathbb{R}^2)$, let $x = w(1)$ and write $\mathbb{P}$ for $\mathbb{P}^{(w)}$. Recall the definitions of Z_1 and Z^D from the previous section and recall also that it suffices to prove (5.6).

PROPOSITION 6.1. *There is a constant* c_{12} (350 *will do*) *such that if* $r(n) = 2^{-2^n}$, *then*

$$\limsup_{n\to\infty} \langle Z^{D_{2^n}}, 1\rangle h(r(n))^{-1} \leq c_{12}, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Use (3.4) to see that if $\lambda = 1/12$, then for $k \in \mathbb{N}$,

$$\lambda 2^{2k} k^{-1} \leq \tfrac{1}{4} G\big(1(D(x,2^{-k})),1\big)^{-1}.$$

Therefore, Lemma 3.1 shows that

$$\begin{aligned}
&\mathbb{P}\big(\exp(\lambda 2^{2k}k^{-1}Z_1(D(x,2^{-k})))\big)\\
&\quad= \exp\left(4\int_0^1 \mathbb{N}_{t,w(t)}\big(\exp(\lambda 2^{2k}k^{-1}X_1(D(x,2^{-k}))) - 1\big)\,dt\right)\\
&\quad\le \exp\left(8\lambda 2^{2k}k^{-1}\int_0^1 P^{w(t)}\big(B_{1-t}\in D(x,2^{-k})\big)\,dt\right) \quad [\text{use (2.5) as well}]\\
&\quad\le \exp\big(8\lambda 2^{2k}k^{-1}G\big(1(D(x,2^{-k})),1\big)\big).
\end{aligned}$$

Use (3.4) again to see that $G(1(D(x,2^{-k})),1)\le(3\ln 2)2^{-2k}k$ and, therefore,

$$\mathbb{P}\big(\exp(\lambda 2^{2k}k^{-1}Z_1(D(x,2^{-k})))\big)\le 4.$$

Using Lemma 5.4, and noting that $u(W)=t$ for $\mathbb{N}_{t,w(t)}$-a.a. W, we have

$$\begin{aligned}
&\mathbb{P}\Big(\exp\big(\lambda 2^{2k}(20k)^{-1}\langle Z^{D_k},1\rangle\big)\Big)\\
&\quad= \exp\left(4\int_0^1 1(t<\tau(w))\mathbb{N}_{t,w(t)}\Big(\exp\big(\lambda 2^{2k}(20k)^{-1}\langle X^{D_k},1\rangle\big) - 1\Big)\,dt\right)\\
&\quad\le \exp\left(4\int_0^1 \mathbb{N}_{t,w(t)}\big(\exp(\lambda 2^{2k}k^{-1}X_1(D(x,2^{-k}))) - 1\big)\,dt\right)\\
&\quad= \mathbb{P}\big(\exp(\lambda 2^{2k}k^{-1}Z_1(D(x,2^{-k})))\big)\\
&\quad\le 4.
\end{aligned}$$

It follows in particular that, for $k\le k_0$,

$$\begin{aligned}
\mathbb{P}\big(\langle Z^{D_k},1\rangle \ge (241)2^{-2k}k\log\log k\big) &\le 4\exp(-(241/240)\log\log k)\\
&= 4(\log k)^{-241/240}.
\end{aligned}$$

If $k=2^n$, this is summable over n and an application of the Borel–Cantelli lemma completes the proof. □

For every $k\in\mathbb{N}$, let $n=n(k)$ be the unique integer such that $2^{n-1}\le k<2^n$. If $W\in\Omega_0$, let $z(W)=W_0(u(W))$ so that $(u(W),z(W))$ denotes the "starting point" of W. Clearly $(u(W),z(W))=(t,z)$, $\mathbb{N}_{t,z}$-a.e.

PROPOSITION 6.2. *For every $a>0$, $A>0$, there is a constant $c_{13}(a)$ in $(0,1]$ and an integer $k_0(a,A)$ such that if $k\ge k_0$ and $n=n(k)$, then*

$$\mathbb{P}\big(\langle Z^{D_{2^n}},1\rangle \ge c_{13}(a)Ah(2^{-2^n})\big|\mathscr{F}^{D_k}\big)\ge \tfrac{3}{4}\quad on\ \big\{Z^{D_k}(U_k)\ge Ah(2^{-k})\big\}.$$

PROOF. We use the special Markov property under $\mathbb{P}=\mathbb{P}^{(w)}$ (Proposition 5.2). Let $\mathscr{N}_p^k$ be as in that result with $D=D_k$, and write τ_k for τ_{D_k}, the exit time from D_k. Decompose $X^{D_{2^n}}(W)$ according to the contributions from excursions outside D_k to see that

$$\begin{aligned}
Z^{D_{2^n}} &\ge \int 1(u(W)<\tau_k(w))X^{D_{2^n}}(W)\mathscr{M}(dW)\\
&= \int X^{D_{2^n}}(W)\mathscr{N}_p^k(dW).
\end{aligned}$$

Let

$$\tilde{Z}^{D_{2^n}} = \int 1_{U_k}(u(W), z(W)) X^{D_{2^n}}(W) \mathcal{N}_p^k(dW)$$

denote the contribution to $X^{D_{2^n}}$ from excursions outside D_k which "start" in $U_k = U_k(a)$. By Proposition 5.2,

$$\begin{aligned}\mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k}) &= \int\int 1_{U_k}(u(W), z(W)) \langle X^{D_{2^n}}, 1\rangle \mathbb{N}_{t,z}(dW) Z^{D_k}(dt, dz)\\ &= \int 1_{U_k}(t, z) \mathbb{N}_{t,z}(\langle X^{D_{2^n}}, 1\rangle) Z^{D_k}(dt, dz).\end{aligned}$$

If $(t, z) \in U_k$, then

$$\begin{aligned}\mathbb{N}_{t,z}(\langle X^{D_{2^n}}, 1\rangle) &= P^{t,z}(\tau_{2^n} < \infty) \quad [\text{see } (2.8)]\\ &\geq P^z\big(B_{1-t} \in D(x, 2^{-2^n})\big)\\ &= P^0\Big(B_1 \in D\big((x-z)(1-t)^{-1/2}, 2^{-2^n}(1-t)^{-1/2}\big)\Big)\\ &\geq c(a) 2^{2(k-2^n)}\end{aligned}$$

for $c(a) = \exp(-(a+1)^2/2)/2$. We have used the fact that, for $(t, z) \in U_k$, $2^{-2^n}(1-t)^{-1/2} \geq 2^{k-2^n}$ and $|x - z|(1-t)^{-1/2} \leq a$. Using this in the above, we obtain

$$\mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k}) \geq c(a) 2^{2(k-2^n)} Z^{D_k}(U_k). \tag{6.1}$$

Then, by Proposition 5.2 and a standard formula for the second moment measure of a Poisson measure,

$$\begin{aligned}&\mathbb{P}\Big(\big(\langle \tilde{Z}^{D_{2^n}}, 1\rangle - \mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k})\big)^2 \Big| \mathscr{F}^{D_k}\Big)\\ &\quad = \int\Big[\int 1_{U_k}(u(W), z(W)) \langle X^{D_{2^n}}, 1\rangle^2 \mathbb{N}_{t,z}(dW)\Big] Z^{D_k}(dt, dz)\\ &\quad = \int 1_{U_k}(t, z) \mathbb{N}_{t,z}(\langle X^{D_{2^n}}, 1\rangle^2) Z^{D_k}(dt, dz)\\ &\quad \leq 80(1 + 2(2^n - k)) 2^{-2\cdot 2^n} \mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k})\\ &\quad \equiv \sigma(k)^2.\end{aligned}$$

In the last inequality we used Lemma 5.3 and the fact that $1 - t \leq 2^{-2k}$ for $(t, z) \in U_k$. Chebyshev's inequality implies

$$\mathbb{P}\Big(\big|\langle \tilde{Z}^{D_{2^n}}, 1\rangle - \mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k})\big| \geq 2\sigma(k) \Big| \mathscr{F}^{D_k}\Big) \leq \tfrac{1}{4}; \tag{6.2}$$

(6.1) shows that on the set $\{Z^{D_k}(U_k) \geq Ah(2^{-k})\}$ and, for $k \geq k_0(a, A)$,

$$\begin{aligned}\mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k}) &\geq c(a) A 2^{2(k-2^n)} h(2^{-k})\\ &\geq \frac{c(a)}{4} A h(2^{-2^n})\end{aligned}$$

and so an easy calculation gives, for $k \geq k_0(a, A)$,

$$2\sigma(k) \leq \tfrac{1}{2}\mathbb{P}(\langle \tilde{Z}^{D_{2^n}}, 1\rangle | \mathscr{F}^{D_k}).$$

Use the above two bounds in (6.2) to prove the result with $c_{13}(a) = (c(a))/8$. □

COROLLARY 6.3. *We have*

$$\limsup_{k\to\infty} Z^{D_k}(U_k(a))h(2^{-k})^{-1} \leq c_{12}c_{13}(a)^{-1}, \qquad \mathbb{P}\text{-}a.s.\ \forall\, a > 0.$$

PROOF. Let $c > c_{12}c_{13}(a)^{-1}$ and let $(k_j(\omega), j \in \mathbb{N})$ be the successive "times" for which $Z^{D_k}(U_k) \geq ch(2^{-k})$, where $k_j = \infty$ if the jth such time does not exist. Since Z^{D_k} is $\mathscr{F}^{D_k}$-measurable [Le Gall (1995)], $\{k_j\}$ are $(\mathscr{F}^{D_k})$-stopping times. It suffices to show that $\mathbb{P}(k_j < \infty\ \forall\, j) = 0$. Let $n_j = n(k_j)$ and

$$A_j = \left\{k_j < \infty, \langle Z^{D_{2^{n_j}}}, 1\rangle \geq c_{13}(a)ch(2^{-2^{n_j}})\right\}.$$

Proposition 6.2 (with $A = c$) and the fact that k_j is a stopping time show that, for j large,

$$\mathbb{P}(A_j) \geq \tfrac{3}{4}\mathbb{P}(k_j < \infty).$$

It follows that

$$\mathbb{P}(\limsup A_j) \geq \limsup \mathbb{P}(A_j) \geq \tfrac{3}{4}\mathbb{P}(k_j < \infty\ \forall\, j).$$

The left side is zero by Proposition 6.1 because $c_{13}(a)c > c_{12}$ and the proof is complete. □

PROPOSITION 6.4. *If $a \geq 8$, then for any $c > 0$,*

$$\sum_{k=1}^{\infty} \mathbb{P}\bigg(\langle Z^{D_{k+1}}, 1\rangle \geq ch(2^{-k-1}), Z^{D_k}(U_k(a)) \leq \frac{c}{1000}h(2^{-k}), \langle Z^{D_k}, 1\rangle \leq ch(2^{-k})\bigg) < \infty.$$

PROOF. Let $\mathscr{N}_p^k$ and τ_k be as in the proof of Proposition 6.2 and fix $c > 0$. For W such that $u(W) < \tau_k(w)$, decompose $X^{D_{k+1}}(W)$ according to the contributions from individual excursions outside D_k. This shows that

$$Z^{D_{k+1}} = \hat{Z}^{D_{k+1}} + \bar{Z}^{D_{k+1}} + \tilde{Z}^{D_{k+1}},$$

where

$$\hat{Z}^{D_{k+1}} = \int 1(\tau_k(w) \leq u(W) < \tau_{k+1}(w))X^{D_{k+1}}(W)\mathscr{M}(dW),$$

$$\bar{Z}^{D_{k+1}} = \int 1_{U_k^c}(u(W), z(W))X^{D_{k+1}}(W)\mathscr{N}_p^k(dW)$$

and

$$\tilde{Z}^{D_{k+1}} = \int 1_{U_k}(u(W), z(W))X^{D_{k+1}}(W)\mathscr{N}_p^k(dW).$$

Consider $\hat{Z}^{D_{k+1}}$ first. Clearly $\tau_k \geq 1 - 2^{-2k}$ and so if $\lambda \in [0, 2^{2k-2}]$, (5.11) shows that

$$\mathbb{P}\left(\exp\left(\frac{\lambda}{20}\langle \hat{Z}^{D_{k+1}}, 1\rangle\right)\right) = \exp\left(4\int_{\tau_k(w)}^{\tau_{k+1}(w)} \mathbb{N}_{t,w(t)}\left(\exp\left(\frac{\lambda}{20}\langle X^{D_{k+1}}, 1\rangle\right) - 1\right) dt\right)$$
$$\leq \exp\left(4\int_{1-2^{-2k}}^{1} 2\lambda P^{w(t)}\left(B_{1-t} \in D(x, 2^{-k-1})\right) dt\right)$$
$$\leq \exp(8\lambda 2^{-2k}).$$

This bound easily implies

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\langle \hat{Z}^{D_{k+1}}, 1\rangle \geq \frac{c}{3}h(2^{-k-1})\right) < \infty. \tag{6.3}$$

Next we bound

$$\mathbb{P}\left(\langle Z^{D_k}, 1\rangle \leq ch(2^{-k}), \langle \bar{Z}^{D_{k+1}}, 1\rangle \geq \frac{c}{3}h(2^{-k-1})\right). \tag{6.4}$$

Use Proposition 5.2 and (5.11) to see that, for $\lambda \leq 2^{2k}/4$,

$$\mathbb{P}\left(\exp\left(\frac{\lambda}{20}\langle \bar{Z}^{D_{k+1}}, 1\rangle\right)\middle|\mathscr{F}^{D_k}\right)$$
$$= \exp\left(\int 1_{U_k^c}(t,z)\mathbb{N}_{t,z}\left(\exp\left(\frac{\lambda}{20}\langle X^{D_{k+1}}, 1\rangle\right) - 1\right) Z^{D_k}(dt, dz)\right) \tag{6.5}$$
$$\leq \exp\left(2\lambda\int 1_{U_k^c}(t,z)P^z\left(B_{1-t} \in D(x, 2^{-k-1})\right)Z^{D_k}(dt, dz)\right).$$

For the last inequality note that $Z^{D_k}(dt, dz)$ is supported on $\partial' D_k = \partial D_k \cap \{(u,z)\colon u < 1\}$ (since X^{D_k} is; see the proof of Lemma 5.4) and so we may assume $1 - t \leq 2^{-2k}$. If $(t,z) \in \partial' D_k \cap U_k(a)$, then some simple algebra leads to

$$(|x - z| - 2^{-k-1})(1-t)^{-1/2} \geq a - \tfrac{1}{2}\sqrt{a^2+1},$$

and, therefore,

$$P^z\left(B_{1-t} \in D(x, 2^{-k-1})\right)$$
$$= P^0\left(B_1 \in D\left((x-z)(1-t)^{-1/2}, 2^{-k-1}(1-t)^{-1/2}\right)\right)$$
$$\leq P^0\left(|B_1| \geq a - \tfrac{1}{2}\sqrt{a^2+1}\right) \equiv \eta(a).$$

Clearly $\eta(a)$ approaches zero as $a \to \infty$ and in fact an elementary calculation gives $\eta(a) \leq 0.001$ if $a \geq 8$. On the set $\{\langle Z^{D_k}, 1\rangle \leq ch(2^{-k})\}$ we get the bound

$$\mathbb{P}\left(\exp\left(\frac{\lambda}{20}\langle \bar{Z}^{D_{k+1}}, 1\rangle\right)\middle|\mathscr{F}^{D_k}\right) \leq \exp(4\lambda\eta(a)ch(2^{-k})), \qquad \lambda \leq \frac{2^{2k}}{4}.$$

This shows that (6.4) is bounded by (take $\lambda = 2^{2k}/4$ in the above)

$$\exp\left(-\frac{\lambda}{60}ch(2^{-k-1}) + 4\lambda\eta(a)ch(2^{-k})\right) \leq \exp\left(-\frac{c}{4}\tilde{\theta}(2^{-k})\left(\frac{1}{240} - 4\eta(a)\right)\right),$$

where $\tilde{\theta}(r) = \log^+(1/r)\log^+\log^+\log^+(1/r)$. For $a \geq 8$, so that $\eta(a) \leq 0.001$, this is summable in $k \geq k_0$ and so

$$(6.6)\qquad \sum_{k=1}^{\infty} \mathbb{P}\Big(\langle Z^{D_k}, 1\rangle \leq ch(2^{-k}), \langle \bar{Z}^{D_{k+1}}, 1\rangle \geq \frac{c}{3}h(2^{-k-1})\Big) < \infty.$$

It remains to bound

$$(6.7)\qquad \mathbb{P}\Big(\langle \tilde{Z}^{D_{k+1}}, 1\rangle \geq \frac{c}{3}h(2^{-k-1}), Z^{D_k}(U_k) \leq \frac{c}{1000}h(2^{-k})\Big).$$

Argue as in (6.5) to see that, for $\lambda \leq 2^{2k-2}$ and on the set $\{Z^{D_k}(U_k) \leq (c/1000)h(2^{-k})\}$,

$$\mathbb{P}\Big(\exp\Big(\frac{\lambda}{20}\langle \tilde{Z}^{D_{k+1}}, 1\rangle\Big)\Big|\mathscr{F}^{D_k}\Big) \leq \exp\Big(4\lambda\frac{c}{1000}h(2^{-k})\Big).$$

Take $\lambda = 2^{2k-2}$ and conclude that (6.7) is bounded by

$$\exp\Big(-\frac{\lambda c}{60}h(2^{-k-1}) + \lambda\frac{c}{250}h(2^{-k})\Big) \leq \exp\Big(-\frac{c}{4}\tilde{\theta}(2^{-k})\Big(\frac{1}{240} - \frac{1}{250}\Big)\Big).$$

This is summable over $k \geq k_0$ and hence

$$\sum_{k=1}^{\infty} \mathbb{P}\Big(\langle \tilde{Z}^{D_{k+1}}, 1\rangle \geq \frac{c}{3}h(2^{-k-1}), Z^{D_k}(U_k) \leq \frac{c}{1000}h(2^{-k})\Big) < \infty.$$

This, together with (6.3) and (6.6), proves the proposition. □

PROOF OF THEOREM 5.1. Set $a = 8$ and write U_k for $U_k(8)$. We first prove that

$$(6.8)\qquad \limsup_{k\to\infty} \langle Z^{D_k}, 1\rangle h(2^{-k})^{-1} \leq 1000c_{12}c_{13}(8)^{-1}, \qquad \mathbb{P}\text{-a.s.}$$

Let $c > 1000c_{12}c_{13}(8)^{-1}$ and suppose the lim sup in (6.8) exceeds c. Assume also we are outside the $\mathbb{P}$-null sets off which the conclusions of Proposition 6.1 and Corollary 6.3 hold. By Proposition 6.1 and the fact that $c > c_{12}$, we see there are infinitely many k for which $\langle Z^{D_k}, 1\rangle h(2^{-k})^{-1}$ crosses from $[0, c]$ to $[c, \infty)$. Combine this with the conclusion of Corollary 6.3 to conclude that for infinitely many values of k we have

$$\langle Z^{D_k}, 1\rangle \leq ch(2^{-k}), \qquad \langle Z^{D_{k+1}}, 1\rangle \geq ch(2^{-k-1}),$$
$$Z^{D_k}(U_k) \leq \frac{c}{1000}h(2^{-k}).$$

Proposition 6.4 shows this last event is $\mathbb{P}$-null and so (6.8) is proved.

To complete the proof of (5.6) write

$$Z_1(D(x, 2^{-k})) = Z_k^{(1)} + Z_k^{(2)},$$

where [again denoting by τ_k the exit time of $(t, w(t))$ from D_k]

$$Z_k^{(1)} = \int 1(u(W) < \tau_k) X_1(W)(D(x, 2^{-k}))\mathscr{M}(dW)$$

and

$$Z_k^{(2)} = \int 1(u(W) \geq \tau_k) X_1(W)(D(x, 2^{-k}))\mathscr{M}(dW).$$

Argue as in the derivation of (6.3) [use Lemma 3.1 in place of (5.11)] to see that

$$\sum_k \mathbb{P}\big(Z_k^{(2)} > \varepsilon h(2^{-k})\big) < \infty, \qquad \forall\, \varepsilon > 0. \tag{6.9}$$

Turning to $Z_k^{(1)}$, note that if $\mathscr{N}_p^k$ is as in the proof of Proposition 6.2,

$$Z_k^{(1)} = \int X_1(W)(D(x, 2^{-k}))\mathscr{N}_p^k(dW)$$

and, therefore, Proposition 5.2 shows that, for $\lambda \leq 2^{2k-2}$,

$$\begin{aligned}\mathbb{P}\big(\exp \lambda Z_k^{(1)} | \mathscr{F}^{D_k}\big) &= \exp\Big(\int \mathbb{N}_{t,z}\big(\exp\big(\lambda X_1(D(x, 2^{-k}))\big) - 1\big) Z^{D_k}(dt, dz)\Big) \\ &\leq \exp(2\lambda\langle Z^{D_k}, 1\rangle) \quad [\text{by } (5.11)].\end{aligned}$$

Taking $\lambda = 2^{2k-2}$, we get

$$\mathbb{P}\big(Z_k^{(1)} \geq 2\langle Z^{D_k}, 1\rangle + k2^{-2k} | \mathscr{F}^{D_k}\big) \leq e^{-k/4}.$$

Use the above, (6.9) and the Borel–Cantelli lemma to conclude that $\forall\, \varepsilon > 0$,

$$Z_1\big(D(x, 2^{-k})\big) \leq 2\langle Z^{D_k}, 1\rangle + k2^{-2k} + \varepsilon h(2^{-k}) \quad \text{for large } k \text{ a.s.}$$

Now (6.8) completes the proof of (5.6) with $c_{11} = 2000 c_{12} c_{13} (8)^{-1}$. □

REMARK. For those keeping score, our value of c_{10} in Theorem 5.1 is $2.24 \times 10^7 \times e^{81/2}$.

7. Proof of Theorem 1.1. By (1.4) it suffices to consider $\gamma = 4$. If $\mu \in M_F(\mathbb{R}^2) - \{0\}$, Theorem 1.1 of Evans and Perkins (1991) shows that $\mathbb{Q}_\mu(Y_t \in \cdot)$ and $\mathbb{Q}_{\delta_0}(Y_1 \in \cdot)$ are equivalent laws. Therefore, we only need consider $t = 1$ and $\mu = \delta_0$ in Theorem 1.1.

Theorem 5.1 and the Rogers–Taylor result [e.g., Perkins (1988), Theorem 1.4] imply that

$$Y_1(A) \leq c_{10} h - m(A \cap S(Y_1)), \qquad \forall\, A \in \mathbb{B}(\mathbb{R}^2),\ \mathbb{Q}_{\delta_0}\text{-a.s.}$$

This, together with Theorem 4.2, shows that Y_1 and $\tilde{Y}_1(\cdot) = h - m(\cdot \cap S(Y_1))$ are $\mathbb{Q}_{\delta_0}$-a.s. equivalent measures on $\mathbb{R}^2$ and the Radon–Nikodym derivative of Y_1 with respect to $\tilde{Y}_1$ satisfies

$$c_9 \leq \frac{dY_1}{d\tilde{Y}_1}(x) \leq c_{10} \tag{7.1}$$

for Y_1-a.a. x, $\mathbb{Q}_{\delta_0}$-a.s.

We have to show that, for a certain constant c_0,

$$\frac{dY_1}{d\tilde{Y}_1}(x) = c_0 \tag{7.2}$$

for Y_1-a.a. x, $\mathbb{Q}_{\delta_0}$-a.s. To do this, we use a zero–one law as in the proof of Theorem 5.2 of Dawson and Perkins (1991). The argument is almost identical to the $d \geq 3$ case treated there and so we only sketch the proof, using the notation of the present work.

Let (r_n) be a fixed deterministic sequence decreasing to 0 and, for every measure ν on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, set

$$f(x, \nu) = \limsup_{n \to \infty} \frac{\nu(B(x, r_n))}{h - m(B(x, r_n) \cap S(\nu))}.$$

By standard results on the derivation of measures [see, e.g., Federer (1969), Theorems 2.9.5 and 2.9.7], we have

$$\frac{dY_1}{d\tilde{Y}_1}(x) = f(x, Y_1)$$

for Y_1-a.a. x, $\mathbb{Q}_{\delta_0}$-a.s. Therefore, we need to check that $f(x, Y_1) = c_0$, for Y_1-a.a. x, $\mathbb{Q}_{\delta_0}$-a.s. The same arguments we have used to derive Theorem 5.1 from (5.3) show that this last statement follows from

$$f(x, X_1) = c_0 \quad \text{for } X_1\text{-a.a. } x, \mathbb{N}_0\text{-a.e.} \tag{7.3}$$

To prove (7.3) we rely on the Palm measure formula (5.4). We start with a result which states that only "close cousins" contribute to $X_1(B(x, r))$ for X_1-a.a. x and r small. Recall the notation in (5.4).

LEMMA 7.1. *There exists a constant c_{14} such that*

$$\begin{aligned} &\mathbb{P}^{(w)}\left(\int 1\left(u(W) \leq 1 - c_{14}\left(\log\frac{1}{r}\right)^{-1}\right) X_1(W)(B(w(1), r))\mathscr{M}(dW) > 0\right) \\ &\quad \leq c_{14}\left(\log\log\frac{1}{r}\right)\left(\log\frac{1}{r}\right)^{-1}, \end{aligned} \tag{7.4}$$

for $0 < r \leq (c_{14})^{-1}$ and all w in $C([0,1], \mathbb{R}^2)$.

Although we have altered the notation to be consistent with that used here, this is the two-dimensional version of (5.7) in Dawson and Perkins (1991). For the proof, note that by the definition of $\mathscr{M}$ the left-hand side of (7.4) is

$$1 - \exp\left(-4\int_0^{1 - c_{14}(\log 1/r)^{-1}} du\, \mathbb{N}_{u, w(u)}(X_1(B(w(1), r)) > 0)\right)$$

and use the bound

$$\mathbb{N}_{u,x}(X_1(B(y,r)) > 0) \le c(1-u)^{-1}(\log(1/r))^{-1}, \qquad \text{for } x,y \in \mathbb{R}^2,\ r \in (0,1/2),\ 1-u > r,$$

from Le Gall [(1994b), Theorem 2].

Let us fix $r_n = \exp(-n^2)$. We then use Lemma 7.1 and the Borel–Cantelli lemma to conclude that, for every $w \in C([0,1],\mathbb{R}^2)$, $\mathbb{P}^{(w)}(d\omega)$-a.s., there is an integer $n_0(\omega)$ such that, for $n \ge n_0$,

$$\begin{aligned}\int X_1(W)(B(w(1),r_n))\mathscr{M}(dW) &= \int 1(1 - c_{14}n^{-2} < u(W) \le 1)X_1(W)(B(w(1),r_n))\mathscr{M}(dW)\\ &= Z_1^n(B(w(1),r_n)),\end{aligned}$$

where $Z_1^n = \int 1(1 - c_{14}n^{-2} < u(W) \le 1)X_1(W)\mathscr{M}(dW)$. We have, therefore, also $\mathbb{P}^{(w)}$-a.s.,

$$\begin{aligned}f\Big(w(1), \int X_1(W)\mathscr{M}(dW)\Big) \\ = \limsup_{n\to\infty} \frac{Z_1^n(B(w(1),r_n))}{h-m(B(w(1),r_n)\cap S(Z_1^n))}.\end{aligned} \tag{7.5}$$

Note that Z_1^n is measurable with respect to the σ-field generated by $1(1 - c_{14}n^{-2} < u(W) \le 1)\mathscr{M}(dW)$. By a standard zero–one law for Poisson point measures, the right-hand side of (7.5) is a constant $c_0(w)$, $\mathbb{P}^{(w)}$-a.s. By the Blumenthal zero–one law applied to the process $w(1) - w(1-\cdot)$, this constant does not depend on w, outside a set of P^0-measure 0. We have thus proved that

$$f\Big(w(1), \int X_1(W)\mathscr{M}(dW)\Big) = c_0, \qquad \mathbb{P}^{(w)}\text{-a.s. for } P^0\text{-a.a. } w.$$

Equation (5.4) and the above imply (7.3), which gives (7.2). Finally, (7.1) shows that $0 < c_0 < \infty$, completing the proof. □

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