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SOME NEW CLASSES OF EXCEPTIONAL TIMES OF LINEAR BROWNIAN MOTION

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We study certain classes of exceptional times of a linear Brownian motion $(B_t,\ t \geq 0)$. In particular, we consider the set K^- of all instants $t \in [0,1]$ such that the value B_t of the Brownian motion at time t is greater than its mean value over all intervals $[s,t]$, $s < t$. We also study the subset K of K^- of all instants t such that in addition B_t is greater than the mean value of B over the intervals $[t,s]$, $t < s \leq 1$. We compute the Hausdorff dimension of K^-, K and some other related sets of exceptional times. These results are closely related to a recent work of Sinai motivated by the analysis of solutions to the Burgers equation with random initial data. The proofs involve studying suitable approximations of the sets K^- and K, and deriving precise estimates for the probability that a given time t belongs to these approximations. A delicate zero–one law argument is also needed to prove that the lower bound on the dimension of K holds with probability 1.

1. Introduction. The aim of this paper is to study certain sets of exceptional times of linear Brownian motion. Throughout this work, we consider a linear Brownian motion $(B_t,\ t \geq 0)$ such that $B_0 = 0$ a.s. under the probability measure P. We are mainly interested in the following three random closed subsets of $[0,1]$:

$$K^- = \left\{ t \in [0,1];\ \frac{1}{t-s}\int_s^t B_u\,du \leq B_t,\ \text{for every } s \in [0,t) \right\},$$

$$K = \left\{ t \in [0,1];\ \frac{1}{t-s}\int_s^t B_u\,du \leq B_t,\ \text{for every } s \in [0,t) \cup (t,1] \right\},$$

$$K' = \left\{ t \in [0,1];\ \frac{1}{t-s}\int_s^t B_u\,du \leq B_t,\ \text{for every } s \in [0,t) \right.$$
$$\left. \text{and } \frac{1}{s-t}\int_t^s B_u\,du \geq B_t,\ \text{for every } s \in (t,1] \right\}.$$

For $t \in [0,1]$, t belongs to K^- if and only if the value of B at time t is greater than its mean value over all intervals $[s,t]$, $s \in [0,t)$. In the case of K, we also require B_t to be greater than the mean value over all intervals $[t,s]$, $s \in (t,1]$. Finally, K' contains the instants t such that B_t is greater than the mean value of B over all intervals $[s,t]$, $s \in [0,t)$, and smaller than the mean value over all intervals $[t,s]$, $s \in (t,1]$.

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It is easy to check that, for every fixed $t \in (0,1]$, $P(t \in K^-) = 0$ (note that $0 \in K^-$ by definition) and a fortiori, $P(t \in K) = P(t \in K') = 0$. Indeed, $t \in K^-$ if and only if, for every $s \in [0,t]$,

$$\int_0^s B_u^{(t)}\, du \geq 0,$$

where $B_u^{(t)} = B_t - B_{t-u}$ is a Brownian motion started at 0 and run on the time interval $[0,t]$. An application of the zero–one law immediately shows that this event has probability 0. In this sense, we say that K^-, K and K' are sets of exceptional times.

On the other hand, it is also very easy to check that K^- contains nonzero times and K is nonempty. In fact, K^- obviously contains the set

$$H = \Big\{t \in [0,1],\ B_t = \sup_{s \leq t} B_s\Big\}.$$

By a famous theorem of Lévy, the set H is distributed as the zero set of B on the time interval $[0,1]$, so that its Hausdorff dimension is 1/2, by a theorem of Taylor (1955). Similarly, K contains the (a.s. unique) instant $\rho \in [0,1]$ such that

$$B_\rho = \sup_{s \in [0,1]} B_s.$$

For reasons that will appear later, it is not so easy to exhibit an element of K'. Note that K' would contain any time t such that $B_t = \sup_{0 \leq s \leq t} B_s = \inf_{t \leq s \leq 1} B_s$. However, such times almost surely do not exist, by the nonexistence of increase points of linear Brownian motion. One may interpret the definition of the elements of K' as a weakened form of the definition of an increase point.

We now state the main result of the present work, which shows in particular that K^- and K are in a sense much bigger than H and $\{\rho\}$, respectively. We denote by $\dim A$ the Hausdorff dimension of a subset A of $\mathbb{R}$.

THEOREM 1. *We have*

$$\dim K^- = \tfrac{3}{4}, \quad a.s.,$$
$$\dim K = \tfrac{1}{2}, \quad a.s.,$$
$$\dim K' \leq \tfrac{1}{2}, \quad a.s.,$$

and

$$P\big(\dim K' = \tfrac{1}{2}\big) > 0,$$
$$P(K' = \varnothing) > 0.$$

The same identities hold if the weak inequalities in the definition of K^-, K and K' are replaced by strict inequalities.

Our original motivation for Theorem 1 came from the recent work of Sinai (1992). Let us first present a more geometric interpretation of the sets K, K^- and K' that will allow us to make the connection with the results of Sinai (1992). For every $t \geq 0$, set

$$W_t = \int_0^t B_s \, ds.$$

Note that the tangent line to the curve $t \to W_t$ at $t = t_0$ is given by the line $t \to W_{t_0} + (t - t_0)B_{t_0}$. As a consequence, $t_0 \in K'$ if and only if the whole curve $(W_t,\ 0 \leq t \leq 1)$ lies above this tangent line. Similarly, $t_0 \in K^-$ if and only if the curve $(W_t,\ 0 \leq t \leq t_0)$ lies above the tangent line at t_0 and t_0 belongs to K if, in addition, the curve $(W_t,\ t_0 \leq t \leq 1)$ lies below this tangent line. Theorem 1 implies in particular that the set of times where the tangent line to the curve $(W_t,\ 0 \leq t \leq 1)$ intersects the curve at only one point has dimension 1/2.

Sinai (1992) considered a related problem motivated by the statistical analysis of discontinuities of solutions of the Burgers equation with random initial data. Sinai replaced the Brownian motion B by a Brownian motion with drift $\beta_t = B_t + t$, which makes it possible to work on the time interval $[0, \infty)$, instead of $[0, 1]$. Precisely, Sinai considered the set S of all positive times t_0 such that the tangent line at t_0 to the curve $t \to \int_0^t \beta_s \, ds,\ t \geq 0$ intersects this curve at only one point. The main theorem of Sinai (1992) states that the Hausdorff dimension of S is almost surely equal to 1/2 [the arguments of Sinai (1992) are not quite complete, as the proofs of some basic lemmas have been postponed to a forthcoming publication]. This is clearly related to our Theorem 1, although none of the results can be deduced from the others. Our arguments are quite different from Sinai's approach.

The present work is organized as follows. In Section 2, we introduce the main notation and we establish certain preliminary estimates. We use in particular an explicit formula for the distribution of hitting times for the process W, which was recently derived by Lachal (1994). In Section 3, we obtain the upper bound for the Hausdorff dimension of the sets K^-, K and K'. To this end, we use the estimates of Section 2 to bound the probability that K^- or K intersects a given interval $[a, b]$. In Section 4, we prove that the lower bound for the Hausdorff dimension holds with positive probability. We use the standard technique of constructing a random measure supported on the random set and applying Frostman's lemma. This random measure is constructed as a limit of the normalized Lebesgue measure on suitable approximate sets of K^-, K and K', in the spirit of Le Gall (1992). We also verify that our results remain unchanged if one replaces the weak inequalities in the definition of K^-, K and K' by strict inequalities. Finally, in Section 5, we complete the proof of Theorem 1, using in particular a suitable zero–one law to prove that the lower bound holds with probability 1 for K^- and K. This is straightforward for K^-, but much more delicate in the case of K. In that case, we need some precise information on the behavior of the Brownian path near the time ρ of its maximum over $[0, 1]$. This information is provided by a decomposition theorem of the Brownian path on the time interval $[0, 1]$ due to Denisov [see Biane and Yor (1988)].

2. Notation and preliminary estimates. We start by defining suitable approximations of the sets K^-, K and K'. For every $\varepsilon \in [0, 1/2]$ and for $a \in [0, 1-\varepsilon]$, $b \in [\varepsilon, 1]$, we set

$$K^-_{\varepsilon,a} = \left\{ t \in [a+\varepsilon, 1];\ \int_s^t B_u\,du \le (t-s)\,B_t,\ \text{for every } s \in [a, t-\varepsilon] \right\},$$

$$K^+_{\varepsilon,b} = \left\{ t \in [0, b-\varepsilon];\ \int_t^s B_u\,du \le (s-t)\,B_t,\ \text{for every } s \in [t+\varepsilon, b] \right\},$$

$$K^*_{\varepsilon,b} = \left\{ t \in [0, b-\varepsilon];\ \int_t^s B_u\,du \ge (s-t)\,B_t,\ \text{for every } s \in [t+\varepsilon, b] \right\}.$$

We also set

$$K_{\varepsilon,a,b} = K^-_{\varepsilon,a} \cap K^+_{\varepsilon,b}, \qquad K'_{\varepsilon,a,b} = K^-_{\varepsilon,a} \cap K^*_{\varepsilon,b}$$

and

$$K^-_\varepsilon = K^-_{\varepsilon,0}, \quad K^+_\varepsilon = K^+_{\varepsilon,1}, \qquad K^*_\varepsilon = K^*_{\varepsilon,1}, \qquad K_\varepsilon = K_{\varepsilon,0,1}, \qquad K'_\varepsilon = K'_{\varepsilon,0,1}.$$

In agreement with the notation of Section 1, we then take $K^- = K^-_0$, $K^+ = K^+_0$, $K^* = K^*_0$, $K = K_0$ and $K' = K'_0$.

We denote by $|A|$ the Lebesgue measure of a Borel subset A of $\mathbb{R}$ and by $(\mathscr{F}_t)$ the canonical filtration of B.

Recall that we have defined $W_t = \int_0^t B_s\,ds$. The pair (W_t, B_t) is a Markov process with values in $\mathbb{R}^2$. It will be convenient to consider this Markov process starting at any fixed point (x, y) of $\mathbb{R}^2$. To this end, we slightly modify the previous definitions by assuming that on our basic probability space there is a collection of probability measures $(P_{x,y}, (x, y) \in \mathbb{R}^2)$ such that $P_{0,0} = P$ and for every $(x, y) \in \mathbb{R}^2$, we have the following under $P_{x,y}$:

1. The process (B_t) is a linear Brownian motion started at y.
2. $W_t = x + \int_0^t B_s\,ds$.

In other words, the law of $(B_t, W_t;\ t \ge 0)$ under $P_{x,y}$ coincides with the law of $(y + \gamma_t, x + yt + \int_0^t \gamma_s\,ds;\ t \ge 0)$, where γ is a linear Brownian motion started at 0. Unless otherwise stated, the subsequent statements hold under the probability measure $P = P_{0,0}$.

We then introduce the following random times. For $T \ge 0$, we set

$$\tau^-_T = \sup\{t \in [0, T];\ W_t = 0\},$$
$$\tau^+_T = \inf\{t \ge T;\ W_t = 0\},$$
$$\tau = \tau^+_0 = \inf\{t \ge 0;\ W_t = 0\}$$

with the usual conventions $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$. Finally, it will be useful to denote by (B'_t) a linear Brownian motion started at 0 independent of B under the probability P.

The next three propositions will play a crucial role in the proof of Theorem 1.

PROPOSITION 2. (a) *For every fixed* $t \in (0, \infty)$,

$$\lim_{\varepsilon \to 0} \left(\frac{t}{\varepsilon} \right)^{1/4} P(\tau_\varepsilon^+ > t) = \frac{3^{3/2}}{4\pi^{5/2}} \Gamma\left(\frac{1}{4} \right)^2. \tag{1}$$

(b) *There exists a positive constant* c_1 *such that, for every* $t,\ \varepsilon \in (0, \infty)$,

$$\left(\frac{t}{\varepsilon} \right)^{1/4} P(\tau_\varepsilon^+ > t) \le c_1. \tag{2}$$

REMARK. The exact value of the limiting constant in (1), or the value of α in the next proposition, is not really important for our purposes.

PROOF OF PROPOSITION 2. (a) If $0 < \varepsilon \le t$, we have, from Lachal [(1994), Corollaire 2, formule 2-(ii)],

$$\begin{aligned} P(\tau_\varepsilon^+ > t) &= P(\tau_t^- < \varepsilon) \\ &= \left(\frac{3}{2} \right)^{3/2} \pi^{-2} \int_0^\infty \frac{dz}{z} e^{2z} K_0\left(4z \sqrt{\frac{t}{\varepsilon}} \right) \int_0^{4z} \frac{d\theta}{\sqrt{\pi\theta}} e^{-3\theta/2}, \end{aligned}$$

where $K_0(u)$ is the modified Bessel function of index 0. Set $A = (t/\varepsilon)^{1/2}$. The change of variables $\theta = \eta^2/(3A)$, $z = u/(4A)$ gives

$$\sqrt{A}\, P(\tau_\varepsilon^+ > t) = \frac{3}{\sqrt{2\pi^5}} \int_0^\infty \frac{du}{u} \exp\left(\frac{u}{2A} \right) K_0(u) \int_0^{\sqrt{3u}} d\eta \, \exp\left(\frac{-\eta^2}{2A} \right). \tag{3}$$

According to Abramowitz and Stegun [(1965), pages 375–378], we have

$$K_0(u) = \sqrt{\frac{\pi}{2u}}\, e^{-u} (1 + O(u^{-1})) \quad \text{as } u \to \infty, \tag{4}$$

$$K_0(u) = \log \frac{1}{u} + O(1) \quad \text{as } u \to 0. \tag{5}$$

This allows us to use dominated convergence to pass to the limit $A \to \infty$ in the right-hand side of (3). The proof of (a) is then completed by the following formula [Abramowitz and Stegun (1965), page 486, formula 11.4.22]:

$$\int_0^\infty \frac{K_0(u)}{\sqrt{u}}\, du = 2^{-3/2} \Gamma\left(\frac{1}{4} \right)^2.$$

(b) The stated bound is trivial if $t \le \varepsilon$. If $t > \varepsilon$, we have $A > 1$ and we can use (3) to get

$$\sqrt{A}\, P(\tau_\varepsilon^+ > t) \le \frac{3^{3/2}}{\sqrt{2\pi^5}} \int_0^\infty \frac{du}{\sqrt{u}} e^{u/2} K_0(u),$$

where the last integral is finite by (4) and (5). □

PROPOSITION 3. *Let* $0 \le a < b \le 1$ *and* $\alpha = (3^{3/2}/8\pi^{5/2})\Gamma(1/4)^2$.
(a) *For every fixed* $t \in (a,b)$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/4}\, P(t \in K^-_{\varepsilon,a}) = \frac{\alpha}{(t-a)^{1/4}}, \tag{6}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^{-1/2}\, P(t \in K_{\varepsilon,a,b}) &= \lim_{\varepsilon\to 0} \varepsilon^{-1/2}\, P(t \in K'_{\varepsilon,a,b})\\ &= \frac{\alpha^2}{(t-a)^{1/4}(b-t)^{1/4}}.\end{aligned} \tag{7}$$

(b) *There exists a positive constant* c_2, *independent of* a, b, *such that, for every* $(t,\varepsilon) \in (a,b) \times (0,1/2]$,

$$\left(\frac{t-a}{\varepsilon}\right)^{1/4} P(t \in K^-_{\varepsilon,a}) \le c_2, \tag{8}$$

$$\begin{aligned}\frac{(t-a)^{1/4}(b-t)^{1/4}}{\varepsilon^{1/2}} P(t \in K_{\varepsilon,a,b}) &= \frac{(t-a)^{1/4}(b-t)^{1/4}}{\varepsilon^{1/2}} P(t \in K'_{\varepsilon,a,b})\\ &\le c_2.\end{aligned} \tag{9}$$

PROOF. (a) We first observe that, on the set $\{W_\varepsilon \le 0\}$, we have $B_{\tau^+_\varepsilon} \ge 0$, P-a.s. It follows easily (by using the strong Markov property and the zero–one law if $B_{\tau^+_\varepsilon} = 0$) that

$$\tau^+_\varepsilon = \inf\{t \ge \varepsilon,\ W_t = 0\} = \inf\{t \ge \varepsilon,\ W_t > 0\}, \qquad P\text{-a.s. on } \{W_\varepsilon \le 0\}.$$

Hence, for $t \ge a + \varepsilon$,

$$P(\forall\, s \in [\varepsilon, t-a],\ W_s \le 0) = P(W_\varepsilon \le 0,\ \tau^+_\varepsilon \ge t-a) = \tfrac{1}{2} P(\tau^+_\varepsilon \ge t-a)$$

by an obvious symmetry argument.

Then, for $t \ge a + \varepsilon$,

$$\begin{aligned}P(t \in K^-_{\varepsilon,a}) &= P\left(\forall\, s \in [a, t-\varepsilon], \int_s^t B_u\, du \le (t-s)B_t\right)\\ &= P\left(\forall\, s \in [\varepsilon, t-a], \int_0^s (B_{t-v} - B_t)\, dv \le 0\right)\\ &= P\left(\forall\, s \in [\varepsilon, t-a], \int_0^s B'_v\, dv \le 0\right)\\ &= \tfrac{1}{2} P(\tau^+_\varepsilon \ge t-a).\end{aligned}$$

Similarly, using the independence of increments of Brownian motion, we have, for $a + \varepsilon \le t \le b - \varepsilon$,

$$\begin{aligned} P(t \in K_{\varepsilon,a,b}) &= E\left(t \in K^-_{\varepsilon,a}, P\left(\forall\, u \in [t+\varepsilon, b],\ \int_t^u (B_v - B_t)\,dv \le 0 \,\Big|\, \mathscr{F}_t\right)\right) \\ &= P(t \in K^-_{\varepsilon,a})\, P\left(\forall\, s \in [\varepsilon, b-t],\ \int_0^s B'_v\,dv \le 0\right) \\ &= \tfrac{1}{2}\, P(t \in K^-_{\varepsilon,a})\, P(\tau^+_\varepsilon \ge b - t) \\ &= \tfrac{1}{4}\, P(\tau^+_\varepsilon \ge t - a)\, P(\tau^+_\varepsilon \ge b - t). \end{aligned}$$

Exactly the same argument shows that

$$P(t \in K'_{\varepsilon,a,b}) = \tfrac{1}{4}\, P(\tau^+_\varepsilon \ge t - a)\, P(\tau^+_\varepsilon \ge b - t) = P(t \in K_{\varepsilon,a,b}).$$

The desired result follows from the previous formulas and part (a) of Proposition 2.

(b) The bounds of (b) follow from the previous formulas and part (b) of Proposition 2. □

PROPOSITION 4. *There exists a positive constant c_3 such that, for $0 < s < t < 1$ and $\varepsilon \in (0, 1/2]$,*

$$P(s \in K^-_\varepsilon,\ t \in K^-_\varepsilon) \le \frac{c_3\, \varepsilon^{1/2}}{s^{1/4}(t-s)^{1/4}}, \tag{10}$$

$$P(s \in K_\varepsilon,\ t \in K_\varepsilon) \le \frac{c_3\, \varepsilon}{s^{1/4}(t-s)^{1/2}(1-t)^{1/4}}, \tag{11}$$

$$P(s \in K'_\varepsilon,\ t \in K'_\varepsilon) \le \frac{c_3\, \varepsilon}{s^{1/4}(t-s)^{1/2}(1-t)^{1/4}}. \tag{12}$$

PROOF. Let us start with the bound (10). Suppose first that $s \ge \varepsilon$ and $t - s \ge \varepsilon$. Then, using again the independence of increments of Brownian motion, we have

$$\begin{aligned} &P(s \in K^-_\varepsilon,\ t \in K^-_\varepsilon) \\ &\quad \le E\left(s \in K^-_\varepsilon,\ P\left(\forall\, u \in [s, t-\varepsilon],\ \int_u^t (B_v - B_t)\,dv \le 0 \,\Big|\, \mathscr{F}_s\right)\right) \\ &\quad = P(s \in K^-_\varepsilon)\, P\left(\forall\, u \in [s, t-\varepsilon],\ \int_u^t (B_v - B_t)\,dv \le 0\right) \\ &\quad = P(s \in K^-_\varepsilon)\, P(t \in K^-_{\varepsilon,s}), \end{aligned}$$

and the desired bound follows from (8). If $t - s < \varepsilon$, we simply bound

$$P(s \in K^-_\varepsilon,\ t \in K^-_\varepsilon) \le P(s \in K^-_\varepsilon) \le P(s \in K^-_\varepsilon)\left(\frac{\varepsilon}{t-s}\right)^{1/4}$$

and we use (8) once again. Similarly, if $s < \varepsilon$, we use

$$P(s \in K_\varepsilon^-,\ t \in K_\varepsilon^-) \le P(t \in K_\varepsilon^-) \le \left(\frac{\varepsilon}{s}\right)^{1/4} P(t \in K_\varepsilon^-).$$

This completes the proof of (10).

Let us now turn to (11). We first assume that $s \ge \varepsilon$, $t - s \ge 2\varepsilon$ and $1 - t \ge \varepsilon$. Then, by the independence of increments,

$$\begin{aligned} P(s \in K_\varepsilon,\ t \in K_\varepsilon) &\le P(s \in K_{\varepsilon,0,(t+s)/2},\ t \in K_{\varepsilon,(t+s)/2,1}) \\ &= P(s \in K_{\varepsilon,0,(t+s)/2})\, P(t \in K_{\varepsilon,(t+s)/2,1}) \end{aligned}$$

and the bound (11) follows in this case from (9). The remaining cases are treated very easily. For instance, if $t - s \le 2\varepsilon$, we write

$$\begin{aligned} P(s \in K_\varepsilon,\ t \in K_\varepsilon) &\le P(s \in K_\varepsilon^-,\ t \in K_\varepsilon^+) \\ &= P(s \in K_\varepsilon^-)\, P(t \in K_\varepsilon^+) \\ &\le P(s \in K_\varepsilon^-)\, P(t \in K_\varepsilon^+) \left(\frac{2\varepsilon}{t-s}\right)^{1/2}. \end{aligned}$$

and we use (8) together with the bound

$$P(t \in K_\varepsilon^+) \le c_2 \left(\frac{\varepsilon}{1-t}\right)^{1/4}$$

that follows from (8) and an obvious time-reversal argument.

The proof of (12) is exactly similar to that of (11). □

We finish this section with a technical lemma that plays a crucial role in the upper bounds of the next section. Recall that τ denotes the first hitting time of 0 by W.

LEMMA 5. *There exists a positive constant c_4 such that, for every $x > 0$ and every $T \ge 1$,*

$$P_{x,x}(\tau > T) \le c_4\, T^{-1/4}(1 + \sqrt{x}). \tag{13}$$

PROOF. Note that $P_{x,y}(\tau > T)$ is a monotone increasing function in both variables x, y on $(0,\infty) \times (0,\infty)$. It is therefore enough to prove (13) for $x \ge 1$. A scaling argument then gives

$$P_{x,x}(\tau > T) = P_{x^{-2},1}\left(\tau > \frac{T}{x^2}\right) \le P_{1,1}\left(\tau > \frac{T}{x^2}\right).$$

The proof of Lemma 5 will be complete if we can check the existence of a constant c_5 such that, for every $a > 0$,

$$P_{1,1}(\tau > a) \le c_5\, a^{-1/4}. \tag{14}$$

Clearly, we may restrict our attention to $a > 2$. Now observe that

$$P\left(B_1 > 1,\ \int_0^1 B_s\,ds > 1\right) = c > 0.$$

Hence using Proposition 2(b) and the Markov property of the process (W_t, B_t), we have

$$\begin{aligned} c_1\,a^{-1/4} \ge P(\tau_1^+ > a) &\ge E\big(1_{(W_1>1,\ B_1>1)}\,P_{W_1,B_1}(\tau > a-1)\big) \\ &\ge E\big(1_{(W_1>1,\ B_1>1)}\,P_{1,1}(\tau > a-1)\big) \\ &= c\,P_{1,1}(\tau > a-1), \end{aligned}$$

and the desired result (14) follows immediately. □

3. Upper bounds on the Hausdorff dimension.

PROPOSITION 6. *We have* $\dim K^- \le \frac{3}{4}$, $\dim K \le \frac{1}{2}$ *and* $\dim K' \le \frac{1}{2}$, *P-a.s.*

The proof of Proposition 6 relies on the following lemma, which is inspired by Lemma 6 in Evans (1985).

LEMMA 7. *For every $a < b$, denote by $\mathscr{F}_{a,b}$ the σ-field generated by the increments of the process B on the time interval $[a,b]$: $\mathscr{F}_{a,b} = \sigma(B_u - B_a,\ u \in [a,b])$. Then, for every $a, b \in (0,1)$ with $a < b$, $1-b \ge b-a$, there exists an $\mathscr{F}_{a,b}$-measurable random variable $U_{a,b}$ such that*

$$P(K^+ \cap [a,b] \ne \varnothing \mid \mathscr{F}_b) \le (b-a)^{1/4}\,U_{a,b} \tag{15}$$

and

$$E\big(U_{a,b}^2\big) \le c_6\,(1-b)^{-1/2}, \tag{16}$$

where c_6 is a positive constant independent of a, b.

PROOF. Fix $a, b \in (0,1)$ with $a < b$, $1-b \ge b-a$. If $K^+ \cap [a,b] \ne \varnothing$, there exists some $t_0 \in [a,b]$ such that, for every $s \in [b,1]$,

$$\int_{t_0}^s B_u\,du \le (s-t_0)\,B_{t_0}.$$

Hence, we have, for every $s \in [b,1]$,

$$(b-t_0)\inf_{[a,b]} B_u + \int_b^s B_u\,du \le (s-t_0)\,B_{t_0}$$

and also

$$(b-a)\left(\inf_{[a,b]} B_u - \sup_{[a,b]} B_u\right) + \int_b^s B_u\,du \le (s-b)\sup_{[a,b]} B_u.$$

Now set $B_u^b = B_{b+u} - B_b$, so that B^b is a Brownian motion independent of $\mathscr{F}_b$. If $\eta = \sup_{[a,b]} B_u - \inf_{[a,b]} B_u$, we get

$$\begin{aligned}
&P(K^+ \cap [a,b] \neq \varnothing \mid \mathscr{F}_b)\\
&\quad \leq P\biggl(\forall\, s \in [b,1],\ \int_0^{s-b} B_u^b\,du - (b-a)\eta\\
&\qquad\qquad - (s-b)\Bigl(\sup_{[a,b]} B_u - B_b\Bigr) \leq 0 \,\Big|\, \mathscr{F}_b\biggr)\\
&\quad \leq P\biggl(\forall\, s \in [b,1],\ \int_0^{s-b} (B'_u - \eta)\,du - (b-a)\eta \leq 0 \,\Big|\, \mathscr{F}_b\biggr)\\
&\quad = P_{-(b-a)\eta,-\eta}(\forall\, s \in [0,1-b],\ W_s \leq 0)\\
&\quad = P_{-(b-a)\eta,-\eta}(\tau_0 \geq 1-b).
\end{aligned}$$

By the symmetry and the scaling property of the process (W_t, B_t), the last probability is equal to

$$P_{(b-a)^{-1/2}\eta,(b-a)^{-1/2}\eta}\biggl(\tau_0 \geq \frac{1-b}{b-a}\biggr).$$

We take $U_{a,b} = (b-a)^{-1/4}\varphi_{a,b}((b-a)^{-1/2}\eta)$, where $\varphi_{a,b}(y) = P_{y,y}(\tau_0 \geq (1-b)/(b-a))$. Obviously, $U_{a,b}$ is $\mathscr{F}_b$-measurable, and it only remains to check the moment condition (16).

By scaling, $(b-a)^{-1/2}\eta$ has the same distribution under P as $\xi = \sup_{[0,1]} B_u - \inf_{[0,1]} B_u$. Let $p_\xi(y)$ denote the probability density of ξ. Then, by Lemma 5,

$$\begin{aligned}
E(U_{a,b}^2) &= (b-a)^{-1/2} \int_0^\infty p_\xi(y) P_{y,y}\biggl(\tau_0 \geq \frac{1-b}{b-a}\biggr)^2 dy\\
&\leq c_4^2\,(b-a)^{-1/2} \int_0^\infty \biggl(\frac{b-a}{1-b}\biggr)^{1/2} (1+\sqrt{y})^2\, p_\xi(y)\,dy\\
&\leq c_6\,(1-b)^{-1/2}
\end{aligned}$$

because the variable ξ has moments of any order. This completes the proof of Lemma 7. □

PROOF OF PROPOSITION 6. We first consider K^-. Let $a, b \in (0,1)$ with $a < b$, $1-b \geq b-a$. By Lemma 7,

$$P(K^+ \cap [a,b] \neq \varnothing) \leq (b-a)^{1/4}\, E(U_{a,b}) \leq c_6^{1/2} \biggl(\frac{b-a}{1-b}\biggr)^{1/4}.$$

Suppose in addition that $a \geq b-a$. For $0 \leq t \leq 1$, let $\overline{B}_t = B_{1-t} - B_1$, so that $\overline{B}$ is also a Brownian motion started at 0 under P, on the time interval

$[0,1]$. Let $\overline{K}^+$ be defined as K^+ but with B replaced by $\overline{B}$. Then t belongs to $[a,b]\cap K^-$ if and only if $1-t$ belongs to $[1-b,1-a]\cap\overline{K}^+$. Hence

$$P(K^-\cap[a,b]\neq\varnothing)=P(K^+\cap[1-b,1-a]\neq\varnothing)\leq c_6^{1/2}\left(\frac{b-a}{a}\right)^{1/4}.$$

The desired upper bound on $\dim K^-$ will now follow from this inequality by standard arguments. For every $n\geq 1$, let Δ_n be the set of all subintervals of $[0,1]$ defined by

$$Q_i^n=\left[\frac{i-1}{2^n},\frac{i}{2^n}\right],\qquad i=1,\dots,2^n.$$

Fix a constant $c\in(0,1/2)$. Notice that, for n sufficiently large and $[a,b]\in\Delta_n$, the condition $[a,b]\cap[c,1-c]\neq\varnothing$ implies $a>c/2$, $b<1-c/2$. Then, by Fatou's lemma and the previous bound,

$$\begin{aligned}
E&\left(\liminf_{n\to\infty}2^{-3n/4}\sum_{i=1}^{2^n}1_{(K^-\cap[c,1-c]\cap Q_i^n\neq\varnothing)}\right)\\
&\leq\liminf_{n\to\infty}2^{-3n/4}\sum_{i=1}^{2^n}P(K^-\cap[c,1-c]\cap Q_i^n\neq\varnothing)\\
&\leq\liminf_{n\to\infty}2^{-3n/4}\sum_{i=1}^{2^n}1_{(Q_i^n\cap[c,1-c]\neq\varnothing)}P(K^-\cap Q_i^n\neq\varnothing)\\
&\leq\liminf_{n\to\infty}2^{-3n/4}\sum_{i=1}^{2^n}c_6^{1/2}(c/2)^{-1/4}2^{-n/4}\\
&<\infty.
\end{aligned}$$

We conclude that

$$\liminf_{n\to\infty}2^{-3n/4}\sum_{i=1}^{2^n}1_{(K^-\cap[c,1-c]\cap Q_i^n\neq\varnothing)}<\infty,\qquad P\text{-a.s.}$$

which, by the very definition of Hausdorff dimension, implies that

$$\dim(K^-\cap[c,1-c])\leq 3/4\quad\text{a.s.}$$

Since this holds for every $c>0$, the proof of the statement concerning K^- is complete.

We now turn to the proof of the second statement of Proposition 6. Let $0<a<b<1$ with $b-a\leq a\wedge(1-b)$. Notice that the event $\{K^-\cap[a,b]\neq\varnothing\}$ is $\mathscr{F}_b$-measurable. We have, from Lemma 7,

$$\begin{aligned}
P([a,b]\cap K^-\cap K^+\neq\varnothing)&\leq E(1_{([a,b]\cap K^-\neq\varnothing)}P([a,b]\cap K^+\neq\varnothing\mid\mathscr{F}_b))\\
&\leq(b-a)^{1/4}E(1_{([a,b]\cap K^-\neq\varnothing)}U_{a,b}).
\end{aligned}$$

We then use the same time-reversal argument as in the beginning of the proof. By applying Lemma 7 to $\overline{B}$, with an obvious notation,

$$\begin{aligned}
&P([a,b]\cap K^-\cap K^+\neq\varnothing)\\
&\quad\le (b-a)^{1/4}\,E(1_{([1-b,1-a]\cap\overline{K}^+\neq\varnothing)}\,U_{a,b})\\
&\quad= (b-a)^{1/4}\,E\big(U_{a,b}\,P([1-b,1-a]\cap\overline{K}^+\neq\varnothing\mid\sigma(\overline{B}_t,0\le t\le 1-a))\big)\\
&\quad\le (b-a)^{1/2}\,E\big(U_{a,b}\,\overline{U}_{1-a,1-b}\big)\\
&\quad\le (b-a)^{1/2}\,\big(E(U_{a,b}^2)\,E(\overline{U}_{1-a,1-b}^2)\big)^{1/2}\\
&\quad\le c_6\,(a(1-b))^{-1/4}\,(b-a)^{1/2}.
\end{aligned}$$

In the second equality, we used the fact that $U_{a,b}$ is $\mathscr{F}_{a,b}$-measurable, hence also $\sigma(\overline{B}_t,\ 0\le t\le 1-a)$-measurable. We have thus obtained

$$P([a,b]\cap K\neq\varnothing)\le c_6\,(a(1-b))^{-1/4}\,(b-a)^{1/2}.$$

The proof of the upper bound on $\dim K$ is now completed by the same standard arguments that we have used to bound $\dim K^-$. The proof of the upper bound for $\dim K'$ is exactly similar (clearly, the statement of Lemma 7 also holds when K^+ is replaced by K^*). □

4. Lower bounds on the Hausdorff dimension.

PROPOSITION 8. *We have*

$$P\big(\dim K^-\ge \tfrac{3}{4}\big)>0,\qquad P\big(\dim K\ge \tfrac{1}{2}\big)>0,\qquad P\big(\dim K'\ge \tfrac{1}{2}\big)>0.$$

PROOF. We will first obtain the statement concerning K. For every $\varepsilon\in(0,1/2)$, let μ_ε be the (random) finite measure on $[0,1]$ defined by

$$\mu_\varepsilon(A)=\varepsilon^{-1/2}\,|K_\varepsilon\cap A|,$$

for any Borel subset A of $[0,1]$. Then μ_ε is a random variable taking values in the set $\mathscr{M}_f$ of all finite measures on $[0,1]$. The space $\mathscr{M}_f$ is a Polish space for the weak convergence of finite measures. For every $\varepsilon\in(0,1/2)$, the law of μ_ε is a probability measure on $\mathscr{M}_f$.

By Proposition 3,

$$E(\mu_\varepsilon([0,1]))=\varepsilon^{-1/2}\int_0^1 P(t\in K_\varepsilon)\,dt\le c_2\int_0^1\frac{dt}{t^{1/4}(1-t)^{1/4}},$$

so that the quantities $E(\mu_\varepsilon([0,1]))$ are uniformly bounded. As a simple consequence of Lemma 4.5 of Kallenberg (1983), it follows that the laws of the random measures μ_ε are relatively compact (for the weak topology on the set of all probability measures on $\mathscr{M}_f$).

Let (ε_n) be a sequence strictly decreasing toward 0, and let $\mathscr{C}([0,1],\mathbb{R})$ denote the space of all continuous functions from $[0,1]$ into $\mathbb{R}$, equipped with the topology of uniform convergence. For every n, introduce the random variable ζ^n taking values in $\mathscr{M}_f \times \mathscr{C}([0,1],\mathbb{R})$ defined by

$$\zeta^n = (\mu_{\varepsilon_n},\ (B_t, 0 \le t \le 1)).$$

Obviously the laws of ζ^n are also relatively compact. Hence, by extracting a suitable subsequence, we may assume that they converge to a certain probability law on $\mathscr{M}_f \times \mathscr{C}([0,1],\mathbb{R})$. By Skorokhod's representation theorem, we may find on a certain probability space a sequence

$$\overline{\zeta}^n = (\mu^n,\ (B^n_t, 0 \le t \le 1))$$

and a variable

$$\overline{\zeta}^\infty = (\mu^\infty,\ (B^\infty_t, 0 \le t \le 1))$$

in such a way that, for every n, $\overline{\zeta}^n$ has the same distribution as ζ^n and $\overline{\zeta}^n$ converges almost surely to $\overline{\zeta}^\infty$. Notice that the processes B^n, B^∞ are linear Brownian motions started at 0. We denote by $K(B^\infty)$ [respectively, by $K_{\varepsilon_n}(B^n)$] the random closed set defined in the same way as K (respectively, as K_{ε_n}) by replacing the Brownian motion B by B^∞ (respectively, by B^n). Since $\overline{\zeta}^n$ has the same distribution as ζ^n, we have also

$$\mu^n(A) = \varepsilon_n^{-1/2}\,|K_{\varepsilon_n}(B^n) \cap A|,$$

for any Borel subset A of $[0,1]$, a.s. In particular, the measure μ^n is almost surely supported on $K_{\varepsilon_n}(B^n)$.

LEMMA 9. *The random measure μ^∞ is a.s. supported on $K(B^\infty)$.*

PROOF. By the definition of $K(B^\infty)$, it suffices to show that, for every $\eta > 0$, $\gamma > 0$,

$$\mu^\infty\left(\left\{t < 1-\eta,\ \sup_{t+\eta < s \le 1} \frac{\int_t^s B^\infty_u\,du}{s-t} > B^\infty_t + \gamma\right\}\right) = 0 \tag{17}$$

and

$$\mu^\infty\left(\left\{t > \eta,\ \sup_{0 \le s < t-\eta} \frac{\int_s^t B^\infty_u\,du}{t-s} > B^\infty_t + \gamma\right\}\right) = 0. \tag{18}$$

We prove only (17), since the proof of (18) is identical. We fix η and γ and set

$$G(\eta,\gamma) = \left\{t < 1-\eta,\ \sup_{t+\eta < s \le 1} \frac{\int_t^s B^\infty_u\,du}{s-t} > B^\infty_t + \gamma\right\}.$$

Notice that $G(\eta,\gamma)$ is an open set. Hence,

$$\mu^\infty(G(\eta,\gamma)) \le \liminf_{n\to\infty} \mu^n(G(\eta,\gamma)) \quad \text{a.s.} \tag{19}$$

However, for every n, the measure μ^n is a.s. supported on $K_{\varepsilon_n}(B^n)$ and, in particular, on

$$\left\{t \in [0, 1-\varepsilon_n],\ \sup_{t+\varepsilon_n \le s \le 1} \frac{\int_t^s B_u^n\,du}{s-t} \le B_t^n\right\}.$$

From the a.s. uniform convergence of B^n toward B^∞, it follows that a.s. for n sufficiently large, $G(\eta, \gamma)$ does not intersect the support of μ^n. The desired result (17) then follows from (19). □

LEMMA 10. *For every $\nu \in [0, 1/2)$, there exists a constant $c_{(\nu)}$ such that, for every $\varepsilon \in (0, 1/2)$,*

$$E\left(\iint_{[0,1]^2} \frac{\mu_\varepsilon(ds)\,\mu_\varepsilon(dt)}{|t-s|^\nu}\right) \le c_{(\nu)}.$$

PROOF. We use the bound (11) to get

$$\begin{aligned} E\left(\iint_{[0,1]^2} \frac{\mu_\varepsilon(ds)\,\mu_\varepsilon(dt)}{|t-s|^\nu}\right) &= 2\,\varepsilon^{-1} \iint_{(0<s<t<1)} \frac{P(s \in K_\varepsilon,\ t \in K_\varepsilon)}{(t-s)^\nu}\,ds\,dt \\ &\le 2\,c_3 \iint_{(0<s<t<1)} \frac{ds\,dt}{s^{1/4}(t-s)^{\nu+1/2}(1-t)^{1/4}}, \end{aligned}$$

and it is easy to verify that the latter integral is finite. □

We can now come back to the proof of Proposition 8. First observe that the measures $\mu^n \otimes \mu^n$ also converge a.s. to $\mu^\infty \otimes \mu^\infty$. Since μ^n has the same distribution as μ_{ε_n}, it follows from Lemma 10 that

$$E\left(\iint \frac{\mu^\infty(ds)\,\mu^\infty(dt)}{|t-s|^\nu}\right) \le \liminf_{n\to\infty} E\left(\iint \frac{\mu_{\varepsilon_n}(ds)\,\mu_{\varepsilon_n}(dt)}{|t-s|^\nu}\right) \le c_{(\nu)}.$$

Thus,

$$\iint \frac{\mu^\infty(ds)\,\mu^\infty(dt)}{|t-s|^\nu} < \infty \quad \text{a.s.} \tag{20}$$

On the other hand, if we take $\nu = 0$ in Lemma 10, we obtain that the sequence $\mu^n([0,1])$ is bounded in L^2-norm. Hence, this sequence is uniformly integrable and we have

$$\begin{aligned} E(\mu^\infty([0,1])) &= \lim_{n\to\infty} E(\mu^n([0,1])) = \lim_{n\to\infty} \int_0^1 \varepsilon_n^{-1/2}\, P(t \in K_{\varepsilon_n})\,dt \\ &= \alpha^2 \int_0^1 \frac{dt}{t^{1/4}(1-t)^{1/4}}, \end{aligned}$$

by (7), using (9) to justify dominated convergence. We get in particular that

$$P(\mu^\infty \ne 0) > 0.$$

Using Lemma 9, we get that the compact set $K(B^\infty)$ supports a measure μ_∞, which is nontrivial with positive probability and such that the bound (20) holds for every $\nu \in [0,1/2)$. By the classical Frostman lemma, this implies that $\dim K(B^\infty) \geq 1/2$ with positive probability. Since the compact sets K and $K(B^\infty)$ have the same distribution, the second statement of Proposition 8 follows.

The proof of the third statement is exactly the same. The proof of the statement concerning K^- is also similar, with some obvious changes: The auxiliary measures μ_ε are defined by

$$\mu_\varepsilon(A) = \varepsilon^{-1/4}\,|K_\varepsilon^- \cap A|.$$

We use again Proposition 3 to get the relative compactness of the laws of μ_ε. In the analogue of Lemma 10, we replace the condition $\nu \in [0,1/2)$ by $\nu \in [0,3/4)$ and we use the bound (10) for the proof. □

We will now strengthen Proposition 8 by showing that this result remains valid if we take a more restrictive definition of the sets K^-, K and K'. Precisely, we denote by $\tilde{K}^-$ (respectively, $\tilde{K}$ and $\tilde{K}'$) the random sets obtained by replacing the weak inequalities in the definition of K^- (respectively, of K, K') (cf. Section 1) by strict inequalities.

PROPOSITION 11. *The results of Propositions 6 and 8 remain valid if K^-, K and K' are replaced by $\tilde{K}^-$, $\tilde{K}$ and $\tilde{K}'$, respectively.*

PROOF. It is obvious that the upper bounds of Proposition 6 also hold for $\tilde{K}^-$, $\tilde{K}$ and $\tilde{K}'$. We will show that the lower bound for K in Proposition 8 also holds for $\tilde{K}$. The same argument can be applied to $\tilde{K}^-$ and $\tilde{K}'$.

Recall the notation of the proof of Proposition 8. We will check that, for every $\delta \in (0,1/4)$,

$$\mu^\infty\left(\left\{t \in [0,1-\delta),\ B_t^\infty = \sup_{t+\delta\leq s\leq 1} \frac{1}{s-t}\int_t^s B_u^\infty\,du\right\}\right) = 0. \tag{21}$$

The same argument applies to the processes B^n and B^∞ reversed at time 1 [i.e., to the processes $(B_{1-t}^n - B_1^n)$ and $(B_{1-t}^\infty - B_1^\infty)$] and implies that

$$\mu^\infty\left(\left\{t \in (\delta,1],\ B_t^\infty = \sup_{0\leq s\leq t-\delta} \frac{1}{t-s}\int_s^t B_u^\infty\,du\right\}\right) = 0. \tag{22}$$

By using (21) and (22) for a sequence (δ_p) decreasing to 0, we get that

$$\mu^\infty(K(B^\infty)\backslash\tilde{K}(B^\infty)) = 0 \quad \text{a.s.}$$

Therefore, μ^∞ is a.s. supported on $\tilde{K}(B^\infty)$, and the same argument as in the end of the proof of Proposition 8 implies that $\dim\tilde{K} \geq 1/2$ with positive probability.

It remains to prove (21). We fix $\delta \in (0, 1/4)$. For $t \in [0, 1-2\delta]$, we set

$$I_t^n = \sup_{t+2\delta \le u \le 1} \int_t^u (B_r^n - B_t^n)\, dr,$$

$$I_t^\infty = \sup_{t+2\delta \le u \le 1} \int_t^u (B_r^\infty - B_t^\infty)\, dr.$$

Let g be a continuous function with compact support from $\mathbb{R}$ into $[0,1]$, and let h be a continuous function from $[0,1]$ into $[0,1]$, such that $h(t) = 0$ if and only if $t \ge 1 - 2\delta$. Recall the almost sure uniform convergence of B^n toward B^∞, the almost sure weak convergence of μ^n toward μ^∞ and the fact that the sequence $\mu^n([0,1])$ is bounded in the L^2-norm. It follows easily that

$$E\Big(\int \mu^\infty(dt)\, h(t)\, g(I_t^\infty)\Big) = \lim_{n\to\infty} E\Big(\int \mu^n(dt)\, h(t)\, g(I_t^n)\Big). \tag{23}$$

On the other hand,

$$\begin{aligned} E\Big(\int \mu^n(dt)\, h(t)\, g(I_t^n)\Big) &= \varepsilon_n^{-1/2} \int_0^1 dt\, h(t)\, E\big(1_{K_{\varepsilon_n}(B^n)}(t) g(I_t^n)\big) \\ &\le \varepsilon_n^{-1/2} \int_0^1 dt\, h(t)\, E\big(1_{K^n_{\varepsilon_n,0,t+\delta}}(t) g(I_t^n)\big), \end{aligned} \tag{24}$$

where $K^n_{\varepsilon_n,0,t+\delta} = K_{\varepsilon_n,0,t+\delta}(B^n)$, with an obvious notation. We may write I_t^n in the form

$$I_t^n = \sup_{t+2\delta \le u \le 1} \Big(\int_t^{t+\delta} (B_r^n - B_t^n)\, dr + \int_{t+\delta}^u (B_r^n - B_t^n)\, dr\Big).$$

Then, since the event $\{t \in K^n_{\varepsilon_n,0,t+\delta}\}$ is measurable with respect to $\sigma(B_u^n,\ u \le t+\delta)$, we can apply the Markov property at time $t+\delta$ to get

$$E\big(1_{K^n_{\varepsilon_n,0,t+\delta}}(t) g(I_t^n)\big) = E\Big(1_{K^n_{\varepsilon_n,0,t+\delta}}(t)\, E_{a_{n,t},b_{n,t}}\Big(g\Big(\sup_{\delta \le u \le 1-(t+\delta)} W_u\Big)\Big)\Big),$$

where $a_{n,t} = \int_t^{t+\delta} (B_r^n - B_t^n)\, dr$, $b_{n,t} = B_{t+\delta}^n - B_t^n$ and we use the notation introduced in Section 2 for the process (W_t) and the probability measures $P_{x,y}$. We have, therefore,

$$E\big(1_{K^n_{\varepsilon_n,0,t+\delta}}(t) g(I_t^n)\big) \le P\big(t \in K^n_{\varepsilon_n,0,t+\delta}\big)\, \psi(t),$$

where

$$\psi(t) = \sup_{a,b \in \mathbb{R}} E_{a,b}\Big(g\Big(\sup_{\delta \le u \le 1-(t+\delta)} W_u\Big)\Big).$$

Hence, from (23) and (24), we obtain

$$\begin{aligned} E\Big(\int \mu^\infty(dt)\, h(t)\, g(I_t^\infty)\Big) &\le \liminf_{n\to\infty} \int_0^1 dt\, h(t)\, \psi(t)\, \varepsilon_n^{-1/2} P(t \in K^n_{\varepsilon_n,0,t+\delta}) \\ &\le c_2\, \delta^{-1/4} \int_0^1 dt\, h(t)\, \psi(t)\, t^{-1/4} \end{aligned}$$

by Proposition 3. Now consider a decreasing sequence (g_p) such that $g_p(0) = 1$ and $g_p(x) = 0$ if $|x| > 1/p$. Denote by ψ_p the associated function ψ. We claim that, for every $t \in [0, 1 - 2\delta]$,

$$\lim_{p\to\infty} \downarrow \psi_p(t) = 0.$$

This follows from the fact that the density under $P_{a,b}$ of $\sup_{[\delta,1-(t+\delta)]} W_u$ is bounded by a constant independent of a, b (to verify this property, write $\sup_{[\delta,1-(t+\delta)]} W_u = W_\delta + V$, then observe that the variables W_δ and V are conditionally independent given B_δ and that the conditional density of W_δ is bounded independently of a, b). Then, using dominated convergence,

$$\begin{aligned} E\Big(\int \mu^\infty(dt)\, h(t)\, 1_{(I_t^\infty=0)}\Big) &= \lim_{p\to\infty} \downarrow E\Big(\int \mu^\infty(dt)\, h(t)\, g_p(I_t^\infty)\Big) \\ &\le \lim_{p\to\infty} \downarrow c_2\, \delta^{-1/4} \int_0^1 dt\, h(t)\, \psi_p(t)\, t^{-1/4} \\ &= 0. \end{aligned}$$

The desired result (21) follows, with δ replaced by 2δ. □

5. Zero–one laws for K^- and for K. In this section, we complete the proof of Theorem 1 by checking that the lower bounds for $\dim K^-$ and $\dim K$ in Proposition 8 and their analogues for $\tilde{K}^-$ and $\tilde{K}$ obtained in Proposition 11 hold almost surely. It is of course sufficient to treat the case of the sets $\tilde{K}^-$ and $\tilde{K}$.

PROPOSITION 12. *The lower bounds* $\dim \tilde{K}^- \ge 3/4$ *and* $\dim \tilde{K} \ge 1/2$ *hold almost surely.*

PROOF. We first treat the case of $\tilde{K}^-$. This case is a very simple application of the classical zero–one law for linear Brownian motion. For every $L \in (0, 1]$, set

$$\tilde{K}^{-,L} = \left\{ t \in [0, L];\ \forall\, s \in [0, t),\ \frac{1}{t-s} \int_s^t B_u\, du < B_t \right\} = \tilde{K}^- \cap [0, L].$$

Let (L_n) be any sequence strictly decreasing to 0. An obvious scaling argument shows that $\tilde{K}^{-,L_n}$ has the same distribution as $L_n \cdot \tilde{K}^-$. In particular, the sets $\{\dim \tilde{K}^{-,L_n} \ge 3/4\}$ all have the same probability, which is also $P(\dim \tilde{K}^- \ge 3/4)$. Since these sets form a decreasing sequence,

$$P\Big(\bigcap_{n=1}^{\infty} \{\dim \tilde{K}^{-,L_n} \ge 3/4\} \Big) = P(\dim \tilde{K}^- \ge 3/4) > 0,$$

by Proposition 11. However, by the zero–one law, the event

$$\bigcap_{n=1}^{\infty} \{\dim \tilde{K}^{-,L_n} \ge 3/4\}$$

must have probability 0 or 1. Hence, it has probability 1, which gives the desired result for $\tilde{K}^-$.

We now turn to the analogous result concerning $\tilde{K}$. The proof will be significantly more difficult, because we cannot use the classical zero–one law. Instead, we need a kind of zero–one law for the behavior of the Brownian paths near their maximum on the time interval $[0,1]$. Roughly speaking, the decomposition theorems of Williams (1974) suggest that the behavior near the maximum can be described by two independent three-dimensional Bessel processes, to which a suitable zero–one law can be applied. The precise formulation of this idea will require certain details that are developed below.

We start with some notation. Note that $\tilde{K} \subset (0,1)$ a.s. For every $\eta > 0$, set

$$\tilde{K}^{(\eta)} = \left\{ t \in \tilde{K} \cap (0,1),\ \frac{1}{t}\int_0^t B_u\,du < B_t - \eta \text{ and } \frac{1}{1-t}\int_t^1 B_u\,du < B_t - \eta \right\}.$$

Notice that, if (η_n) is a sequence decreasing to 0, $\tilde{K}$ is (a.s.) the countable union of the sets $\tilde{K}^{(\eta_n)}$.

Let $R = (R_t,\ t \geq 0)$ be a three-dimensional Bessel process started at 0 [in short, a $\mathrm{Bes}_3(0)$ process, see, e.g., Revuz and Yor (1991), Chapter XI]. Let R' be another $\mathrm{Bes}_3(0)$ independent of R. The process $U = (U_t,\ t \in \mathbb{R})$ defined by

$$U_t = \begin{cases} -R_t, & \text{if } t \geq 0, \\ -R'_{-t}, & \text{if } t \leq 0, \end{cases}$$

will be called a two-sided $\mathrm{Bes}_3(0)$ process. If $Z = (Z_t,\ t \in \mathbb{R})$ is a continuous process indexed by $t \in \mathbb{R}$, we define

$$\tilde{K}(Z) = \left\{ t \in \mathbb{R};\ \forall\, s \in (-\infty, t) \cup (t, \infty),\ \frac{1}{s-t}\int_t^s Z_u\,du < Z_t \right\}.$$

Finally we also recall the notation introduced in Section 1: ρ is the almost surely unique time in $[0,1]$ such that $B_\rho = \sup_{[0,1]} B_t$.

LEMMA 13. *Let $\delta \in (0, 1/2)$ and $\eta > 0$. There exists a process $(Z_t,\ t \in \mathbb{R})$, defined possibly on an enlarged probability space, such that the law of Z is absolutely continuous with respect to the law of the two-sided* $\mathrm{Bes}_3(0)$ *process, and the following property holds a.s.*:

$$\{t - \rho;\ t \in \tilde{K}^{(\eta)} \cap [\delta, 1-\delta]\} \subset (\tilde{K}(Z) \cap [\delta - \rho, 1 - \delta - \rho]) \subset \{t - \rho;\ t \in \tilde{K}\}.$$

We postpone the proof of Lemma 13, which is basically an application of a decomposition theorem for the Brownian path, and we complete the proof of Proposition 12. By Proposition 11, we may choose $\delta \in (0, 1/2)$ and $\eta > 0$ so small that

$$P\big(\dim(\tilde{K}^{(\eta)} \cap [\delta, 1-\delta]) \geq 1/2\big) > 0.$$

The first inclusion of Lemma 13 then implies that

$$P\big(\dim(\tilde{K}(Z) \cap [-1, 1]) \geq 1/2\big) > 0.$$

Since the distribution of Z is absolutely continuous with respect to that of the process U previously defined, we have also

$$P\big(\dim(\tilde{K}(U) \cap [-1,1]) \geq 1/2\big) > 0.$$

The point is that by the scaling properties of the $\mathrm{Bes}_3(0)$ process, the quantity

$$P\big(\dim(\tilde{K}(U) \cap [-a,a]) \geq 1/2\big) > 0$$

is independent of the choice of $a > 0$. If (a_n) is a sequence decreasing to 0, we have, therefore,

$$P\bigg(\bigcap_{n=1}^{\infty}\big\{\dim(\tilde{K}(U) \cap [-a_n,a_n]) \geq 1/2\big\}\bigg) > 0.$$

We claim that the event

$$H = \bigcap_{n=1}^{\infty}\big\{\dim(\tilde{K}(U) \cap [-a_n,a_n]) \geq 1/2\big\}$$

must have probability 0 or 1. Let us briefly justify this claim. For every $b > 0$, set

$$\lambda_b = \sup\{t \geq 0,\ R_t = b\}, \qquad \lambda'_b = \sup\{t \geq 0,\ R'_t = b\}.$$

Then H is measurable with respect to $\mathscr{G}_b = \sigma(U_t,\ -\lambda'_b \leq t \leq \lambda_b)$. Now notice that $\lambda_b \downarrow 0$ as $b \downarrow 0$, and that the process $(R_{\lambda_b+t} - b, t \geq 0)$ is a $\mathrm{Bes}_3(0)$ process independent of $\mathscr{G}_b$, by a well-known property of $\mathrm{Bes}_3(0)$ processes [see, e.g., Revuz and Yor (1991), Proposition 3.9, page 236]. A slight modification of the proof of the standard zero–one law for Brownian motion shows that the σ-field $\bigcap_{b>0} \mathscr{G}_b$ is trivial, which gives the claim.

We conclude that H has probability 1. Using the fact that the law of Z is absolutely continuous with respect to that of U, we have also

$$\dim(\tilde{K}(Z) \cap [-a,a]) \geq 1/2, \qquad \forall\, a > 0, \text{ a.s.}$$

Then, the second inclusion of Lemma 13 shows that

$$\dim \tilde{K} \geq 1/2 \quad \text{a.s. on } \{\delta < \rho < 1 - \delta\}.$$

To complete the proof, it suffices to note that $P(\delta < \rho < 1-\delta) \uparrow 1$ as $\delta \downarrow 0$. □

REMARK. The previous argument gives the more precise statement

$$\dim(\tilde{K} \cap [\rho - a, \rho + a]) \geq \tfrac{1}{2}, \qquad \forall\, a > 0, \text{ a.s.}$$

PROOF OF LEMMA 13. We first recall a decomposition theorem for the Brownian path due to Denisov, in the form stated in Biane and Yor [(1988), Théorème 10]. Let (X'_t) and (X''_t) be the two processes indexed by $t \in [0,1]$ defined by

$$X'_t = (1-\rho)^{-1/2}(B_\rho - B_{\rho+(1-\rho)t}),$$
$$X''_t = \rho^{-1/2}(B_\rho - B_{\rho-\rho t}).$$

Then X' and X'' are two independent Brownian meanders, which are also independent of ρ. According to Biane and Yor [(1988), Théorème 3], we also know that the law of the Brownian meander is absolutely continuous with respect to the law of the $\mathrm{Bes}_3(0)$ process considered on the time interval $[0,1]$. Therefore, using the scaling property of the $\mathrm{Bes}_3(0)$ process, we obtain that the law of the triple $(\rho,(B_\rho - B_{(\rho+t)\wedge 1})_{t\geq 0},(B_\rho - B_{(\rho-t)\vee 0})_{t\geq 0})$ is absolutely continuous with respect to the law of $(\rho,(R'_{t\wedge(1-\rho)})_{t\geq 0},(R''_{t\wedge\rho})_{t\geq 0})$, where R' and R'' are two independent $\mathrm{Bes}_3(0)$ processes, independent of ρ.

For $x \geq 0$, denote by Q_x the law of a three-dimensional Bessel process $(R_t)_{t\geq 0}$, with $R_0 = x$. For $a > 0$, let $L_a = \sup\{t \geq 0,\ R_t = a\}$ $(\sup\varnothing = 0)$ and denote by Q_x^a the law of the process $(R_t)_{t\geq 0}$ conditional on the event $\{a\,L_a < \delta\eta\}$. Notice that this event has positive probability, so that Q_x^a is (obviously) absolutely continuous with respect to Q_x.

Set $M = B_\rho - \inf_{[0,1]} B_s$. By enlarging the probability space if necessary, we may introduce a pair of processes (Y', Y'') such that the conditional distribution of (Y',Y'') knowing $(B_t)_{0\leq t\leq 1}$ is $Q^M_{B_\rho - B_1} \otimes Q^M_{B_\rho}$. We then set

$$V'_t = \begin{cases} B_\rho - B_{\rho+t}, & \text{if } 0 \leq t \leq 1-\rho,\\ Y'_{t-(1-\rho)}, & \text{if } t > 1-\rho,\end{cases}$$

$$V''_t = \begin{cases} B_\rho - B_{\rho-t}, & \text{if } 0 \leq t \leq \rho,\\ Y''_{t-\rho}, & \text{if } t > \rho.\end{cases}$$

Notice that, if $\lambda'_M = \sup\{t \geq 0, Y'_t = M\}$, $\lambda''_M = \sup\{t\geq 0,\ Y''_t = M\}$, we have $M\lambda'_M \leq \eta\delta$ and $M\lambda''_M \leq \eta\delta$ by construction.

By using the fact that $Q_x^a \ll Q_x$ for every $a \geq 0$, $x \geq 0$, together with the previous observation on the law of the triple $(\rho,(B_\rho - B_{(\rho+t)\wedge 1})_{t\geq 0},(B_\rho - B_{(\rho-t)\vee 0})_{t\geq 0})$, one easily checks that the law of the pair (V', V'') is absolutely continuous with respect to the law of two independent $\mathrm{Bes}_3(0)$ processes.

We take

$$Z_t = \begin{cases} -V'_t, & \text{if } t \geq 0,\\ -V''_{-t}, & \text{if } t \leq 0.\end{cases}$$

It only remains to verify the two inclusions of Lemma 13. The second one is clear because $Z_t = B_{\rho+t} - B_\rho$ for every $t \in [-\rho, 1-\rho]$. The first one will follow from our construction and, in particular, from the conditioning used to define V' and V''. Let $t \in \tilde{K}^{(\eta)} \cap [\delta, 1-\delta]$. We have to verify that, for $u \neq t$,

$$\frac{1}{u-t}\int_{t-\rho}^{u-\rho} Z_s\,ds < Z_{t-\rho} = B_t - B_\rho.$$

Consider the case $u > t$ (the case $u < t$ is symmetric). If $u \in (t, 1]$, the result is immediate:

$$\int_{t-\rho}^{u-\rho} Z_s\,ds = \int_t^u (B_s - B_\rho)\,ds < (u-t)(B_t - B_\rho),$$

because $t \in \tilde{K}^{(\eta)} \subset \tilde{K}$. Note that, for $u = 1$, the definition of $\tilde{K}^{(\eta)}$ gives a little more:

$$\int_{t-\rho}^{1-\rho} Z_s\,ds = \int_t^1 (B_s - B_\rho)\,ds < (1-t)(B_t - B_\rho) - (1-t)\eta \le (1-t)(B_t - B_\rho) - \delta\eta.$$

Then, if $u > 1$,

$$\begin{aligned}
\int_{t-\rho}^{u-\rho} Z_s\,ds &= \int_{t-\rho}^{1-\rho} Z_s\,ds + \int_{1-\rho}^{u-\rho} Z_s\,ds \\
&< (1-t)(B_t - B_\rho) - \delta\eta - \int_0^{u-1} Y'_r\,dr \\
&\le (1-t)(B_t - B_\rho) - \delta\eta - (u - 1 - \lambda'_M)_+ M \\
&\le (1-t)(B_t - B_\rho) - \delta\eta - \left(u - 1 - \frac{\delta\eta}{M}\right)_+ (B_\rho - B_t) \\
&\le (u-t)(B_t - B_\rho) - \delta\eta + \frac{\delta\eta}{M}(B_\rho - B_t) \\
&\le (u-t)(B_t - B_\rho),
\end{aligned}$$

which completes the proof. □

The different assertions of Theorem 1 follow from Propositions 6, 8, 11 and 12, except for the fact that $P(K' \neq \varnothing) > 0$. This fact can be easily derived as follows. Let $\varepsilon \in (0, 1/8)$. The event

$$-\varepsilon - t \le B_t \le \varepsilon - t, \qquad \forall\, t \in [0,1], \tag{25}$$

has positive probability. However, if (25) holds, we have, for $t \in [0, 1/2]$,

$$\frac{1}{1-t}\int_t^1 B_u\,du \le \varepsilon - \frac{1}{2}(1+t) < -\varepsilon - t \le B_t$$

and, for $t \in [1/2, 1]$,

$$\frac{1}{t}\int_0^t B_u\,du \ge -\varepsilon - \frac{t}{2} > \varepsilon - t \ge B_t.$$

This shows that $K' = \varnothing$ whenever (25) holds.

Note, however, that $K' \neq \varnothing$ as soon as $B_1 > 0$. In fact, K' contains any time $t_0 \in (0,1)$ such that $W_{t_0} = \inf_{[0,1]} W_t$ and such a time clearly exists a.s. on $\{B_1 > 0\}$. This suggests the following plausible conjecture.

CONJECTURE.

$$\dim K' = \tfrac{1}{2} \quad \text{a.s. on } \{B_1 > 0\}.$$

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