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THE ATTRACTIVENESS OF THE FIXED POINTS OF A $\cdot/GI/1$ QUEUE

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We consider an infinite tandem of first-come-first-served queues. The service times have unit mean, and are independent and identically distributed across queues and customers. Let $\mathbf{I}$ be a stationary and ergodic interarrival sequence with marginals of mean $\tau > 1$, and suppose it is independent of all service times. The process $\mathbf{I}$ is said to be a fixed point for the first, and hence for each, queue if the corresponding interdeparture sequence is distributed as $\mathbf{I}$. Assuming that such a fixed point exists, we show that it is the distributional limit of passing an arbitrary stationary and ergodic interarrival process of mean τ through the infinite queueing tandem.

1. Introduction. Consider an infinite series of $\cdot/GI/1$ queues indexed by $\mathbb{Z}^+$. Such a series is usually defined by an i.i.d.\sequence of non-negative random variables $\{S(n,k)\}_{n\in\mathbb{Z},k\in\mathbb{Z}^+}$, where $S(n,k)$ is the service time of the nth customer at the kth node. It is assumed that $\mathbb{E}(S(1,1)) = 1$, and that the service distribution is a fixed, but otherwise arbitrary, probability measure σ on $\mathbb{R}^+$. To avoid trivialities we will suppose that σ is not a point mass concentrated at 1 (otherwise, it is easy to see that every departure process from the queue is a fixed point for the queue). At each queue, the buffers are assumed to have infinite capacity and the service discipline is assumed to be first-come-first-served.

We study the effect of passing customers through this infinite queueing tandem. The arrivals process to this tandem, $\mathbf{A}^1 = \{A(n,1)\}_{n\in\mathbb{Z}}$, is assumed to be stationary and ergodic. The variable $A(n,1)$ is the inter-arrival time between the nth and $(n+1)$st customers. We assume $\mathbb{E}(A(1,1)) = \tau > 1$. This ensures stability at the first queue: that waiting times of customers, $\{W^A(n,1)\}_{n\in\mathbb{Z}}$, form an almost surely finite stationary and ergodic sequence [9]. (Details of Loynes' construction from which the previously mentioned stability follows may be found, e.g., in the book by Baccelli and Brémaud [3].) In terms of the arrival and service processes, the waiting time of the nth customer is given by the equation

$$W^A(n,1) = \sup_{j\le n-1}\left\{\sum_{i=j}^{n-1} S(i,1) - A(i,1),\ 0\right\}. \tag{1}$$

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Lindley's recursion relates the inter-departure times to the inter-arrival times and service times via waiting times as follows:

$$
\begin{aligned}
W^A(n+1,1) &= [W^A(n,1)+S(n,1)-A(n,1)]^+,\\
A(n,2) &= [A(n,1)-S(n,1)-W^A(n,1)]^+ + S(n+1,1).
\end{aligned} \tag{2}
$$

The process $\mathbf{A}^2$ is input to the second queue from which we obtain $\mathbf{A}^3$ as the departure process, and so on. In general, $\mathbf{A}^k = \{A(n,k)\}_{n\in\mathbb{Z}}$ is the arrivals process at the kth queue with $\{W^A(n,k)\}_{n\in\mathbb{Z}}$ as the corresponding set of waiting times. Thus, $A(n,k)$ is the inter-arrival time between the nth and the $(n+1)$st customers at the kth queue and $W^A(n,k)$ is the waiting time of the nth customer at the kth queue. Using the result of Loynes [9] inductively, one obtains that $\mathbf{A}^k$ is stationary and ergodic for each k, with $\mathbb{E}(A(1,k))=\tau$. Let $\mathcal{T}$ denote the queueing operator and represent the queueing tandem as $\mathbf{A}^{k+1}=\mathcal{T}^k(\mathbf{A}^1),\ k\geq 1$.

DEFINITION. A stationary and ergodic arrivals process $\mathbf{I}=\{I(n)\}_{n\in\mathbb{Z}}$ with $\mathbb{E}(I(1))=\tau>1$ is said to be a fixed point or an invariant distribution at rate $1/\tau$ for a $\cdot/GI/1$ queue whose service times are distributed as σ, if $\mathcal{T}(\mathbf{I})$ equals $\mathbf{I}$ in distribution.

The following theorem is the main result of the paper.

THEOREM 1. *Suppose that a $\cdot/GI/1$ queue with service distribution σ admits a rate $1/\tau$ fixed point $\mathbf{I}$. Let $\mathbf{A}^1$ be a rate $1/\tau$ ergodic stationary arrival process to an infinite tandem of independent copies of the $\cdot/GI/1$ queue. Then $\mathcal{T}^k(\mathbf{A}^1)\to\mathbf{I}$ in distribution as $k\to\infty$.*

We shall prove the theorem by coupling $\mathbf{A}^1$ with an independent process $\mathbf{I}^1$, which has the same distribution as $\mathbf{I}$. Similar methods, but different couplings, were used in [12–14] to establish the distributional convergence of departures in a tandem of queues with different assumptions on the service distribution. Chang [5] used the couplings of this paper to show that a queue which offers i.i.d. services of unbounded support can have at most one fixed point of a given rate. The related literature is surveyed in more detail at the end of the paper.

2. The coupling. The process $\mathbf{A}^1$ is coupled with a process $\mathbf{I}^1$, distributed as the fixed point $\mathbf{I}$. The assumptions are that $\mathbf{A}^1$ and $\mathbf{I}^1$ are mutually independent and also independent of the service processes $\{S(n,k)\}_{n\in\mathbb{Z},k\in\mathbb{Z}^+}$. Let $\mathbf{I}^k=\{I(n,k)\}_{n\in\mathbb{Z}}$ be the input at node k when $\mathbf{I}^1$ is input at node 1, and let $W^I(n,k)$ be the waiting time of the nth customer of $\mathbf{I}^k$. Then, we have the following recursions:

$$
\begin{aligned}
W^I(n+1,k) &= [W^I(n,k)+S(n,k)-I(n,k)]^+,\\
I(n,k+1) &= [I(n,k)-S(n,k)-W^I(n,k)]^+ + S(n+1,k).
\end{aligned} \tag{3}
$$

A similar recursion for the process $\mathbf{A}^k$ is given by

$$\begin{aligned} W^A(n+1,k) &= [W^A(n,k)+S(n,k)-A(n,k)]^+, \\ A(n,k+1) &= [A(n,k)-S(n,k)-W^A(n,k)]^+ + S(n+1,k). \end{aligned} \tag{4}$$

From (3) we obtain

$$I(n,k+1)-W^I(n+1,k) = I(n,k)-S(n,k)-W^I(n,k)+S(n+1,k), \tag{5}$$

and from (4) we obtain

$$A(n,k+1)-W^A(n+1,k) = A(n,k)-S(n,k)-W^A(n,k)+S(n+1,k). \tag{6}$$

Subtracting (5) from (6) we get

$$\begin{aligned} &[A(n,k+1)-I(n,k+1)]-[W^A(n+1,k)-W^I(n+1,k)] \\ &\quad = [A(n,k)-I(n,k)]-[W^A(n,k)-W^I(n,k)]. \end{aligned} \tag{7}$$

A little algebra now yields

$$\begin{aligned} &[W^A(n+J,k)-W^I(n+J,k)] \\ &\quad = \Biggl(\sum_{j=0}^{J-1}[A(n+j,k+1)-I(n+j,k+1)] \\ &\qquad\qquad - [A(n+j,k)-I(n+j,k)]\Biggr) \\ &\qquad + [W^A(n,k)-W^I(n,k)]. \end{aligned} \tag{8}$$

Equation (8) will form the basis of the coupling argument.

Throughout the rest of the paper, it is helpful to imagine that there are two queues at each node k, one for the $\mathbf{A}$ customers and one for the $\mathbf{I}$ customers. This makes explicit the notion that customers of one process do not influence the waiting of the customers of the other process. The coupling between the two processes merely consists of providing customers numbered n with identical service times, $S(n,k)$, distributed i.i.d. over n and k. We will refer to each of the two queues as the A-queue and the I-queue, respectively.

2.1. *A coloring scheme.* The next step is to introduce a coloring scheme for our processes. Since $\mathbf{A}^1$ and $\mathbf{I}^1$ are both of rate $1/\tau$ and are not identical (else Theorem 1 is trivially true), there must exist "points of crossing." That is, there exist disjoint random sets of integers

$$\mathcal{A}^1 = \{n \in \mathbb{Z} : A(n,1) > I(n,1)\},$$
$$\mathcal{I}^1 = \{n \in \mathbb{Z} : I(n,1) > A(n,1)\},$$

where $\mathbf{A}^1$ dominates $\mathbf{I}^1$ or the other way around. Call these the *sets of domination*. The stationarity and ergodicity of $(\mathbf{A}^1, \mathbf{I}^1)$ implies that of $(\mathcal{A}^1, \mathcal{I}^1)$; therefore, the density of $\mathcal{A}^1$,

$$d(\mathcal{A}^1) = \lim_{N\to\infty} \frac{\text{\# of points of } \mathcal{A}^1 \text{ in } [-N, N]}{2N+1},$$

is well defined and almost surely equal to a positive constant. Similarly, $d(\mathcal{I}^1)$ is almost surely a positive constant.

Let $r = \sup\{m < 0 : m \in \mathcal{A}^1\}$ and define

$$b(1, 1) = \inf\{m > r : m \in \mathcal{I}^1\},$$
$$r(1, 1) = \inf\{m > b(1, 1) : m \in \mathcal{A}^1\}.$$

For $n \ge 2$, recursively define $r(n, 1)$ and $b(n, 1)$ as follows:

$$\begin{aligned} b(n, 1) &= \inf\{m > r(n-1, 1) : m \in \mathcal{I}^1\}, \\ r(n, 1) &= \inf\{m > b(n, 1) : m \in \mathcal{A}^1\}. \end{aligned} \tag{9}$$

Let $\hat{r}(0, 1) = \sup\{m < b(1, 1) : m \in \mathcal{A}^1\}$ and $\hat{b}(0, 1) = \sup\{m \le \hat{r}(0, 1) : m \in \mathcal{I}^1\}$. For $n \le -1$, define $\hat{r}(n, 1)$ and $\hat{b}(n, 1)$ as

$$\hat{r}(n, 1) = \sup\{m < \hat{b}(n+1, 1) : m \in \mathcal{A}^1\},$$
$$\hat{b}(n, 1) = \sup\{m < \hat{r}(n, 1) : m \in \mathcal{I}^1\}.$$

Finally, for $n \le 0$ define

$$\begin{aligned} r(n, 1) &= \inf\{m : \hat{b}(n, 1) < m \le \hat{r}(n, 1) \text{ and } m \in \mathcal{A}^1\}, \\ b(n, 1) &= \inf\{m : \hat{r}(n-1, 1) < m \le \hat{b}(n, 1) \text{ and } m \in \mathcal{I}^1\}. \end{aligned} \tag{10}$$

Of interest to us throughout this paper are the quantities $r(n, 1)$ and $b(n, 1)$; see Figure 1 for an illustration.

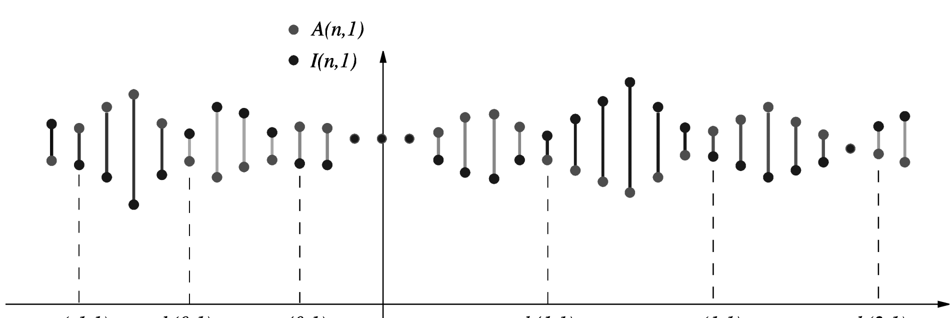

FIG. 1. *A realization.*

Since $d(\mathcal{A}^1) > 0$ and $d(\mathcal{I}^1) > 0$ a.s., $r(n,1)$ and $b(n,1)$ are almost surely finite for every n, and the infima in the above definitions are minima. Define

$$\begin{aligned} \mathcal{R}(n,1) &= \{k \in \mathbb{Z} : r(n,1) \le k < b(n+1,1)\}, \\ \mathcal{B}(n,1) &= \{k \in \mathbb{Z} : b(n,1) \le k < r(n,1)\} \end{aligned} \tag{11}$$

to be nonoverlapping intervals of integers which almost surely partition $\mathbb{Z}$. Finally, let

$$\begin{aligned} R(n,1) &= \sum_{k \in \mathcal{R}(n,1)} A(k,1) - I(k,1) \quad \text{and} \\ B(n,1) &= \sum_{k \in \mathcal{B}(n,1)} I(k,1) - A(k,1). \end{aligned} \tag{12}$$

We are now ready to introduce the coloring scheme. One thinks of the sets of domination $\mathcal{A}^1$ and $\mathcal{I}^1$ as the "support of red and blue bubbles," respectively. That is, $r(n,1)$ is the point at which the nth red bubble begins, $\mathcal{R}(n,1)$ is the interval over which it is supported, and $R(n,1)$ is its volume. With reference to Figure 1, $R(1,1)$ is the sum of the lengths of the vertical red lines in $\mathcal{R}(1,1) = [r(1,1), b(2,1))$. A similar interpretation may be made for the blue bubbles.

It is crucial that for $m \ne n$ the shades of the mth and the nth red bubbles are distinct. Thus, we think of the nth red bubble as being colored with an nth shade of red. Similarly, one is able to distinguish between the various shades of blue. See Figure 1.

2.2. *Sketch of the proof.* By the ergodicity of $(\mathbf{A}^1, \mathbf{I}^1)$, the densities of the red and blue bubbles at the first stage are exactly equal. This follows from the fact that $\lim_{n\to\infty}((\sum_{j=-n}^{n} A(j,1) - I(j,1))/(2n+1)) = 0$, which implies

$$\begin{aligned} &\lim_{n\to\infty} \frac{\sum_{j=-n}^{n}(A(j,1) - I(j,1))\mathbb{1}_{\{j\in\mathcal{A}^1\}}}{2n+1} \\ &\qquad = \lim_{n\to\infty} \frac{\sum_{j=-n}^{n}(I(j,1) - A(j,1))\mathbb{1}_{\{j\in\mathcal{I}^1\}}}{2n+1}, \end{aligned}$$

that is,

$$\lim_{n\to\infty} \frac{\text{volume of red in } [-n,n]}{2n+1} = \lim_{n\to\infty} \frac{\text{volume of blue in } [-n,n]}{2n+1} \triangleq d(1),$$

where $d(1)$ is the average volume of red (or blue) per arrival at stage 1.

At any stage k, the arrival processes $(\mathbf{A}^k, \mathbf{I}^k)$ are jointly ergodic. Hence, by the ergodic theorem,

$$d(k) = \frac{E(|A(1,k) - I(1,k)|)}{2}$$

is almost surely a constant and equals the average volume of red (or blue) per arrival at stage k. Chang [5] has shown

$$E\big(|A(1,k) - I(1,k)|\big) \geq E\big(|A(1,k+1) - I(1,k+1)|\big)$$

for each k. Hence the $d(k)$ are monotonically nonincreasing. Given that $\mathbf{I}^1$ is a fixed point for the queue, the desired weak convergence will follow from showing that $d(k)$ converges to zero almost surely as $k \to \infty$.

We shall do this by observing the evolution of *each individual* red and blue bubble as the two processes pass through the series of queues. Thus, for each n, the quantities $r(n,k)$, $b(n,k)$, $\mathcal{R}(n,k)$, $\mathcal{B}(n,k)$, $R(n,k)$ and $B(n,k)$ are derived from the corresponding quantities at stage $k-1$ and the service process at stage $k-1$. For each k, the nth red (resp., blue) bubble is imagined to be colored with the nth shade of red (resp., blue). We then show that the red and blue bubbles cancel each other out and therefore that $R(n,k)$ and $B(n,k)$ decrease monotonically to zero as $k \to \infty$.

The main difficulty in establishing the monotone decrease of $R(n,k)$ and $B(n,k)$ to zero is that it is possible for the red bubbles to accumulate far away from the blue bubbles and not cancel them out. We address this problem by showing that (i) the ordering between red and blue bubbles is always maintained, that is, they cannot overtake each other, and (ii) since the services are independent the two types of bubbles will be forced to interact and must therefore cancel each other out. The details follow.

3. Preliminary lemmas.

We simplify the notation as follows: Let

$$\begin{aligned} d^w(n,k) &= W^A(n,k) - W^I(n,k) \qquad \forall n \in \mathbb{Z}, k \in \mathbb{Z}^+, \\ d^a(n,k) &= A(n,k) - I(n,k) \qquad \forall n \in \mathbb{Z}, k \in \mathbb{Z}^+. \end{aligned} \tag{13}$$

Mnemonically, $d^w(\cdot,\cdot)$ is "the difference in waiting times" between customers of the two processes. In this notation (8) reads:

$$d^w(n+J,k) = \left(\sum_{j=0}^{J-1} d^a(n+j,k+1) - d^a(n+j,k)\right) + d^w(n,k). \tag{14}$$

LEMMA 1. *The following hold for any n and k*:

(i) $A(n,k) \leq S(n,k) \Rightarrow A(n,k+1) = S(n+1,k)$ *and* $d^a(n,k+1) \leq 0$;

(ii) $I(n,k) \leq S(n,k) \Rightarrow I(n,k+1) = S(n+1,k)$ *and* $d^a(n,k+1) \geq 0$.

PROOF. Given that $A(n,k) \leq S(n,k)$, it follows from (4) and the nonnegativity of $W^A(n,k)$ that $A(n,k+1) = S(n+1,k)$. From (3) we get $I(n,k+1) \geq S(n+1,k)$. Therefore, $d^a(n,k+1) \leq 0$, and similarly with (ii). $\square$

LEMMA 2. *For any n and k*:

(i) $d^w(n,k) \geq d^a(n,k) \Rightarrow d^a(n,k+1) \leq 0$ *and* $d^w(n+1,k) \geq 0$;
(ii) $d^w(n,k) \leq d^a(n,k) \Rightarrow d^a(n,k+1) \geq 0$ *and* $d^w(n+1,k) \leq 0$.

Additionally,

(iii) *if under* (i), $S(n,k) \geq I(n,k)$, *then* $d^a(n,k+1) = 0$;
(iv) *if under* (ii), $S(n,k) \geq A(n,k)$, *then* $d^a(n,k+1) = 0$.

Hence, *under* (iii) *and* (iv), (14) *implies* $d^w(n+1,k) = d^w(n,k) - d^a(n,k)$.

PROOF. Equations (3) and (4) imply (i). Consider (iii). From part (i) we have that $d^a(n,k+1) \leq 0$. From part (ii) of Lemma 1 we have that $d^a(n,k+1) \geq 0$. Therefore, $d^a(n,k+1) = 0$. The proofs of (ii) and (iv) are similar. □

LEMMA 3. *The following hold for any n and k*:

(i) $d^w(n,k) \geq 0$, $d^a(n,k) \geq 0 \Rightarrow d^a(n,k+1) \leq d^a(n,k)$, $d^w(n+1,k) \leq d^w(n,k)$;
(ii) $d^w(n,k) \leq 0$, $d^a(n,k) \leq 0 \Rightarrow d^a(n,k+1) \geq d^a(n,k)$, $d^w(n+1,k) \geq d^w(n,k)$.

PROOF. We prove (i). From (3) and (4) we get that

$$d^a(n,k+1) = [A(n,k) - W^A(n,k) - S(n,k)]^+ - [I(n,k) - W^I(n,k) - S(n,k)]^+.$$

If $[A(n,k) - W^A(n,k) - S(n,k)]^+ = 0$, then $d^a(n,k+1) \leq 0 \leq d^a(n,k)$.

On the other hand, if $A(n,k) - W^A(n,k) - S(n,k) > 0$, then

$$\begin{aligned} d^a(n,k+1) &= A(n,k) - W^A(n,k) - S(n,k) - [I(n,k) - W^I(n,k) - S(n,k)]^+ \\ &\leq A(n,k) - W^A(n,k) - S(n,k) - \big(I(n,k) - W^I(n,k) - S(n,k)\big) \\ &= d^a(n,k) - d^w(n,k) \leq d^a(n,k). \end{aligned}$$

Using this in (14), we get

$$d^w(n+1,k) = d^a(n,k+1) - d^a(n,k) + d^w(n,k) \leq d^w(n,k).$$

The same argument with reversed inequalities proves (ii). □

LEMMA 4. *The following hold for any n and k*:

(i) $d^w(n+1,k) > 0 \Rightarrow d^a(n,k+1) \leq 0$;
(ii) $d^a(n,k+1) < 0 \Rightarrow d^w(n+1,k) \geq 0$;
(iii) $d^w(n+1,k) < 0 \Rightarrow d^a(n,k+1) \geq 0$;
(iv) $d^a(n,k+1) > 0 \Rightarrow d^w(n+1,k) \leq 0$.

PROOF. We verify (i) and (ii). From (3) and (4) we get

$$\begin{aligned} d^w(n+1,k) > 0 \quad &\Longrightarrow \quad A(n,k) - W^A(n,k) < I(n,k) - W^I(n,k) \\ &\Longrightarrow \quad A(n,k+1) \le I(n,k+1) \\ &\Longrightarrow \quad d^a(n,k+1) \le 0 \end{aligned}$$

and

$$\begin{aligned} d^a(n,k+1) < 0 \quad &\Longrightarrow \quad A(n,k) - W^A(n,k) < I(n,k) - W^I(n,k) \\ &\Longrightarrow \quad W^A(n+1,k) \ge W^I(n+1,k) \\ &\Longrightarrow \quad d^w(n+1,k) \ge 0. \end{aligned}$$ □

LEMMA 5. *The following hold for any n and k*:

(i) $d^w(n+1,k) > 0,\ d^a(n,k) \ge 0 \Rightarrow d^w(n,k) \ge d^w(n+1,k)$;
(ii) $d^w(n+1,k) < 0,\ d^a(n,k) \le 0 \Rightarrow d^w(n,k) \le d^w(n+1,k)$.

PROOF. Under hypothesis (i), Lemma 4 implies $d^a(n,k+1) \le 0$. Using this in $d^w(n,k) = d^w(n+1,k) - d^a(n,k+1) + d^a(n,k)$ we get $d^w(n,k) \ge d^w(n+1,k)$. The proof of (ii) is similar. □

LEMMA 6. *The following hold for any n and k*:

(i) $d^w(n,k) \ge 0, d^a(n,k) \le 0 \Rightarrow d^w(n+1,k) \ge 0, d^a(n,k+1) \le 0$;
(ii) $d^w(n,k) \le 0, d^a(n,k) \ge 0 \Rightarrow d^w(n+1,k) \le 0, d^a(n,k+1) \ge 0$.

The proof follows from Lemma 2.

4. The evolution of the bubbles. From the processes $(\mathbf{A}^1, \mathbf{I}^1)$ and $(\mathbf{A}^2, \mathbf{I}^2)$ and the service process at node 1, we deduce the status of the red and blue bubbles at node 2. It eases the exposition to do this gradually, to first consider the evolution of bubbles in certain simple situations. One can isolate three basic possibilities for bubble evolution and all other possibilities can be described in terms of these three.

Possibility α: bubbles can only move to the right. Consider only the first red bubble and ignore all others, that is, from the processes $\mathbf{A}^1$ and $\mathbf{I}^1$ construct the process $\tilde{\mathbf{A}}^1 = \{\tilde{A}(n,1)\}_{n\in\mathbb{Z}}$ as follows: $\tilde{A}(n,1) = A(n,1)$ for all $n \in \mathcal{R}(1,1)$ and $\tilde{A}(n,1) = I(n,1)$ for all $n \notin \mathcal{R}(1,1)$. We then modify the definition of $\mathcal{R}(1,1)$ and let it equal the set $\{l : r(1,1) \le l \le q\}$, where $q = \min\{n \ge r(1,1) : \tilde{A}(k,1) = I(k,1),\ \forall k \ge n\}$. Since $\tilde{A}(n,1) \ge I(n,1)$ for each $n \in \mathbb{Z}$, $W^{\tilde{A}}(n,1) \le W^I(n,1)$ and $\tilde{A}(n,2) \ge I(n,2)$. For $k = 1,2$, let $\tilde{d}^a(n,k) = \tilde{A}(n,k) - I(n,k)$ and $\tilde{d}^w(n,k) = W^{\tilde{A}}(n,k) - W^I(n,k)$.

Parenthetically, although the process $\tilde{\mathbf{A}}^1$ is not stationary, it is *rate stable*. That is,

$$\lim_{n\to\infty} \frac{\sum_{j=-n}^{n} \tilde{A}(j,1)}{2n+1} = \tau > 1,$$

which, via equation (1), implies that for any $N \in \mathbb{Z}$ there is an $n > N$ such that $W^{\tilde{A}}(n,1) = 0$.

Define $q' = \inf\{n : \tilde{A}(k,2) = I(k,2),\ \forall k > n\}$. To see that $q' < \infty$, let $i = \inf\{n > q : W^I(n,1) = 0\}$. Note that the stability of the I-queue implies that $i < \infty$ a.s. Since $W^I(n,1) \ge W^{\tilde{A}}(n,1)$ for every n, $W^I(i,1) = 0$ implies $W^{\tilde{A}}(i,1) = 0$. This and the fact that $\tilde{A}(n,1) = I(n,1)$ for all $n \ge i > q$, give us recursively via (14) that $\tilde{A}(n,2) = I(n,2)$ and $W^I(n,1) = W^{\tilde{A}}(n,1)$ for all $n > i$. In particular, we obtain $q' \le i < \infty$ a.s.

Now define $r(1,2) = \inf\{n \le q' : \tilde{A}(n,2) > I(n,2)\}$. The following lemma will show that $r(1,2)$ is well defined; indeed, it lies in the interval $[r(1,1), q']$. Define $\mathcal{R}(1,2)$ to be the set of all integers in $[r(1,2), q']$. Now the set of n for which $\tilde{A}(n,2) > I(n,2)$ is included in $\mathcal{R}(1,2)$. Therefore one may think of $\mathcal{R}(1,2)$ as the support of the red bubble at node 2. For $k = 1,2$, let $R(1,k) = \sum_{n\in\mathcal{R}(1,k)} \tilde{d}^a(n,k)$.

LEMMA 7. *The following hold*:

(i) $r(1,2) \ge r(1,1)$;
(ii) $R(1,1) = R(1,2)$.

PROOF. Since $\tilde{A}(n,1) = I(n,1)$ for $n < r(1,1)$ it follows that $\tilde{A}(n,2) = I(n,2)$ for $n < r(1,1)$. Therefore $r(1,2) \ge r(1,1)$.

Next, we know that $\tilde{d}^w(n,1) = 0$ for $n \le r(1,1)$ and for $n \ge i$. Since $\max\{q,q'\} \le i$, it follows that $\mathcal{R}(1,1) \subset [r(1,1), i]$ and $\mathcal{R}(1,2) \subset [r(1,1), i]$. Therefore, from (14) we get

$$\tilde{d}^w(i+1,1) = \left(\sum_{n=r(1,1)}^{i} \tilde{d}^a(n,2) - \tilde{d}^a(n,1)\right) + \tilde{d}^w\bigl(r(1,1),1\bigr),$$

which implies

$$R(1,2) = \sum_{n\in\mathcal{R}(1,2)} \tilde{d}^a(n,2) = \sum_{n\in\mathcal{R}(1,1)} \tilde{d}^a(n,1) = R(1,1). \tag{15}$$

$\square$

REMARK. We shall interpret $r(1,2) \ge r(1,1)$ as "a bubble can only move to the right," and $R(1,1) = R(1,2)$ as "the volume of the red bubble is preserved." However, the next basic possibility shows that when there are red and blue bubbles present, one of them can move to the right and cancel some or all of the volume of the other.

Possibility β: neighboring bubbles of opposite color may cancel parts of each other. This time consider only the first blue and the first red bubbles. That is, set $\tilde{A}(n,1) = A(n,1)$ for all $n \in \mathcal{B}(1,1) \cup \mathcal{R}(1,1)$ and $\tilde{A}(n,1) = I(n,1)$ otherwise. We then modify the definition as follows: $\mathcal{R}(1,1) = [r(1,1), q]$, where $q = \min\{n \geq r(1,1) : \tilde{A}(k,1) = I(k,1)\ \forall k \geq n\}$ and $\mathcal{B}(1,1) = [b(1,1), r(1,1) - 1]$.

LEMMA 8. *Let $\tilde{b} = \sup\{n : I(n,2) > \tilde{A}(n,2)\}$ and $\tilde{r} = \inf\{n : \tilde{A}(n,2) > I(n,2)\}$, with the conventions $\sup\{\varnothing\} = -\infty$ and $\inf\{\varnothing\} = \infty$. Suppose that $\tilde{b}$ and $\tilde{r}$ are finite. Then*:

(i) *$\tilde{b} < \tilde{r}$ and*
(ii) *$I(n,2) \leq \tilde{A}(n,2)$ for $n \geq \tilde{r}$ and $I(n,2) \geq \tilde{A}(n,2)$ for $n \leq \tilde{b}$.*

PROOF. To prove (i) by contradiction, suppose that $\tilde{b} \geq \tilde{r}$. We claim $\tilde{r} \geq r(1,1)$, and establish it as follows. Since $\tilde{d}^w(b(1,1),1) = 0$ [because $\tilde{d}^a(n,1) = 0$ for $n < b(1,1)$] and $\tilde{d}^a(n,1) \leq 0$ for all $n \in [b(1,1), r(1,1) - 1]$, it follows recursively from Lemma 6 that $\tilde{d}^a(n,2) \leq 0$ for all $n \in [b(1,1), r(1,1) - 1]$. Therefore $\tilde{r} \geq r(1,1)$.

Now, $\tilde{d}^a(\tilde{r},2) > 0$ implies $\tilde{d}^w(\tilde{r}+1,1) \leq 0$, by Lemma 4, and since $\tilde{r} \geq r(1,1)$, we have that $\tilde{d}^a(n,1) \geq 0$ for all $n \geq \tilde{r}$. Therefore, from Lemma 6 we get that $\tilde{d}^a(n,2) \geq 0$ for $n \geq \tilde{r}$. This contradicts $\tilde{b} \geq \tilde{r}$.

In the above we have shown that $\tilde{d}^a(n,2) \geq 0$ for $n \geq \tilde{r}$. This proves the first part of (ii). Since $\tilde{b} < \tilde{r}$ by part (i), the definition of $\tilde{r}$ implies $I(n,2) \geq \tilde{A}(n,2)$ for $n \leq \tilde{b}$. $\square$

The above lemma shows that "bubbles do not overtake each other." The next two lemmas make this clearer.

Suppose that $\tilde{b}$ and $\tilde{r}$ defined above are both finite. Define $b(1,2) = \inf\{n \leq \tilde{b} : I(n,2) > \tilde{A}(n,2)\}$ and $r(1,2) = \tilde{r}$. Also define $\mathcal{B}(1,2) = \{n \in [b(1,2), r(1,2) - 1]\}$ and $\mathcal{R}(1,2) = \{n \in [r(1,2), q']\}$, where $q' = \min\{n \geq r(1,2) : \tilde{A}(k,2) = I(k,2),\ \forall k > n\}$. As in Case α, it is easy to see that $q' < \infty$. Let $B(1,2) = -\sum_{n \in \mathcal{B}(1,2)} \tilde{d}^a(n,2)$ and $R(1,2) = \sum_{n \in \mathcal{R}(1,2)} \tilde{d}^a(n,2)$.

LEMMA 9. *Suppose that $\tilde{b}$ and $\tilde{r}$ are both finite, then*:

(i) *$b(1,2) \geq b(1,1)$ and $r(1,2) \geq r(1,1)$,*
(ii) *$B(1,2) \leq B(1,1)$ and $R(1,2) \leq R(1,1)$ and*
(iii) *$B(1,2) - B(1,1) = R(1,2) - R(1,1)$.*

PROOF. It follows from Case α that $b(1,2) \geq b(1,1)$, and from the proof of Lemma 8 we know that $r(1,1) \leq \tilde{r} = r(1,2)$. This proves (i).

Now since $\tilde{d}^w(b(1,1),1) = 0$ and $\tilde{d}^a(n,1) \leq 0$ for $n \in [b(1,1), r(1,1) - 1]$, Lemma 6 recursively implies that $\tilde{d}^w(r(1,1),1) \geq 0$.

CASE 1 [$\tilde{d}^w(r(1,1),1) = 0$]. This and the fact that $\tilde{d}^a(n,1) \geq 0$ for $n \geq r(1,1)$ imply recursively via Lemma 6 that $\tilde{d}^a(n,2) \geq 0$ and $\tilde{d}^w(n,1) \leq 0$ for $n \geq r(1,1)$. Therefore $B(1,2) = -\sum_{n=b(1,1)}^{r(1,1)-1} \tilde{d}^a(n,2)$ and from

$$\tilde{d}^w(r(1,1),1) = \sum_{n=b(1,1)}^{r(1,1)-1} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w(b(1,1),1)$$

we get that $B(1,2) = -\sum_{n=b(1,1)}^{r(1,1)-1} \tilde{d}^a(n,1) = B(1,1)$. Let $i > q'$ be the first time that the I-queue idles. Since $\tilde{d}^w(n,1) \leq 0$ for $n \geq r(1,1)$ the fact that $W^I(i,1) = 0$ implies that $W^A(i,1) = 0$. Therefore from

$$\tilde{d}^w(i,1) = \sum_{n=r(1,1)}^{i-1} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w(r(1,1),1)$$

we get that $R(1,2) = \sum_{n=r(1,1)}^{i-1} \tilde{d}^a(n,2) = \sum_{n=r(1,1)}^{i-1} \tilde{d}^a(n,1) = R(1,1)$. In the preceding step we have used that both $\mathcal{R}(1,1)$ and $\mathcal{R}(1,2)$ are contained in $[r(1,1), i-1]$. Thus when $\tilde{d}^w(r(1,1),1) = 0$, we have proved both (ii) and (iii).

CASE 2 [$\tilde{d}^w(r(1,1),1) > 0$]. From (14) we get

$$\begin{aligned} 0 < \tilde{d}^w(r(1,1),1) &= \sum_{n=b(1,1)}^{r(1,1)-1} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w(b(1,1),1) \\ &= -B'(1,2) + B(1,1), \end{aligned} \tag{16}$$

where $B'(1,2) = -\sum_{n=b(1,1)}^{r(1,1)-1} \tilde{d}^a(n,2)$ is the volume of blue at the second stage in $[b(1,1), r(1,1)-1]$. Note that $\tilde{d}^a(n,1) \leq 0$ and $\tilde{d}^a(n,2) \leq 0$ for $n \in [b(1,1), r(1,1)-1]$, the first by definition of $b(1,1)$ and $r(1,1)$ and the second because $r(1,2) \geq r(1,1)$. So, we only have blue in this interval on both the input and output sides as shown in (16).

Let $\tilde{b}$ be as defined in Lemma 8, and recall that $\tilde{b} < \tilde{r} = r(1,2)$. If $\tilde{b} < r(1,1)$, then since $\tilde{d}^a(n,2) \geq 0$ for $n > \tilde{b}$, it follows that $B'(1,2) = B(1,2)$ and (16) gives $B(1,2) < B(1,1)$.

Else, let $B''(1,2) = -\sum_{n=r(1,1)}^{\tilde{b}} \tilde{d}^a(n,2)$ be the amount of blue volume to the right of $r(1,1)$ and note that $B(1,2) = B'(1,2) + B''(1,2)$. We shall show that $B''(1,2) < \tilde{d}^w(r(1,1),1)$, which when used at (16) gives $B(1,1) > B'(1,2) + B''(1,2) = B(1,2)$. Accordingly, consider

$$\tilde{d}^w(\tilde{b}+1,1) - \sum_{n=r(1,1)}^{\tilde{b}} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) = \tilde{d}^w(r(1,1),1),$$

$$\tilde{d}^w(\tilde{b}+1,1) + B''(1,2) + \sum_{n=r(1,1)}^{\tilde{b}} \tilde{d}^a(n,1) = \tilde{d}^w(r(1,1),1).$$

Since $\tilde{d}^a(\tilde{b}, 2) < 0$, by Lemma 4, $\tilde{d}^w(\tilde{b}+1, 1) \geq 0$ and $\sum_{n=r(1,1)}^{\tilde{b}} \tilde{d}^a(n, 1) > 0$, by definition of $r(1, 1)$. Therefore, it follows that $B''(1, 2) < \tilde{d}^w(r(1, 1), 1)$ and that $B(1, 2) < B(1, 1)$.

We shall now prove that $B(1, 1) - B(1, 2) = R(1, 1) - R(1, 2)$. This will establish both part (iii), and in conjunction with $B(1, 2) < B(1, 1)$ it will also show that $R(1, 2) < R(1, 1)$.

Let $i = \min\{n > q : W^I(n, 1) = 0\}$. By the stability of the I-queue, it again follows that $i < \infty$. It also follows (as before) that $\tilde{d}^w(i, 1) = 0$. We use this and the fact that $\tilde{d}^w(b(1, 1), 0) = 0$ as follows:

$$\begin{aligned}0 = \tilde{d}^w(i, 1) &= \sum_{n=b(1,1)}^{i-1} \tilde{d}^a(n, 2) - \tilde{d}^a(n, 1) + \tilde{d}^w\bigl(b(1, 1), 1\bigr)\\ &= R(1, 2) - B(1, 2) - R(1, 1) + B(1, 1)\end{aligned}$$

or

$$R(1, 1) - R(1, 2) = B(1, 1) - B(1, 2).$$

This concludes the proof of the lemma. □

REMARK. Again, the lemma establishes that bubbles only move to the right, that their volumes do not increase, and that they do not overtake one another. Volume cancellations are equal and happen, in this case, when the blue bubble moves into the red one. This movement is manifested by the condition $\tilde{d}^w(r(1, 1), 1) > 0$. For, this is the only condition under which $B(1, 2) < B(1, 1)$.

To conclude Possibility β, we need to consider the case that at least one of $\tilde{b}$ and $\tilde{r}$ is not finite. Accordingly, we state the following definitions:

(a) If $\tilde{b} = -\infty$ and $\tilde{r} < \infty$, define $r(1, 2) = \tilde{r}$, $b(1, 2) = r(1, 2)$, $\mathcal{B}(1, 2) = \varnothing$ and $\mathcal{R}(1, 2) = [r(1, 2), q']$, where q' is as defined earlier.

(b) If $\tilde{b} > -\infty$ and $\tilde{r} = \infty$, define $b(1, 2) = \tilde{b}$, $r(1, 2) = \infty$, $\mathcal{B}(1, 2) = [b(1, 2), q'']$, where $q'' = \min\{n > b(1, 2) : \tilde{A}(k, 2) = I(k, 2)\ \forall k \geq n\}$ and $\mathcal{R}(1, 2) = \varnothing$.

(c) If $|\tilde{b}| = |\tilde{r}| = \infty$, define $b(1, 2) = r(1, 2) = \infty$ and $\mathcal{B}(1, 2) = \mathcal{R}(1, 2) = \varnothing$.

These definitions ensure that bubble movements are always to the right. The next lemma shows that bubble volumes do not increase in this case either.

LEMMA 10. *Suppose at least one of $\tilde{b}$ and $\tilde{r}$ is not finite. Then* $B(1, 2) < B(1, 1)$, $R(1, 2) < R(1, 1)$ *and* $B(1, 1) - B(1, 2) = R(1, 1) - R(1, 2)$.

PROOF. Since either $B(1, 2)$ or $R(1, 2)$ is 0 in this case, the lemma is proved if we establish $B(1, 1) - B(1, 2) = R(1, 1) - R(1, 2)$. But, this is simple.

Let $i = \min\{n : \tilde{d}^a(k, 0) = 0 = \tilde{d}^w(k, 0)\ \forall k > n\}$. Note that the stability of both queues ensures that $i < \infty$. From

$$0 = \tilde{d}^w(i, 1) = \sum_{n=b(1,1)}^{i-1} \tilde{d}^a(n, 2) - \tilde{d}^a(n, 1) + \tilde{d}^w\big(b(1, 1), 1\big),$$
$$= R(1, 2) - B(1, 2) - R(1, 1) + B(1, 1)$$

or

$$R(1, 1) - R(1, 2) = B(1, 1) - B(1, 2),$$

we obtain a proof of the lemma. $\square$

The last of the three possibilities concerns the mixing of two neighboring blue (red) bubbles, after the red (resp. blue) bubble between them has been cancelled.

Possibility γ: the mixing of bubbles of the same color. This time consider the first two blue bubbles and the first red bubble only. That is, set $\tilde{A}(n, 1) = A(n, 1)$ for $n \in (\mathcal{B}(1, 1) \cup \mathcal{R}(1, 1) \cup \mathcal{B}(2, 1))$ and $\tilde{A}(n, 1) = I(n, 1)$ otherwise. Again, we modify the definition of $\mathcal{B}(2, 1)$, setting it equal to $\{b(2, 1) \le n \le q\}$, where $q = \min\{n \ge b(2, 1) : \tilde{A}(k, 1) = I(k, 1)\ \forall k \ge n\}$.

Given the preceding discussion of possibilities α and β, the dynamics of bubble evolution under γ are easy to understand.

Define $\tilde{b}_1 = \inf\{n : A(n, 2) < I(n, 2)\}$, $\tilde{b}_2 = \sup\{n : A(n, 2) < I(n, 2)\}$, $\tilde{r}_1 = \inf\{n : A(n, 2) > I(n, 2)\}$, and $\tilde{r}_2 = \sup\{n : A(n, 2) > I(n, 2)\}$. Following earlier conventions, the infimum (supremum) of the empty set equals ∞ (resp. $-\infty$).

LEMMA 11. *Suppose $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{r}_1$ and $\tilde{r}_2$ are all finite. Then exactly one of the following must be true*:

(i) $\tilde{b}_1 \le \tilde{b}_2 < \tilde{r}_1 \le \tilde{r}_2$,
(ii) $\tilde{b}_1 < \tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_2$ *and* $\tilde{d}^a(n, 2) \ge 0$ *for all* $n \in [\tilde{r}_1, \tilde{r}_2]$, *and*
(iii) $\tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_1 \le \tilde{b}_2$.

PROOF. First suppose that $\tilde{b}_1 < \tilde{r}_1$, but that $\tilde{r}_1 < \tilde{b}_2 < \tilde{r}_2$. From the proof of Lemma 8 we know that $\tilde{r}_1 \ge r(1, 1)$.

CASE 1 [$\tilde{r}_1 \ge b(2, 1)$]. This implies $\tilde{r}_2 > \tilde{b}_2 > b(2, 1)$. Since $\tilde{d}^a(\tilde{b}_2, 2) < 0$, by Lemma 4 we get that $\tilde{d}^w(\tilde{b}_2 + 1, 1) \ge 0$. This together with the two facts: $\tilde{b}_2 > b(2, 1)$, and $\tilde{d}^a(n, 1) \le 0$ for all $n \ge b(2, 1)$ implies, recursively via Lemma 6, that $\tilde{d}^a(n, 2) \le 0$ for all $n \ge b(2, 1)$. This contradicts $\tilde{r}_2 > \tilde{b}_2$.

CASE 2 [$r(1,1) \le \tilde{r}_1 < b(2,1)$]. Lemma 4 implies $\tilde{d}^w(\tilde{r}_1+1,1) \le 0$. Since $\tilde{d}^a(n,1) \ge 0$ for $n \in [r(1,1), b(2,1)-1]$, Lemma 6 implies $\tilde{d}^a(n,2) \ge 0$ for such n. This further implies $\tilde{b}_2 \ge b(2,1)$. Now a similar argument to the one in Case 1 makes it impossible for there to be an $n > \tilde{b}_2$ such that $\tilde{d}^a(n,2) > 0$. Again, this contradicts $\tilde{r}_2 > \tilde{b}_2$.

Therefore, if $\tilde{b}_1 < \tilde{r}_1$, either $\tilde{b}_1 \le \tilde{b}_2 < \tilde{r}_1 \le \tilde{r}_2$ [this proves (i)] or $\tilde{b}_1 < \tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_2$ [this proves part of (ii)].

To finish (ii), simply note that if there is a $b \in [\tilde{r}_1, \tilde{r}_2]$ such that $\tilde{d}^a(b,2) < 0$, replacing $\tilde{b}_2$ with b in the arguments of Cases 1 and 2 above will imply that either $b < \tilde{r}_1$ or $b > \tilde{r}_2$.

Finally, suppose that $\tilde{r}_1 < \tilde{b}_1$. For contradiction suppose that $\tilde{b}_1 < \tilde{r}_2$. The preceding arguments make it clear that $\tilde{r}_1 \ge r(1,1)$, hence $\tilde{b}_1 > r(1,1)$. We first claim that $\tilde{b}_1 \ge b(2,1)$. If not $r(1,1) \le \tilde{r}_1 < \tilde{b}_1 < b(2,1)$. But, $\tilde{d}^a(\tilde{r}_1,1) > 0$ implies $\tilde{d}^w(\tilde{r}_1+1,1) \le 0$ (by Lemma 4); and, in conjunction with $\tilde{d}^a(n,1) \ge 0$ for $n \in [r(1,1), b(2,1)-1]$ this further implies (via Lemma 6) that $\tilde{d}^a(n,2) \ge 0$ for $n \in [r(1,1), b(2,1)-1]$. This contradicts $\tilde{b}_1 < b(2,1)$.

Thus, our assumption that $\tilde{b}_1 < \tilde{r}_2$ leads to the conclusion $b(2,1) \le \tilde{b}_1 < \tilde{r}_2$. Since $\tilde{d}^a(\tilde{b}_1,2) < 0$, Lemma 4 implies $\tilde{d}^w(\tilde{b}_1+1,1) \ge 0$. This and the fact $\tilde{d}^a(n,1) \le 0$ for $n \ge b(2,1)$ imply (via Lemma 6) that $\tilde{d}^a(n,2) \le 0$ for all such n. This contradicts $\tilde{r}_2 > \tilde{b}_1$. Therefore, if $\tilde{r}_1 < \tilde{b}_1$, it must be that $\tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_1 \le \tilde{b}_2$ and the lemma is proved. $\square$

REMARK. The essence of the lemma is that bubbles do not overtake or intersperse between one another. That is, there are uninterrupted runs of blue and red volumes whenever these are not zero. This intuitive statement is made more precise in the next few lemmas.

Suppose that $\tilde{r}_1, \tilde{r}_2, \tilde{b}_1$ and $\tilde{b}_2$ are all finite and $\tilde{b}_1 < \tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_2$. Define $b(1,2) = \tilde{b}_1$, $r(1,2) = \tilde{r}_1$ and $b(2,2) = \min\{n : \tilde{r}_2 < n \le \tilde{b}_2,\ \tilde{d}^a(n,2) < 0\}$. Also define $B(1,2) = -\sum_{n=b(1,2)}^{r(1,2)-1} \tilde{d}^a(n,2)$ $R(1,2) = \sum_{n=r(1,2)}^{b(2,2)-1} \tilde{d}^a(n,2)$ and $B(2,2) = -\sum_{n=b(2,2)}^{\tilde{b}_2} \tilde{d}^a(n,2)$. The previous lemma implies $B(1,2) \ge 0$, $R(1,2) \ge 0$ and $B(2,2) \ge 0$.

LEMMA 12. *Suppose that $\tilde{r}_1, \tilde{r}_2, \tilde{b}_1$ and $\tilde{b}_2$ are all finite and $\tilde{b}_1 < \tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_2$. Then*:

(i) $b(1,2) \ge b(1,1)$, $r(1,2) \ge r(1,1)$ *and* $b(2,2) \ge b(2,1)$,
(ii) $B(1,2) \le B(1,1)$, $R(1,2) \le R(1,1)$ *and* $B(2,1) \le B(2,2)$, *and*
(iii) $R(1,1) - R(1,2) = B(1,1) + B(2,1) - B(1,2) - B(2,2)$.

PROOF. Statement (i) follows from Possibilities α and β and the proof of Lemma 11.

It again follows from Possibility β that $B(1,2) \le B(1,1)$. We shall prove the lemma by showing that $B(2,2) \le B(2,1)$ and $R(1,1) - R(1,2) = B(1,1) + B(2,1) - B(1,2) - B(2,2)$. We proceed by establishing some preliminary claims.

CLAIM 1. For $b = \max\{n \in [b(1,2), r(1,2)-1] : \tilde{d}^a(n,2) < 0\}$, it holds that $b < b(2,1)$.

Suppose to the contrary $b \ge b(2,1)$. Then, since $\tilde{d}^a(b,2) < 0$, Lemma 4 implies $\tilde{d}^w(b+1,1) \ge 0$. And since $\tilde{d}^a(n,1) \le 0$ for all $n \ge b(2,1)$, this further implies via Lemma 6 that $\tilde{d}^a(n,2) \le 0$ for all $n \ge b$. This contradicts the finiteness of $\tilde{r}_2$ and establishes the claim.

CLAIM 2. $\tilde{d}^w(b(2,1),1) \le 0$.

Suppose $\tilde{d}^w(b(2,1),1) > 0$. Since $\tilde{d}^a(n,2) \le 0$ for all $n \ge b(2,1)$, Lemma 6 tells us that $\tilde{d}^a(n,2) \le 0$ for all $n \ge b(2,1)$. Therefore $r(1,2) < b(2,1)$. Now, given that $\tilde{d}^a(r(1,2),2) > 0$, Lemma 4 implies $\tilde{d}^w(r(1,2)+1,1) \le 0$, and since $r(1,2) \ge r(1,1)$, this further implies (via Lemma 6) that $\tilde{d}^w(n,1) \le 0$ for $n \in [r(1,2)+1, b(2,1)]$. This contradicts our assumption that $\tilde{d}^w(b(2,1),1) > 0$ and proves the claim.

Now, given that $b < b(2,1)$, it follows from the definition of $b(2,2)$ that $\tilde{d}^a(n,2) \ge 0$ for all $n \in [b(2,1), b(2,2)-1]$. Therefore (14) gives

$$\tilde{d}^w(i+1,1) = \sum_{n=b(2,1)}^{i} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w\big(b(2,1),1\big), \tag{17}$$

where $i = \min\{n \ge b(2,1) : \tilde{d}^w(k,1) = 0\ \forall\, k \ge n\}$. Rewriting (17) we get

$$0 = \sum_{n=b(2,1)}^{b(2,2)-1} \tilde{d}^a(n,2) - B(2,2) + B(2,1) + \tilde{d}^w\big(b(2,1),1\big)$$

or

$$B(2,1) - B(2,2) = -\tilde{d}^w\big(b(2,1),1\big) - \sum_{n=b(2,1)}^{b(2,2)-1} \tilde{d}^a(n,2).$$

If $\tilde{d}^a(n,2) = 0$ for all $n \in [b(2,1), b(2,2)-1]$, the above equation immediately gives $B(2,1) - B(2,2) = -\tilde{d}^w(b(2,1),1) \ge 0$.

Else, $\tilde{r}_2 \ge b(2,1)$ for $\tilde{r}_2$ as defined earlier. In this case note that $\sum_{n=b(2,1)}^{b(2,2)-1} \tilde{d}^a(n,2) = \sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,2)$. Therefore it suffices to show $-\tilde{d}^w(b(2,1),1) - \sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,2) \ge 0$ in order to conclude $B(2,1) \ge B(2,2)$.

But, this follows immediately from considering

$$\begin{aligned}\tilde{d}^w\big(b(2,1),1\big) &= \tilde{d}^w(\tilde{r}_2+1,1) - \sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,2) + \sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,1)\\ &\Leftrightarrow \tilde{d}^w\big(b(2,1),1\big) + \sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,2)\\ &= \tilde{d}^w(\tilde{r}_2+1,1) + \sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,1) \le 0,\end{aligned}$$

since $\tilde{d}^a(\tilde{r}_2,2) > 0$ implies $\tilde{d}^w(\tilde{r}_2+1,1) \le 0$ (from Lemma 4), and $\sum_{n=b(2,1)}^{\tilde{r}_2} \tilde{d}^a(n,1) \le 0$ because, to the right of $b(2,1)$ on the input side, blue is all there is.

Finally, from

$$0 = \tilde{d}^w(i+1,1) = \sum_{n=b(1,1)}^{i} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w\big(b(1,1),1\big)$$

and the fact that $\tilde{d}^w(b(1,1),1) = 0$ it immediately follows that $R(1,1) - R(1,2) = B(1,1) + B(2,1) - B(1,2) - B(2,2)$. This completes the proof of the lemma. $\square$

Suppose $\tilde{r}_1, \tilde{r}_2, \tilde{b}_1$ and $\tilde{b}_2$ are all finite and $\tilde{b}_1 \le \tilde{b}_2 < \tilde{r}_1 \le \tilde{r}_2$. Then define $b(1,2) = \tilde{b}_1$, $r(1,2) = \tilde{r}_1$ and $b(2,2) = \infty$. Also define $B(1,2) = -\sum_{n=b(1,2)}^{r(1,2)-1} \tilde{d}^a(n,2)$, $R(1,2) = \sum_{n=r(1,2)}^{\tilde{r}_2} \tilde{d}^a(n,2)$ and set $B(2,2) = 0$.

LEMMA 13. *If $\tilde{r}_1, \tilde{r}_2, \tilde{b}_1$ and $\tilde{b}_2$ are all finite and $\tilde{b}_1 \le \tilde{b}_2 < \tilde{r}_1 \le \tilde{r}_2$, then*:

(i) $b(1,2) \ge b(1,1)$, $r(1,2) \ge r(1,1)$ *and* $b(2,2) \ge b(2,1)$,
(ii) $B(1,2) \le B(1,1)$ *and* $R(1,2) \le R(1,1)$,
(iii) $R(1,1) - R(1,2) = B(1,1) + B(2,1) - B(1,2) - B(2,2)$.

PROOF. Part (i) follows from Possibilities α and β, and clearly $b(2,2) = \infty > b(2,1)$.

It follows as in the proof of Lemma 12 that $\tilde{d}^w(b(2,1),1) \le 0$. (In words, this means no blue volume from the first blue bubble enters the second blue bubble.) As a consequence, it is straightforward to infer from Possibility β that $B(1,2) \le B(1,1)$. Since $B(2,2) = 0$, the rest of the lemma will follow from showing $R(1,1) - R(1,2) = B(1,1) + B(2,1) - B(1,2)$. Because $\tilde{d}^w(b(1,1),1) = 0$, this follows trivially from

$$0 = \tilde{d}^w(i+1,1) = \sum_{n=b(1,1)}^{i} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w\big(b(1,1),1\big),$$

where $i = \min\{n \ge b(2,1) : \tilde{d}^w(k,1) = 0\ \forall\, k \ge n\}$. $\square$

Next suppose $\tilde{b}_1, \tilde{b}_2, \tilde{r}_1$ and $\tilde{r}_2$ are all finite and $\tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_1 \le \tilde{b}_2$. Then define $b(1,2) = r(1,2) = \tilde{r}_1$ and $b(2,2) = \tilde{b}_1$. Set $B(1,2) = 0$, define $R(1,2) = \sum_{n=r(1,2)}^{\tilde{r}_2} \tilde{d}^a(n,2)$ and $B(2,2) = -\sum_{n=b(2,2)}^{\tilde{b}_2} \tilde{d}^a(n,2)$.

LEMMA 14. *If $\tilde{r}_1, \tilde{r}_2, \tilde{b}_1$ and $\tilde{b}_2$ are all finite and $\tilde{r}_1 \le \tilde{r}_2 < \tilde{b}_1 \le \tilde{b}_2$, then*:

(i) $b(1,2) \ge b(1,1)$, $r(1,2) \ge r(1,1)$ *and* $b(2,2) \ge b(2,1)$,
(ii) $R(1,2) \le R(1,1)$ *and* $B(2,2) \le B(2,1)$,
(iii) $R(1,1) - R(1,2) = B(1,1) + B(2,1) - B(1,2) - B(2,2)$.

The proof follows from Case β, similarly as in the proof of Lemma 13.

Possibility γ': only the red bubble survives. Suppose that $\tilde{b}_1$ and $\tilde{b}_2$ are not finite. In this case, there is no blue at the second stage. Define $b(1,2) = r(1,2) = \tilde{r}_1$ and set $b(2,2) = \infty$. Also set $B(1,2) = B(2,2) = 0$ and define $R(1,2) = \sum_{n=r(1,2)}^{\tilde{r}_2} \tilde{d}^a(n,2)$.

It is clear that $b(1,2) \ge b(1,1)$, $r(1,2) \ge r(1,1)$ and $b(2,2) \ge b(2,1)$.

LEMMA 15. *Suppose that $\tilde{b}_1$ and $\tilde{b}_2$ are not finite. Then $R(1,2) = R(1,1) - B(1,1) - B(2,1)$. In particular, $B(1,2) \le B(1,1)$, $R(1,2) < R(1,1)$ and $B(2,2) \le B(2,1)$.*

PROOF. Consider the equation

$$0 = \tilde{d}^w(i+1,1) = \sum_{n=b(1,1)}^{i} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w\bigl(b(1,1),1\bigr),$$

where $i = \min\{n \ge b(2,1) : \tilde{d}^w(k,1) = 0 \ \forall k \ge n\}$. Given that $\tilde{d}^w(b(1,1),1) = 0$, it follows immediately that $R(1,2) = R(1,1) - B(1,1) - B(2,1)$. $\square$

Possibility γ^*: the red bubble is completely cancelled. Suppose that $\tilde{r}_1$ and $\tilde{r}_2$ are not finite. In this case, there is no red at the second stage. Define $b(1,2) = r(1,2) = b(2,2) = \tilde{b}_1$ and

$$B(1,2) = B(1,1)\left(\frac{-\sum_{n=b(1,2)}^{\tilde{b}_2} \tilde{d}^a(n,2)}{B(1,1) + B(2,1)}\right),$$

$$B(2,2) = B(2,1)\left(\frac{-\sum_{n=b(1,2)}^{\tilde{b}_2} \tilde{d}^a(n,2)}{B(1,1) + B(2,1)}\right),$$

$$R(1,2) = 0.$$

Note that it is possible for $r(1,2) < r(1,1)$ and $b(2,2) < b(2,1)$. This is a crucial deviation from all previous cases, necessitated by reasons detailed in the remark after the lemma.

LEMMA 16. *Suppose that $\tilde{r}_1$ and $\tilde{r}_2$ are not finite. Then*:

(i) $b(1,2) \le r(1,2) \le b(2,2)$,
(ii) $B(i,2) \le B(i,1)$ *for* $i = 1,2$ *and* $R(1,2) \le R(1,1)$ *and*
(iii) $B(1,1) + B(2,1) - B(1,2) - B(2,2) = R(1,1)$.

PROOF. Statement (i) follows by definition, while (ii) and (iii) follow from

$$0 = \tilde{d}^w(i+1,1) = \sum_{n=b(1,1)}^{i} \tilde{d}^a(n,2) - \tilde{d}^a(n,1) + \tilde{d}^w\big(b(1,1),1\big),$$

where $i = \min\{n \ge b(2,1) : \tilde{d}^w(k,1) = 0\ \forall k \ge n\}$. Observing that $\tilde{d}^w(b(1,1), 1) = 0$ completes the proof of the lemma. $\square$

REMARK. The definitions of quantities at the second stage are so as to preserve orderings between bubble start points and to ensure that bubble volumes do not grow. The above lemma shows that when the red bubble is fully cancelled, it nullifies an equal amount of blue from the blue bubbles put together. The amount of volume taken out of each blue bubble is proportional to its original size. Thus, we do not keep an account of whether or not a specific blue bubble contributed to the cancelling of the red. This is both unnecessary and can lead to needless complication, as seen below.

First, without elaboration, here are the ways (and concomitant conditions) in which the red bubble can be cancelled: (i) the first blue bubble cancels all of the red bubble [$\tilde{d}^w(r(1,1),1) \ge R(1,1)$ and $\tilde{d}^w(b(2,1),1) \ge 0$], (ii) each of the blue bubbles contributes to the cancellation of the red bubble [$\tilde{d}^w(r(1,1),1) > 0$, $\tilde{d}^w(b(2,1),1) = \tilde{d}^w(r(1,1),1) - R(1,1) < 0$, and $\tilde{d}^a(n,2) \le 0$ for $n \ge b(2,1)$] and (iii) only the second bubble cancels the red bubble [$\tilde{d}^w(r(1,1),1) = 0$, $-\tilde{d}^w(b(2,1),1) = R(1,1)$ and $\tilde{d}^a(n,2) \ge 0$ for $n \ge b(2,1)$].

Note that in situations (ii) and (iii) above $\tilde{d}^w(b(2,1),1) \le 0$; or, in words, no blue volume enters the second blue bubble. This ensures that the two shades of blue do not mix. But, if $\tilde{d}^w(b(2,1),1) > 0$, as can happen in (i), the two shades of blue do mix. This can make it impossible to decide $b(2,2)$ so as to satisfy the following conditions simultaneously: (a) $B(1,2) \le B(1,1)$ and (b) $B(2,2) \le B(2,1)$.

For example, suppose that $B(1,1) = B(2,1) = 100$, $R(1,1) = 50$, $\tilde{d}^w(r(1,1), 1) = 100$ and $\tilde{d}^w(b(2,1),1) = 25$. Also suppose that $\tilde{d}^a(l,2) = 25$ for some $l \in [r(1,1), b(2,1)-1]$ and that $\tilde{d}^a(m,2) = 100$, $\tilde{d}^a(n,2) = 25$ for some $m < n \in [b(2,1), \infty)$. Observe that no choice of $b(2,2)$ can satisfy conditions (a) and (b) above.

A simple way out is to set $r(1,2) = b(2,2) = b(1,2)$, and divide the volume of blue on the output side proportionately among the two blue bubbles. Although this choice can cause $r(1,2) < r(1,1)$, it must be seen as a consequence of convenience. It will be clear that this causes no problems in the rest of the argument.

4.1. *The equilibrium evolution.* Using the ideas of the previous section, we will now describe the evolution of the bubbles in equilibrium. Consider the processes $\mathbf{A}^1$, $\mathbf{I}^1$ and $\{S(n,1)\}_{n\in\mathbb{Z}}$. The quantities $r(n,1)$, $b(n,1)$, $R(n,1)$, $B(n,1)$, $\mathcal{R}(n,1)$ and $\mathcal{B}(n,1)$ are as defined in (9), (11) and (12).

We describe the procedure for determining $r(n,2)$ and $b(n,2)$ for each $n\in\mathbb{Z}$. From these one can deduce the quantities $R(n,2)$, $B(n,2)$, $\mathcal{R}(n,2)$ and $\mathcal{B}(n,2)$. As in the previous section the sequence $\{d^w(n,1)\}_{n\in\mathbb{Z}}$ plays a key role in the determination of $r(n,2)$ and $b(n,2)$. For what follows, it is helpful to make a connection between the sign of $d^w(n,1)$ and what it means for bubble movements at n. Accordingly, depending on whether $d^w(n,1)=0$, $d^w(n,1)<0$ or $d^w(n,1)>0$, there is a movement from $n-1$ to n, respectively, of nothing, red or blue of volume $|d^w(n,1)|$.

Consider $\mathcal{S}=\{n: d^a(n,2)>0 \text{ infinitely often}\}$. By the joint ergodicity of $\mathbf{A}^2$ and $\mathbf{I}^2$, $P(\mathcal{S})=0$ or 1. If $P(\mathcal{S})=0$ then $d^a(n,2)\le 0$ for every n a.s. [ergodicity clearly rules out that $d^a(n,2)\le 0$ for finitely many n with positive probability]. But this last fact together with $E(A(n,2))=E(I(n,2))$ implies that $\mathbf{A}^2=\mathbf{I}^2$ a.s. Thus, if $P(\mathcal{S})=0$ the proof of Theorem 1 is complete.

Therefore, suppose $P(\mathcal{S})=1$. Note that this implies the co-existence of infinitely many blue and red bubbles at the second stage. We will now give a procedure for determining $b(1,2)$ and $r(1,2)$, and hence for $b(n,2)$ and $r(n,2)$ for every n.

First consider the processes $\mathbf{A}^2$ and $\mathbf{I}^2$. These are jointly ergodic, and hence it is possible to apply the procedure of Section 2.1 and obtain bubbles. Let $\tilde{b}(n,2)$ and $\tilde{r}(n,2)$ be the start points of the bubbles, and let $\tilde{B}(n,2)$ and $\tilde{R}(n,2)$ be the corresponding bubble volumes. Note that $\tilde{r}(n-1,2)<\tilde{b}(n,2)<\tilde{r}(n,2)$ for every n. We need variables $\tilde{e}(n,2)$ and $\tilde{f}(n,2)$ which mark the end points of the bubbles in order to proceed. Thus, let

$$\tilde{e}(n,2)=\max\{k\in[\tilde{b}(n,2),\tilde{r}(n,2)-1]: d^a(n,2)<0\},$$

$$\tilde{f}(n,2)=\max\{k\in[\tilde{r}(n,2),\tilde{b}(n+1,2)-1]: d^a(n,2)>0\}.$$

Note that $\tilde{b}(n,2)\le\tilde{e}(n,2)<\tilde{r}(n,2)\le\tilde{f}(n,2)<\tilde{b}(n+1,2)$ for every n. With these definitions, the following procedure relates $r(\cdot,2)$ and $b(\cdot,2)$ to $\tilde{r}(\cdot,2)$ and $\tilde{b}(\cdot,2)$.

Determining $b(1,2)$. Clearly $b(1,1)\in[\tilde{b}(k,2),\tilde{b}(k+1,2)-1]$ for some k.

(a) If $b(1,1)\in[\tilde{b}(k,2),\tilde{e}(k,2)]$, set $b(1,2)=\tilde{b}(k,2)$.

(b) If $b(1,1)\in[\tilde{e}(k,2)+1,\tilde{r}(k,2)]$ and $r(1,1)\le\tilde{f}(k,2)$, set $b(1,2)=\tilde{r}(k,2)$.

(b$'$) If $b(1,1)\in[\tilde{e}(k,2)+1,\tilde{r}(k,2)]$ and $r(1,1)>\tilde{f}(k,2)$, set $b(1,2)=\tilde{b}(k+1,2)$.

(c) If $b(1,1)\in[\tilde{r}(k,2)+1,\ \tilde{f}(k,2)]$ and $r(1,1)\in[b(1,1)+1,\ \tilde{f}(k,2)]$, set $b(1,2)=\tilde{r}(k,2)$.

(c′) If $b(1,1) \in [\tilde{r}(k,2)+1, \tilde{f}(k,2)]$ and $r(1,1) \notin [b(1,1)+1, \tilde{f}(k,2)]$, set $b(1,2) = \tilde{b}(k+1,2)$.

(d) If $b(1,1) \in [\tilde{f}(k,2)+1, \tilde{b}(k+1,2)-1]$, set $b(1,2) = \tilde{b}(k+1,2)$.

Determining $r(1,2)$. Clearly $r(1,1) \in [\tilde{r}(k,2), \tilde{r}(k+1,2)-1]$ for some k.

(e) If $r(1,1) \in [\tilde{r}(k,2), \tilde{f}(k,2)]$, set $r(1,2) = \tilde{r}(k,2)$.

(f) If $r(1,1) \in [\tilde{f}(k,2)+1, \tilde{b}(k+1,2)]$ and $b(2,1) \le \tilde{e}(k+1,2)$, set $r(1,2) = \tilde{b}(k+1,2)$.

(f′) If $r(1,1) \in [\tilde{f}(k,2)+1, \tilde{b}(k+1,2)]$ and $b(2,1) > \tilde{e}(k+1,2)$, set $r(1,2) = \tilde{r}(k+1,2)$.

(g) If $r(1,1) \in [\tilde{b}(k+1,2)+1, \tilde{e}(k+1,2)]$ and $b(2,1) \in [r(1,1)+1, \tilde{e}(k+1,2)]$, set $r(1,2) = \tilde{b}(k+1,2)$.

(g′) If $r(1,1) \in [\tilde{b}(k+1,2)+1, \tilde{e}(k+1,2)]$ and $b(2,1) \notin [r(1,1)+1, \tilde{e}(k+1,2)]$, set $r(1,2) = \tilde{r}(k+1,2)$.

(h) If $r(1,1) \in [\tilde{e}(k,2)+1, \tilde{r}(k+1,2)-1]$, set $r(1,2) = \tilde{r}(k+1,2)$.

LEMMA 17. *If $b(1,1) \le \tilde{b}(p,2)$ then $b(1,2) \le \tilde{b}(p,2)$.*

PROOF. If $p = k$, for k as defined in the above procedure, then from case (a), $b(1,1) = \tilde{b}(p,2) = b(1,2)$. If $p \ge k+1$, then note that $\tilde{r}(k,2) < \tilde{b}(k+1,2) \le \tilde{b}(p,2)$, and hence by the procedure for determining $b(1,2)$, it follows that $b(1,2) \le \tilde{b}(k+1,2) \le \tilde{b}(p,2)$. □

LEMMA 18. *For every $n \in \mathbb{Z}$, $r(n,2) \le b(n+1,2) \le r(n+1,2)$.*

PROOF. We prove $r(n,2) \le b(n+1,2)$, the other inequality is similarly established. By construction of $r(n,2)$ and $b(n,2)$, these points are always at the start point of a red or a blue bubble at the second stage [i.e., they equal some $\tilde{r}(k,2)$ or $\tilde{b}(m,2)$]. Further, they each move either to the nearest start point on the left, or to one of the nearest two start points on the right.

Now, $r(n,1) < b(n+1,1)$. Therefore, every time $r(n,2) \le r(n,1)$ [i.e., $r(n,1)$ moved to its nearest left start point], it follows that $r(n,2) \le b(n+1,2)$. This covers cases (e) and (g), which are the cases when $r(n,2) \le r(n,1)$.

Under (f), $r(n,2) = \tilde{b}(k+1,2)$ for some k. And our procedure for $b(n+1,2)$ [cases (a) and (d)] sets $b(n+1,2) = \tilde{b}(k+1,2)$.

Under (f′), $r(n,2) = \tilde{r}(k+1,2)$ for some k. Our procedure [cases (b), (b′), (c) and (c′)] determines that $b(n+1,2) \ge \tilde{r}(k+1,2) = r(n,2)$.

Under (g′), since $b(n+1,1) > \tilde{e}(k+1,2) \ge r(n,1)$, it follows from cases (b), (b′), (c) and (c′) that $b(n+1,2) \ge \tilde{r}(k+1,2) = r(n,2)$.

Finally, under (h), we again see that $b(n+1,1) > r(n,1) \ge \tilde{e}(k+1,2)$. Again from cases (b), (b′), (c) and (c′) it follows that $b(n+1,2) \ge \tilde{r}(k+1,2) = r(n,2)$.

This concludes the proof of the lemma. □

LEMMA 19. *At the start of a red bubble at the second stage, there is always exactly one more $r(\cdot, 2)$ than there are $b(\cdot, 2)$'s. Similarly, at the start of a blue bubble at the second stage, there is always exactly one more $b(\cdot, 2)$ than there are $r(\cdot, 2)$'s.*

PROOF. For concreteness, consider $\tilde{r}(1, 2)$. By the order-preservation established in Lemma 18, it suffices to show that $\tilde{r}(1, 2) = r(k, 2) = b(k + 1, 2) = \cdots = r(k + m, 2)$ for some k and $m \geq 0$.

Consider $\tilde{e}(1, 2)$. If $\max\{p : b(p, 1) \leq \tilde{e}(1, 2)\} > \max\{q : r(q, 1) \leq \tilde{e}(1, 2)\}$, then we claim that there exists an $r(j, 1) \in [\tilde{e}(1, 2) + 1, \tilde{r}(1, 2)]$. Suppose not. This means $d^a(n, 1) \leq 0$ for all $n \in [\tilde{e}(1, 2) + 1, \tilde{r}(1, 2)]$. By Lemma 4, $d^w(\tilde{e}(1, 2) + 1, 1) \geq 0$ and from the fact that $d^a(n, 1) \leq 0$ for all $n \in [\tilde{e}(1, 2) + 1, \tilde{r}(1, 2)]$, we get from recursively using Lemma 6 that $d^a(\tilde{r}(1, 2), 2) \leq 0$. This contradiction establishes the claim.

Let $r(J, 1) = \min\{r(j, 1) \in [\tilde{e}(1, 2) + 1, \tilde{r}(1, 2)]\}$. By our procedure, $r(J, 2) = \tilde{r}(1, 2)$ and $b(J, 1) = \tilde{b}(1, 2)$. By the order-preservation established in Lemma 18 this identifies $r(J, 1)$ as the smallest bubble start point on the input side that gets mapped to $\tilde{r}(1, 2)$.

[We shall find it useful later to note that $d^w(r(J, 1), 1) \geq 0$. This is because $d^w(\tilde{e}(1, 2) + 1, 1) \geq 0$, by Lemma 4. Since $d^a(n, 1) \leq 0$ for $n \in [\tilde{e}(1, 2) + 1, r(J, 1) - 1]$, by a recursive use of Lemma 6, we get that $d^w(r(J, 1), 1) \geq 0$).]

On the other hand, suppose that $\max\{p : b(p, 1) \leq \tilde{e}(1, 2)\} < \max\{q : r(q, 1) \leq \tilde{e}(1, 2)\}$. Let $r(Q, 1) = \max\{r(q, 1) \leq \tilde{e}(1, 2)\}$. We claim that $r(Q, 1) \in [\tilde{f}(0, 2) + 1, \tilde{e}(1, 2)]$. Suppose not. Then, by Lemma 4 $d^w(\tilde{f}(0, 2) + 1, 1) \leq 0$. And since $d^a(n, 1) \geq 0$ for all $n \in [\tilde{f}(0, 2) + 1, \tilde{e}(1, 2)]$ (because $b(Q, 1) < r(Q, 1) \leq \tilde{f}(0, 2) + 1$ and $b(Q + 1, 1) > \tilde{e}(1, 2)$) it follows recursively from Lemma 6 that $d^a(n, 2) \geq 0$ for all $n \in [\tilde{f}(0, 2) + 1, \tilde{e}(1, 2)]$. This contradicts $\tilde{b}(1, 2) \in [\tilde{f}(0, 2) + 1, \tilde{e}(1, 2)]$.

By our procedure [cases (f) and (f′)], $r(Q, 2) = \tilde{r}(1, 2)$. And from the procedure for $b(Q, 1)$, we get that $b(Q, 2) < \tilde{r}(1, 2)$. This also identifies $r(Q, 1)$ as the smallest bubble start point on the input side that gets mapped to $\tilde{r}(1, 2)$.

[We again note that $d^w(r(Q, 1), 1) \geq 0$. Suppose not. Then, since $d^a(n, 1) \geq 0$ for all $n \in [r(Q, 1), \tilde{e}(1, 2)]$, a repeated use of Lemma 6 implies that $d^a(n, 2) \geq 0$ for all such n. This contradicts $d^a(\tilde{e}(1, 2), 2) < 0$.]

Thus, in both cases, we see that there exists a k such that $r(k, 2) = \tilde{r}(1, 2) > b(k, 2)$.

Now, we shall identify $r(k + m, 2)$. This is easy to see from our procedure: $k + m = \max\{l : r(l, 1) \geq r(k, 1) \text{ and } r(l, 1) \leq \tilde{f}(1, 2)\}$. Note that it follows from our procedure that $b(k + m + 1, 2) > \tilde{r}(1, 2)$.

This concludes the proof of the lemma. □

Note from the above proof that $r(k, 1) \leq \tilde{r}(1, 2)$. We also claim that $d^w(r(k, 1), 1) \geq 0$.

COROLLARY 1. *For $r(k,2)$ as defined in the proof of Lemma 19, $r(k,1) \leq \tilde{r}(1,2)$ and $d^w(r(k,1),1) \geq 0$.*

Both statements have been established during the proof of Lemma 19.

REMARK. While Lemma 18 demonstrates that our procedure for determining $r(n,2)$ and $b(n,2)$ preserves order, Lemma 19 is the more important. It records precisely the identity of the bubbles in the input process that contribute to the volume of a bubble at the output. That is, consider $\tilde{r}(1,2) = r(k,2) = \cdots = r(k+m,2)$. This implies (as in case γ^*) that the only possible shades in the volume $\tilde{R}(1,2)$ are the red shades k to $k+m$. It also implies that all the intermediate blue shades have been completely cancelled. This influences the following definition of bubble volumes at the output stage.

Determining $B(1,2)$. Consider $b(1,2)$. If $b(1,2) = \tilde{r}(n,2)$ for some n, then set $B(1,2) = 0$. Else, $b(1,2) = \tilde{b}(n,2)$ for some n. Suppose $\tilde{b}(n,2) = b(k,2) = \cdots = b(1,2) = \cdots = b(k+m,2)$. Then set

$$B(1,2) = B(1,1)\frac{\tilde{B}(n,2)}{\sum_{i=k}^{k+m} B(i,1)}.$$

As in Possibility γ^*, this credits each of the blue bubbles proportionately for vanquishing the intermediate red bubbles.

Determining $R(1,2)$. Similarly as above.

LEMMA 20. *For every n, $R(n,2) \leq R(n,1)$ and $B(n,2) \leq B(n,1)$.*

PROOF. We establish $R(1,2) \leq R(1,1)$. There is nothing to prove if $R(1,2) = 0$. Else, there is a p such that $\tilde{r}(p,2) = r(k,2) = \cdots = r(1,2) = \cdots = r(k+m,2)$ for some k and m. And

$$R(1,2) = R(1,1)\frac{\tilde{R}(p,2)}{\sum_{i=k}^{k+m} R(i,1)}.$$

Therefore, it suffices to prove that

$$\tilde{R}(p,2) \leq \sum_{i=k}^{k+m} R(i,1). \tag{18}$$

First note that $r(k,1) \leq \tilde{r}(p,2)$ and from Corollary 1 $d^w(r(k,1),1) \geq 0$. Also note that $d^w(\tilde{f}(p,2)+1,1) \leq 0$ and $d^w(\tilde{e}(p,2)+1,1) \geq 0$, using Lemma 4. We consider two cases.

CASE 1 [$r(k,1) \geq \tilde{e}(p,2)+1$]. Then (14) gives

$$d^w\big(\tilde{f}(p,2)+1,1\big) = \sum_{i=r(k,1)}^{\tilde{f}(p,2)} d^a(n,2) - d^a(n,1) + d^w\big(r(k,1),1\big).$$

From this we get that $\sum_{i=r(k,1)}^{\tilde{f}(p,2)} d^a(n,2) \leq \sum_{i=r(k,1)}^{\tilde{f}(p,2)} d^a(n,1)$, since $d^w(\tilde{f}(p,2)+1,1) \leq 0$ and $d^w(r(k,1),1) \geq 0$. But $\sum_{i=r(k,1)}^{\tilde{f}(p,2)} d^a(n,2) = \tilde{R}(p,2)$ and $\sum_{i=r(k,1)}^{\tilde{f}(p,2)} d^a(n,1) \leq \sum_{i=k}^{k+m} R(i,1)$, since $r(k+m+1,1) \geq \tilde{f}(p,2)$.

CASE 2 [$r(k,1) < \tilde{e}(p,2)+1$]. From (14) we get

$$d^w\big(\tilde{f}(p,2)+1,1\big) = \sum_{i=\tilde{e}(p,2)+1}^{\tilde{f}(p,2)} d^a(n,2) - d^a(n,1) + d^w\big(\tilde{e}(p,2)+1,1\big).$$

Again this gives

$$\begin{aligned}\tilde{R}(p,2) = \sum_{i=\tilde{e}(p,2)+1}^{\tilde{f}(p,2)} d^a(n,2) &\leq \sum_{i=\tilde{e}(p,2)+1}^{\tilde{f}(p,2)} d^a(n,1) \\ &\leq \sum_{i=r(k,1)}^{\tilde{f}(p,2)} [d^a(n,1)]^+ \leq \sum_{i=k}^{k+m} R(i,1).\end{aligned}$$

□

4.1.1. *Summary of equilibrium evolution.* We have just described the evolution of bubbles from the first stage to the second and it is easy to see that the same description holds at each stage k, $k > 1$. We summarize the conclusions in the following lemma.

LEMMA 21. *The following hold for each $n \in \mathbb{Z}$ and $k \geq \mathbb{Z}^+$*:

1. *Bubbles do not overtake each other*: $r(n-1,k) \leq b(n,k) \leq r(n,k) \leq b(n+1,k)$.
2. *Bubble volumes do not grow*: $R(n,k+1) \leq R(n,k)$ *and* $B(n,k+1) \leq B(n,k)$. *Therefore, for each n*, $\lim_{k\to\infty} R(n,k)$ *and* $\lim_{k\to\infty} B(n,k)$ *exist.*
3. *And*, $d(k+1) \leq d(k)$, *where $d(k)$ is defined in Section* 2.2. *Therefore*, $\lim_{k\to\infty} d(k) = d$ *exists.*

5. Proof that $d(k)\to 0$. Given that $d(k)$ is nonincreasing, the proof of Theorem 1 is complete if $d = \lim_{k\to\infty} d(k) = 0$. We shall argue this by contradiction and hence assume that $P(d > \alpha) > 0$ for some $\alpha > 0$. Equally, letting $ER(n) = \lim_{k\to\infty} R(n,k)$ and $EB(n) = \lim_{k\to\infty} B(n,k)$, the assumption for contradiction implies that $ER(n)$ and $EB(n)$ are not zero for all $n \in \mathbb{Z}$ a.s.

Thus, there are some red and blue bubbles which never vanish. Call these bubbles "everred" and "everblue," respectively. We proceed by following the method of [12].

Consider the process of the limiting volumes, $\{(ER(n), EB(n)), n \in \mathbb{Z}\}$, of everred and everblue bubbles. Since this process is the decreasing limit of red and blue volumes $\{(R(n,k), B(n,k)), n \in \mathbb{Z}\}$, we may imagine that within each red and blue bubble there lives an everred or an everblue bubble which is colored with the same shade of red or blue. Specifically, consider the process $\{(ER(n,1), EB(n,1)), n \in \mathbb{Z}\}$ of everred and everblue volumes present in the initial arrival process. We imagine that $ER(n,1)$ is the volume of the nth everred, which is colored with the nth shade of red, and $ER(n,1) = ER(n) \le R(n,1)$. And similarly for $EB(n,1)$. Note that we allow everred and everblue bubbles to have zero volume; when this happens, it is to be understood that the original blue and red bubbles will be completely cancelled out eventually.

For each n, and for $l \in \mathcal{R}(n,1)$, define

$$X^k(l) = d^a(l,1)\frac{R(n,k)}{R(n,1)},$$

and for $l \in \mathcal{B}(n,1)$, define

$$X^k(l) = d^a(l,1)\frac{B(n,k)}{B(n,1)}.$$

Given that bubbles volumes do not increase, it follows that $X^k(l)$ is a nonincreasing (nondecreasing) sequence for $l \in \mathcal{R}(n,1)$ (resp. for $l \in \mathcal{B}(n,1)$). Let $X(l) = \lim_k X^k(l)$. Since $R(1,k) = \sum_{\mathcal{R}(1,1)} X^k(l)$, one thinks of $[X^k(l)]^+$ as the amount of red of shade 1 at location l that survives through to the kth stage. Likewise, the process $\mathbf{X} = \{X(l), l \in \mathbb{Z}\}$ may be interpreted as the process of everred and everblue volumes present in the original arrival processes at location l: if $X(l) > 0$, then some everred is present at location l and if $X(l) < 0$ some everblue is present at location l in the original arrival processes.

By the translation-invariant nature of the queueing operation, $\{X^k(l), l \in \mathbb{Z}\}$ is ergodic for each k. Therefore,

$$\lim_{n\to\infty} \frac{\sum_{l=-n}^{n} [X^k(l)]^+}{2n+1} = \lim_{n\to\infty} \frac{\sum_{l=-n}^{n} [X^k(l)]^-}{2n+1} = d(k),$$

where $d(k)$ was defined in Section 2.2. Since $d(k)$ is nonincreasing and almost surely a constant for each k, $d = \lim_k d(k)$ is almost surely constant.

As the decreasing limit of stationary processes, $\mathbf{X}$ is, a priori, a stationary (but not necessarily ergodic) process. Therefore,

$$x = \lim_{n\to\infty} \frac{\sum_{l=-n}^{n} [X(l)]^+}{2n+1}$$

exists a.s., and must be treated as a random quantity. But

$$d = \lim_k d(k) = \lim_k \lim_n \frac{\sum_{l=-n}^n [X^k(l)]^+}{2n+1} = \lim_n \lim_k \frac{\sum_{l=-n}^n [X^k(l)]^+}{2n+1} = x,$$

where the limit interchange is due to the following. First, observe that

$$d(k) \geq \limsup_n \frac{\sum_{l=-n}^n [X(l)]^+}{2n+1}$$

for every k, due to the monotonicity of $[X^k(l)]^+$. By Fatou's lemma, for each n we get

$$\sum_{l=-n}^{n} \liminf_k \frac{[X^1(l)]^+ - [X^k(l)]^+}{2n+1} \leq \liminf_k \sum_{l=-n}^{n} \frac{[X^1(l)]^+ - [X^k(l)]^+}{2n+1},$$

which implies $\liminf_n((\sum_{l=-n}^n [X(l)]^+)/(2n+1)) \geq \limsup_k d(k)$. Therefore $d = x$, making x an almost sure constant.

Thus, the assumption $d(k) \nrightarrow 0$ leads to the co-existence, with probability 1, of everblue and everred bubbles in $\mathbf{A}^1$ and $\mathbf{I}^1$ of strictly positive volume per arrival equal to d. Since the shades of all red and blue bubbles are distinct, the everreds and everblues have distinct shades. By Lemma 21, the ordering of the red and blue bubbles, and hence of the everred and everblue bubbles, is preserved at each stage k.

Consider the subprocess $\{X(l)\mathbb{1}_{|X(l)|>\varepsilon}, l \in \mathbb{Z}\}$ and its support set $\mathcal{E}(1) = \{l : |X(l)| > \varepsilon\}$. Call this the "process of chosen everred and everblue segments" and observe that it is stationary since $\{X(l), l \in \mathbb{Z}\}$ is stationary. Now $\mathcal{E}(1)$ can be written as the disjoint union of two sets: $\mathcal{E}_r(1) = \{l : X(l) > \varepsilon\}$ and $\mathcal{E}_b(1) = \{l : X(l) < -\varepsilon\}$, which support chosen everred and everblue segments respectively. Given that $d > 0$, for any $\varepsilon \in (0, d)$ we get that the density of points in $\mathcal{E}_b(1)$ and $\mathcal{E}_r(1)$ for this choice of ε is strictly positive a.s. (but possibly random). Fix one such ε and observe that the process of chosen segments appears as an alternating sequence of everred and everblue segments. Consider the left endpoints of a run of chosen everblue segments, and let $\mathcal{L}\mathcal{E}_b(1) \subset \mathcal{E}_b(1)$ be the set of integers which support these segments. The process of left chosen everblue segments is also stationary (since chosen everblue segments are stationary) and therefore the set $\mathcal{L}\mathcal{E}_b(1)$ has a possibly random density which must be strictly positive a.s. [else $\mathcal{E}_b(1)$ cannot have strictly positive density].

A crucial consequence of our construction is that any two chosen left everblue segments must be shaded with different colors of blue, since they are separated by chosen everred segments. Now consider the starting point of the blue bubbles to which the chosen left everblue segments belong, and write $\mathcal{S}_b(1)$ for the integers which form these starting points. Note that the set of points in $\mathcal{S}_b(1)$ form a stationary sequence. Since there is a one-to-one correspondence between points in $\mathcal{LE}_b(1)$ and $\mathcal{S}_b(1)$, their densities are almost surely equal. In particular, the density of $\mathcal{S}_b(1)$, denoted by the random variable C, is almost surely strictly positive.

Thus, we have obtained at stage 1 the existence of blue bubbles with the following properties:

(a) their volumes are at least ε;
(b) between any two of them there is a red bubble with volume at least ε; and
(c) their start points have density $C > 0$ a.s.

By definition, for each n the nth everred and everblue bubbles retain their volume at every stage k. And by Lemma 21 the original ordering between everreds and everblues is preserved throughout. This allows us to use the same argument as above at each stage k and obtain, under our assumption that $d(k) \nrightarrow 0$, the existence of blue bubbles satisfying properties (a), (b) and (c) listed above at every stage k.

Proceeding, choose $\delta > 0$ such that $P(C > \delta) > \delta$. Write $C = C_l + C_g$, where C_l is the density of the start points, l, of blue bubbles satisfying property (a) above with the additional property that there is another blue with volume at least ε and start point l' such that $l' \le l + 2/\delta$ and there is a red bubble with volume at least ε between them. Let C_g be the density of the start points of the remaining blue bubbles satisfying properties (a) and (b). By definition, $C_g \le \delta/2$ a.s. Therefore $C_l \ge \delta/2$ whenever $C > \delta$. Choose δ so that $2/\delta$ is an integer.

Therefore, our above arguments imply that for any k there exist blue bubbles satisfying the following properties:

(1) their volumes are at least ε;
(2) there is a red bubble with volume at least ε contained in the interval $[l, l+2/\delta]$, where l is the start point of the blue bubble; and
(3) the density of their start points is at least $\delta/2$ with probability at least δ.

For any k consider the event E:

$$\{\text{the density of the start points of blue bubbles satisfying 1 and 2} \ge \delta/2\}.$$

This event is shift-invariant and contained in the jointly ergodic processes $(\mathbf{A}^k, \mathbf{I}^k)$. By property 3, $P(E) \ge \delta$. Therefore $P(E) = 1$. We record this in the following lemma.

LEMMA 22. *If $d > 0$, then there exist strictly positive ε and δ not depending on k such that there exist blue bubbles in $(\mathbf{A}^k, \mathbf{I}^k)$ satisfying properties 1 and 2 above. Further, the start points of these blue bubbles have, with probability 1, a density at least $\delta/2$.*

5.1. *Unbounded service times.* We specialize to service times whose support is unbounded and deal with bounded service times in the next section. By the independence of $\mathbf{I}^k$ from the service process, the i.i.d. nature and unbounded support of the service times $\{S(n,k), n \in \mathbb{Z}\}$, we obtain for the nth arrival of the process $\mathbf{I}^k$

$$P\left(S(i,k) > I(i,k) \text{ for all } i \in \left[n, n+1+\frac{2}{\delta}\right] \middle| \mathbf{I}^k\right) > 0 \qquad \text{a.s.} \tag{19}$$

Given that the above conditional probability is strictly positive for each arrival of $\mathbf{I}^k$, a small enough choice of γ gives the following lemma.

LEMMA 23. *If $d > 0$, then there exist strictly positive ε, δ and γ not depending on k such that there exist blue bubbles in $(\mathbf{A}^k, \mathbf{I}^k)$ with the following properties*:

(A) *their volumes are at least ε*;
(B) *there is a red bubble with volume at least ε contained in the interval $[l, l+2/\delta]$, where l is the start point of the blue bubble*; *and*
(C) *$P(S(i,k) > I(i,k)$ for all $i \in [l, l+1+2/\delta]|\mathbf{I}^k) > \gamma$ and whose start points have density at least $\delta/3$ a.s.*

We shall prove Theorem 1 after stating the following lemma.

LEMMA 24. *Let $l = b(n,k)$ be the start point of a blue bubble whose volume $B(n,k) \geq \varepsilon$, and let $r(m,k) > b(n,k)$, $m \geq n$, start a red bubble whose volume $R(m,k) \geq \varepsilon$. Further suppose that both bubbles are contained in the interval $[l, l+L]$ (i.e., $\mathcal{R}(m,k) \subset [l, l+L]$). If $S(i,k) > I(i,k)$ for all $i \in [l, l+L]$, then either $B(n,k+1) = 0$ or $R(m,k+1) = 0$.*

PROOF. From the proof of Lemma 18 we know that $b(n,k+1)$ equals either the start of a red bubble or the start of a blue bubble at stage $k+1$. First suppose that $b(n,k+1)$ equals the start of a red bubble. Then, by the procedure for determining $B(n,k+1)$, it follows that $B(n,k+1) = 0$.

Next suppose that $b(n,k+1)$ equals the start of a blue bubble. We claim $r(m,k+1) = b(n,k+1)$; hence $r(m,k+1)$ equals the start of a blue bubble which implies $R(m,k+1) = 0$. To establish the claim, first observe that $d^a(n,k+1) \geq 0$ for all $n \in [l, l+L]$ by (ii) of Lemma 1. Therefore, one of the following must be true: (1) $b(n,k+1) < l$ or (2) $b(n,k+1) > l+L$.

Under case (1), since $b(n,k+1) < b(n,k)$ [i.e., the start point of the blue bubble moved left to the start point, say $\tilde{b}(p,k+1)$, of a blue bubble at stage $k+1$] we are in Case (a) of the procedure for determining $b(n,k+1)$. Accordingly, $b(n,k) \in [\tilde{b}(p,k+1), \tilde{e}(p,k+1)]$, where $\tilde{e}(p,k+1)$ is as defined in the procedure for deciding $b(n,k+1)$. Since $d^a(n,k+1) \geq 0$ for all $n \in [l, l+L]$, it follows from the definition of $\tilde{e}(p,k+1)$ that $\tilde{e}(p,k+1) > l+L$ and in fact that

$d^a(n, k+1) = 0$ for all $n \in [l, l+L]$. Now, since $\mathcal{R}(m, k) \subset [l, l+L]$, it follows that $b(m+1, k) \leq \tilde{e}(p, k+1)$. Therefore, by case (g) for determining $r(m, k+1)$, we get that $r(m, k+1) = \tilde{b}(p, k+1) = b(n, k+1)$.

Under case (2), let $b(n, k+1) = \tilde{b}(p, k+1) > l + L$. Again since $\mathcal{R}(m, k) \subset [l, l+L]$, it follows that $b(m+1, k) \leq \tilde{b}(p, k+1)$. It follows from Lemma 17 that $b(m+1, k+1) \leq \tilde{b}(p, k+1)$. But, by the order-preservation of start points, it follows that $b(n, k+1) = r(m, k+1) = b(m+1, k+1) = \tilde{b}(p, k+1)$. $\square$

To complete the proof of Theorem 1, consider a blue bubble at stage k satisfying properties (A)–(C) of Lemma 23. Lemma 24 shows that there is a reduction in the sum of blue and red volumes for every such blue bubble by an amount at least ε. Since $d(k)$ equals half of the average of the sum of blue and red volumes per arrival, we have shown that $d(k) - d(k+1) \geq \frac{1}{2}\frac{\delta}{3}\gamma\varepsilon$ for every k. This contradiction proves Theorem 1 when the service times have unbounded support.

5.2. *Bounded service times.* In this section we show how the argument of the previous section can be extended to handle the case of bounded service times. Observe that the boundedness of service times affects only properties (C) of Lemma 23, properties (A) and (B) continue to hold since they do not depend on service times. The key observation is that although the lack of unbounded services may not guarantee the interaction of blue and red bubbles at a single stage, the i.i.d. nature of the services can be used to force the bubbles to interact over *several* stages as shown below.

Suppose that properties (A) and (B) of Lemma 23 hold. Forcing the cancellation of an ε amount of blue volume over several stages consists of two parts: (i) ensuring that the red bubble stays within the interval $[l, l+2/\delta]$, and (ii) forcing the blue bubble to move to the right of this interval. We shall show how each of these parts can be accomplished in turn.

Since services are nonconstant, there exist $0 \leq a < b$ such that $P(S(1,1) \leq a)\, P(S(1,1) \geq b) > 0$. Consider the event

$$F = \left\{ \sum_{i=l}^{l+1+2/\delta} I(i, k) < K(b-a) \right\}.$$

Since the average density of points belonging to $\mathbf{I}^k$ equals $1/\tau$, given ν_1 $(0 < \nu_1 < 1/\tau)$, we may choose K large enough that the density of customers l in $\mathbf{I}^k$ for whom F holds is bigger than $1/\tau - \nu_1$. Fix K so that the above is true and define the event

$$G_k = \left\{ S(i, k) \leq \min\{I(i, k), a\} \text{ for all } i \in \left[l+1, l+1+\frac{2}{\delta} \right] \right\}.$$

LEMMA 25. *Suppose G_k holds. If $d^w(L+1, k) < 0$ for some $L \in [l, l+2/\delta]$, then $d^a(i, k+1) \geq 0$ for all $i \in [l, L]$.*

REMARK. In words, the above lemma states that under the event G_k if there is any movement of red volume to the right at location L in the interval $[l, l+2/\delta]$ ($d^w(L+1,k) < 0$), there cannot be any blue volume left in $[l, L]$ ($d^a(i,k+1) \geq 0$ for all $i \in [l, L]$). That is, if any red volume moves to the right under G_k we may infer that the blue bubble of volume at least ε has been fully cancelled. The event G_k, therefore, ensures that red volume stays in $[l, l+2/\delta]$ so long as there is any blue volume in this interval to the *left of* (or, ahead of) the red bubble.

PROOF OF LEMMA 25. If $d^w(L+1,k) < 0$ then $W^I(L+1,k) > W^A(L+1,k) \geq 0$. But, if $W^I(L+1,k) > 0$ then since $I(i,k) \geq S(i,k)$ for all $i \in [l+1, L]$, it follows inductively from the recursion

$$W^I(i+1,k) = [W^I(i,k) + S(i,k) - I(i,k)]^+$$

that $W^I(i,k) > 0$ for all $i \in [l+1, L]$. Or, equally, that $I(i,k) - S(i,k) - W^I(i,k) < 0$ for all $i \in [l, L]$. Now, from the equation

$$I(i,k+1) = [I(i,k) - S(i,k) - W^I(i,k)]^+ + S(i,k+1)$$

we deduce that $I(i,k+1) = S(i+1,k)$ for all $i \in [l, L]$. Since it is always true that $A(i,k+1) \geq S(i+1,k)$ [see equation (4)], we get that $d^a(i,k+1) = A(i,k+1) - I(i,k+1) \geq 0$ for all $i \in [l, L]$. $\square$

COROLLARY 2. *Let* $G = \bigcap_{p=k}^{k+K} G_p$. *If, under* G, $d^w(L+1,p) < 0$ *for some* $L \in [l, l+2/\delta]$ *and* $p \in [k, k+K]$, *then* $d^a(i,p+1) \geq 0$ *for all* $i \in [l, L]$.

The event G ensures that the red bubble does not move out of the interval $[l, l+2/\delta]$ so long as there is blue volume in it during stages k through $k+K$ to the left of the red bubble.

We now consider the second part: ensuring the blue bubble is forced to the right and cancels the red volume. Toward this end consider the event

$$H = \{S(l,m) \geq b \text{ for all } m \in [k, k+K]\}.$$

On the event $G \cap H$ the service time of customer l is greater than the service times of all the subsequent $1+2/\delta$ customers by at least $b-a$ during stages k through $k+K$. Given the first-come-first-served nature of the service discipline, this implies customer l will be "slowed down" during stages k through $k+K$ allowing customers $l+1$ through $l+1+2/\delta$ to "catch up." Now the event F bounds the separation between customers l through $l+1+2/\delta$. Therefore, we are guaranteed that under $F \cap G \cap H$ customers l through $l+1+2/\delta$ will be served in one busy cycle at stage $k+K$. That is, the interdeparture times from stage $k+K$ for the I-process will all equal service times: $I(i,k+K+1) = S(i+1,k+K)$ for all $i \in [l, l+2/\delta]$. We establish the above formally in the following lemma.

LEMMA 26. *Assume the event $F \cap G \cap H$ holds. Then there exists a $K' \leq K$ such that $d^a(i, k + K' + 1) \geq 0$ for all $i \in [l, l + 2/\delta]$.*

PROOF. Recursively from (5) we obtain

$$\sum_{i=l}^{l+2/\delta} I(i, k+1) - W^I(l + 1 + 2/\delta, k)$$
$$= \sum_{i=l}^{l+2/\delta} I(i, k) - S(l, k) + S(l + 1 + 2/\delta, k) - W^I(l, k).$$

Setting $C_j = \sum_{i=l}^{l+2/\delta} I(i, j)$, the above becomes

$$C_{k+1} = C_k - S(l, k) + S(l + 1 + 2/\delta, k) - W^I(l, k) + W^I(l + 1 + 2/\delta, k).$$

Suppose that $W^I(l + 1 + 2/\delta, j) = 0$ for all $j \in [k, k + K]$. Then, applying the previous equation recursively, we get that $C_{k+K} \leq C_k - K(b - a)$. This implies $C_{k+K} < 0$ on the event $F \cap H$, which is a contradiction. Therefore, it must be that there is a $K' \leq K$ such that $W^I(l + 1 + 2/\delta, K') > 0$.

Now from $W^I(n + 1, j) = [W^I(n, j) + S(n, j) - I(n, j)]^+$ we get that $W^I(i, K') > 0$ for all $i \in [l+1, l+1+2/\delta]$, since $S(i, K') \leq I(i, K')$ for all such i (under the event G). This implies via (3) that $I(i, k+K'+1) = S(i+1, k+K')$ for all $i \in [l, l+2/\delta]$. Since $A(n, m+1) \geq S(n+1, m)$ for every n and m [from (4)], it follows that $d^a(i, k + K' + 1) \geq 0$ for all $i \in [l, l + 2/\delta]$. $\square$

Therefore, under the event $F \cap G \cap H$, we have ensured that during stages k through $k + K'$ (i) no red volume leaves $[l, l+2/\delta]$ (event G), and (ii) and only red volume remains in the interval $[l, l+2/\delta]$ at stage $k+K'$ (under event $F \cap G \cap H$). This implies one of the following must have occurred by stage $k + K'$: (i) the blue bubble of volume ε wiped out the red bubble and moved outside the interval, (ii) the blue bubble moved into and got cancelled by the red bubble or (iii) some red volume entered the interval from the left and cancelled the blue bubble. In all cases it follows that either a red or a blue volume of ε was cancelled between stages k and $k+K'$. (We omit a tedious argument, similar to the one in the proof of Lemma 24, that identifies the start points of the red and blue bubbles over multiple stages and infers the above volume loss.)

Continuing, since $\mathbf{I}^k$ is a fixed point, it must satisfy all properties of any departure process. In particular, departure processes stochastically dominate the service process: $D(n, k) \geq S(n + 1, k - 1)$ for all n and k. Hence $\mathbf{I}^k$ must stochastically dominate the services. From this and the independence of services from arrivals it follows that for each customer l the conditional probability of the event G given the process $\mathbf{I}^k$ is strictly positive. Therefore, a small enough choice of ν_2 ensures that the density of customers $l \in \mathbf{I}^k$ for which $P(F \cap G \cap H | \mathbf{I}^k) > \nu_2$ is at least $1/\tau - \nu_2$.

We record the above development in the following lemma, which is analogous to Lemma 23.

LEMMA 27. *If $d > 0$, there exist strictly positive ε, δ, ν and K not depending on k such that there exist blue bubbles in $(\mathbf{A}^k, \mathbf{I}^k)$ with the following properties*:

(A) *their volumes are at least ε*;
(B) *there is a red bubble with volume at least ε contained in the interval $[l, l + 2/\delta]$, where l is the start point of the blue bubble*; *and*
(C) *$P(F \cap G \cap H|\mathbf{I}^k) > \nu$ and whose start points have density at least $\delta/3$ a.s.*

To conclude, we have shown under the hypotheses of Lemma 27 that $d(k) - d(k+K) \geq \frac{1}{2}\frac{\delta}{3}\varepsilon\nu$ for every k. This proves Theorem 1 for bounded services as well.

5.3. *Corollaries.*

DEFINITION 1. The $\bar{\rho}$ distance between two stationary and ergodic sequences $X = \{X_n, n \in \mathbb{Z}\}$ and $Y = \{Y_n, n \in \mathbb{Z}\}$ of mean τ is given by

$$\bar{\rho}(X, Y) = \inf_{\gamma} E_{\gamma}|\hat{X}_1 - \hat{Y}_1|,$$

where γ is a distribution on $M_e^\tau \times M_e^\tau$—the space of jointly stationary and ergodic sequences $(\hat{X}, \hat{Y})$, with marginals $\hat{X}_1$ and $\hat{Y}_1$ distributed as X_1 and Y_1. (See, e.g., [5] or [8], Definition 2.3, for further details of the $\bar{\rho}$ metric.)

Chang [5] has shown that the $\cdot/GI/1$ queue is a contraction in the $\bar{\rho}$ distance. That is, if $\mathbf{A}^1$ and $\mathbf{I}^1$ are two ergodic inputs to a $\cdot/GI/1$ queue with corresponding outputs equal to $\mathbf{A}^2$ and $\mathbf{I}^2$, then $\bar{\rho}(\mathbf{A}^2, \mathbf{I}^2) \leq \bar{\rho}(\mathbf{A}^1, \mathbf{I}^1)$. He also showed that this inequality is strict when the service times have unbounded support.

For each k let μ^k be the joint distribution of the processes $(\mathbf{A}^k, \mathbf{I}^k)$. Choosing $\mathbf{A}^1$ and $\mathbf{I}^1$ to be independent as in the previous sections, μ_1 equals the product measure—clearly a member of $M_e^\tau \times M_e^\tau$. The translation invariant nature of the queueing operation preserves joint ergodicity, implying $\mu^k \in M_e^\tau \times M_e^\tau$ for every k.

COROLLARY 3. *$\bar{\rho}(\mathbf{A}^k, \mathbf{I}^k) \to 0$ as $k \to \infty$.*

PROOF. Now $2d(k) = E_{\mu^k}|A(1,k) - I(1,k)|$. Therefore

$$\begin{aligned} 2\bar{\rho}(\mathbf{A}^k, \mathbf{I}^k) &= \inf_{\gamma} E_{\gamma}|\hat{A}(1,k) - \hat{I}(1,k)| \\ &\leq E_{\mu^k}|A(1,k) - I(1,k)| \\ &= 2d(k) \overset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

This proves the corollary. $\square$

COROLLARY 4. *If a stationary and ergodic fixed point exists at mean τ, then it is unique.*

COROLLARY 5. *Suppose $\mathbf{I}_1$ and $\mathbf{I}_2$ are two stationary and ergodic fixed points for a $\cdot/GI/1$ queue at means τ_1 and τ_2, respectively. If $\tau_1 < \tau_2$, then $\mathbf{I}_2$ stochastically dominates $\mathbf{I}_1$; that is, there exists a joint distribution γ of $\mathbf{I}_1$ and $\mathbf{I}_2$ such that under γ, $\mathbf{I}_1(n) \le \mathbf{I}_2(n)$ for every n a.s.*

PROOF. Let $\mathbf{F}^1 = \{F(n,1), n \in \mathbb{Z}\}$ be distributed as $\mathbf{I}_2$ and define $\mathbf{A}^1 = \{A(n,1), n \in \mathbb{Z}\}$, where $A(n,1) = (\tau_1/\tau_2)F(n,1)$, to be another ergodic arrival process of mean τ_1. Pass $\mathbf{F}^1$ and $\mathbf{A}^1$ through a tandem of $\cdot/GI/1$ queues giving the nth customers of both processes the same service time, $S(n,k)$, at each stage k. Since $A(n,1) \le F(n,1)$ for all n, we get [from (3) and (4)] that $A(n,k) \le F(n,k)$ for all n and k. Let γ_k be the joint distribution of $(\mathbf{A}^k, \mathbf{F}^k)$. By Theorem 1, $(\mathbf{A}^k, \mathbf{F}^k)$ converges to $(\mathbf{A}^\infty, \mathbf{F}^\infty) \stackrel{d}{=} (\mathbf{I}_1, \mathbf{I}_2)$ in distribution. Let γ_∞ be the joint distribution of $(\mathbf{A}^\infty, \mathbf{F}^\infty)$ and note that $\mathbf{I}_2$ stochastically dominates $\mathbf{I}_1$ under γ_∞. To finish, set $\gamma = \gamma_\infty$. $\square$

6. Conclusions and related work. Assuming the existence of ergodic fixed points for a first-come-first-served $\cdot/GI/1$ queue, we have used coupling arguments to show that they are attractors. As a consequence we have also seen that an ergodic fixed point at mean $\tau > 1$ is unique. We note that while the arguments do not place any restrictions on the service time distribution (other than that it have a finite mean and be nonconstant), they rely crucially on the first-come-first-served nature of the service discipline.

Earlier work on the uniqueness and attractiveness of fixed points for various queues can be found in [1, 5], Section 9.4 of [6], [12–15]. While this list of papers is not exhaustive, combined with the references contained in them, they give a fuller picture of earlier work. The references [5] and [12] are particularly relevant for the present paper.

The existence of ergodic fixed points at some rates $1/\tau < 1$ has been established by Mairesse and Prabhakar [10] assuming that the service times satisfy $\int P(S(0,0) \ge u)^{1/2}\, du < \infty$. We refer the reader to [10] for precise details and for a statement of further work. The result of [10] relies on the work of Baccelli, Borovkov and Mairesse [2], Glynn and Whitt [7] and Martin [11] concerning the limiting behavior of waiting times in a tandem of $\cdot/GI/1$ queues with arbitrary input processes.

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