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SECOND-ORDER LINEAR HYPERBOLIC SPDES DRIVEN BY ISOTROPIC GAUSSIAN NOISE ON A SPHERE

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We study a class of linear hyperbolic stochastic partial differential equations in bounded domains, which includes the wave equation and the telegraph equation, driven by Gaussian noise that is white in time but not in space. We give necessary and sufficient conditions on the spatial correlation of the noise for the existence (and uniqueness) of square-integrable solutions. In the particular case where the domain is a ball and the noise is concentrated on a sphere, we characterize the isotropic Gaussian noises with this property. We also give explicit necessary and sufficient conditions when the domain is a hypercube and the Gaussian noise is concentrated on a hyperplane.

1. Introduction. The study of stochastic partial differential equations (SPDEs) has become an active area of research following the seminal articles of [12] and [31]. The most studied equations are the heat and wave equations, and parabolic and hyperbolic generalizations of these, driven by space–time white noise. For linear equations driven by such noise, there are generally solutions in the space of real-valued stochastic processes when the spatial dimension is 1, but only distribution-valued solutions in dimensions greater than 1. The study of non-linear forms of these equations has been therefore mostly limited to dimension 1, though there are some attempts toward notions of solutions in higher dimensions (see [22]).

A different approach to the study of SPDEs in dimensions greater than 1 is to consider noise that is somewhat smoother than space–time white noise. While "white in time" is a property that is motivated by physical considerations, introducing spatial correlations is also natural in many physical applications (see [18]). With this type of Gaussian noise, it is possible to establish existence and regularity properties of solutions to many SPDEs in higher dimensions. This has mostly been done under the additional assumption that the noise is *spatially homogeneous,* which is a natural hypothesis that makes it possible to use techniques from Fourier analysis. Indeed, the covariance function of the noise must be nonnegative definite, and the Bochner–Schwartz theorem (see [28], Chapter 7,

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Theorem 17) states that this function is the Fourier transform of a nonnegative tempered measure, termed the *spectral measure* of the process. Existence and regularity properties can then be established under a condition on this spectral measure (see [5], [7], [14], [19], [24] and [25]).

In this paper, we shall study a class of linear hyperbolic SPDEs in bounded domains, driven by Gaussian noise that is not spatially homogeneous: typically, it will be concentrated on a lower-dimensional set, such as the boundary of a ball. Stochastic partial differential equations driven by such noises can be viewed (see Remark 3.5) as a generalization of the class of SPDEs with random boundary conditions, as have been considered, mainly in the parabolic case, in [3], [10], [11], Chapter 13, [16], [17] and [30].

An interesting class of noises is that with isotropic covariances (see Section 3.2). This class of noises is natural in contexts where spherical symmetry is present and appears not to have been previously considered in the literature. Our main objective will be to give necessary and sufficient conditions on the isotropic covariance for existence of a square-integrable solution to the SPDE.

More precisely, let $d \geq 1$ and let D be a bounded domain in $\mathbb{R}^d$ whose boundary ∂D is a C^∞ manifold and such that D is locally on one side of ∂D. For $a, b \in \mathbb{R}$, we consider the following class of linear hyperbolic SPDEs:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) + 2a\frac{\partial u}{\partial t}(t,x) + bu(t,x) - \Delta u(t,x) = \dot{F}^D(t,x), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in \mathbb{R}_+ \times D, \\
&\frac{\partial u}{\partial \nu}(t,x) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad (t,x) \in \mathbb{R}_+ \times \partial D, \\
&u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in D,
\end{aligned}
\tag{1.1}
$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative of u on ∂D, u_0, v_0 are two given functions on D and $\dot{F}^D = \{\dot{F}^D(t,x), (t,x) \in \mathbb{R}_+ \times D\}$ is a generalized centered Gaussian process whose covariance is formally given by

$$
\mathbb{E}\big(\dot{F}^D(t,x)\dot{F}^D(s,y)\big) = \delta_0(t-s)\Gamma_D(x,y),
$$

where δ_0 is the usual Dirac measure on $\mathbb{R}$ and Γ_D is a nonnegative definite distribution on $D \times D$, in a sense that will be made precise in Section 2.2. The case $a = b = 0$ is the wave equation. When $a \neq 0$, a can be interpreted as a damping coefficient. When $b = 0$, this is the telegraph equation, and when $a = 0$ and $b \neq 0$, this is the Klein–Gordon equation.

In order for the noise to be white in both time and space, one should have $\Gamma_D(x,y) = \delta_0(x-y)$; that is, the covariance function would have a singularity at the origin. In the case of spatially homogeneous noise on $\mathbb{R}^d$, the covariance function is of the form $\Gamma_D(x,y) = f(x-y)$ for some function $f : \mathbb{R}^d \to \mathbb{R}$.

The regularity or irregularity of the noise is then related to the nature of the singularity of f at the origin, and the answer to the question of the existence of a square-integrable solution can be given in terms of the nature of this singularity; see [5], [14] and [25].

In Section 2, for a general class of bounded domains D and covariances Γ_D, we give a formal definition of the Gaussian noise process and of a notion of the solution to the equation (which uses little more than stochastic integrals with respect to Brownian motion), and we establish the existence and uniqueness of the solution to (1.1) under a necessary and sufficient condition on the covariance function (Assumption B of Section 2.5). This condition is expressed in terms of the eigenvalues and eigenfunctions of the Laplacian in the domain D. It is therefore natural to particularize the problem to specific bounded domains where this condition can be made more explicit (the case of unbounded domains is rather different and is considered in [8]).

In Section 3, we consider the case where the domain is a ball and the Gaussian noise is isotropic and concentrated on the sphere which bounds the ball. Continuous functions which are isotropic and nonnegative definite are characterized by Schönberg's theorem (see [27] and Theorem 3.1). From this result, a wide class of isotropic covariances with a singularity at the origin can be exhibited [see (3.8)]. In this context, Assumption B furnishes, in principle, a necessary and sufficient condition for the existence of a process solution to the equation [see (3.20)], but this condition is expressed in terms of spherical harmonics and is not easy to verify. Using relatively recent estimates [4] on the zeros of Bessel functions and a classical trace theorem for Sobolev spaces, we obtain equivalent explicit conditions that are easy to check (see Theorem 3.10 and, in the case $d = 2$, Proposition 3.14). The case where the noise is concentrated on a sphere with smaller radius than the ball is considered in Section 3.4.

Finally, in Section 4, we examine the analogous problem in the case where the domain D is a hypercube instead of a ball. A related problem for the wave equation on the torus, with spatially homogeneous noise, was considered in [14]. Here, we consider noise concentrated on a hyperplane inside the cube and obtain in Theorem 4.1 the same types of results as those of Section 3.

2. Linear equation in a bounded domain driven by Gaussian noise.

General existence and uniqueness results can be obtained without additional effort for a wide class of domains and Gaussian noises, and we proceed to do so. The key ingredient in the resolution of (1.1) is the spectral theorem, which we now recall.

2.1. *Spectral theorem.* Let $\mathcal{S}(D)$ be the set of $\varphi \in C^\infty(\overline{D})$ such that $\frac{\partial\varphi}{\partial\nu}|_{\partial D} = 0$. We shall denote by $L^2(D)$ the space of measurable and square-integrable functions on D, equipped with the inner product

$$\langle u, v\rangle_0 = \int_D dx\, u(x)v(x),$$

and the corresponding norm $\|\cdot\|_0$; $H^1(D)$ will denote the Sobolev space of functions in $L^2(D)$ whose first partial derivatives also belong to $L^2(D)$, equipped with the inner product $\langle u, v\rangle_1 = \langle u, v\rangle_0 + \langle \nabla u, \nabla v\rangle_0$, and the corresponding norm $\|\cdot\|_1$ [note that $\mathcal{S}(D) \subset H^1(D)$]; $H^{-1}(D)$ will be the dual of $H^1(D)$, equipped with the norm

$$|||u|||_{-1} = \sup_{\varphi\in H^1(D),\, \varphi\neq 0} \frac{|\langle u, \varphi\rangle_{-1,1}|}{\|\varphi\|_1},$$

where $\langle\cdot,\cdot\rangle_{-1,1}$ denotes the duality product between $H^{-1}(D)$ and $H^1(D)$. Note that if $u \in L^2(D)$, then $\langle u, \varphi\rangle_{-1,1} = \langle u, \varphi\rangle_0$ for all $\varphi \in H^1(D)$.

We shall use the following spectral theorem from classical analysis for the Laplacian operator on D with Neumann boundary conditions (see, for instance, [31], Example 3, page 336).

THEOREM 2.1. *There exists an orthonormal basis $\{e_n, n \in \mathbb{N}\} \subset \mathcal{S}(D)$ of $L^2(D)$ and $\{\lambda_n, n \in \mathbb{N}\} \subset \mathbb{R}_+$ such that $\Delta e_n + \lambda_n e_n = 0$ for all $n \in \mathbb{N}$, $\lambda_n \uparrow +\infty$ and, for all $p > d/2$,*

$$\sum_{n\in\mathbb{N}} (1 + \lambda_n)^{-p} < \infty. \tag{2.1}$$

The fact that $e_n \in \mathcal{S}(D)$ for all $n \in \mathbb{N}$ is verified in [29], Theorem 2.1. Note, moreover, that $\lambda_0 = 0$ and $e_0(x) \equiv |D|^{-1/2}$, since we consider Neumann boundary conditions.

A direct consequence of the above theorem is that $\{(1 + \lambda_n)^{-1/2} e_n, n \in \mathbb{N}\}$ is an orthonormal basis of $H^1(D)$. Moreover,

$$\|u\|_0^2 = \sum_{n\in\mathbb{N}} |\langle u, e_n\rangle_0|^2, \qquad \|u\|_1^2 = \sum_{n\in\mathbb{N}} (1 + \lambda_n)|\langle u, e_n\rangle_0|^2,$$

and the norm $|||\cdot|||_{-1}$ on $H^{-1}(D)$ is equivalent to

$$\|u\|_{-1}^2 = \sum_{n\in\mathbb{N}} (1 + \lambda_n)^{-1} |\langle u, e_n\rangle_{-1,1}|^2,$$

since

$$\langle u, \varphi\rangle_{-1,1} = \sum_{n\in\mathbb{N}} \langle u, e_n\rangle_{-1,1} \langle e_n, \varphi\rangle_0 \qquad \text{for all } \varphi \in H^1(D).$$

More generally, for $r \in \mathbb{R}$ and $\varphi \in L^2(D)$, we set

$$\|\varphi\|_r^2 = \sum_{n\in\mathbb{N}} (1 + \lambda_n)^r |\langle \varphi, e_n\rangle_0|^2,$$

and we let $H^r(D)$ be the completion of $\{\varphi \in L^2(D) : \|\varphi\|_r < \infty\}$ in $\|\cdot\|_r$.

2.2. *Gaussian noise.* We shall define a process $F^D = \{F_t^D(\varphi), t \in \mathbb{R}_+, \varphi \in \mathcal{S}(D)\}$, which is informally related to $\dot{F}^D(t,x)$ in (1.1) by

$$F_t^D(\varphi) = \int_0^t ds \int_D dx\, \dot{F}^D(s,x)\varphi(x), \qquad t \in \mathbb{R}_+, \varphi \in \mathcal{S}(D). \tag{2.2}$$

In order to define F^D rigorously, we assume that the covariance Γ_D is a semi-inner product on $\mathcal{S}(D)$; that is, Γ_D is bilinear, symmetric and

$$\Gamma_D(\varphi,\varphi) \geq 0 \qquad \text{for all } \varphi \in \mathcal{S}(D).$$

By the Kolmogorov extension theorem (see [21], Proposition 3.4), there exist a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a centered Gaussian process $F^D = \{F_t^D(\varphi), t \in \mathbb{R}_+, \varphi \in \mathcal{S}(D)\}$ defined on this space, whose covariance is given by

$$\mathbb{E}\big(F_t^D(\varphi)F_s^D(\psi)\big) = (t \wedge s)\Gamma_D(\varphi,\psi).$$

ASSUMPTION A. There exist $r \geq 0$ and $K > 0$ such that, for all $\varphi \in \mathcal{S}(D)$, $\Gamma_D(\varphi,\varphi) \leq K\|\varphi\|_r^2$.

By the Cauchy–Schwarz inequality for $\Gamma_D(\cdot,\cdot)$, this implies that Γ_D is *separately continuous* with respect to the H^r-norm, that is, for all $\varphi, \psi \in \mathcal{S}(D)$, $\Gamma_D(\varphi,\psi) \leq K\|\varphi\|_r\|\psi\|_r$. Moreover, under Assumption A, Theorem 4.1, page 332 of [31] implies (see Example 3, page 336, of [31]) that $F_t^D(\cdot)$ has a modification with values in $H^{-(r+1)-d/2}(D)$. Clearly,

$$\mathbb{E}\big(\big(F_t^D(\varphi) - F_s^D(\varphi)\big)^2\big) \leq K(t-s)\|\varphi\|_r^2,$$

so, by [9], Proposition 3.15, the process $\{F_t^D, t \in \mathbb{R}_+\}$ has a continuous modification $\{\tilde{F}_t^D, t \in \mathbb{R}_+\}$ with values in $H^{-(r+1)-d/2}(D)$. In particular, for all $\varphi \in \mathcal{S}(D)$, the map $t \mapsto \tilde{F}_t^D(\varphi)$ is continuous and is therefore a continuous Brownian motion with speed $\Gamma_D(\varphi,\varphi)$. In the following, we assume that $\{F_t^D, t \in \mathbb{R}_+\}$ *is* this modification.

2.3. *Weak formulation of the equation.* We now give a rigorous meaning to (1.1). First, setting formally $v(t,x) = \frac{\partial u}{\partial t}(t,x)$, we obtain the following two formal equations, after integration in t of (1.1):

$$u(t,x) = u_0(x) + \int_0^t ds\, v(s,x),$$
$$v(t,x) = v_0(x) + \int_0^t ds\,\big(-2av(s,x) - bu(s,x) + \Delta u(s,x) + \dot{F}^D(s,x)\big).$$

We now multiply both sides of these two equations by a test function $\varphi \in \mathcal{S}(D)$ and integrate them over $x \in D$. We then formally apply Green's theorem to the term with the Laplacian, taking into account the Neumann boundary conditions for u and φ. Assuming that $(u_0, v_0) \in L^2(D) \oplus H^{-1}(D)$, considering that (u, v) takes its values in $L^2(D) \oplus H^{-1}(D)$ and using the informal relationship (2.2), we get the following rigorous formulation: a *weak solution* of (1.1) is a process $(u, v) = \{(u(t), v(t)), t \in \mathbb{R}_+\}$ with values in $L^2(D) \oplus H^{-1}(D)$ such that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that, for all $\omega \notin \mathcal{N}$ and $\varphi \in \mathcal{S}(D)$, the map $t \mapsto (\langle u(t,\omega), \varphi\rangle_0, \langle v(t,\omega), \varphi\rangle_{-1,1})$ is continuous on $\mathbb{R}_+$ and satisfies

$$\begin{aligned}
&\langle u(t), \varphi\rangle_0 = \langle u_0, \varphi\rangle_0 + \int_0^t ds\, \langle v(s), \varphi\rangle_{-1,1},\\
&\langle v(t), \varphi\rangle_{-1,1} = \langle v_0, \varphi\rangle_{-1,1} + \int_0^t ds\, \big(-2a\langle v(s), \varphi\rangle_{-1,1}\\
&\qquad\qquad - b\langle u(s), \varphi\rangle_0 + \langle u(s), \Delta\varphi\rangle_0\big)\\
&\qquad + F_t^D(\varphi).
\end{aligned} \tag{2.3}$$

We will often refer to u, instead of (u, v), as the solution of (2.3).

REMARK 2.2. A solution u of (2.3) is termed a "weak" solution of (1.1), because it takes its values in $L^2(D)$, and therefore neither Δu nor $\frac{\partial u}{\partial \nu}|_{\partial D}$ is defined. We will see in Remark 3.11 that when the noise is concentrated on a sphere, there never exists a *regular* solution of (2.3), that is, with values in $H^1(D) \oplus L^2(D)$.

2.4. *Properties of the Green kernel.* The solution of the deterministic linear equation corresponding to (2.3) can be expressed in terms of the *Green kernel* of the equation, which in turn can be decomposed into the eigenmodes of the Laplacian given in Theorem 2.1. We define here these components of the Green kernel and establish some basic inequalities in Lemma 2.3.

Let $n \in \mathbb{N}$ and let $G_n : \mathbb{R} \to \mathbb{R}$ be the solution of the differential equation

$$G_n''(t) + 2aG_n'(t) + (b + \lambda_n)G_n(t) = 0, \qquad G_n(0) = 0, \qquad G_n'(0) = 1. \tag{2.4}$$

One easily checks that G_n is given by

$$G_n(t) = \begin{cases} e^{-at}\dfrac{\sin(t\sqrt{\lambda_n + b - a^2})}{\sqrt{\lambda_n + b - a^2}}, & \text{if } \lambda_n > a^2 - b,\\ e^{-at}t, & \text{if } a^2 - b \geq 0 \text{ and } \lambda_n = a^2 - b,\\ e^{-at}\dfrac{\sinh(t\sqrt{a^2 - b - \lambda_n})}{\sqrt{a^2 - b - \lambda_n}}, & \text{if } a^2 - b > 0 \text{ and } \lambda_n < a^2 - b. \end{cases} \tag{2.5}$$

Note that the first of these three expressions actually contains the other two, since we have $\lim_{u\to 0} \sin(u)/u = 1$ and $\sin(iu) = i\sinh(u)$.

LEMMA 2.3. (a) *For $t > 0$, there exist $C(t) > 0$ and $n_0(t) \in \mathbb{N}$ such that, for all $s \in [0, t]$ and $n \geq n_0(t)$,*

$$|G_n(s)| \leq \frac{C(t)}{\sqrt{1 + \lambda_n}} \quad \textit{and} \quad |G_n'(s)| \leq C(t). \tag{2.6}$$

(b) *For $t > 0$, there exist $C_+(t) \geq C_-(t) > 0$ and $n_0(t) \in \mathbb{N}$ such that*

$$\frac{C_-(t)}{1+\lambda_n} \leq \int_0^t ds\, G_n(s)^2 \leq \frac{C_+(t)}{1+\lambda_n} \qquad \textit{for all } n \geq n_0(t).$$

PROOF. (a) For n sufficiently large, there is $C > 0$ such that, for $s \leq t$,

$$|G_n(s)| \leq (\lambda_n + b - a^2)^{-1/2} e^{2a^- t} \leq C(1+\lambda_n)^{-1/2} e^{2a^- t},$$

where $a^- = \max(0, -a)$. The second estimate is obtained in a similar manner.

(b) The upper bound is an immediate consequence of (a). In order to prove the lower bound, set $a^+ = \max(0, a)$. Then

$$\begin{aligned}\int_0^t ds\, G_n(s)^2 &= \int_0^t ds\, e^{-2a\, s} \frac{\sin^2(s\sqrt{\lambda_n + b - a^2}\,)}{\lambda_n + b - a^2} \\ &\geq \frac{e^{-2a^+ t}}{\lambda_n + b - a^2} \frac{t}{2} \biggl(1 - \frac{\sin(2t\sqrt{\lambda_n + b - a^2}\,)}{2t\sqrt{\lambda_n + b - a^2}}\biggr).\end{aligned}$$

For large n, the factor in parentheses is greater than or equal to $1/2$ and $(1+\lambda_n)/(\lambda_n + b - a^2) \geq 1/2$, so

$$\int_0^t ds\, G_n(s)^2 \geq \frac{t e^{-2a^+ t}}{8(1 \vee (b - a^2))(1+\lambda_n)}.$$

This completes the proof. $\square$

The solution of the (2.3) will make use of the following functions. For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, set $H_n(t) = G_n'(t) + 2aG_n(t)$. From (2.4), H_n satisfies

$$H_n''(t) + 2aH_n'(t) + (b + \lambda_n)H_n(t) = 0, \qquad H_n(0) = 1,\ H_n'(0) = 0. \tag{2.7}$$

Explicit formulas for $H_n(t)$ and $H_n'(t)$ are

$$H_n(t) = e^{-at} \cos\bigl(t\sqrt{\lambda_n + b - a^2}\,\bigr) + a e^{-at} \frac{\sin(t\sqrt{\lambda_n + b - a^2}\,)}{\sqrt{\lambda_n + b - a^2}} \tag{2.8}$$

and

$$\begin{aligned}H_n'(t) = &-e^{-at}\sqrt{\lambda_n + b - a^2} \sin\bigl(t\sqrt{\lambda_n + b - a^2}\,\bigr) \\ &- a^2 e^{-at} \frac{\sin(t\sqrt{\lambda_n + b - a^2}\,)}{\sqrt{\lambda_n + b - a^2}}.\end{aligned}$$

Clearly, for $t > 0$, there exist $C(t) > 0$ and $n_0(t) \in \mathbb{N}$ such that, for all $s \in [0, t]$ and $n \geq n_0(t)$,

$$|H_n(s)| \leq C(t) \quad \text{and} \quad |H_n'(s)| \leq C(t)\sqrt{1+\lambda_n}. \tag{2.9}$$

2.5. *Existence and uniqueness of the solution.* We will show that there exists a unique weak solution to (1.1) under the following assumption. Let e_n be the elements of the orthonormal basis described in Theorem 2.1 and set

$$\gamma_{n,m} = \Gamma_D(e_n, e_m), \qquad \gamma_n = \gamma_{n,n}, \qquad n, m \in \mathbb{N}.$$

ASSUMPTION B. The following condition is satisfied:

$$\sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+\lambda_n} < \infty.$$

For ease of reference, we recall the following special case of the stochastic Fubini theorem (see, for instance, [26], Chapter 4, Theorem 46).

THEOREM 2.4. *If* $W = \{W_s, s \in \mathbb{R}_+\}$ *is a standard Brownian motion*, $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}_+)$ *and* $t \in \mathbb{R}_+$, *then*, $\mathbb{P}$-*a.s.*,

$$\int_0^t ds \int_0^s dW_r\, g(r,s) = \int_0^t dW_r \int_r^t ds\, g(r,s).$$

Let us now state the two main results of this section.

THEOREM 2.5. *Let* $(u_0, v_0) \in L^2(D) \oplus H^{-1}(D)$. *Under Assumptions* A *and* B, *the process* $(u,v) = \{(u(t), v(t)), t \in \mathbb{R}_+\}$ *with values in* $L^2(D) \oplus H^{-1}(D)$ *defined by*

$$u(t) = u^0(t) + p(t) \quad \textit{and} \quad v(t) = v^0(t) + q(t), \tag{2.10}$$

where

$$u^0(t) = \sum_{n\in\mathbb{N}} u_n^0(t) e_n, \qquad p(t) = \sum_{n\in\mathbb{N}} p_n(t) e_n,$$
$$v^0(t) = \sum_{n\in\mathbb{N}} v_n^0(t) e_n, \qquad q(t) = \sum_{n\in\mathbb{N}} q_n(t) e_n$$

and

$$u_n^0(t) = H_n(t)\langle u_0, e_n\rangle_0 + G_n(t)\langle v_0, e_n\rangle_{-1,1}, \qquad p_n(t) = \int_0^t dF_s^D(e_n)\, G_n(t-s),$$

$$v_n^0(t) = H_n'(t)\langle u_0, e_n\rangle_0 + G_n'(t)\langle v_0, e_n\rangle_{-1,1}, \qquad q_n(t) = \int_0^t dF_s^D(e_n)\, G_n'(t-s),$$

admits a modification $(\tilde{u}, \tilde{v})$ *which is the unique weak solution of* (1.1). *Moreover*, $\mathbb{E}(\|\tilde{u}(t)\|_0^2) < \infty$ *and* $\mathbb{E}(\|\tilde{v}(t)\|_{-1}^2) < \infty$ *for all* $t \in \mathbb{R}_+$.

THEOREM 2.6. *Let* $(u_0, v_0) \in L^2(D) \oplus H^{-1}(D)$. *If there exists a weak solution* (u, v) *to* (1.1) *such that* $\mathbb{E}(\|u(t_0)\|_0^2) < \infty$ *for some* $t_0 > 0$, *then Assumption* B *is satisfied*.

REMARK 2.7. Note that Theorem 2.5 is, in principle, part of the general theory developed in [9] and [11]. Indeed, the wave equation (though not with Neumann boundary conditions) is mentioned in [9], Example 5.8, and some boundary noises are treated in [11], Chapter 13. Assumption B is formally equivalent to (5.14) in [9]. However, Theorem 2.6 shows that Assumption B is necessary for the existence of a solution. Moreover, Theorem 2.5 covers a wider class of Gaussian noises, including boundary noises, which will be considered in Section 3.

PROOF OF THEOREM 2.5. We first show existence. By (2.6) and (2.9),

$$\|u^0(s)\|_0^2 = \sum_{n\in\mathbb{N}} |u_n^0(s)|^2 < \infty \quad \text{and} \quad \|v^0(s)\|_{-1}^2 = \sum_{n\in\mathbb{N}} \frac{|v_n^0(s)|^2}{1+\lambda_n} < \infty,$$

so the deterministic process (u^0, v^0) takes its values in $L^2(D) \oplus H^{-1}(D)$, and, in fact, the supremum over $0 \le s \le t$ of $\|u^0(s)\|_0$ and $\|v^0(s)\|_{-1}$ is finite. By a direct calculation using (2.4) and (2.7), we see that, for each $n \in \mathbb{N}$,

$$u_n^0(t) = \langle u_0, e_n\rangle_0 + \int_0^t ds\, v_n^0(s),$$

$$v_n^0(t) = \langle v_0, e_n\rangle_{-1,1} - \int_0^t ds\, \bigl(2av_n^0(s) + (b+\lambda_n)u_n^0(s)\bigr).$$

Multiplying the first equation by $\langle \varphi, e_n\rangle_0$ and summing over $n \in \mathbb{N}$ leads to the following equation for (u^0, v^0):

$$\langle u^0(t), \varphi\rangle_0 = \langle u_0, \varphi\rangle_0 + \int_0^t ds\, \langle v^0(s), \varphi\rangle_{-1,1},$$

$$\langle v^0(t), \varphi\rangle_{-1,1} = \langle v_0, \varphi\rangle_{-1,1} + \int_0^t ds\, \bigl(-2a\langle v^0(s), \varphi\rangle_{-1,1} - b\langle u^0(s), \varphi\rangle_0 + \langle u^0(s), \Delta\varphi\rangle_0\bigr) \tag{2.11}$$

for all $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{S}(D)$. On the other hand, by the fundamental theorem of calculus,

$$\int_0^t dF_s^D(e_n)\, G_n(t-s) = \int_0^t dF_s^D(e_n) \int_s^t dr\, G_n'(r-s),$$

while integrating (2.4) with respect to t, then with respect to the Brownian motion $F_\cdot^D(e_n)$, gives

$$\int_0^t dF_s^D(e_n)\, G_n'(t-s) = F_t^D(e_n) - \int_0^t dF_s^D(e_n) \int_s^t dr\, \bigl(2aG_n'(r-s) + (b+\lambda_n)G_n(r-s)\bigr).$$

Apply the stochastic Fubini theorem (Theorem 2.4) to the integral terms, to see that the process (p_n, q_n) satisfies

$$
\begin{aligned}
&p_n(t) = \int_0^t dr\, q_n(r),\\
(2.12)\qquad &q_n(t) = F_t^D(e_n) - \int_0^t dr\, \bigl(2aq_n(r) + (b+\lambda_n)p_n(r)\bigr).
\end{aligned}
$$

We now check that $p(t) \in L^2(D)$ and $q(t) \in H^{-1}(D)$ for all $t \in \mathbb{R}_+$. Indeed,

$$
\mathbb{E}\bigl(\|p(t)\|_0^2\bigr) = \sum_{n\in\mathbb{N}} \mathbb{E}\bigl(p_n(t)^2\bigr) = \sum_{n\in\mathbb{N}} \gamma_n \int_0^t ds\, G_n(t-s)^2 < \infty
$$

by the upper bound in Lemma 2.3(b) and Assumption B. Furthermore,

$$
\mathbb{E}\bigl(\|q(t)\|_{-1}^2\bigr) = \sum_{n\in\mathbb{N}} \frac{\mathbb{E}(q_n(t)^2)}{1+\lambda_n} = \sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+\lambda_n} \int_0^t ds\, G_n'(t-s)^2 < \infty
$$

by (2.6) and Assumption B.

We now verify that $(p(t), q(t))$ solves (2.3) with $u^0 = v^0 \equiv 0$. Using the fact that the Laplacian is symmetric on $\mathcal{S}(D)$, we have, for $\varphi \in \mathcal{S}(D)$,

$$
\begin{aligned}
\langle p(t), \Delta\varphi\rangle_0 &= \sum_{n\in\mathbb{N}} p_n(t)\langle e_n, \Delta\varphi\rangle_0 = \sum_{n\in\mathbb{N}} p_n(t)\langle \Delta e_n, \varphi\rangle_0,\\
&= -\sum_{n\in\mathbb{N}} \lambda_n p_n(t)\langle e_n, \varphi\rangle_0, \qquad \text{a.s.}
\end{aligned}
$$

by Theorem 2.1. Therefore, multiplying the two equations in (2.12) by $\langle e_n, \varphi\rangle_0$ and summing over $n \in \mathbb{N}$ leads, for all $t \in \mathbb{R}^+$, to the following equation for (p, q):

$$
\begin{aligned}
&\langle p(t), \varphi\rangle_0 = \int_0^t ds\, \langle q(s), \varphi\rangle_{-1,1}, \qquad \text{a.s.},\\
(2.13)\quad &\langle q(t), \varphi\rangle_{-1,1} = F_t^D(\varphi) + \int_0^t ds\, \bigl(-2a\langle q(s), \varphi\rangle_{-1,1}\\
&\qquad\qquad - b\langle p(s), \varphi\rangle_0 + \langle p(s), \Delta\varphi\rangle_0\bigr), \qquad \text{a.s.},
\end{aligned}
$$

where we have used the fact that

$$
\sum_{n\in\mathbb{N}} F_t^D(e_n)\langle e_n, \varphi\rangle_0 = F_t^D(\varphi), \qquad \text{a.s.}
$$

by Assumption A and the fact that $\varphi \in H^r(D)$ for all $r > 0$. Note that the $\mathbb{P}$-null set involved in (2.13) can depend on t and φ. Therefore, we still have to check that the process $(p, q) = \{(p(t), q(t)), t \in \mathbb{R}_+\}$ admits a modification $(\tilde{p}, \tilde{q})$ which is continuous on $\mathbb{R}_+$ with values in $H^{-1}(D) \oplus H^{-2}(D)$. To this end, we will use the

Kolmogorov test for Gaussian processes with values in a Hilbert space (see [9], Proposition 3.15). Let us therefore compute, for fixed $T > 0$ and $0 \le t \le t+h \le T$,

$$\begin{aligned}
&\mathbb{E}\big(\| p(t+h) - p(t)\|_{-1}^2\big)\\
&\quad = \sum_{n\in\mathbb{N}} \frac{\mathbb{E}((p_n(t+h) - p_n(t))^2)}{1+\lambda_n}\\
&\quad = \sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+\lambda_n}\bigg(\int_0^t ds\,\big(G_n(t+h-s) - G_n(t-s)\big)^2\\
&\qquad\qquad + \int_t^{t+h} ds\, G_n(t+h-s)^2\bigg)\\
&\quad = \sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+\lambda_n}\bigg(\int_0^t ds\,\bigg(\int_{t-s}^{t+h-s} dr\, G_n'(r)\bigg)^2 + \int_0^h dr\, G_n(r)^2\bigg)\\
&\quad \le \sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+\lambda_n}\big(Th^2 C(T)^2 + hC(T)^2\big)\\
&\quad \le C_T h
\end{aligned}$$

by (2.6) and Assumption B. In a similar manner, we conclude that

$$\mathbb{E}\big(\|q(t+h) - q(t)\|_{-2}^2\big) \le C_T h \qquad \text{for all } 0 \le t \le t+h \le T,$$

so there exists a continuous modification $(\tilde{p}, \tilde{q})$ of the process (p, q) by [9], Proposition 3.15. Combining (2.11) and (2.13) and setting $(\tilde{u}, \tilde{v}) = (u^0 + \tilde{p}, v^0 + \tilde{q})$, we conclude that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that, for all $\omega \notin \mathcal{N}$ and $\varphi \in \mathcal{S}(D)$, the map $t \mapsto (\langle \tilde{u}(t,\omega), \varphi\rangle_0, \langle \tilde{v}(t,\omega), \varphi\rangle_{-1,1})$ is continuous on $\mathbb{R}_+$ and solves (2.3).

In order to prove uniqueness, let $(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$ be two solutions of (2.3) and define $(\bar{u}, \bar{v}) = (u^{(1)} - u^{(2)}, v^{(1)} - v^{(2)})$. Then there exists a $\mathbb{P}$-null set such that, outside this set, for all $\varphi \in \mathcal{S}(D)$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}
\langle \bar{u}(t), \varphi\rangle_0 &= \int_0^t ds\, \langle \bar{v}(s), \varphi\rangle_{-1,1},\\
\langle \bar{v}(t), \varphi\rangle_{-1,1} &= \int_0^t ds\,\big(-2a\langle \bar{v}(s), \varphi\rangle_{-1,1} - b\langle \bar{u}(s), \varphi\rangle_0 + \langle \bar{u}(s), \Delta\varphi\rangle_0\big).
\end{aligned}$$

Fix $n \in \mathbb{N}$ and, for $t \in \mathbb{R}_+$, define $\bar{u}_n(t) = \langle \bar{u}(t), e_n\rangle_0$ and $\bar{v}_n(t) = \langle \bar{v}(t), e_n\rangle_{-1,1}$. Replacing φ by e_n in the preceding equation and using the symmetry of the Laplacian on $\mathcal{S}(D)$, we obtain, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}
\bar{u}_n(t) &= \int_0^t ds\, \bar{v}_n(s),\\
\bar{v}_n(t) &= -\int_0^t ds\,\big(2a\bar{v}_n(s) + (b+\lambda_n)\bar{u}_n(s)\big).
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, $\bar{u}_n(t) = \bar{v}_n(t) = 0$. The conclusion follows. □

PROOF OF THEOREM 2.6. Let (u, v) be a solution of (2.3) and let $t_0 > 0$ be such that $\mathbb{E}(\|u(t_0)\|_0^2) < \infty$. By Theorem 2.5, $u(t) = u^0(t) + p(t)$, so

$$\mathbb{E}\big(\|p(t_0)\|_0^2\big) \le 2\big(\mathbb{E}\big(\|u(t_0)\|_0^2\big) + \|u^0(t_0)\|_0^2\big).$$

The right-hand side is finite by assumption and because $\|u^0(t_0)\|_0^2 < \infty$. On the other hand, a direct calculation shows that

$$\mathbb{E}\big(\|p(t_0)\|_0^2\big) = \sum_{n\in\mathbb{N}} \gamma_n \int_0^{t_0} ds\, G_n(t-s)^2 \ge C_-(t_0) \sum_{n\ge n_0(t_0)} \frac{\gamma_n}{1+\lambda_n}$$

by the lower bound in Lemma 2.3(b), so Assumption B must be satisfied. This completes the proof. □

REMARK 2.8. Performing the same kind of analysis as above, one sees that if there exists a solution (u, v) to (2.3) with values in $H^1(D) \oplus L^2(D)$, then the following condition (stronger than Assumption B) must be satisfied:

$$\sum_{n\in\mathbb{N}} \gamma_n < \infty. \tag{2.14}$$

2.6. *The heat equation.* If, instead of the hyperbolic equation considered above, we consider the following heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\Delta u(t,x) = \dot{F}^D(t,x), && (t,x) \in \mathbb{R}_+ \times D,\\
&\frac{\partial u}{\partial \nu}(t,x) = 0, && (t,x) \in \mathbb{R}_+ \times \partial D,\\
&u(0,x) = u_0(x), && x \in D,
\end{aligned} \tag{2.15}$$

then we can reproduce the analysis of the preceding paragraphs. The only difference will consist of the fact that the weak formulation is simpler to write [there is only one process u taking its values in $L^2(D)$] and the G_n are now solutions of $G_n'(t) + \frac{\lambda_n}{2} G_n(t) = 0$, with $G_n(0) = 1$. They are therefore given by $G_n(t) = \exp(-\lambda_n t/2)$. The analysis is similar to that of the hyperbolic case because these G_n also satisfy the conclusions of Lemma 2.3, so the methods and conclusions of Theorems 2.5 and 2.6 remain valid in the case of the heat equation.

3. Linear equation in a ball, driven by isotropic noise on a sphere. Let $d > 1$. In this section, we shall study the existence of a weak solution to the hyperbolic equation (1.1) [in the sense defined in (2.3)], in the specific case where the domain is $D = B(0,1)$, the centered unit ball in $\mathbb{R}^d$, and the noise is concentrated on the sphere $S^{d-1} = \partial B(0,1)$. Our objective is to obtain explicit and easily verifiable conditions under which the conclusions of Theorems 2.5 and 2.6 are valid. For this, we need rather detailed information about the orthonormal basis described in Theorem 2.1, which we now recall.

3.1. *Eigenvalues and eigenfunctions of the Laplacian in* $B(0,1)$. In this section, we recall the classical explicit formulas for the eigenvalues and eigenfunctions of the Laplacian in $B(0,1)$. These will be used to reformulate Assumption B into an easily verifiable condition. Recall the definition of the Bessel functions $J_l(d,\cdot)$ for $l \in \mathbb{N}$ and $d \geq 2$ (see [20], formula (§27.2)):

$$J_l(d,r) = \Gamma\left(\frac{d}{2}\right)\left(\frac{r}{2}\right)^{(2-d)/2} \mathcal{J}_{l+(d-2)/2}(r), \qquad r > 0, \tag{3.1}$$

where Γ is the Euler Gamma function defined by

$$\Gamma(\nu) = \int_0^\infty dt\, t^{\nu-1} e^{-t}, \qquad \nu > 0, \tag{3.2}$$

and $\mathcal{J}_\nu$ is the regular Bessel function of the first kind and of order ν (see [1], formula 9.1.10, or [20], formula (§27.3), or [13]) defined by

$$\mathcal{J}_\nu(r) = \left(\frac{r}{2}\right)^\nu \sum_{n\in\mathbb{N}} \frac{(-r^2/4)^n}{n!\Gamma(\nu+n+1)}, \qquad r \in \mathbb{R}_+.$$

Clearly, the derivative of $J_l(d,\cdot)$ in r is given by

$$J_l'(d,r) = \Gamma\left(\frac{d}{2}\right)\left(\frac{r}{2}\right)^{(2-d)/2} \left(\mathcal{J}'_{l+(d-2)/2}(r) - \frac{d-2}{2r}\mathcal{J}_{l+(d-2)/2}(r)\right). \tag{3.3}$$

In addition to using the abstract result of Theorem 2.1, we now describe explicitly the solutions of the eigenvalue problem

$$\Delta\varphi + \lambda\varphi = 0 \qquad \text{in } B(0,1) \quad \text{and} \quad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial B(0,1)} = 0. \tag{3.4}$$

It is well known (see [20], (§22)) that they are of the form $\varphi(x) = f(r)Y(\theta)$, where $r = |x|$ and $\theta \in S^{d-1}$ represents the angular part of x. The function Y is a solution of the eigenvalue problem $\Delta_\theta Y(\theta) + \Lambda Y(\theta) = 0$, where Δ_θ denotes the Laplace–Beltrami operator on S^{d-1} (see [29], page 255). The solutions of this problem are given (see [20], (§15), Lemma 1) by $\{\Lambda_l, Y_l^m, l \in \mathbb{N}, 1 \leq m \leq N(d,l)\}$, where $\Lambda_l = l(l+d-2)$, $\{Y_l^m, 1 \leq m \leq N(d,l)\}$ is the list of generalized complex-valued spherical harmonics of order l on S^{d-1} and $N(d,l)$ is the number of these harmonics. This set forms an orthonormal basis of $L^2(S^{d-1})$. Moreover, note that, when $d=2$, $N(2,l)=2$ and $Y_l^\pm(\theta) = \exp(\pm il\theta)$; when $d=3$, $N(3,l) = 2l+1$ and the Y_l^m are the standard spherical harmonics on S^2. More generally, $N(d,l) \sim l^{d-2}$ as $l \to \infty$, where we write $a_l \sim b_l$ when there exists $C \in]0,\infty[$ such that $\lim_{l\to\infty} a_l/b_l = C$.

For a fixed $l \in \mathbb{N}$, f is a solution of the eigenvalue problem

$$f''(r) + \frac{d-1}{r} f'(r) + \left(\lambda - \frac{l(l+d-2)}{r^2}\right) f(r) = 0, \qquad f'(1) = 0.$$

The solutions of this problem are given (see [20], (§22)) by $\{\lambda_{kl}, f_{kl}, k \in \mathbb{N}\}$, where $\lambda_{kl} = \mu_{kl}^2$, and, for a fixed $l \in \mathbb{N}$, $\{\mu_{kl}, k \in \mathbb{N}\}$ is the ascending list of zeros of the derivative of the Bessel function $J_l(d, \cdot)$ defined by (3.1) and f_{kl} is the function defined for fixed $k, l \in \mathbb{N}$ by

$$f_{kl}(r) = \frac{J_l(d, \mu_{kl} r)}{\sqrt{\int_0^1 dq\, q^{d-1} J_l(d, \mu_{kl} q)^2}}. \tag{3.5}$$

This leads to the following solutions of (3.4):

$$\{\lambda_{kl}, e_{klm} = f_{kl} \otimes Y_l^m, k, l \in \mathbb{N}, 1 \le m \le N(d,l)\},$$

the above "tensor product" meaning $e_{klm}(x) = f_{kl}(r) Y_l^m(\theta)$. Note that these eigenfunctions are normalized in $L^2(B(0,1))$, that is,

$$\int_{B(0,1)} dx |e_{klm}(x)|^2 = 1 \qquad \text{for all } k, l \in \mathbb{N}, 1 \le m \le N(d,l).$$

3.2. *Covariance of the noise and Schönberg's theorem.* We now define a class of isotropic Gaussian noises on the sphere S^{d-1}. In order to obtain a general form for the covariance of such noises, we consider first the case of a continuous covariance. Let $f : S^{d-1} \times S^{d-1} \to \mathbb{R}$ be a continuous, symmetric and nonnegative definite function, that is,

$$\sum_{i,j=1}^{m} c_i c_j f(x^{(i)}, x^{(j)}) \ge 0, \tag{3.6}$$

for all $m \ge 1$, $c_1, \dots, c_m \in \mathbb{R}$, $x^{(1)}, \dots, x^{(m)} \in S^{d-1}$. This function f is the covariance of a centered Gaussian process indexed by the elements of S^{d-1}. If there exists a continuous function $g : [-1, +1] \to \mathbb{R}$ such that $f(x, y) = g(x \cdot y)$ for all $x, y \in S^{d-1}$, where $x \cdot y$ is the Euclidean inner product, then we say that the Gaussian process is *isotropic*. In particular, condition (3.6) becomes

$$\sum_{i,j=1}^{m} c_i c_j g(x^{(i)} \cdot x^{(j)}) \ge 0 \tag{3.7}$$

for all $m \ge 1$, $c_1, \dots, c_m \in \mathbb{R}$, $x^{(1)}, \dots, x^{(m)} \in S^{d-1}$.

In order to characterize the functions g that satisfy (3.7), consider the generalized Legendre polynomials (see [20], (§2), Lemma 4) defined for $d \ge 2$ by

$$P_l(d,t) = \left(-\frac{1}{2}\right)^l \frac{\Gamma(\frac{d-1}{2})}{\Gamma(l + \frac{d-1}{2})} (1-t^2)^{(3-d)/2} \left(\frac{d}{dt}\right)^l (1-t^2)^{l+(d-3)/2},$$

$$l \in \mathbb{N}, t \in [-1, +1],$$

where Γ is the Euler Gamma function. Notice that these are simply the Chebyshev polynomials when $d = 2$ and the standard Legendre polynomials when $d = 3$.

Schönberg's theorem (see [27], Theorem 1), similar to Bochner's theorem concerning nonnegative definite functions on $\mathbb{R}^d$, states the following.

THEOREM 3.1. *Let* $g : [-1, +1] \to \mathbb{R}$ *be a continuous function. Then* g *is nonnegative definite on* S^{d-1} [*in the sense of* (3.7)] *if and only if there exists a sequence* $\{a_l, l \in \mathbb{N}\}$ *of nonnegative numbers such that*

$$g(t) = \sum_{l \in \mathbb{N}} a_l P_l(d, t), \qquad t \in [-1, +1], \quad \textit{and} \quad \sum_{l \in \mathbb{N}} a_l < \infty.$$

In order to be able to consider generalized processes, indexed by test functions on S^{d-1}, rather than by S^{d-1} itself, we need a wider class of (not necessarily continuous) covariances. Observe that, under assumption (3.7), the functional $\Gamma_S : C^\infty(S^{d-1}) \times C^\infty(S^{d-1}) \to \mathbb{R}$, defined by

$$\Gamma_S(\varphi, \psi) = \int_{S^{d-1}} d\sigma(x) \int_{S^{d-1}} d\sigma(y) \varphi(x) g(x \cdot y)\, \psi(y), \qquad \varphi, \psi \in C^\infty(S^{d-1}),$$

where σ is the uniform surface measure on the unit sphere S^{d-1}, is a semi-inner product on $C^\infty(S^{d-1})$. Moreover, Γ_S is *isotropic*, that is,

$$\Gamma_S(R\varphi, R\psi) = \Gamma_S(\varphi, \psi) \qquad \text{for all } \varphi, \psi \in C^\infty(S^{d-1}),$$

for any rotation R on the sphere S^{d-1} [where, by definition, $R\varphi(x) = \varphi(R^{-1}x)$].

In view of Theorem 3.1, it is natural to consider functionals Γ_S of the form

$$\Gamma_S(\varphi, \psi) = \sum_{l \in \mathbb{N}} a_l \Gamma_l(\varphi, \psi), \qquad \varphi, \psi \in C^\infty(S^{d-1}), \tag{3.8}$$

where

$$\Gamma_l(\varphi, \psi) = \int_{S^{d-1}} d\sigma(x) \int_{S^{d-1}} d\sigma(y)\, \varphi(x) P_l(d, x \cdot y) \psi(y),$$

and $a_l \geq 0$, but the condition $\sum_{l \in \mathbb{N}} a_l < \infty$ is replaced by

$$\sum_{l \in \mathbb{N}} \frac{a_l}{(1 + l)^{r_0}} < \infty \qquad \text{for some } r_0 > 0. \tag{3.9}$$

Condition (3.9) is analogous to the growth condition required of tempered measures. The following lemma shows that the series in (3.8) converges.

LEMMA 3.2. *If* (3.9) *holds, then* $\Gamma_S(\varphi, \psi) < \infty$ *for all* $\varphi, \psi \in C^\infty(S^{d-1})$.

PROOF. By the Cauchy–Schwarz inequality, it is sufficient to check that $\Gamma_S(\varphi, \varphi) < \infty$ for each $\varphi \in C^\infty(S^{d-1})$. Consider the following Sobolev spaces on S^{d-1}. For $r \geq 0$, let $H^r(S^{d-1})$ be the domain of the operator $(I - \Delta_\theta)^{r/2}$ in

$L^2(S^{d-1})$, where Δ_θ is the Laplace–Beltrami operator on the sphere S^{d-1}. By the spectral decomposition of $L^2(S^{d-1})$ in spherical harmonics,

$$H^r(S^{d-1}) = \left\{ v = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} c_{lm} Y_l^m \,\middle|\, \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} (1+\Lambda_l)^r |c_{lm}|^2 < \infty \right\}.$$

Since $\Lambda_l = l(l+d-2)$, we can equip this space with the equivalent norm

$$\|v\|_r^2 = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} (1+l)^{2r} |c_{lm}|^2. \tag{3.10}$$

Using the fact that $C^\infty(S^{d-1}) \subset \bigcap_{r\geq 0} H^r(S^{d-1})$, a function $\varphi \in C^\infty(S^{d-1})$ can be written as

$$\varphi = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} c_{lm} Y_l^m, \quad \text{with} \quad \sum_{l\in\mathbb{N}} (1+l)^{2r} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 < \infty \qquad \forall r > 0. \tag{3.11}$$

Using the following additivity property (see [20], (§2), Theorem 2):

$$P_l(d, x\cdot y) = \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} \overline{Y_l^m(x)} Y_l^m(y), \tag{3.12}$$

and the orthonormality of the spherical harmonics, we see that

$$\begin{aligned} \Gamma_l(\varphi,\varphi) &= \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} \left| \int_{S^{d-1}} d\sigma(x)\, \varphi(x) \overline{Y_l^m(x)} \right|^2 \\ &= \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} |c_{lm}|^2, \end{aligned} \tag{3.13}$$

and therefore, using (3.8), we obtain

$$\begin{aligned} \Gamma_S(\varphi,\varphi) &= \sum_{l\in\mathbb{N}} a_l \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \\ &= |S^{d-1}| \sum_{l\in\mathbb{N}} \frac{a_l}{(1+l)^{r_0}} \left[\frac{(1+l)^{r_0}}{N(d,l)} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \right]. \end{aligned} \tag{3.14}$$

Because $N(d,l) \geq 1$ and (3.11) holds, the expression in brackets is bounded, so the series converges by (3.9). □

REMARK 3.3. (a) In the representation (3.8), white noise on the sphere is obtained by setting $a_l = N(d,l)/|S^{d-1}| \sim l^{d-2}$. Indeed, for white noise,

$$\Gamma_S(\varphi,\psi) = \int_{S^{d-1}} d\sigma(x)\, \varphi(x)\psi(x) \qquad \text{for all } \varphi, \psi \in C^\infty(S^{d-1}),$$

and therefore $\Gamma_S(Y_l^m, Y_l^m) = 1$. From (3.8) and (3.12), $\Gamma_S(Y_l^m, Y_l^m) = a_l \times |S^{d-1}|/N(d,l)$, which proves the claim.

(b) It is natural to ask whether the analog in this context of the Bochner–Schwartz theorem holds, that is, whether *every* isotropic semi-inner product Γ on $C^\infty(S^{d-1})$, with some additional continuity property, is of the form Γ_S given above. To our knowledge, this is an open problem.

In order to relate the particular covariance Γ_S on S^{d-1} defined above to the general covariance Γ_D which was considered in Section 2 and defined on the entire domain D [here, $D = B(0,1)$], we define Γ_D by

$$\Gamma_D(\varphi, \psi) = \Gamma_S(\varphi|_{S^{d-1}}, \psi|_{S^{d-1}}), \qquad \varphi, \psi \in C^\infty(\overline{B(0,1)}). \tag{3.15}$$

LEMMA 3.4. *The class of covariances defined in* (3.15) *satisfies Assumption* A.

PROOF. We will use here the following fact (see [2], Theorem 7.53): for all $r > 0$ the trace operator γ_0, defined for $\varphi \in C^\infty(\overline{B(0,1)})$ by $\gamma_0(\varphi) = \varphi|_{S^{d-1}}$, can be extended continuously from $H^{r+1/2}(B(0,1))$ to $H^r(S^{d-1})$. That is, there exists $C > 0$ such that

$$\|\gamma_0(u)\|_r^2 \le C\|u\|_{r+1/2}^2 \qquad \text{for all } u \in H^{r+1/2}(B(0,1)). \tag{3.16}$$

(Note that the definition of $H^r(S^{d-1})$ in [2] coincides with the definition we have given above; see, for instance, [29], page 255.) It suffices therefore to check that, for some $r > 0$, there exists $C > 0$ such that

$$|\Gamma_S(v,v)| \le C\|v\|_r^2 \qquad \text{for all } v \in C^\infty(S^{d-1}), \tag{3.17}$$

since this will imply, by (3.16), that, for all $\varphi \in C^\infty(\overline{B(0,1)})$,

$$|\Gamma_D(\varphi,\varphi)| = |\Gamma_S(\gamma_0(\varphi), \gamma_0(\varphi))| \le C\|\gamma_0(\varphi)\|_r^2 \le \tilde{C}\|\varphi\|_{r+1/2}^2.$$

We therefore check (3.17). By (3.9), there exists $C > 0$ such that $a_l \le C(1+l)^{r_0}$ for all $l \in \mathbb{N}$. Let $\varphi \in C^\infty(S^{d-1})$. As in (3.11), φ can be written as $\varphi = \sum_{l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} c_{lm} Y_l^m$. Therefore, by (3.14),

$$\begin{aligned}
\Gamma_S(\varphi,\varphi) &= \sum_{l\in\mathbb{N}} a_l \frac{|S^{d-1}|}{N(d,l)} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \\
&\le C|S^{d-1}| \sum_{l\in\mathbb{N}} (1+l)^{r_0} \sum_{m=1}^{N(d,l)} |c_{lm}|^2 \\
&= C|S^{d-1}| \|\varphi\|_{r_0/2}^2.
\end{aligned}$$

This completes the proof. $\square$

REMARK 3.5. As mentioned in Section 1, (2.3) can also be interpreted as the homogeneous p.d.e. with stochastic boundary condition

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(t,x) + 2a\frac{\partial u}{\partial t}(t,x) + bu(t,x) - \Delta u(t,x) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in \mathbb{R}_+ \times D, \\ &\frac{\partial u}{\partial \nu}(t,x) = \dot{F}^S(t,x), \qquad\qquad\qquad\qquad\qquad (t,x) \in \mathbb{R}_+ \times \partial D, \\ &u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in D. \end{aligned} \tag{3.18}$$

Indeed, the noise term $F_t^D(\varphi)$ can be formally rewritten here as

$$F_t^D(\varphi) = \int_0^t ds \int_{\partial B(0,1)} d\sigma(x)\, \dot{F}^S(s,x)\varphi(x), \qquad \varphi \in C^\infty(\overline{B(0,1)}),$$

where $\dot{F}^S$ is a generalized centered Gaussian process concentrated on the sphere $\partial B(0,1) = S^{d-1}$ with covariance

$$\mathbb{E}\big(\dot{F}^S(t,x)\dot{F}^S(s,y)\big) = \delta_0(t-s)\Gamma_S(x,y). \tag{3.19}$$

Stochastic partial differential equations with this type of boundary term have been considered in [3], [10], [11] and [30].

3.3. *Explicit conditions.* In the following, we give a necessary and sufficient condition on the coefficients a_l which guarantees the existence of a weak solution to (3.18) or, equivalently, (1.1) driven by noise with spatial covariance Γ_D defined in (3.8) and (3.15).

In the present setting, Assumption B of the preceding section becomes

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \frac{\gamma_{klm}}{1+\lambda_{kl}} < \infty, \tag{3.20}$$

where $\gamma_{klm} = \Gamma_D(e_{klm}, e_{klm})$. Our objective is now to translate this condition into an explicit condition on the a_l in (3.8). We rewrite it first in a different manner.

LEMMA 3.6. *The expression in* (3.20) *is equal to* $\sum_{l\in\mathbb{N}} a_l b_l$, *where*

$$b_l = |S^{d-1}| \sum_{k\in\mathbb{N}} \frac{f_{kl}(1)^2}{1+\lambda_{kl}}.$$

PROOF. Clearly,

$$\begin{aligned} \gamma_{klm} &= \Gamma_D(e_{klm}, e_{klm}) = \Gamma_S\big(f_{kl}(1)Y_l^m, f_{kl}(1)Y_l^m\big) \\ &= f_{kl}(1)^2 \sum_{n\in\mathbb{N}} a_n \Gamma_n(Y_l^m, Y_l^m). \end{aligned}$$

By (3.13) and the orthonormality of the Y_l^m, $\Gamma_n(Y_l^m, Y_l^m) = |S^{d-1}|\delta_{nl}/N(d,n)$. Therefore,

$$\gamma_{klm} = f_{kl}(1)^2 a_l \frac{|S^{d-1}|}{N(d,l)}, \tag{3.21}$$

and the conclusion follows. $\square$

We now estimate the behavior of b_l as $l \to \infty$. For this, we need to relate $f_{kl}(1)$ to the eigenvalue λ_{kl} and to estimate $\mu_{kl} = \sqrt{\lambda_{kl}}$, which we do in the following two lemmas.

LEMMA 3.7. *For all $k, l \in \mathbb{N}$,*

$$f_{kl}(1)^2 = \frac{2\lambda_{kl}}{\lambda_{kl} - l^2 - l(d-2)}.$$

PROOF. By (3.5),

$$f_{kl}(1)^2 = \frac{J_l(d, \mu_{kl})^2}{\int_0^1 dq\, q^{d-1} J_l(d, \mu_{kl} q)^2}. \tag{3.22}$$

By (3.3), the μ_{kl} are the positive solutions of the equation

$$\mathcal{J}_{l+(d-2)/2}(x) + \frac{2x}{d-2}\mathcal{J}'_{l+(d-2)/2}(x) = 0.$$

Therefore, by the definition of $J_l(d, \cdot)$ and [1], formula 11.4.5,

$$\begin{aligned}
&\int_0^1 dq\, q^{d-1} J_l(d, \mu_{kl} q)^2 \\
&\qquad = \Gamma\left(\frac{d}{2}\right)^2 \left(\frac{\mu_{kl}}{2}\right)^{2-d} \int_0^1 dq\, q\, \mathcal{J}_{l+(d-2)/2}(\mu_{kl} q)^2 \\
&\qquad = \frac{1}{2}\Gamma\left(\frac{d}{2}\right)^2 \left(\frac{\mu_{kl}}{2}\right)^{2-d} \frac{\lambda_{kl} - l^2 - l(d-2)}{\lambda_{kl}} \mathcal{J}_{l+(d-2)/2}(\mu_{kl})^2 \\
&\qquad = \frac{1}{2}\frac{\lambda_{kl} - l^2 - l(d-2)}{\lambda_{kl}} J_l(d, \mu_{kl})^2.
\end{aligned}$$

Together with (3.22) above, this proves the lemma. $\square$

LEMMA 3.8. *For all $k, l \in \mathbb{N}$,*

$$l + \frac{d-2}{2} + \pi(k-2) \le \mu_{kl} \le \frac{\pi}{2}\left(l + \frac{d-2}{2}\right) + \pi(k+2). \tag{3.23}$$

PROOF. Denote by $\{\mu^0_{k\nu}, k \in \mathbb{N}\}$ the ascending list of zeros of the standard Bessel function $\mathcal{J}_\nu$. For $\nu = l + \frac{d-2}{2}$, by (3.1), these are also the zeros of $J_l(d, \cdot)$. By [4], Theorem 1 and Lemma 2,

$$\nu + \pi(k-1) \le \mu^0_{k\nu} \le \frac{\pi}{2}\nu + \pi(k+1). \tag{3.24}$$

Since $J_l(d, \cdot)$ is proportional to $\mathcal{J}_{l+(d-2)/2}(\cdot)$, the interlacing property of the zeros of Bessel functions and their derivatives (i.e., if ν_{kl} are the zeros of $J_l(d, \cdot)$, then $\nu_{k-1,l} < \mu_{kl} < \nu_{k+1,l}$, see [1], formula 9.5.2) implies that

$$\mu^0_{k-1,l+(d-2)/2} \le \mu_{kl} \le \mu^0_{k+1,l+(d-2)/2},$$

so (3.23) follows from (3.24). $\square$

With these two lemmas, we obtain the following estimate.

LEMMA 3.9. *There exist $C_1, C_2 > 0$ such that, for sufficiently large l, the coefficients b_l defined in Lemma 3.6 satisfy*

$$\frac{C_1}{1+l} \le b_l \le \frac{C_2}{1+l}.$$

PROOF. By Lemmas 3.6 and 3.7,

$$b_l = 2|S^{d-1}| \sum_{k\in\mathbb{N}} \frac{\lambda_{kl}}{1+\lambda_{kl}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)}. \tag{3.25}$$

We first prove the lower bound. Since $\lambda_{kl}/(1+\lambda_{kl}) \in [1/2, 1]$ when $\lambda_{kl} \ge 1$, b_l is bounded below by

$$|S^{d-1}| \sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl} - l^2 - l(d-2)} \ge |S^{d-1}| \sum_{k\in\mathbb{N}} \frac{1}{\lambda_{kl}}.$$

By Lemma 3.8, this is in turn bounded below by

$$\begin{aligned}
|S^{d-1}| &\sum_{k\in\mathbb{N}} \left(\frac{\pi}{2}\left(l + \frac{d-2}{2}\right) + \pi(k+2)\right)^{-2} \\
&\ge \frac{|S^{d-1}|}{\pi^2} \int_2^\infty dx \left(\frac{l}{2} + \frac{d-2}{4} + x\right)^{-2} \\
&= \frac{|S^{d-1}|}{\pi^2} \left(\frac{l}{2} + \frac{d-2}{4} + 2\right)^{-1} \ge \frac{C}{1+l}.
\end{aligned}$$

In order to prove the upper bound, we check first that $b_l < \infty$ for each $l \in \mathbb{N}$. By (3.25) and the lower bound in Lemma 3.8,

$$b_l \le |S^{d-1}| \left(\sum_{k\le 2} \frac{f_{kl}(1)^2}{1+\lambda_{kl}} + 2\sum_{k>2} \frac{1}{(l + (d-2)/2 + \pi(k-2))^2 - l^2 - l(d-2)} \right),$$

which is clearly finite. Fix now l_0 and m_0 and consider the function u on $B(0,1)$ whose Fourier components in the orthonormal basis (e_{klm}) are given by

$$u_{klm} = \frac{f_{kl}(1)}{1+\lambda_{kl}} \delta_{ll_0} \delta_{mm_0}.$$

This function belongs to $H^1(B(0,1))$, since

$$\|u\|_1^2 = \sum_{klm} (1+\lambda_{kl}) |u_{klm}|^2 = \sum_{k \in \mathbb{N}} \frac{f_{kl_0}(1)^2}{1+\lambda_{kl_0}} = \frac{b_{l_0}}{|S^{d-1}|} < \infty,$$

as was shown above. Using (3.10) and (3.16) with $r = 1/2$, we obtain

$$\sum_{l \in \mathbb{N}} \sum_{m=1}^{N(d,l)} (1+l) \left| \sum_{k \in \mathbb{N}} u_{klm} f_{kl}(1) \right|^2 \le C \sum_{klm} (1+\lambda_{kl}) |u_{klm}|^2,$$

that is,

$$(1+l_0) \left(\sum_{k \in \mathbb{N}} \frac{f_{kl_0}(1)^2}{1+\lambda_{kl_0}} \right)^2 \le C \sum_{k \in \mathbb{N}} \frac{f_{kl_0}(1)^2}{1+\lambda_{kl_0}}.$$

Since the sums are finite, we obtain by cancellation that

$$\sum_{k \in \mathbb{N}} \frac{f_{kl_0}(1)^2}{1+\lambda_{kl_0}} \le \frac{C}{1+l_0},$$

which completes the proof. $\square$

We can now state the following theorem, which is a reformulation of Theorems 2.5 and 2.6 in the present setting.

THEOREM 3.10. *Let* $(u_0, v_0) \in L^2(B(0,1)) \oplus H^{-1}(B(0,1))$. *Consider* (1.1), *where the covariance of* $\dot{F}^D(t,x)$ *is given by* (3.15) *and* (3.8) [*or, equivalently*, (3.18), *where the covariance of* $\dot{F}^S(t,x)$ *is given by* (3.19) *and* (3.8)]. *If*

$$\sum_{l \in \mathbb{N}} \frac{a_l}{1+l} < \infty, \tag{3.26}$$

then the equation has a unique weak solution $u = \{u(t), t \in \mathbb{R}_+\}$, *and* $\mathbb{E}(\|u(t)\|_0^2) < \infty$ *for all* $t \in \mathbb{R}_+$. *On the other hand, if there exists a weak solution u to the equation such that* $\mathbb{E}(\|u(t_0)\|_0^2) < \infty$ *for some* $t_0 > 0$, *then condition* (3.26) *is satisfied.*

PROOF. Let us first prove the sufficiency of (3.26). By Theorem 2.5, we simply have to check that this condition implies Assumptions A and B of Section 2. Assumption A has already been checked in Lemma 3.4, and Assumption B is a direct consequence of condition (3.26), Lemma 3.6 and the upper bound in Lemma 3.9.

To show that condition (3.26) is necessary, we use Theorem 2.6. Indeed, by Lemma 3.6 and the lower bound in Lemma 3.9, Assumption B implies (3.26), so the theorem is proven. □

REMARK 3.11. (a) By Remark 3.3(a), (3.26) is not satisfied for white noise on the sphere.

(b) Following Remark 2.8 of the preceding section, we see that if the solution u of (2.3) took its values in $H^1(B(0,1))$, then condition (2.14) would be satisfied. In the present case, (2.14) can be rewritten as

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \gamma_{klm} < \infty.$$

By (3.21),

$$\sum_{k,l\in\mathbb{N}} \sum_{m=1}^{N(d,l)} \gamma_{klm} = |S^{d-1}| \sum_{l\in\mathbb{N}} a_l \left(\sum_{k\in\mathbb{N}} f_{kl}(1)^2 \right) = \infty,$$

since the term in parentheses is infinite by Lemma 3.7. Therefore, (2.14) is never satisfied, so there never exists a solution with values in $H^1(B(0,1))$, when the noise under consideration is a (nonvanishing) boundary noise.

3.4. *Noise on a sphere of smaller radius.* We assume in this section that the noise is concentrated on a sphere of lower radius $r_0 \in]0,1[$, therefore interior to the domain $B(0,1)$. The preceding analysis generally carries over to this case, with the following changes. The general form for the covariance of the noise is

$$\Gamma_S(\varphi,\psi) = \sum_{l\in\mathbb{N}} a_l \Gamma_l(\varphi,\psi),$$

where

$$\Gamma_l(\varphi,\psi) = \int_{S(r_0)} d\sigma(x) \int_{S(r_0)} d\sigma(y)\, \varphi(x) P_l\left(d, \frac{x\cdot y}{r_0^2}\right)\psi(y),$$

and the a_l and P_l are as before and $S(r_0)$ is the sphere of radius r_0. The covariance Γ_D is related to Γ_S by

$$\Gamma_D(\varphi,\psi) = \Gamma_S\big(\varphi|_{S(r_0)}, \psi|_{S(r_0)}\big), \qquad \varphi,\psi \in \mathcal{S}(D). \tag{3.27}$$

Note that the noise can no longer be interpreted as a stochastic boundary condition, as in (3.18). However, we have the following theorem.

THEOREM 3.12. *Let* $(u_0, v_0) \in L^2(B(0,1)) \oplus H^{-1}(B(0,1))$. *Consider* (1.1), *where the covariance of* $\dot{F}^D(t,x)$ *is given by* (3.27). *If condition* (3.26) *is satisfied, then there exists a unique weak solution* $u = \{u(t), t \in \mathbb{R}_+\}$ *of* (1.1), *and* $\mathbb{E}(\|u(t)\|_0^2) < \infty$ *for all* $t \in \mathbb{R}_+$.

PROOF. By Theorem 2.5, it suffices to check that Assumptions A and B are satisfied. Using the trace operator γ_{r_0} on $S(r_0)$, instead of γ_0, we easily prove as in the proof of Lemma 3.4 that Assumption A is satisfied. In order to check Assumption B, notice that, as in (3.21),

$$\gamma_{klm} = \Gamma_D(e_{klm}, e_{klm}) = f_{kl}(r_0)^2 a_l \frac{|S^{d-1}|}{N(d,l)}.$$

We need therefore to check that

$$\sum_{l \in \mathbb{N}} a_l \left(\sum_{k \in \mathbb{N}} \frac{f_{kl}(r_0)^2}{1 + \lambda_{kl}} \right) < \infty.$$

We first prove that, for each $l \in \mathbb{N}$, the term in parentheses is finite: by a calculation similar to that of Lemma 3.7, we have

$$f_{kl}(r_0)^2 = \frac{2\lambda_{kl}}{\lambda_{kl} - l^2 - l(d-2)} \left(\frac{J_l(d, \mu_{kl} r_0)}{J_l(d, \mu_{kl})} \right)^2.$$

Moreover, for fixed $l \in \mathbb{N}$,

$$J_l(d,r) = C_d r^{(1-d)/2} \cos\left(r - \left(l + \frac{d-2}{2} \right) \frac{\pi}{2} - \frac{\pi}{4} \right) + O(r^{-d/2}) \tag{3.28}$$

(see [1], formula 9.2.1), so one expects

$$J_l(d, \mu_{kl})^2 \sim \mu_{kl}^{1-d} \qquad \text{as } k \to \infty. \tag{3.29}$$

We assume this for the moment. Then, for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\frac{J_l(d, \mu_{kl} r_0)}{J_l(d, \mu_{kl})} \right)^2 \le r_0^{1-d} + \varepsilon \qquad \text{for all } k \ge k_0.$$

Therefore,

$$\sum_{k \in \mathbb{N}} \frac{f_{kl}(r_0)^2}{1 + \lambda_{kl}} \le \sum_{k \le k_0} \frac{f_{kl}(r_0)^2}{1 + \lambda_{kl}} + \sum_{k > k_0} \frac{2}{\lambda_{kl} - l^2 - l(d-2)} (r_0^{1-d} + \varepsilon),$$

which is finite by Lemma 3.8. We now proceed as in the last part of the proof of Lemma 3.9, replacing again the operator γ_0 by γ_{r_0}. This leads to the conclusion that there exists $C(r_0) > 0$ such that

$$\sum_{k \in \mathbb{N}} \frac{f_{kl}(r_0)^2}{1 + \lambda_{kl}} \le \frac{C(r_0)}{1 + l}, \tag{3.30}$$

which proves Assumption B.

We now check (3.29). Set $\nu = l - 1 + d/2$. By (3.3), μ_{kl} solves

$$\mathcal{J}'_\nu(r) = \frac{d-2}{2r}\mathcal{J}_\nu(r). \tag{3.31}$$

By [1], 9.2.11, 9.2.15 and 9.2.16,

$$\mathcal{J}'_\nu(r) = \sqrt{\frac{2}{\pi r}}\big(-R(\nu, r)\sin(\chi_r) - S(\nu, r)\cos(\chi_r)\big),$$

where $\chi_r = r - \pi r/2 - \pi/4$, and $R(\nu, r)$ and $S(\nu, r)$ satisfy $\lim_{r\to\infty} R(\nu, r) = 1$ and $\lim_{r\to\infty} S(\nu, r) = 0$. Further, by [1], 9.2.1,

$$\mathcal{J}_\nu(r) = \sqrt{\frac{2}{\pi r}}\big(\cos(\chi_r) + O(r^{-1})\big).$$

Equation (3.31) can be rewritten as

$$-R(\nu, r)\sin(\chi_r) - S(\nu, r)\cos(\chi_r) = \frac{d-2}{2r}\big(\cos(\chi_r) + O(r^{-1})\big),$$

so (μ_{kl}) verifies $\lim_{k\to\infty}\sin(\chi_{\mu_{kl}}) = 0$, and, therefore, $\lim_{k\to\infty}\cos(\chi_{\mu_{kl}}) = 1$. Together with (3.28), this proves (3.29). $\square$

REMARK 3.13. The question of whether condition (3.26) is not only sufficient for the conclusion of Theorem 3.12, but also necessary, remains an open problem. Indeed, when $0 < r_0 < 1$, it is not clear whether the inequality in (3.30) can be reversed [with $C(r_0)$ replaced by a different constant].

3.5. *An integral test* ($d = 2$). We consider here the case $d = 2$, that is, the linear hyperbolic equation in two space dimensions driven by noise concentrated on the unit circle S^1. Our aim is to reformulate condition (3.26) as an integral test. For this, we shall make the additional assumption that the covariance Γ_S is given by a nonnegative measure on the unit circle. To make this precise, recall that $P_l(2, \cos\theta) = \cos(l\theta)$ and therefore the covariance Γ_S from (3.8) is given by

$$\begin{aligned}\Gamma_S(\varphi, \psi) &= \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta_x \int_{-\pi}^{\pi} d\theta_y\, \varphi(\theta_x)\cos\big(l(\theta_x - \theta_y)\big)\psi(\theta_y)\\ &= \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\cos(l\theta)\int_{-\pi}^{\pi} d\theta_x\, \varphi(\theta_x)\psi(\theta_x - \theta),\end{aligned}$$

by the change of variable $\theta = \theta_x - \theta_y$ (the addition/subtraction operations are identified with the group operations on S^1, which we identify with $[-\pi, \pi]$). This can be rewritten as

$$\Gamma_S(\varphi, \psi) = \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta\cos(l\theta)(\varphi * \tilde{\psi})(\theta),$$

where

$$(\varphi * \psi)(\theta) = \int_{-\pi}^{\pi} d\vartheta\, \varphi(\vartheta)\psi(\theta - \vartheta)$$

is the convolution product on S^1 and $\tilde{\psi}(\theta) = \psi(-\theta)$. The map

$$\varphi \mapsto \sum_{l \in \mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta \cos(l\theta)\varphi(\theta), \qquad \varphi \in C^\infty(S^1),$$

defines a distribution on S^1 (see [28], Chapter 7, Section I). Let us now assume that this distribution is nonnegative. By the fundamental theorem of Radon–Riesz (see, e.g., [15], Chapter 2, Theorem 2.2), there exists therefore a nonnegative Borel measure Γ on S^1 such that

$$\int_{-\pi}^{\pi} \Gamma(d\theta)\,\varphi(\theta) = \sum_{l \in \mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta \cos(l\theta)\varphi(\theta) \qquad \text{for all } \varphi \in C^\infty(S^1). \tag{3.32}$$

We now have the following reformulation of condition (3.26) as a condition on the measure Γ.

PROPOSITION 3.14. *Set*

$$K(\vartheta) = \int_{-\pi}^{\pi} \Gamma(d\theta) \ln\left(\frac{2\pi}{|\vartheta - \theta|}\right), \qquad \vartheta \in [-\pi, \pi].$$

Then (3.26) *holds if and only if*

$$\sup_{\vartheta \in [-\pi,\pi]} K(\vartheta) < \infty. \tag{3.33}$$

REMARK 3.15. Condition (3.33) is slightly stronger than the condition $K(0) < \infty$, that is,

$$\int_{-\pi}^{\pi} \Gamma(d\theta) \ln\left(\frac{2\pi}{|\theta|}\right) < \infty.$$

However, (3.33) is implied by the condition "K is continuous at 0," since K is nonnegative definite. Condition (3.33) should be compared with the conditions in the spatially homogeneous setting; compare [5]–[7], [23] and [25].

PROOF OF PROPOSITION 3.14. Set

$$h(\theta) = \ln\left(\frac{2\pi}{|\theta|}\right) = \frac{c_0}{2} + \sum_{l \in \mathbb{N}^*} c_l \cos(l\theta), \qquad \theta \in [-\pi, \pi] \setminus \{0\},$$

and $h(0) = +\infty$. Note that since h belongs to $L^2([-\pi, \pi])$, the above Fourier series also converges in $L^2([-\pi, \pi])$. Moreover, computing the Fourier coefficients c_l gives, for $l \in \mathbb{N}^*$,

$$c_l = \frac{2}{\pi} \int_0^{\pi} d\theta \ln\left(\frac{2\pi}{\theta}\right) \cos(l\theta) = \frac{2}{\pi l} \int_0^{l\pi} du\, \frac{\sin(u)}{u}$$

by integration by parts and change of variable $u = l\theta$. Since the integral converges to $\pi/2$ as $l \to \infty$, (3.26) is equivalent to

$$\sum_{l\in\mathbb{N}} a_l c_l < \infty. \tag{3.34}$$

Suppose now that (3.26) or, equivalently, (3.34) holds. Since $\theta \mapsto \ln(2\pi/\theta)$ does not belong to $C^\infty(S^1)$, we cannot simply set $\varphi(\theta) = \ln(2\pi/\theta)$ in (3.32): some smoothing is required. For $t > 0$, set

$$\begin{aligned}\psi_t(\theta) &= \frac{1}{\pi} + \frac{2}{\pi}\sum_{l\in\mathbb{N}^*} e^{-lt}\cos(l\theta) = \frac{1}{\pi} + \frac{2}{\pi}\mathrm{Re}\sum_{l\in\mathbb{N}^*} e^{l(-t+i\theta)} \\ &= \frac{1}{\pi}\frac{\sinh(t)}{\cosh(t)-\cos(\theta)}, \qquad \theta \in [-\pi, \pi],\end{aligned}$$

by summing the geometric series. Then ψ_t is a probability density on $[-\pi, \pi]$, and if we set $h_t(\theta) = (h * \psi_t)(\theta)$ for $\theta \in [-\pi, \pi]$, then $h_t \geq 0$ and $\lim_{t\downarrow 0} h_t(\theta) = h(\theta)$ for all $\theta \in [-\pi, \pi]$ (the ψ_t play the role of approximations to the Dirac δ_0 distribution). By Parseval's identity,

$$h_t(\theta) = c_0 + 2\sum_{l\in\mathbb{N}^*} c_l e^{-lt}\cos(l\theta), \qquad \theta \in [-\pi, \pi],$$

and $h_t \in C^\infty(S^1)$ for all $t > 0$, since the coefficients $c_l e^{-lt}$ are rapidly decreasing in l. Therefore,

$$\begin{aligned}\int_{-\pi}^{\pi} \Gamma(d\theta)\, h_t(\vartheta - \theta) &= \sum_{l\in\mathbb{N}} a_l \int_{-\pi}^{\pi} d\theta \cos(l\theta) h_t(\vartheta - \theta) \\ &= \pi \sum_{l\in\mathbb{N}} a_l c_l e^{-lt}\cos(l\vartheta),\end{aligned} \tag{3.35}$$

and, by Fatou's lemma,

$$K(\vartheta) \leq \liminf_{t\downarrow 0} \int_{-\pi}^{\pi} \Gamma(d\theta) h_t(\vartheta - \theta) \leq \pi \sum_{l\in\mathbb{N}} a_l c_l < \infty$$

by (3.34). It follows that (3.33) is satisfied.

Suppose now that (3.33) holds. By Fubini's theorem,

$$\int_{-\pi}^{\pi} d\vartheta\, \psi_t(\vartheta) K(\vartheta) = \int_{-\pi}^{\pi} \Gamma(d\theta)(h * \psi_t)(\theta) = \pi \sum_{l\in\mathbb{N}} a_l c_l e^{-lt},$$

and, by assumption, the left-hand side is bounded by $\sup_{\vartheta\in[-\pi,\pi]} K(\vartheta)$. Applying the monotone convergence theorem to the right-hand side, we conclude that (3.34), and therefore (3.26), holds. $\square$

4. Linear equation in a hypercube driven by homogeneous noise on a hyperplane. In this section, we carry out an analysis similar to that of Section 3, in the case where $D = [0, \pi]^d$ $(d > 1)$ and the noise is concentrated on $K = [0, \pi]^{d-1} \times \{\alpha\}$, where $\alpha \in [0, \pi]$. We identify K with $[0, \pi]^{d-1}$.

4.1. *Noise on K.* Let us first define a class of covariances on K. As in Section 3.2, we begin with a continuous and symmetric function $f : [-\pi, \pi]^{d-1} \to \mathbb{R}$, which is nonnegative definite on K, that is,

$$\sum_{i,j=1}^{m} c_i \overline{c_j} f\bigl(x^{(i)} - x^{(j)}\bigr) \geq 0$$

$$\text{for all } m \geq 1, c_1, \dots, c_m \in \mathbb{C}, x^{(1)}, \dots, x^{(m)} \in K.$$

This function is the covariance of a Gaussian process indexed by the elements of K, which is spatially homogeneous. Examples of such functions are

$$f(x_1, \dots, x_{d-1}) = \sum_{n_1, \dots, n_{d-1} \in \mathbb{N}} a_{n_1, \dots, n_{d-1}} \cos(n_1 x_1) \cdots \cos(n_{d-1} x_{d-1}),$$

where $(x_1, \dots, x_{d-1}) \in [-\pi, \pi]^{d-1}$, $a_{n_1, \dots, n_{d-1}} \geq 0$ and

$$\sum_{n_1, \dots, n_{d-1} \in \mathbb{N}} a_{n_1, \dots, n_{d-1}} < \infty. \tag{4.1}$$

Using the addition identity for $\cos(m(x - y))$, one easily checks that such an f satisfies the required properties. Note that we cannot apply the classical Bochner theorem to conclude that any covariance f has the preceding form, because K is not a group.

As in Section 3.2, we consider a covariance functional given by

$$\Gamma_K(\varphi, \psi) = \sum_{n_1, \dots, n_{d-1} \in \mathbb{N}} a_{n_1, \dots, n_{d-1}} \Gamma_{n_1, \dots, n_{d-1}}(\varphi, \psi), \qquad \varphi, \psi \in C^\infty(K),$$

where $a_{n_1, \dots, n_{d-1}} \geq 0$ and

$$\begin{aligned} &\Gamma_{n_1, \dots, n_{d-1}}(\varphi, \psi) \\ &\quad = \int_K dx_1 \cdots dx_{d-1} \int_K dy_1 \cdots dy_{d-1} \varphi(x_1, \dots, x_{d-1}) \cos\bigl(n_1(x_1 - y_1)\bigr) \cdots \\ &\qquad \times \cos\bigl(n_{d-1}(x_{d-1} - y_{d-1})\bigr) \psi(y_1, \dots, y_{d-1}), \end{aligned}$$

with condition (4.1) replaced by another one [see (4.2)], under which we can easily check that $\Gamma_K(\varphi, \psi)$ is well defined for each $\varphi, \psi \in C^\infty(K)$.

Let us finally define the covariance Γ_D by

$$\Gamma_D(\varphi, \psi) = \Gamma_K(\varphi|_K, \psi|_K), \qquad \varphi, \psi \in \mathcal{S}(D).$$

where $\mathcal{S}(D)$ is defined in Section 2.

4.2. *Conditions for existence.* Consider the condition

$$\sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} \frac{a_{n_1,\dots,n_{d-1}}}{\sqrt{1+n_1^2+\cdots+n_{d-1}^2}} < \infty. \tag{4.2}$$

The main result of this section is the following (note that for the heat equation similar results were already obtained in [10] and [11], Theorem 13.3.1).

THEOREM 4.1. *Let* $(u_0, v_0) \in L^2(D) \oplus H^{-1}(D)$. *There exists a unique weak solution* u *of* (1.1) *such that* $\mathbb{E}(\|u(t)\|_0^2) < \infty$ *for all* $t \in \mathbb{R}_+$ *if and only if condition* (4.2) *is satisfied.*

PROOF. The solutions of the eigenvalue problem

$$\Delta\varphi + \lambda\varphi = 0 \text{ in } D \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu}\bigg|_{\partial D} = 0$$

have the following simple expressions:

$$e_{\underline{n}}(x) = \left(\frac{2}{\pi}\right)^{d/2} \cos(n_1x_1)\cdots\cos(n_dx_d), \qquad \lambda_{\underline{n}} = n_1^2+\cdots+n_d^2,$$

where $\underline{n} = (n_1,\dots,n_d) \in \mathbb{N}^d$. Notice that the $e_{\underline{n}}$ are C^∞, even though ∂D is not.

Let us now compute the coefficients $\gamma_{\underline{n}} = \Gamma_D(e_{\underline{n}}, e_{\underline{n}})$:

$$\gamma_{\underline{n}} = \sum_{m_1,\dots,m_{d-1}\in\mathbb{N}} a_{m_1,\dots,m_{d-1}} \Gamma_{m_1,\dots,m_{d-1}}(e_{\underline{n}}|_K, e_{\underline{n}}|_K).$$

Since

$$e_{\underline{n}}|_K(x_1,\dots,x_{d-1}) = \left(\frac{2}{\pi}\right)^{d/2} \cos(n_1x_1)\cdots\cos(n_{d-1}x_{d-1})\cos(n_d\alpha),$$

and, using the addition identity for $\cos(m(x-y))$, we obtain

$$\gamma_{\underline{n}} = \frac{2}{\pi} a_{n_1,\dots,n_{d-1}} \cos^2(n_d\alpha).$$

In order to check that condition (4.2) is sufficient, we simply need to check that it implies Assumptions A and B of Section 2, which in turn imply the desired existence and uniqueness result by Theorem 2.5. To see that Assumption A is satisfied, we follow the proof of Lemma 3.4 with $r = \frac{1}{2}$; that is, we check that there exists $K > 0$ such that, for all $\varphi \in H^{1/2}(K)$,

$$\Gamma_K(\varphi,\varphi) \le K\|\varphi\|_{1/2}^2, \tag{4.3}$$

where $\|\cdot\|_{1/2}$ is the $H^{1/2}$-norm on K defined by

$$\|\varphi\|_{1/2}^2 = \sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} \sqrt{1+n_1^2+\cdots+n_{d-1}^2}\,|c_{n_1,\dots,n_{d-1}}|^2$$

for

$$\varphi(x_1,\dots,x_{d-1}) = \sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} c_{n_1,\dots,n_{d-1}} \cos(n_1 x_1)\cdots\cos(n_{d-1}x_{d-1}).$$

We then compute

$$\Gamma_K(\varphi,\varphi) = \left(\frac{2}{\pi}\right)^{d-1} \sum_{m_1,\dots,m_{d-1}\in\mathbb{N}} a_{m_1,\dots,m_{d-1}} \left|c_{m_1,\dots,m_{d-1}}\right|^2.$$

Under condition (4.2), there exists $C>0$ such that

$$a_{m_1,\dots,m_{d-1}} \le C\sqrt{1+m_1^2+\cdots+m_{d-1}^2};$$

therefore, (4.3) holds as in the proof of Lemma 3.4.

Let us now verify Assumption B. We compute

$$\begin{aligned}\sum_{\underline{n}\in\mathbb{N}^d} \frac{\gamma_{\underline{n}}}{1+\lambda_{\underline{n}}} &= \frac{2}{\pi} \sum_{n_1,\dots,n_d\in\mathbb{N}} \frac{a_{n_1,\dots,n_{d-1}}\cos^2(n_d\alpha)}{1+n_1^2+\cdots+n_d^2} \\ &= \frac{2}{\pi} \sum_{n_1,\dots,n_{d-1}\in\mathbb{N}} a_{n_1,\dots,n_{d-1}} \left(\sum_{n_d\in\mathbb{N}} \frac{\cos^2(n_d\alpha)}{1+n_1^2+\cdots+n_d^2}\right).\end{aligned} \tag{4.4}$$

Since

$$\sum_{n_d\in\mathbb{N}} \frac{\cos^2(n_d\alpha)}{a^2+n_d^2} \le \frac{1}{a^2} + \int_0^\infty dx\, \frac{1}{a^2+x^2} = \frac{1}{a^2}+\frac{\pi}{2a} \le \frac{C}{a}, \qquad a \ge 1,$$

we see that (4.2) implies Assumption B.

To see that (4.2) is necessary, if there exists a weak solution of (1.1) such that $\mathbb{E}(\|u(t)\|_0^2) < \infty$ for all $t\in\mathbb{R}$, then, by Theorem 2.6, Assumption B is satisfied. By (4.4), it suffices therefore to check that there exists $C>0$ such that

$$\sum_{n\in\mathbb{N}} \frac{\cos^2(n\alpha)}{a^2+n^2} \ge \frac{C}{a}, \tag{4.5}$$

since this will prove the necessity of condition (4.2). Because $\cos^2(n\alpha) = \cos^2(n(\pi-\alpha))$, we may as well assume that $\alpha\in[0,\pi/2]$. In this case, noting that $\{x\in[-\pi,\pi]\colon \cos(x)\ge\sqrt{2}/2\} = [-\pi/4,\pi/4]$, and this interval has length $\pi/2$, set $N_0=1$, $N_\alpha = [2\pi/\alpha]+1$ if $\alpha>0$, and observe that the set $\{n\in\mathbb{N}\colon \cos(n\alpha)\ge\sqrt{2}/2\}$ contains an increasing sequence $(m_i, i\in\mathbb{N})$ such that $0<m_{i+1}-m_i\le N_\alpha$. Since $x\mapsto(a^2+x^2)^{-1}$ is a decreasing function of $x\in\mathbb{R}_+$, we conclude that

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{\cos^2(n\alpha)}{a^2+n^2} &\ge \sum_{i\in\mathbb{N}} \frac{\cos^2(n\alpha)}{a^2+m_i^2} \ge \frac{\sqrt{2}}{2}\sum_{n\in\mathbb{N}} \frac{1}{a^2+(N_\alpha n)^2} \\ &\ge \frac{\sqrt{2}}{2}\int_0^\infty dx\, \frac{1}{a^2+(N_\alpha x)^2} = \frac{\pi\sqrt{2}}{4N_\alpha}\frac{1}{a},\end{aligned}$$

which proves (4.5) and concludes the proof. $\square$

4.3. *An integral test* ($d = 2$). As in Section 3.5, in the case where $d = 2$, we shall reformulate condition (4.2) as an integral test, when the functional Γ_K is given by a nonnegative measure on the segment $K = [0, 2\pi] \times \{\alpha\}$. With $d = 2$, if φ and ψ are 2π-periodic, then

$$\begin{aligned}\Gamma_K(\varphi, \psi) &= \sum_{n\in\mathbb{N}} a_n \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, \varphi(x) \cos\bigl(n(x-y)\bigr)\psi(y)\\ &= \sum_{n\in\mathbb{N}} a_n \int_{-\pi}^{\pi} dx \cos(nx)\, (\varphi * \tilde{\psi})(x).\end{aligned}$$

As in Section 3.5, the map

$$\varphi \mapsto \sum_{n\in\mathbb{N}} a_n \int_{-\pi}^{\pi} d\theta \cos(n\theta)\varphi(\theta), \qquad \varphi \in C^\infty(S^1),$$

defines a distribution on S^1, which we assume to be nonnegative, so there exists a Borel measure Γ on S^1 such that

$$\int_{-\pi}^{\pi} \Gamma(d\theta)\, \varphi(\theta) = \sum_{n\in\mathbb{N}} a_n \int_{-\pi}^{\pi} d\theta \cos(n\theta)\varphi(\theta) \qquad \text{for all } \varphi \in C^\infty(S^1).$$

Condition (4.2) can now be reformulated as a condition on this measure Γ. The proposition below shows that the critical singularity at the origin of this measure is the same for noise on a segment as for noise on a circle.

PROPOSITION 4.2. *Let $K(\vartheta)$ be as in Proposition* 3.14. *Then*

$$\sum_{n\in\mathbb{N}} \frac{a_n}{\sqrt{1+n^2}} < \infty$$

if and only if (3.33) *holds*.

The proof is identical to that of Proposition 3.14 and is therefore omitted.

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