The Annals of Probability
1996, Vol. 24, No. 3, 1647–1652

BOOK REVIEW

MARK A. PINSKY, *Lectures on Random Evolutions*. World Scientific, Singapore, 1991, 150 pages, \$36.00.

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In mathematical terms, an *evolution* is extraordinarily simple to state:

$$\dot{x} = \phi(x). \tag{1}$$

In the absence of any context for (1), little of value can be added. Any situation in which ϕ is random can make a rightful claim for being called a *random evolution*. As before, unless we have some additional structure, little in the way of analysis can be performed.

On the other hand, large areas in the theory of probability and random processes begin with a suitable form and interpretation of ϕ. Perhaps the most well known choice is

$$\phi(x, \dot{w}) = b(x) + \sigma(x)\dot{w}. \tag{2}$$

Here, x and $b(x)$ are vectors of the same dimension, say n, and $\dot{w}$ is a vector perhaps of a second dimension, say d. Finally, $\sigma(x)$ is an $n \times d$ matrix. We begin the discussion of stochastic differential equations by inserting a d-dimensional white noise for $\dot{w}$. Nowadays, we study two interpretations of (2): the Itô formulation and the Stratonovich formulation. Excellent books from all four corners of the globe have been written on stochastic differential equations.

A second choice for ϕ is $\phi_t(x_t) = A_t x_t$. In this situation, the index t for the random process is frequently the non-negative integers, and the dot is taken to be the difference

$$\dot{x}_t = x_{t+1} - x_t. \tag{3}$$

Thus, we are studying the product of random square matrices

$$x_{t+1} = (A_t + I) \cdots (A_0 + I) x_0. \tag{4}$$

For the case that $\{A_t; t \in \mathbf{N}\}$ forms a stationary sequence of random matrices, this line of investigation has lead us to an ergodic theory of matrix products and, in particular, to the Oseledec multiplicative ergodic theorem with its companion Lyapunov exponents.

The probabilistic techniques used in the development of a theory of stochastic differential equations and a theory of random matrix products are quite distinct. However, these two topics cover common ground in their

Received March 1996.

respective generalizations to stochastic flows and to compositions of random transformations.

When Griego and Hersh (1969) introduced the term random evolution, they placed the subject in a context similar to control theory by making the choice of ϕ having the form $\phi(x, v)$. (This, of course, did not preclude the development of the theory of stochastic control.) Unlike control theory, the process v is not selected to control the process x with a view toward optimizing some functional of x. However, by the structure of the evolution equation, the effect of the *random* process v is to control the behavior of x. Consequently, we sometimes refer to v as the *driving process* and to x as the *driven process*.

A simple example of a random evolution results by taking $\phi(x, v) = v$ and letting v to be a two-state Markov chain with state space $\{v_1, v_2\} \subset \mathbf{R}$. The chain remains in either of its two states for an exponential length of time, parameter λ. This example was introduced by Goldstein (1951) and popularized by Kac (1956), who used this telegraph process to give a stochastic representation formula for the solution to the telegraph equation.

Beginning in Chapter 0 with this example, Mark Pinsky presents his rather personal view on the development of the subject. Because the author has made regular substantial contributions to the subject for more than two decades, this point of view is a privileged one. The aim of Chapter 0 is a description of the telegraph process by explicit computation. Pinsky's methodology is largely analytical, using the Fourier and Laplace transforms and techniques from differential equations and linear algebra. For example, he takes the Fourier transform of the telegraph equation to find explicit expressions for the distribution function of (x_t, v_t) in terms of Bessel functions. He computes the Laplace transform of the distribution of the first exit time from a set. If v is a Markov process having generator Q_v, then the pair (x, v) is also a Markov process having generator

$$L = \phi(x, v) \cdot \nabla_x + Q_v. \tag{5}$$

With this motivation, Pinsky makes use of the techniques available from the theory of analytical semigroups by using, for example, the properties of the resolvent to verify the semigroup property of the telegraph process.

When the driving process v is fluctuating rapidly compared to the fluctuations of x, the emphasis moves to a study of the asymptotic properties of the random evolution. For the analog to the law of large numbers, we consider

$$\dot{x}_t^{\varepsilon} = \phi\big(x_t^{\varepsilon}, v_{t/\epsilon}\big) \tag{6}$$

in the limit as $\varepsilon \to 0$. If the appropriate centering conditions hold, the analog to the central limit theorem uses a more rapid time scaling for the driving process:

$$\dot{x}_t^{\varepsilon} = \frac{1}{\varepsilon}\phi\big(x_t^{\varepsilon}, v_{t/\varepsilon^2}\big). \tag{7}$$

Pinsky establishes these limit theorems for the telegraph process in Theorem 0.3.1 by verifying a Feynman–Kac formula and taking limits for matrix eigenvalues obtained by explicit computation.

In Chapter 1, we consider the case in which v is a finite-state irreducible Markov chain, ϕ is a function of v and $x_0 = 0$. Thus, x is a continuous additive functional of v,

$$x_t = \int_0^t \phi(v_s)\, ds. \tag{8}$$

The techniques and aims of Chapter 1 are similar to those in Chapter 0. Pinsky adds a renewal argument to the Laplace transform calculation to establish the Kolmogorov forward equation. The asymptotic theory is now based on *estimates* of the eigenvalue and eigenvector for the matrix $Q + \xi\phi$ in a neighborhood of $\xi = 0$.

The proof of the central limit theorem alludes to a method involving perturbed test functions that will be more completely described in Chapter 4. To introduce this method, note that if L^ε, $\varepsilon > 0$, and L generate Markov processes and $L^\varepsilon f \to Lf$ suitably as $\varepsilon \to 0$ for a sufficiently large class of test functions f, then we can conclude that the semigroups associated to L^ε converge to the semigroup associated to L. Consequently, the finite-dimensional distributions of the corresponding Markov processes converge.

This method has little chance of success in the case of random evolutions because L^ε is an operator on functions of x and v, whereas the desired generator L is frequently an operator on functions of x only. The effects of the Markov process v appear in L via some of its averaging properties. We try to achieve the same result by choosing perturbations f^ε of f so that $f^\varepsilon \to f$ and $L^\varepsilon f^\varepsilon$ converge as $\varepsilon \to 0$. For the situation described in (7), the generator of the Markov process $(x_t^\varepsilon, v_{t/\varepsilon^2})$ is

$$L^\varepsilon = \frac{1}{\varepsilon}\phi(x,v)\cdot\nabla_x + \frac{1}{\varepsilon^2}Q_v. \tag{9}$$

If we choose

$$f^\varepsilon(x,v) = f(x) + \varepsilon h(x,v)\cdot\nabla_x f(x), \tag{10}$$

then

$$L^\varepsilon f^\varepsilon(x,v) = \phi(x,v)\cdot\nabla_x\big(h(x,v)\cdot\nabla_x f(x)\big) \tag{11}$$

provided that

$$Q_v h(x,v) = -\phi(x,v). \tag{12}$$

For the situation in Chapter 1, ϕ and hence h are functions of v alone. Thus

$$L^\varepsilon f^\varepsilon(x,v) = \phi(v)\cdot h(v)\frac{\partial^2}{\partial x^2}f(x). \tag{13}$$

The process v is a finite-state irreducible Markov process and hence has a unique stationary distribution π. If $\int\phi(v)\pi(dv) = 0$, then (12) has a solution. Call the solution operator H. Now

$$L^{\varepsilon}f^{\varepsilon}(x,v) = \phi(v)\cdot H\phi(v)\frac{\partial^2}{\partial x^2}f(x). \tag{14}$$

The final step involves justifying the averaging over the stationary distribution π to obtain

$$Lf(x) = \left(\int \phi(v)\cdot H\phi(v)\pi(dv)\right)\frac{\partial^2}{\partial x^2}f(x). \tag{15}$$

Thus the limit process is a Brownian motion without drift.

For the symmetric telegraph process $v_2 = -v_1$. Take $h(v) = v/2\lambda$ to obtain $Qh(v) = -v$. Then

$$L^{\varepsilon}f^{\varepsilon}(x,v) = v\cdot\frac{1}{2\lambda}v\frac{\partial^2}{\partial x^2}f(x) = \frac{v_1^2}{2\lambda}\frac{\partial^2}{\partial x^2}f(x). \tag{16}$$

The chapter concludes with a criterion for recurrence of x_t and a generalization to additive functionals ϕ having jumps.

The topic of Chapter 2 is general random evolutions. The ingredients are a finite-state irreducible Markov chain v on $\{1,2,\dots,N\}$, and a set of generators $\{A_1, A_2,\dots, A_N\}$ with their corresponding semigroups $\{T_1,T_2,\dots,T_N\}$. The random evolution equation under investigation is

$$\dot{x}_t = A_{v_t}x_t. \tag{17}$$

To set the stage for this abstract setting, the author begins the chapter with a brief, but general, introduction on semigroups of operators.

Let $\tau_1 < \tau_2 < \cdots$ be the jump times of v and let $N(t) = \max\{n\colon \tau_n \le t\}$. Pinsky bases his analysis on the multiplicative operator functional

$$\begin{aligned} M(s,t] &= T_{v(\tau_{N(t)})}(t-\tau_{N(t)})\cdots T_{v(\tau_{N(s)})}(\tau_{N(s)+1}-\tau_{N(s)}) \\ &\quad\times T_{v(\tau_{N(s)-1})}(\tau_{N(s)}-s). \end{aligned} \tag{18}$$

(Note that the order of the products is reversed here from the convention chosen in the lecture notes.)

The proof of the limit theorems in this general context uses results on the convergence of resolvents. The chapter also includes the case of discontinuous random evolutions and places the concept of a multiplicative operator functional in the context of the martingale problem.

In moving from the two-state velocity model to the operator based random evolution, the author concludes his general investigation on random evolutions. For a more extensive treatment on Markov processes and the convergence of random processes, see the book by Ethier and Kurtz (1986). For more on random evolutions, consult the monographs and papers of the Ukrainian school. The book by Koroliuk and Svishchuk (1995) is a good place to start.

The lecture notes conclude with three chapters that serve as introductions to applications of the ideas in random evolutions—the linear Boltzmann equation, isotropic transport on manifolds and the stability analysis of a linear system of stochastic equations with a jump Markov process serving as the driving process. Because the purpose has shifted from instruction to overview, these latter chapters are less self-contained. Mark Pinsky has made significant contributions to each of the areas.

The linearized Boltzmann equation has the form

$$\frac{\partial h}{\partial t} + v \cdot \frac{\partial h}{\partial x} = \nu(v) \int_{\mathbf{R}^3} [k(v, \eta) h(t, x, \eta) - h(t, x, v)] \rho(d\eta). \tag{19}$$

Let v be a jump Markov process with generator

$$Lf(v) = \nu(v) \int_{\mathbf{R}^3} (f(\eta) - f(v)) \rho(d\eta) \tag{20}$$

and define the multiplicative operator functional

$$M(s, t] = \prod_{s < u \le t} k(v_{u-}, v_{u+}). \tag{21}$$

Chapter 3 develops the apparatus to prove the stochastic representation formula

$$h(t, x, v) = E\left[M(0, t]\, h\left(0, x - \int_0^t v_s\, ds, v_t\right) \middle| v_0 = v\right] \tag{22}$$

for the solution to (19) and concludes with an asymptotic analysis of the linearized Boltzmann equation.

Chapter 4 begins with the Rayleigh model of random flight. In this Markov model, the author chooses a complete Riemannian manifold for the state space of the driven process. The driving process is a jump Markov process. The time between jumps is exponential with parameter 1. The state between jumps is a direction chosen according to a rotationally invariant distribution on the space of unit vectors. The driven process is motion at speed 1 along the geodesic determined by the driving process. Under central limit theorem scaling, Pinsky shows, using the perturbed test function method, that the resolvents of these random flight models converge to the resolvents of Brownian motion on the manifold. Borrowing from a Lyapunov method, he then goes on to give two results on the recurrence properties of this limit process determined by a condition on the curvature of the manifold.

The concluding chapter focuses on the detailed analysis of the two-dimensional system,

$$\dot{x}_t = Ax_t + \varepsilon F(v_t) Bx_t, \tag{23}$$

where v is a finite irreducible Markov chain on $\{1, 2, \dots, N\}$ having stationary distribution π. The "noise" is centered, that is, $\int F(v)\pi(dv) = 0$ and the noise term parameter ε is small.

Pinsky notes that the small noise analysis in (23) can be separated into three cases. The harmonic oscillator is the case in which the 2×2 matrix A has complex conjugate eigenvalues. The free particle and saddle point models correspond to cases in which A has real eigenvalues which are, respectively, repeated and distinct.

Call the radial process ρ and the angular process θ. Pinsky's principle aim is the determination of the Lyapunov exponent

$$\lim_{t \uparrow \infty} \frac{\rho(t)}{t}. \tag{24}$$

Because the equation is linear, the pair (θ, v) is a Markov process. For determining the Lyapunov exponent, this property suggests a trial function of the form $f(\rho, \theta, v) = \rho + \hat{f}(\theta, v)$. The analysis proceeds with expansion of $\hat{f}$ in powers of ε. A comparison is then made of the Lyapunov exponent of this system and an analogous system stated as a linear stochastic differential equation in Stratonovich form.

The case of a free particle is a nilpotent system. A scaling argument shows that the appropriate power series expansion for the Lyapunov exponent is in powers of $\varepsilon^{2/3}$. In this case, Pinsky uses the analogous stochastic differential equation to approximate the Lyapunov exponent in (23).

Overall, the lectures notes provide a good introduction to the subject of random evolutions. I was able to read each of the six lectures in a single sitting and gained from this reading a historical perspective on the development of stochastic analysis. Unfortunately, the notes suffer from having a few too many typos and confusing choices of notation. Most of these glitches are easy to fix. However, they are likely to be frustrating for anyone new to the subject. My personal choice for the notes would be to have them serve as the basis of a reading course with a graduate student. The book repeatedly entices the student to explore the many areas of analysis and probability that have had an impact on the theory of random evolutions.

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