The Annals of Probability
1996, Vol. 24, No. 3, 1643–1646

BOOK REVIEW

D. W. Stroock, *Probability Theory. An Analytic View*. Cambridge University Press, 1993.

Review by A. de Acosta

Case Western Reserve University

"In fact, I myself enjoy probability theory best when it is inextricably interwoven with other branches of mathematics and not when it is presented as an entity unto itself." True to this statement in the preface, Stroock has written a very interesting, distinctly personal volume on probability theory with a strong analytic flavor. The book covers many standard topics in a first graduate course on probability as well as a number of more advanced probabilistic subjects, but also contains several deep excursions into pure analysis. Broadly speaking, the analytical results presented in the book either have significant probabilistic ingredients in their proofs or exhibit illuminating analogies to certain probabilistic results.

Here is a brief description of the book's contents. Chapters I and II deal with limit theorems for sums of independent random variables. Weak convergence, Wiener measure and Lévy processes are presented in Chapter III. Chapter IV is devoted to a deeper study of Wiener measure. Chapters V and VI present discrete parameter martingales and various applications. Continuous martingales are discussed in Chapter VII. Finally, Chapter VIII is an introduction to classical potential theory.

We will now try to highlight some of the novel features of the book: proofs or approaches that appear to be new or at least not well known and topics not usually included in books of comparable character.

Cramér's theorem on large deviations, with an emphasis on inequalities (rather than just logarithmic asymptotics), is proved in Chapter I. Then it is used to provide the tail estimates needed to prove the Hartman–Wintner law of the iterated logarithm for bounded random variables. The general finite variance case is reduced to the bounded case using ideas originating in the law of the iterated logarithm in Banach spaces (see Ledoux and Talagrand [4]; for another contemporary elementary proof, see Griffin and Kuelbs [3]).

In Chapter II the Berry–Esséen theorem is proved using Bolthausen's adaptation of Stein's method [5], rather than the classical Fourier-analytic technique. Section 2.3 contains the first analytical excursion. The central limit theorem is used to prove a criterion for the hypercontractivity of Hermite multipliers [if μ is a probability measure and $1 \le p \le q$, then a linear operator of norm 1 from $L^p(\mu)$ into $L^q(\mu)$ is said to be hypercontrac-

Received March 1996.

tive], and Nelson's hypercontractivity theorem and Beckner's sharp form of the Hausdorff–Young inequality for Fourier transforms are deduced.

In Chapter III the existence of a Lévy process (independent stationary increments and cadlag paths) associated to a generalized Poisson measure is proved by a direct constructive argument (for a different approach to the Lévy–Khinchine representation which is valid for infinitely divisible laws in Banach spaces, see de Acosta, Araujo and Gine [2]). The author emphasizes the role of a real variable inequality of Garsia, Rodemich and Rumsey in the tightness argument required for the simultaneous proof of the existence of Wiener measure on $C([0,\infty),\mathbf{R}^N)$ and of Donsker's invariance principle.

There is a nice heuristic presentation of Wiener measure in Chapter IV as the probability measure on $C([0,\infty),\mathbf{R}^N)$ "with a density whose value at $f \in C([0,\infty),\mathbf{R}^N)$ is

$$C\exp\left\{-\tfrac{1}{2}\int_{[0,\infty)}(f'(t))^2\,dt\right\}."$$

Then Wiener measure is discussed in the context of the abstract Wiener spaces introduced by L. Gross. The Cameron–Martin translation formula and the Feynman–Kac formula are proved.

The discussion of martingales in Chapter V covers vector-valued martingales and μ-martingales, where μ is a σ-finite measure. μ-martingales are used to prove the celebrated Hardy–Littlewood inequality for the maximal function of an L^1 function. Then the maximal function is used to derive another major analytical result, the Calderon–Zygmund decomposition of an $L^1(\mathbf{R}^N)$ function. The author emphasizes the analogy between the role of Doob's maximal inequality in martingale theory and that of the Hardy–Littlewood maximal inequality in real analysis.

The proof of Birkhoff's ergodic theorem in Section 6.1 (for σ-finite measures and for both discrete and continuous semigroups of measure-preserving transformations) makes use of Hardy's inequality (a consequence of the Hardy–Littlewood inequality). Section 6.2 is another deep analytical excursion. The tools developed in Chapter V are used to prove continuity properties of Riesz transforms, which are then applied to study the relation between the L^p norm of a function and that of related square functions. The section includes a proof of the Marcinkiewicz interpolation theorem. Section 6.3 contains two proofs of Burkholder's inequality comparing the L^p norm of a martingale with that of its square function. The first proof is essentially the original proof and the author emphasizes the analogy with the derivation of the analytical square function inequalities; the second proof is Burkholder's recent refinement (see [1]) in which the norm in the comparison is the (dimension-free) best constant and which holds for Hilbert-valued martingales.

The exponential martingale associated to Brownian motion in $\mathbf{R}^N$ is introduced early in Chapter VII and it is shown that it generates many other martingales by using the Fourier inversion formula. Wiener measure on $C([0,\infty),\mathbf{R}^N)$ is characterized as the unique solution of a martingale problem, and Lévy's characterization of Brownian motion is proved. Sections 7.3–7.5 are devoted to the study of the class of diffusions obtained by transforming Wiener measure via the deterministic map $\psi \to X^b(\cdot, \psi)$ [$\psi \in C([0,\infty), \mathbf{R}^N)$] defined by the integral equation

$$X^b(t, \psi) = \psi(t) + \int_0^t b\big(X^b(s, \psi)\big)\, ds,$$

where $b\colon \mathbf{R}^N \to \mathbf{R}^N$ is a (smooth) vector field; that is, the class of diffusions with generators $L^b = (1/2)\Delta + b \cdot \nabla$ (a condition precluding explosions in finite time is imposed). Recurrence, transience and ergodic properties are studied. In the last section the drift b is taken to be a gradient field and then an invariant measure can be described explicitly.

The book closes with a celebrated theme, "the miraculous connections between Wiener paths and classical potential theory" (page xiii). Brownian motion is used to represent the solution to several boundary-value problems for the operator $(1/2)\Delta$: the Cauchy initial value problem for the heat equation with Dirichlet data, the Dirichlet problem and the Poisson problem. The subject of the concluding section is Green potentials and capacity.

The book is well organized and very carefully written and the details of proofs are spelled out. There are a large number of substantial exercises; many are extensions of the theory or even fundamental results (for example, Exercise 3.1.18 is Kolmogorov's consistency theorem and Exercise 3.1.19 is Lévy's continuity theorem for characteristic functions). Some standard topics in a graduate course on probability theory are omitted: for example, the Glivenko–Cantelli theorem, Kolmogorov's three series theorem, general random walks, triangular arrays, Poisson convergence, stable laws as well as a fuller discussion of characteristic functions. Although the notation is very precise, the reader may occasionally be puzzled by the formulation of certain results: for example, since conditional expectation is defined in Chapter V and Brownian motion (the process) is not introduced until Chapter VII, the Markov and strong Markov properties are stated in Chapter IV in terms of double integrals with respect to Wiener measure, a formulation which, though of course technically correct, appears to lack the clarity and intuitive appeal of the (mathematically equivalent) standard formulation (which emerges to an extent in Exercises 5.1.30 and 5.1.32).

A prospective reader must possess solid knowledge of (abstract) measure and integration and may sometimes need to consult other sources for certain topics in analysis. The book may be used as a text for an analytically oriented graduate course on probability theory or as a supplement to other more purely probabilistic expositions (the author suggests in the Preface several possible plans for using the book; also, a table of dependence is provided). It is also an excellent reference for many topics, both in the text and in the exercises.

This is a rich and demanding book with a definite point of view. It will be of great value for students of probability theory, analysts with an interest in the subject, and professional probabilists.

REFERENCES

[1] BURKHOLDER, D. (1991). *Explorations in Martingale Theory and Its Applications*. *Lecture Notes in Math*. **1464** 1–66. Springer, New York.

[2] DE ACOSTA, A., ARAUJO, A. and GINE, E. (1978). On Poisson measures, Gaussian measures and the general central limit theorem in Banach spaces. *Adv. in Probab. Related Topics* **4** 1–68. Dekker, New York.

[3] GRIFFIN, P. and KUELBS, J. (1991). Some extensions of the LIL via self-normalizations. *Ann. Probab*. **19** 380–395.

[4] LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces*. Springer, New York.

[5] STEIN, C. (1986). *Approximate Computation of Expectations*. Institute of Mathematical Statistics, Hayward, CA.

DEPARTMENT OF MATHEMATICS
CASE WESTERN RESERVE UNIVERSITY
10900 EUCLID AVENUE
CLEVELAND, OHIO 44106-7058