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DISTANCE FLUCTUATIONS AND LYAPOUNOV EXPONENTS

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We associate certain translation invariant random metrics on $\mathbb{R}^d$ to Brownian motion evolving in a truncated Poissonian potential. These metrics behave over large distances, in an appropriate sense, like certain deterministic norms (the so-called Lyapounov exponents). We prove here upper bounds on the size of fluctuations of the metrics around their mean. Under an additional assumption of rotational invariance, we also derive upper bounds on the difference between the mean of the metrics and the Lyapounov norms.

Introduction. We consider in this article a Brownian motion in dimension $d \geq 1$, evolving in a truncated Poissonian potential and conditioned to reach a remote location. This conditioned Brownian motion "feels the presence of soft Poissonian obstacles" and can be viewed as a type of polymer in a random environment. Our purpose here is to study the fluctuation properties of certain naturally defined random distance functions. These metrics roughly describe the cost attached to connecting two points of $\mathbb{R}^d$ for this polymer in a random environment.

For $x \in \mathbb{R}^d$, we denote by P_x the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting from x and let $\mathbb{P}$ stand for the Poisson law with constant intensity $\nu > 0$ on the space Ω of simple pure point measures on $\mathbb{R}^d$. Our soft obstacles are modelled on a shape function $W(\cdot) \geq 0$, which is assumed to be bounded, measurable, compactly supported and not a.e. equal to 0. The truncated potential is then defined as

$$V(x, \omega) = \Big(\sum_i W(x - x_i)\Big) \wedge M = \Big(\int_{\mathbb{R}^d} W(x-y)\,\omega(dy)\Big) \wedge M \tag{I.1}$$

for $x \in \mathbb{R}^d$ and $\omega = \sum_i \delta_{x_i} \in \Omega$. The positive constant M determines the truncation level.

Central objects of interest in the present work are the nonnegative functions

$$d_\lambda(x, y, \omega) = \max\big(a_\lambda(x, y, \omega), a_\lambda(y, x, \omega)\big) \tag{I.2}$$

for $x, y \in \mathbb{R}^d$, $\lambda \geq 0$ and $\omega \in \Omega$, where

$$a_\lambda(x, y, \omega) = -\inf_{\overline{B}(x,1)} \log e_\lambda(\cdot, y, \omega) \tag{I.3}$$

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and

$$\text{(I.4)}\quad e_\lambda(x,y,\omega) = E_x\left[\exp\left\{-\int_0^{H(y)}(\lambda+V)(Z_s,\omega)\,ds\right\},\, H(y)<\infty\right].$$

Here $Z_.$ stands for the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$ and $H(y)$ stands for the entrance time of $Z_.$ in $\overline{B}(y,1)$, the closed ball of radius 1 around y. It was observed in [12] that the nonnegative functions $a_\lambda(\cdot,\cdot,\omega)$ satisfy the triangle inequality. In fact, under mild assumptions [see (1.7) below], the $d_\lambda(\cdot,\cdot,\omega)$ are distance functions on $\mathbb{R}^d$.

From our results in [12] and as is recalled in Section 1, there are norms $\alpha_\lambda(\cdot)$ on $\mathbb{R}^d$ for which

$$\text{(I.5)}\qquad \mathbb{P}\text{-a.s.}\quad d_\lambda(0,y,\omega)\sim \alpha_\lambda(y)\quad \text{as } y\to\infty.$$

These norms are the so-called Lyapounov exponents which govern the $\mathbb{P}$-almost sure exponential directional decay of $e_\lambda(0,\cdot,\omega)$ and of the λ-Green function $[-\frac{1}{2}\Delta+\lambda+\mathrm{V}(\cdot,\omega)]^{-1}(0,\cdot)$.

The model we study here has very much the flavour of models of first passage percolation (see [7]) or of directed polymers (see [3] and [9]). For instance the function $d_\lambda(\cdot,\cdot,\omega)$ should be viewed as natural analogues of the point to point passage times and the Lyapounov coefficients of the directional time constants. Both in the case of first passage percolation and directed polymers, one does not know too much about the analogues of the Lyapounov coefficients. In several instances one makes assumptions on the curvature of the unit spheres of the corresponding norms (see [11] and [10]) which usually cannot be checked directly. One possible interesting feature of the random polymer model studied here is that the $\alpha_\lambda(\cdot)$ are proportional to the Euclidean norm when $W(\cdot)$ is invariant under rotation, and one has of course a good control over the unit ball or sphere for the $\alpha_\lambda(\cdot)$ norm.

Our purpose here is to derive upper bounds on the size fluctuations of $d_\lambda(0, y, \omega)$ around its mean (which is finite; see Section 1),

$$\text{(I.6)}\qquad D_\lambda(0,y)=\mathbb{E}[d_\lambda(0,y,\omega)],\qquad y\in\mathbb{R}^d,$$

for $W(\cdot)$ as above, and on the difference $D_\lambda(0,y)-\alpha_\lambda(y)$ for rotationally invariant $W(\cdot)$. These bounds are not expected to capture the true size of these fluctuations (for instance when $d=2$, $\lambda>0$, the considerations developed in Krug and Spohn [9] would lead us to expect fluctuations of order $|y|^{1/3}$). Nevertheless this type of bounds can be very very useful (see [11] and [10]).

Let us now describe how the present article is organized. Section 1 recalls certain useful facts from [12] and [13] and develops suitable estimates on the random metrics. The role of the truncation parameter M in (I.1) is to simplify things by providing certain uniform Harnack inequalities.

Section 2 studies the fluctuations of $d_\lambda(0, y, \omega)$ around $D_\lambda(0, y)$. The general approach is similar to Kesten [8]. That is, we use the martingale method and derive some exponential estimates on the distribution of $d_\lambda(0, y, \omega) - D_\lambda(0, y)$ under $\mathbb{P}$. Our main results are Theorem 2.1 and Corollary 2.4.

In Section 3 we assume $W(\cdot)$ is rotationally invariant. Theorem 3.1 and Corollary 3.4 provide bounds on the difference $D_\lambda(0, y) - \alpha_\lambda(y)$. As follows from a straightforward subadditivity argument $D_\lambda(0, y) \geq \alpha_\lambda(y)$, so that our main concern is the derivation of lower bounds for $\alpha_\lambda(y)$ in terms of $D_\lambda(0, y)$. The general scheme proposed in Alexander [2] does not seem to be easily applicable here. Our line of approach is more in the spirit of Alexander [1]. We construct certain approximately submultiplicative quantities $g_\beta(m)$, $m \geq 1$ [see (3.10)] based on moments of small order $\beta \in (0, 1)$ of $e_\lambda(0, \cdot, \omega)$. In contrast to [1], the proof bypasses the Van den Berg–Kesten inequality and uses instead a "splitting technique," which is given in Lemma 3.3. The approximate submultiplicative property forces lower estimates [see (3.33) and (3.34)] on the norm $\alpha_\lambda(\cdot)$ in terms of $-(1/m)\log g_\beta(m)$. On the other hand, the exponential estimates of Section 2 are used to relate $g_\beta(\cdot)$ and $D_\lambda(0, \cdot)$. The combination of these arguments produces the adequate lower bounds for $\alpha_\lambda(\cdot)$.

1. Setting and preliminary estimates. We first introduce some notation. We denote by $a = a(W) > 0$ the smallest possible a such that $W(\cdot) = 0$ on $\overline{B}(0, a)^c$. For a closed subset A of $\mathbb{R}^d$, H_A stands for the entrance time of $Z_\cdot$ in A,

$$H_A = \inf\{s \geq 0, Z_s \in A\}, \tag{1.1}$$

and for an open subset U of $\mathbb{R}^d$, T_U stands for the exit time from U,

$$T_U = \inf\{s \geq 0, Z_s \notin U\}. \tag{1.2}$$

When $z \in \mathbb{R}^d$, we also define

$$B(z) = \overline{B}(z, 1) \quad \text{and} \quad H(z) = H_{B(z)}. \tag{1.3}$$

We now recall some results from [12]. The present situation as compared to [12] is in fact simplified by the truncation of the Poissonian potential. For $\lambda \geq 0$, $e_\lambda(x, y, \omega)$ is continuous in x and measurable in ω ([12], Lemma 1.1) and $a_\lambda(x, y, \omega)$ is jointly continuous in x, y and measurable in ω ([12], after (1.10)). Moreover, from [12] (Theorems 1.4 and 1.7), there are certain nonnegative Lyapounov coefficients $\alpha_\lambda(x)$, $\lambda \geq 0$, $x \in \mathbb{R}^d$, jointly continuous in λ, x such that

$$\text{for } x \in \mathbb{R}^d, \qquad \lambda \geq 0 \to \alpha_\lambda(x) \text{ is concave increasing}, \tag{1.4}$$

$$\text{for } \lambda \geq 0, \qquad x \in \mathbb{R}^d \to \alpha_\lambda(x) \text{ is a norm on } \mathbb{R}^d, \tag{1.5}$$

$$\mathbb{P}\text{-a.s. for } A > 0, \quad \lim_{y \to \infty} \sup_{x \in B(0)} \sup_{0 \leq \lambda \leq A} \frac{1}{|y|} |-\log e_\lambda(x, y, \omega) - \alpha_\lambda(y)| = 0, \tag{1.6}$$

and the convergence in (1.6) takes place in $L^1(\mathbb{P})$ as well.

The nonnegative continuous functions $a_\lambda(\cdot, \cdot, \omega)$ and $d_\lambda(\cdot, \cdot, \omega)$ ([12], (1.10) and (1.16)) satisfy the triangle inequality. Of course $d_\lambda(\cdot, \cdot, \omega)$ is a symmetric function of its two variables. These properties are not specific to Poissonian potentials and in fact show up in a variety of situations.

Now when $d \geq 3$ or $\lambda > 0$ or ω is such that for any $z \in \mathbb{R}^d$, $V(\cdot, \omega)$ is not a.e. equal to 0 on $B(z)^c$ (which of course occurs $\mathbb{P}$-a.s.), $d_\lambda(x, y, \omega) = 0$ forces $x = y$. In this case

$$d_\lambda(\cdot, \cdot, \omega) \text{ is a distance function on } \mathbb{R}^d \text{ which induces the usual topology.} \tag{1.7}$$

EXAMPLE 1.1. When $d = 3$, $\lambda \geq 0$ and $\omega = 0$, for $x, y \in \mathbb{R}^d$,

$$\begin{aligned} e_\lambda(x, y) &= \frac{1}{|x-y| \vee 1} \exp\{-\sqrt{2\lambda}\,(|x-y| - 1)_+\}, \\ d_\lambda(x, y) &= \sqrt{2\lambda}\,|x-y| + \log(1 + |x-y|). \end{aligned} \tag{1.8}$$

It is easy to check that there are no "genuine flat triangles" for $d_\lambda(\cdot, \cdot)$, that is, $d_\lambda(x, z) + d_\lambda(z, y) = d_\lambda(x, y) \Rightarrow x = z$ or $y = z$.

This feature is different from the distance functions which show up in first passage percolation, for instance, and somehow corresponds to the fact that "Brownian motion has more than one way of going from x to y."

We still need some further notation. For $U \subseteq \mathbb{R}^d$ an open subset of $\mathbb{R}^d$, we introduce the "Schrödinger heat kernel"

$$\begin{aligned} r_U(t, x, y, \omega) &= (2\pi t)^{-d/2} \exp\left\{-\frac{(y-x)^2}{2t}\right\} \\ &\quad \times E^t_{x,y}\left[\exp\left\{-\int_0^t V(Z_s, \omega)\, ds\right\}, T_U > t\right], \\ &\qquad\qquad t > 0,\ x, y \in \mathbb{R}^d,\ \omega \in \Omega, \end{aligned} \tag{1.9}$$

where $E^t_{x,y}$ stands for the expectation with respect to the Brownian bridge measure in time t from x to y.

When U is nonvoid, r_U is known to be the kernel of the self-adjoint semigroup on $L^2(U, dx)$ generated by $-\frac{1}{2}\Delta + V$ with Dirichlet boundary conditions. We also introduce the $(\lambda + V)$-Green function relative to U:

$$g_{\lambda, U}(x, y, \omega) = \int_0^\infty e^{-\lambda s} r_U(s, x, y, \omega)\, ds \in (0, \infty]. \tag{1.10}$$

When $U = \mathbb{R}^d$, the subscript $\mathbb{R}^d$ will be dropped from the notation. We shall also omit the ω dependence in the notation when this causes no confusion. Throughout this work we shall use "positive constants" in our estimates, usually denoted by $c_1, c_2, \dots$ or $\gamma_1, \gamma_2, \dots$ or sometimes by const, which solely depend on the parameters of our model, namely, d, ν, $W(\cdot)$, M and λ.

The following lemma which we shall often use in the sequel relates the distance function d_λ to the decay properties of e_λ and g_λ. For $\lambda \geq 0$, $z \in \mathbb{R}^d$, $\omega \in \Omega$, we define

$$F_\lambda(z, \omega) = \log^+\left(\int_{B(z) \times B(z)} g_\lambda(x_1, x_2, \omega)\, dx_1\, dx_2\right) \in (0, \infty]. \tag{1.11}$$

LEMMA 1.2. *For* $|x-y|>4$ *and* $\omega \in \Omega$,

$$\max\{|d_\lambda(x,y)+\log g_\lambda(x,y)|, |d_\lambda(x,y)+\log e_\lambda(x,y)|, |d_\lambda(x,y)-a_\lambda(x,y)|\} \le c_1(1+F_\lambda(x)+F_\lambda(y)). \tag{1.12}$$

For $x \in \mathbb{R}^d$ *and* $\omega \in \Omega$,

$$F_\lambda(x) \le \begin{cases} c(d,\lambda), & \text{if } d>3 \text{ or } \lambda>0,\\ c(W,M)(1+\log^+(\log \operatorname{dist}(x,\operatorname{supp}\omega))), & \text{if } d=2,\ \lambda=0,\\ c(W,M)(1+\log^+(\operatorname{dist}(x,\operatorname{supp}\omega))), & \text{if } d=1,\ \lambda=0,\end{cases} \tag{1.13}$$

provided supp ω *denotes the support of* ω.

PROOF. We begin with the proof of (1.12). Without loss of generality, we assume that $d>3$ or $\lambda>0$ or $\omega \neq 0$. Otherwise $F_0(\cdot, \omega=0)=\infty$ and there is nothing to prove. Then the methods of Chung ([4], pages 208–218) apply [see also [12], (1.37)] and there are bounded positive measures $e_y^{\lambda,\omega}(dz)$ on $B(y)$ such that for each $x \in \mathbb{R}^d$,

$$e_\lambda(x,y,\omega) = \int_{B(y)} g_\lambda(x,z,\omega)e_y^{\lambda,\omega}(dz) \tag{1.14}$$

[the measure $e_y^{\lambda,\omega}$ is the $(\lambda+V)$-equilibrium measure of $B(y)$]. Consider now x, y in $\mathbb{R}^d$ with $|x-y|>4$. It follows from Harnack's inequality applied to both variables (see for instance [12], before Remark 1.8) that for $x' \in B(x)$, $y' \in B(y)$ and a constant $c(d,\lambda+M)>1$,

$$c^{-1}(d,\lambda+M) \le g_\lambda(x',y',\omega)/g_\lambda(x,y,\omega) \le c(d,\lambda+M). \tag{1.15}$$

This and (1.14) show that for $x' \in B(x)$,

$$|-\log e_\lambda(x',y,\omega)+\log g_\lambda(x,y,\omega)| \le \log c + |\log e_y^{\lambda,\omega}(B(y))|, \tag{1.16}$$

$$|-\log e_\lambda(x',y,\omega)+\log e_\lambda(x,y,\omega)| \le 2\log c. \tag{1.17}$$

Therefore, from the symmetry of $g_\lambda(\cdot,\cdot,\omega)$ and (1.16) we deduce

$$|d_\lambda(x,y,\omega)+\log g_\lambda(x,y,\omega)| \le 2\log c + |\log e_x^{\lambda,\omega}(B(x))| + |\log e_y^{\lambda,\omega}(B(y))|. \tag{1.18}$$

Our claim (1.12) will now follow after we provide a suitable upper bound on $|\log e_z^{\lambda,\omega}(B(z))|$, when $z \in \mathbb{R}^d$. From Theorem 6 in Section 5.2 of Chung [4], the $(\lambda+V)$-capacity of $B(z)$ equals

$$e_z^{\lambda,\omega}(B(z)) = 1/\inf\left\{\int_{B(z)\times B(z)} g_\lambda(z_1,z_2,\omega)\mu(dz_1)\mu(dz_2),\ \mu \in M_1(B(z))\right\}, \tag{1.19}$$

where $M_1(B(z))$ denotes the set of probability measures on $B(z)$. We know from (1.43) of [12] that

$$e_z^{\lambda,\omega}(B(z)) \le c(d,\lambda,M). \tag{1.20}$$

Moreover, if $\omega_d = |B(0)|$ stands for the volume of the unit ball of $\mathbb{R}^d$,

$$e_z^{\lambda,\omega}(B(z)) \geq \omega_d^2 \Big/ \left(\int_{B(z)\times B(z)} g_\lambda(z_1, z_2, \omega)\, dz_1, dz_2 \right). \tag{1.21}$$

Our claim (1.12) now follows. As for (1.13), the first inequality, when $d \geq 3$ or $\lambda > 0$ simply comes from

$$g_\lambda(\cdot, \cdot, \omega) \leq g_\lambda(\cdot, \cdot, \omega = 0) \leq g_0(\cdot, \cdot, \omega = 0).$$

The last two inequalities, when $d = 1, 2$, $\lambda = 0$ and $\omega \neq 0$, come from $g_0(\cdot, \cdot, \omega) \leq g_0(\cdot, \cdot, \delta_{x_i})$, where $x_i \in \operatorname{supp} \omega$ is such that $d(x, x_i) = d(x, \operatorname{supp} \omega)$, together with rather standard estimates on

$$E_0\left[\int_0^\infty 1_{B(0,2)}(Z_s) \exp\left\{ -\int_0^s W(Z_u - y) \wedge M\, du \right\} ds\right]. \qquad \square$$

As an application of Lemma 1.2, we have the following proposition:

PROPOSITION 1.3. *On a set of full $\mathbb{P}$-measure, for $\lambda \geq 0$,*

$$\lim_{y\to\infty} \frac{1}{|y|} |d_\lambda(0, y, \omega) - \alpha_\lambda(y)| = 0, \tag{1.22}$$

the convergence holds in $L^1(\mathbb{P})$ as well, and one can replace $d_\lambda(0, y, \omega)$ by $a_\lambda(y, 0, \omega)$, $-\log g_\lambda(0, y, \omega)$ or $-\log e_\lambda(y, 0, \omega)$.

PROOF. When $d \geq 3$ or $\lambda > 0$, this is an immediate consequence of (1.6) and Lemma 1.2. When $d = 1, 2$ and $\lambda = 0$, the L^1 convergence follows easily from Lemma 1.2 and (1.6). As for the almost sure convergence, observe that when $c = c(d, \nu)$ is large enough,

$$\sum_{q \in \mathbb{Z}^d \setminus \{0\}} \mathbb{P}\left[d(q, \operatorname{supp} \omega) \geq c(\log|q|)^{1/d} \right] < \infty.$$

It follows that on a set of full $\mathbb{P}$-measure,

$$\limsup_{y\to\infty} d(y, \operatorname{supp} \omega)/(\log|y|)^{1/d} \leq c \tag{1.23}$$

and our claim now follows. $\square$

We now recall some estimates from [13] which will be of use. We first introduce a paving of $\mathbb{R}^d$. Namely, for $q \in \mathbb{Z}^d$, we introduce the cube of size l and center lq,

$$C(q) = \left\{ z \in \mathbb{R}^d, -\frac{l}{2} \leq z^i - lq^i < \frac{l}{2}, i = 1, \dots, d \right\}, \tag{1.24}$$

and pick a large enough

$$l(d, \nu, a) \in (d(4 + 8a), \infty), \tag{1.25}$$

so that

$$9^{nd} p_n(l, \nu) \le 2^{-n}, \qquad n \ge 1, \tag{1.26}$$

provided $p_n(l, \nu)$ stands for the probability that a binomial variable with parameter n and success probability $p = 1 - \exp(-\nu l^d / 4^d)$ takes a value smaller than $n/2$.

Such a choice of l can be made, as can be seen from standard exponential estimates on the binomial distribution, with success probability close to 1. Moreover, the factor $9^{dn} = (3^d)^{2n}$ represents a rough upper bound on the number of $\| \, \|$-lattice animals Γ (i.e., finite connected sets) on $\mathbb{Z}^d$ of size n, containing 0, if the adjacency relation of two sites $q, q' \in \mathbb{Z}^d$ is defined as $\|q - q'\| \le 1$. We use the notation $\|z\| = \sup_{i=1,\dots,d} |z^i|$ for $z \in \mathbb{R}^d$.

We refer to [13] or Lemma 1 of Cox, Gandolfi, Griffin and Kesten [5]. The quantity $9^{dn} p_n(l, \nu)$ is then an upper bound on the $\mathbb{P}$-probability that there exists an animal Γ containing 0 of size n, such that

$$\begin{aligned} \sum_{q \in \Gamma} 1\{&\text{the open cube of size } l/4 \text{ centered at} \\ &lq \text{ receives a point of } \omega\} \le n/2. \end{aligned} \tag{1.27}$$

We then introduce

$$\begin{aligned} N_0(\omega) = {}&\text{the smallest } n \ge 1 \text{ such that for } k \ge n \text{ and } \Gamma \\ &\text{a } \| \, \|\text{-lattice animal containing 0 with } |\Gamma| = k, \\ &\sum_{q \in \Gamma} 1\left\{ \omega\left(lq + \left(-\frac{l}{8}, \frac{l}{8} \right)^d \right) \neq 0 \right\} \ge \frac{k}{2}. \end{aligned} \tag{1.28}$$

It follows from (1.26) and the above discussion that for $n \ge 1$,

$$\mathbb{P}[N_0(\omega) \ge n] \le \sum_{k=n}^{\infty} 2^{-k} = 2^{-(n-1)}. \tag{1.29}$$

We now introduce for $y \in \mathbb{R}^d$ and $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ with $H(y) < \infty$, the random lattice animal

$$\mathscr{A}(w) = \left\{ q \in \mathbb{Z}^d, H_{\overline{C}(q)} < H(y) \right\}. \tag{1.30}$$

We can now apply the estimates of [13] after (1.38) [let us mention that the supermartingale argument of [13] works as well in the $d = 1$ case and that the presence in the present article of the truncation level M simply modifies the constant χ which appears in (1.33) of [13], where one should replace $W(\cdot)$ by $M \wedge W(\cdot)$]. It now follows from the above reference that picking $c_3(d, \nu, W, M) > 0$ small enough, we have for $x \in C(0)$ and $y \in \mathbb{R}^d$,

$$E_x\left[\exp\left\{ c_3 |\mathscr{A}| - \int_0^{H(y)} V(Z_s, \omega)\, ds \right\}, H(y) < \infty \right] \le 2^{N_0(\omega)/2} \tag{1.31}$$

and of course from (1.29), $\mathbb{E}[2^{N_0/2}] < \infty$.

Let us mention the following two consequences of the exponential estimates (1.31). For $u > 0$, we define

$$\Lambda_u = \{z \in \mathbb{R}^d, \|z\| < u\}. \tag{1.32}$$

If $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ is such that $w(0) \in C(0)$ and $H(y) < \infty$, then

$$|\mathscr{A}| > c(d,l)\|y\| - c'(d,l).$$

Moreover, if $T_{\Lambda_u} \le H(y)$,

$$|\mathscr{A}| > \tilde{c}(d,l)u - \tilde{c}'(d,l),$$

as follows from simple geometric considerations. Therefore, for $x \in C(0)$, $u > 0$, $y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$E_x\left[\exp\left\{-\int_0^{H(y)} V(Z_s,\omega)\,ds\right\}, H(y) < \infty\right] \le c_4 2^{N_0(\omega)/2} \exp\{-c_5\|y\|\}, \tag{1.33}$$

$$\begin{aligned} &E_x\left[\exp\left\{-\int_0^{H(y)} V(Z_s,\omega)\,ds\right\}, T_{\Lambda_u} < H(y) < \infty\right] \\ &\quad \le c_4 2^{N_0(\omega)/2} \exp\{-c_5 u\} \end{aligned} \tag{1.34}$$

for suitable constants $c_4 > 1$ and $c_5 > 0$.

Finally we close this section with a tubular estimate for Brownian motion which provides a lower bound on $e_\lambda(x, y, \omega)$:

$$\begin{aligned} e_\lambda(x,y,\omega) \ge E_x\Bigg[&\exp\{-(\lambda+M)H(y)\}, \\ &\sup_{0\le s\le |y-x|}\left|Z_s - \left(x + \frac{s(y-x)}{|y-x|}\right)\right| < 1\Bigg] \\ \ge\ & c_6 \exp\{-c_7|y-x|\} \end{aligned} \tag{1.35}$$

with constants $c_6 \in (0,1)$ and $c_7 > 0$ (see for instance (1.11) of [12]).

2. Fluctuations around the mean value. We want to derive in this section some upper bounds on the size of fluctuations of quantities like $d_\lambda(x, y, \omega)$ or $-\log e_\lambda(x, y, \omega)$ around their mean value. Our main tool here will be the martingale method as in Kesten [8]. As mentioned in the Introduction, this is only expected to produce very rough upper bounds, which are nevertheless useful.

THEOREM 2.1. *Assume $d \ge 3$ or $\lambda > 0$. For $|y| > 4$ and $0 \le u \le c_7|y|$,*

$$\mathbb{P}\left[\left|\log e_\lambda(0,y) - \mathbb{E}[\log e_\lambda(0,y)]\right| \ge u\sqrt{|y|}\right] \le c_8 \exp\{-c_9 u\}. \tag{2.1}$$

Analogous estimates hold with $e_\lambda(0, y)$ replaced by $a_\lambda(0, y)$ or $d_\lambda(0, y)$.

PROOF. In view of Lemma 1.2, it clearly suffices to prove (2.1). We now introduce an enumeration q_k, $k \geq 1$, of $\mathbb{Z}^d$ and the filtration $\mathscr{F}_k$, $k \geq 0$:

$$\mathscr{F}_0 = \{\phi, \Omega\},$$
$$\mathscr{F}_k = \left\{\sigma(\omega(A));\ A \in B(\mathbb{R}^d) \text{ and } A \subseteq \bigcup_{i=1}^{k} C(q_i)\right\} \quad \text{when } k \geq 1. \tag{2.2}$$

We now pick a fixed y in $\mathbb{R}^d$ with $|y| > 4$ and introduce the nonnegative martingale

$$M_k = \mathbb{E}[-\log e_\lambda(0, y) | \mathscr{F}_k], \qquad k \geq 0.$$

In view of (1.35), $-\log e_\lambda(0, y)$ is bounded and M_k converges $\mathbb{P}$-a.s. and in L^p, $p \in [1, \infty)$, converges to $-\log e_\lambda(0, y)$. We have, in fact,

$$M_\infty - M_0 = -\log e_\lambda(0, y) + \mathbb{E}[\log e_\lambda(0, y)]. \tag{2.3}$$

Our main task is now to derive upper bounds on the martingale increments $\Delta M_k \overset{\text{def}}{=} M_k - M_{k-1}$, $k \geq 1$. We shall define [see (2.16) below] certain $\mathscr{F}_\infty$-measurable nonnegative variables U_k with $\mathbb{E}[(\Delta M_k)^2 | \mathscr{F}_{k-1}] \leq \mathbb{E}[U_k | \mathscr{F}_{k-1}]$, $k \geq 1$.

Our claim will then follow from suitable exponential estimates on $\sum_{k \geq 1} U_k$ derived in Lemma 2.3 and from Theorem 3 of Kesten [8].

We shall use the following notation: for $k \geq 0$, $x \in \mathbb{R}^d$ and $\omega, \sigma \in \Omega$ (two cloud configurations), we define

$$V_k(x, \omega, \sigma) = M \wedge \left(\sum_{m \leq k} \int_{C(q_m)} W(x - y)\, \omega(dy) + \sum_{m > k} \int_{C(q_m)} W(x - y)\, \sigma(dy) \right) \tag{2.4}$$

as well as

$$e_{\lambda, k}(x, y) = E_x\left[\exp\left\{-\int_0^{H(y)} (\lambda + V_k)(Z_s)\, ds\right\}, H(y) < \infty\right], \tag{2.5}$$

where we have dropped the ω, σ dependence from the notation. We can now represent M_k, $k \geq 0$, through

$$M_k = \mathbb{E}^\sigma[-\log e_{\lambda, k}(0, y)], \qquad k \geq 0, \tag{2.6}$$

where $\mathbb{E}^\sigma$ denotes the integration over the variable σ with respect to the measure $\mathbb{P}$. It is also convenient to introduce the path measures on $C(\mathbb{R}_+, \mathbb{R}^d)$ for $x \in \mathbb{R}^d$, $k \geq 0$ and $\omega, \sigma \in \Omega$,

$$\hat{P}_x^k = e_{\lambda, k}(x, y)^{-1} 1\{H(y) < \infty\} \exp\left\{-\int_0^{H(y)} (\lambda + V_k)(Z_s)\, ds\right\} P_x, \tag{2.7}$$

as well as the analogously defined measure $\hat{P}_x$, where $V_k(\cdot, \omega, \sigma)$ is being replaced by $V(x, \omega)$. To simplify notation, we shall write for $k \geq 1$,

$$C_k = C(q_k), \qquad \tilde{C}_k = \text{the closed } a\text{-neighborhood of } C_k,$$
$$H_k = H_{\tilde{C}_k} \quad \left(\text{the entrance time of } Z_\cdot \text{ in } \tilde{C}_k\right). \tag{2.8}$$

Observe that $W(\cdot) = 0$ outside $\overline{B}(0, a)$ and therefore

$$V_k(\cdot, \omega, \sigma) = V_{k-1}(\cdot, \omega, \sigma) \quad \text{on } \tilde{C}_k^c, \text{ for } k \geq 1. \tag{2.9}$$

From the strong Markov property it now follows that when $k \geq 1$, with a slight abuse of notation,

$$\frac{e_{\lambda, k-1}}{e_{\lambda, k}}(0, y) = \hat{P}_0^k[H_k > H(y)] + \hat{E}_0^k\left[H_k \leq H(y), \frac{e_{\lambda, k-1}}{e_{\lambda, k}}(Z_{H_k}, y)\right]$$

and, therefore,

$$\frac{e_{\lambda, k-1}}{e_{\lambda, k}}(0, y) - 1 = \hat{E}_0^k\left[H_k \leq H(y), \left(\frac{e_{\lambda, k-1}}{e_{\lambda, k}}(Z_{H_k}, y) - 1\right)\right]. \tag{2.10}$$

In an analogous fashion we have

$$\frac{e_{\lambda, k}}{e_{\lambda, k-1}}(0, y) - 1 = \hat{E}_0^{k-1}\left[H_k \leq H(y), \left(\frac{e_{\lambda, k}}{e_{\lambda, k-1}}(Z_{H_k}, y) - 1\right)\right]. \tag{2.11}$$

We shall say that a box C_k is a "neighbor of y" if $\|q_k - q_m\| \leq 1$, for some q_m with $C_m \cap B(y) \neq \emptyset$. Of course the number of neighboring boxes of y is bounded independently of y. Finally for $\lambda \geq 0$ and $k \geq 0$ we shall denote the $(\lambda + V_k)$-Green function of $B(y)^c = U$ by

$$g_{\lambda, k}^y(\cdot, \cdot) \text{ defined as in (1.10) with } V \text{ replaced by } V_k. \tag{2.12}$$

LEMMA 2.2. *For $x \in \mathbb{R}^d$, $k \geq 1$ and $\lambda \geq 0$,*

$$\begin{aligned} e_{\lambda, k-1}(x, y) - e_{\lambda, k}(x, y) &= \int_{\tilde{C}_k} g_{\lambda, k}^y(x, z)(V_k - V_{k-1})(z) e_{\lambda, k-1}(z, y)\, dz \\ &= \int_{\tilde{C}_k} g_{\lambda, k-1}^y(x, z)(V_k - V_{k-1})(z) e_{\lambda, k}(z, y)\, dz. \end{aligned} \tag{2.13}$$

Moreover, we can pick a constant $c_{10}(d, \nu, \lambda, W, M)$ such that

$$|\Delta M_k| \leq c_{10}, \qquad k \geq 1, \tag{2.14}$$

$$\mathbb{E}\left[|\Delta M_k|^2 | \mathscr{F}_{k-1}\right] \leq \mathbb{E}[U_k | \mathscr{F}_{k-1}], \qquad k \geq 1, \tag{2.15}$$

provided

$$U_k = \begin{cases} c_{10}, & \text{if } C_k \text{ is a neighbor of } y, \\ c_{10} \hat{P}_0[H_k < H(y)]^2, & \text{otherwise.} \end{cases} \tag{2.16}$$

PROOF. The argument to prove (2.13) is classical. For $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ with $H(y) < \infty$, we have

$$\begin{aligned}\exp\left\{-\int_0^{H(y)}(V_{k-1}-V_k)(Z_s)\,ds\right\} \\ = 1 + \int_0^{H(y)}(V_k - V_{k-1})(Z_s) \\ \times\exp\left\{-\int_s^{H(y)}(V_{k-1}-V_k)(Z_u)\,du\right\}ds.\end{aligned} \tag{2.17}$$

Multiplying both members of (2.17) with $\exp\{-\int_0^{H(y)}V_k(Z_s)\,ds\}$ and integrating with respect to $1\{H(y)<\infty\}P_x$, we obtain the first equality of (2.13). The second equality is obtained by exchanging the role of k and $k-1$.

Let us now prove (2.14) and (2.15). From (2.6), (2.10) and (2.11), we have

$$\begin{aligned}\Delta M_k &= \mathbb{E}^\sigma\left[\log\left(1+\hat{E}_0^k\left[H_k \le H(y), \left(\frac{e_{\lambda,k-1}}{e_{\lambda,k}}(Z_{H_k}, y) - 1\right)\right]\right)\right] \\ &= -\mathbb{E}^\sigma\left[\log\left(1+\hat{E}_0^{k-1}\left[H_k \le H(y), \left(\frac{e_{\lambda,k}}{e_{\lambda,k-1}}(Z_{H_k}, y) - 1\right)\right]\right)\right].\end{aligned} \tag{2.18}$$

When C_k is a neighbor of y, we have

$$\begin{aligned}|\Delta M_k| &\le \max\left(\mathbb{E}^\sigma\left[\sup_{\tilde{C}_k}\log e_{\lambda,k}^{-1}(\cdot, y)\right], \mathbb{E}^\sigma\left[\sup_{\tilde{C}_k}\log e_{\lambda,k-1}^{-1}(\cdot, y)\right]\right) \\ &\le \log\left(c_6^{-1}\right) + c_7\left(2\sqrt{d}\,l + 1 + a\right), \quad \text{using (1.35)}.\end{aligned} \tag{2.19}$$

On the other hand, when C_k is not a neighbor of y, it follows from (2.13) and (2.18) that

$$\begin{aligned}|\Delta M_k| \le \max\Bigg\{&\mathbb{E}^\sigma\Bigg[\log\Bigg(1+\hat{E}^k\Bigg[H_k \le H(y), \\ &\quad \int_{\hat{C}_k} g_{\lambda,k-1}^y(Z_{H_k}, z)(V_k - V_{k-1})(z) \\ &\quad \times\frac{e_{\lambda,k}(z,y)}{e_{\lambda,k}(Z_{H_k}, y)}\,dz\Bigg]\Bigg)\Bigg], \\ &\mathbb{E}^\sigma\Bigg[\log\Bigg(1+\hat{E}_0^{k-1}\Bigg[H_k \le H(y), \\ &\quad \int_{\tilde{C}_k} g_{\lambda,k}^y(Z_{H_k}, z)(V_{k-1} - V_k)(z) \\ &\quad \times\frac{e_{\lambda,k-1}(z,y)}{e_{\lambda,k-1}(Z_{H_k}, y)}\,dz\Bigg]\Bigg)\Bigg]\Bigg\}.\end{aligned} \tag{2.20}$$

Now from Harnack's inequality [see, for instance [12] after (1.28)],

$$\sup_{\tilde{C}_k} e_{\lambda, j}(\cdot, y) \Big/ \inf_{\tilde{C}_k} e_{\lambda, j}(\cdot, y) \le c_{11}(d, \lambda, M, a, l) \quad \text{for } j \ge 0$$

and, therefore,

$$\begin{aligned} |\Delta M_k| \le \max\Bigg\{ & \mathbb{E}^\sigma\Bigg[\log\Bigg(1 + Mc_{11}\hat{E}_0^k\Bigg[H_k \le H(y), \int_{\tilde{C}_k} g_{\lambda, k-1}^y(Z_{H_k}, z)\, dz\Bigg]\Bigg)\Bigg], \\ & \mathbb{E}^\sigma\Bigg[\log\Bigg(1 + Mc_{11}\hat{E}_0^{k-1}\Bigg[H_k \le H(y), \\ & \qquad \int_{\tilde{C}_y} g_{\lambda, k-1}^y(Z_{H_k}, z)\, dz\Bigg]\Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{2.21}$$

Now since $d \ge 3$ or $\lambda > 0$, $g_{\lambda, k-1}^y(\cdot, \cdot)$ is smaller than $g_\lambda(\cdot, \cdot, \omega = 0)$, the λ-Green function of Brownian motion. It now easily follows that c_{10} can be adjusted so that (2.14) holds. Squaring both members of (2.21), we also find that when C_k is not a neighbor of y,

$$\begin{aligned} |\Delta M_k|^2 &\le c_{12} \max\Big(\mathbb{E}^\sigma\Big[\hat{P}_0^k[H_k \le H(y)]\Big]^2, \mathbb{E}^\sigma\Big[\hat{P}_0^{k-1}[H_k \le H(y)]\Big]^2\Big) \\ &\le c_{12} \max\Big(\mathbb{E}^\sigma\Big[\hat{P}_0^k[H_k \le H(y)]^2\Big], \mathbb{E}^\sigma\Big[\hat{P}_0^{k-1}[H_k \le H(y)]^2\Big]\Big). \end{aligned}$$

Conditioning with respect to $\mathscr{F}_{k-1}$, we find

$$\begin{aligned} \mathbb{E}\Big[(\Delta M_k)^2 \mid \mathscr{F}_{k-1}\Big] &\le c_{12}\mathbb{E}\Big[\mathbb{E}^\sigma\Big[\hat{P}_0^k[H_k \le H(y)]^2 \\ &\qquad + \hat{P}_0^{k-1}[H_k \le H(y)]^2\Big]\Big|\mathscr{F}_{k-1}\Big] \\ &= 2c_{12}\mathbb{E}^\sigma\Big[\hat{P}_0^{k-1}[H_k \le H(y)]^2\Big] \\ &= 2c_{12}\mathbb{E}\Big[\hat{P}_0[H_k \le H(y)]^2\Big|\mathscr{F}_{k-1}\Big]. \end{aligned} \tag{2.22}$$

From this and (2.19) it follows that c_{10} can be adjusted so that (2.15) holds.

Our next step is to derive exponential bounds on $\sum_k U_k$.

LEMMA 2.3. *For* $|y| > 4$,

$$\mathbb{E}\Bigg[\exp\Bigg\{c_{13} \sum_{k>1} U_k\Bigg\}\Bigg] \le \exp\{c_{14}|y|\}. \tag{2.23}$$

PROOF. In view of the definition of the U_k, $k \ge 1$, it clearly suffices to prove an estimate like (2.23) with U_k replaced by $\hat{P}_0[H_k \le H(y)]^2$. Now observe that on the event $\{H_k \le H(y)\}$, one of the q with $\|q - q_k\| \le 1$ belongs to $\mathscr{A}$ defined in (1.30) and, consequently,

$$\sum_{k\ge 1} \hat{P}_0[H_k \le H(y)]^2 \le \sum_{k\ge 1} \hat{P}_0[H_k \le H(y)] \le 3^d \hat{E}_0[|\mathscr{A}|]. \tag{2.24}$$

We are therefore reduced to proving an estimate like (2.23) with ΣU_k replaced by $\hat{E}_0[|\mathscr{A}|]$. From Jensen and Cauchy–Schwarz's inequalities together with (1.31) and (1.35), we find

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\frac{c_3}{2}\hat{E}_0[|\mathscr{A}|]\right\}\right] &\le \mathbb{E}\left[\hat{E}_0\left[\exp\left\{\frac{c_3}{2}|\mathscr{A}|\right\}\right]\right] \\ &\le \mathbb{E}[2^{N_0/2}]^{1/2} c_6^{-1} \exp\{c_7|y|\}.\end{aligned} \tag{2.25}$$

Our claim follows.

Let us now finish the proof of Theorem 2.1. As a consequence of (2.14), (2.15) and (2.23), we can apply Theorem 3 of Kesten [8]. The role of x_0 in the notations of [8] is played by $\max(e^2c_{10}^2, 2(c_{14}/c_{13})|y|)$. Our claim (2.1) now follows from (1.32) of [8]. □

An interesting consequence of Theorem 2.1 in view of Section 3 is the following corollary:

Corollary 2.4. *Assume $d \ge 3$ or $\lambda > 0$. Then*

$$\sup_{|y|\ge 1} \mathbb{E}\left[\exp\left\{\frac{c_{15}}{\sqrt{|y|}}\left(\mathbb{E}[a_\lambda(0,y)] - a_\lambda(0,y)\right)\right\}\right] < \infty \tag{2.26}$$

and analogous estimates hold with $d_\lambda(0, y)$ or $-\log e_\lambda(0, y)$ instead of $a_\lambda(0, y)$.

Proof. In view of Lemma 1.2 and (1.35) it suffices to prove the estimate for a_λ and $|y| > 4$. From (1.35) we also have

$$a_\lambda(0,y) \le \log(1/c_6) + c_7(|y|+1) \le c_{16}|y|.$$

From Theorem 2.1 we also know that

$$\mathbb{P}\left[\mathbb{E}[a_\lambda(0,y)] - a_\lambda(0,y) \ge u\sqrt{|y|}\right] \le c_8' \exp\{-c_9' u\} \tag{2.27}$$

for $u \le c_7'|y|$. The left member of (2.27) is a decreasing function of u and equals 0 for $u > c_{16}\sqrt{|y|}$. As a consequence, (2.27) holds for all u provided $c_7'|y| > c_{16}\sqrt{|y|}$, that is, $|y| > (c_{16}/c_7')^2$. This and (1.35) easily imply our claim. □

We shall now briefly discuss how the results and proofs are adapted in the slightly more singular situation where $d = 1, 2$ and $\lambda = 0$.

THEOREM 2.5. *Assume $d = 1$ or 2 and $\lambda = 0$. For $|y| > 4$,*

$$\mathbb{P}\Big[\big|\log e_\lambda(0,y) - \mathbb{E}[\log e_\lambda(0,y)]\big| \ge u\sqrt{|y|}\log|y|\Big] \le c_{17}\exp(-c_{18}u) \tag{2.28}$$

for $0 \le u \le c_{19}|y|$ and

$$\sup_{|y|\ge 2}\mathbb{E}\left[\exp\left\{\frac{c_{20}}{\sqrt{|y|}\log|y|}\big(\mathbb{E}[-\log e_\lambda(0,y)] + \log e_\lambda(0,y)\big)\right\}\right] < \infty. \tag{2.29}$$

Analogous estimates hold with $d_\lambda(0,y)$ or $a_\lambda(0,y)$ instead of $-\log e_\lambda(0,y)$.

PROOF. In view of Lemma 1.2, together with the fact that $F_0(x,\omega)$ has finite exponential moments of any order which do not depend on x, it suffices to prove (2.28) and (2.29), the extension to $d_\lambda(0,y)$ and $a_\lambda(0,y)$ being straightforward. Moreover, (2.29) follows from (2.28) by a similar argument as in Corollary 2.4.

Now to prove (2.28), one introduces $\tilde{e}_0(x,y)$ defined as in (I.4) except that $V(\cdot,\omega)$ is replaced by $\tilde{V}(\cdot,\omega)$ which equals M on a complement of a box $\Lambda_{c|y|}$ [see (1.32) for the notation] and coincides with V on $\Lambda_{c|y|}$. Here c stands for the constant $4c_7/c_5$. Observe that

$$e_0(0,y) \ge \tilde{e}_0(0,y) \ge c_6\exp\{-c_7|y|\} \tag{2.30}$$

and

$$\begin{aligned}
&\mathbb{P}[-\log\tilde{e}_0(0,y) + \log e_0(0,y) > \log 2]\\
&\quad = \mathbb{P}\left[\frac{1}{2} \ge \frac{\tilde{e}_0}{e_0}(0,y)\right]\\
&\quad \le \mathbb{P}\left[\hat{P}_0\big[T_{\Lambda_{c|y|}} \le H(y)\big] \ge \frac{1}{2}\right]\\
&\quad \le \mathbb{E}\times E_0\left[\exp\left\{-\int_0^{H(y)} V(Z_s,\omega)\,ds\right\}, T_{\Lambda_{c|y|}} < H(y) < \infty\right]^{1/2}\\
&\qquad \times \mathbb{E}\big[e_0(0,y)^{-2}\big]^{1/2}\\
&\quad \le c_4^{1/2}\exp\left\{-\frac{c_5}{2}c|y|\right\}\mathbb{E}[2^{N_0/2}]^{1/2}\cdot c_6^{-1}\exp\{c_7|y|\}\\
&\quad \le c_{21}\exp\{-c_7|y|\},
\end{aligned} \tag{2.31}$$

using (1.34), (1.35) and our choice of c.

Observe that (2.30) and (2.31) also imply that

$$0 \le \mathbb{E}[-\log\tilde{e}(0,y) + \log e(0,y)] \le c_{22}.$$

It therefore suffices to prove (2.28) with $e_0(0,y)$ replaced by $\tilde{e}_0(0,y)$. For this we proceed as in the proof of Theorem 2.1. We now only have to take into account the cubes C_k which intersect $\Lambda_{c|y|}$. For such cubes we have

$$\sup_{x \in \tilde{C}_k} \int_{\tilde{C}_k} g^y_{\lambda,k}(x,z)\,dz \le \begin{cases} c_{23}\log|y|, & \text{if } d = 2,\\ c_{24}|y|, & \text{if } d = 1,\end{cases}$$

as follows from standard Green function estimates.

As a consequence, we find that for cubes C_k intersecting $\Lambda_{c|y|}$, $|\Delta M_k| \le c_{25}\log|y|$. Moreover, we define U_k as $c_{25}^2(\log|y|)^2$ when $d = 1$, and when $d = 2$ as

$$U_k = \begin{cases} c_{26}, & \text{when } C_k \text{ is a neighbor of } |y|,\\ c_{27}(\log|y|)^2 \hat{P}_0[H_k \le H(y)]^2, & \text{otherwise},\end{cases}$$

where the constants c_{26} and c_{27} are suitably adjusted so that (2.15) holds (of course we only consider boxes C_k which intersect $\Lambda_{c|y|}$). In the exponential estimate (2.23), c_{13} is now replaced by $c_{28}/(\log|y|)^2$ and our claim (2.28) follows again by an application of Theorem 3 of Kesten with x_0 in the notation of [8] being picked equal to $\max(e^2c_{25}^2(\log|y|)^2, 2(c'_{14}/c_{28})(\log|y|)^2|y|)$. □

3. Fluctuations to the Lyapounov norms. In this section we shall derive upper bounds on the difference $D_\lambda(0,y) - \alpha_\lambda(y)$ [see (I.6) and (1.6)]. These estimates combined with the results of Section 2 will provide bounds on the difference $d_\lambda(0,y) - \alpha_\lambda(y)$. Throughout this section we assume that

$$W(\cdot) \text{ is rotationally invariant.} \tag{3.1}$$

As a consequence, both $D_\lambda(0,\cdot)$ and $\alpha_\lambda(\cdot)$ are rotationally invariant. In fact, the norm $\alpha_\lambda(\cdot)$ is of the form $\alpha(\lambda)|\cdot|$, where $\alpha(\lambda)$ is a continuous concave increasing function $\mathbb{R}_+ \to (0,+\infty)$. It follows from translation invariance and the triangle inequality that

$$D_\lambda(0,Nz) \le ND_\lambda(0,z) \quad \text{for } \lambda \ge 0,\ z \in \mathbb{R}^d,\ N \ge 1.$$

Dividing by N and letting N tend to infinity, we conclude from Proposition 1.3 that

$$\alpha_\lambda(z) \le D_\lambda(0,z) \quad \text{for } \lambda \ge 0,\ z \in \mathbb{R}^d. \tag{3.2}$$

Therefore, our main task in this section is to provide lower bounds for $\alpha_\lambda(z)$ in terms of $D_\lambda(0,z)$.

THEOREM 3.1. *Assume* (3.1) *and let* $\lambda \ge 0$ *be given. If* $\kappa(\cdot)\colon R_+ \to (0,\frac{1}{2})$ *is decreasing and satisfies*

$$A \overset{\text{def}}{=} \sup_{z \in \mathbb{R}^d} \mathbb{E}\big[\exp\{\kappa(|z|)(\mathbb{E}[a_\lambda(0,z)] - a_\lambda(0,z))\}\big] < \infty \tag{3.3}$$

then

$$D_\lambda(0,z) - \frac{\delta_1}{\kappa(\gamma_1|z|)}\left(1 + \log\frac{1}{\kappa(\gamma_1|z|)} + \log(1+|z|)\right) \le \alpha_\lambda(z) \le D_\lambda(0,z), \qquad z \in \mathbb{R}^d, \tag{3.4}$$

for suitable constants $\delta_1(d,\nu,W,M,\lambda,A) > 0$ *and* $\gamma_1(d,\nu,W,M,\lambda) > 0$.

PROOF. In the proof below the constants $\gamma_1, \gamma_2, \dots$ will follow the convention we introduced after (1.10), whereas the constants $\delta_1, \delta_2, \dots$ may additionally depend on A defined in (3.3). Thanks to rotational invariance, we assume without loss of generality that $z = |z|e_1$, where $e_1, \dots, e_d$ stands for the canonical basis of $\mathbb{R}^d$. We define

$$R = d(a+2) < \frac{l}{2} \quad [\text{see (1.25)}] \tag{3.5}$$

and introduce for $\lambda \ge 0$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\bar{e}_\lambda(x,y,\omega) = \sup_{x' \in B_R(x)} E_{x'}\left[\exp\left\{-\int_0^{H_R(y)} (\lambda + V)(Z_s,\omega)\right\}, H_R(y) < \infty\right], \tag{3.6}$$

provided for $z \in \mathbb{R}^d$, $B_R(z) = \overline{B}(z,R)$ and $H_R(z) = H_{B_R(z)}$. Given $m \in \mathbb{Z}$, we consider the affine hyperplane $\mathscr{H}_m$ orthogonal to e_1, passing through $3Rme_1$:

$$\mathscr{H}_m = \{z \in \mathbb{R}^d : z \cdot e_1 = 3Rm\} \tag{3.7}$$

(when $d = 1$, $\mathscr{H}_m = \{3Rme_1\}$). We introduce the sublattice of $\mathscr{H}_m$:

$$\mathscr{D}_m = \left\{z = 3mRe_1 + \sum_{i=2}^{d} (a+1)k_i e_i,\ k_i \in \mathbb{Z},\ i = 2,\dots,d\right\}. \tag{3.8}$$

We have picked R so that

$$\bigcup_{z \in \mathscr{D}_m} B_R(z) \supset \{z' \in \mathbb{R}^d, \operatorname{dist}(z', \mathscr{H}_m) \le a+1\}.$$

We let $\mathscr{H}_m^a$ stand for the closed a-neighborhood of $\mathscr{H}_m$ and define the open neighborhood of $\mathscr{H}_m^a$:

$$O_m = \bigcup_{z \in \mathscr{D}_m} B(z,R) \supset \mathscr{H}_m^a. \tag{3.9}$$

The open sets O_m, $m \in \mathbb{Z}$, are then pairwise disjoint. We finally introduce for $\beta \in (0,1)$ and $m \ge 1$,

$$g_\beta(m) = \sum_{m \in \mathscr{D}_m} \mathbb{E}\left[\bar{e}_\lambda(0,z)^\beta\right]. \tag{3.10}$$

Our strategy to derive a lower bound on $\alpha_\lambda(z)$ is to show (in Lemma 3.3) that $g_\beta(m)$ is approximately submultiplicative. It then follows that $-m^{-1}\log g_\beta(m)$ converges to a limit as m tends to infinity and cannot be "substantially smaller" than this limit when m is large [see (3.33)]. This limit value is easily seen to be smaller than $\beta 3R\alpha_\lambda(e_1)$. The exponential estimates (3.3) then enable us to relate $-m^{-1}\log g_\beta(m)$ and $(\beta/2)\mathbb{E}[\alpha_\lambda(0, 6Rme_1)]$ and provide a way to derive a lower estimate of $\alpha_\lambda(6Rme_1)$ [see (3.38)] in terms of $\mathbb{E}[\alpha_\lambda(0, 6Rme_1)]$. The extension to the case of a general z is then easy.

LEMMA 3.2. *There are constants* $\gamma_2 > 3R + 1$ *and* $\gamma_3, \gamma_4 > 1$, *such that for* $m \geq 1$ *and* $\beta \in (0,1)$,

$$
\begin{aligned}
g_\beta(m) &\leq \frac{\gamma_2}{\beta^d} \sum_{\substack{\|z\| \leq \gamma_2 m \\ z \in \mathscr{D}_m}} \mathbb{E}\left[\bar{e}_\lambda(0,z)^\beta\right] \\
&\leq \frac{\gamma_4}{\beta^d} m^{d-1} \sum_{\substack{\|z\| \leq \gamma_2 m \\ z \in \mathscr{D}_m}} \mathbb{E}\left[\tilde{e}_\lambda(0,z)^\beta\right],
\end{aligned} \tag{3.11}
$$

where

$$
\tilde{e}_\lambda(0,z) = \sup_{x \in B_R(0)} E_x\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < T\right] \tag{3.12}
$$

and T *stands for the exit time of* $\Lambda_{\gamma_2 m}$ [*see* (1.32)].

PROOF. We have, on the one hand, a lower bound of $g_\beta(m)$. Indeed from (1.35), for $\beta \in (0,1)$ and $m \geq 1$, assuming $\gamma_2 \geq 3R + 1$,

$$
\begin{aligned}
g_\beta(m) &\geq \mathbb{E}\left[\tilde{e}_\lambda(0, 3mRe_1)^\beta\right] \geq c_6^\beta \exp\{-\beta c_7 3mR\} \\
&\geq \exp\{-\beta\gamma_5 m\}.
\end{aligned} \tag{3.13}
$$

On the other hand, we can cover any closed ball of radius R by finitely many balls of radius 1. As a consequence and in view of (1.33) [note that $B_R(0) \subseteq C(0)$], for $z \in \mathbb{R}^d$ and $\beta \in (0,1)$,

$$
\mathbb{E}\left[\bar{e}_\lambda(0,z)^\beta\right] \leq \gamma_6 \exp\{-\beta\gamma_7\|z\|\}. \tag{3.14}
$$

Moreover, if we let for the time being T stand for T_{Λ_u}, with $u = \gamma_2 m$ and $\gamma_2 \geq 3R + 1$ to be defined below, we know by (1.34) that for $z \in \mathbb{R}^d$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{x \in B_R(0)} E_x\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, T < H(z) < \infty\right]^\beta\right] \\
&\qquad \leq \gamma_8 \exp\{-\beta\gamma_9 u\}.
\end{aligned} \tag{3.15}
$$

Now for $z \in \mathscr{D}_m$, $z - 3mRe_1 \overset{\text{def}}{=} z_\perp$ satisfies

$$
\|z\| - 3mR \leq \|z_\perp\| \leq \|z\|.
$$

It follows that for $A > 3Rm + 1$,

$$
\begin{aligned}
\sum_{\substack{z\in\mathscr{D}_m\\ \|z\|>A}} \mathbb{E}\Big[\tilde{e}_\lambda(0,z)^\beta\Big] &\le \gamma_6 \sum_{\substack{z\in\mathscr{D}_m\\ \|z\|>A}} \exp\{-\beta\gamma_7\|z\|\}\\
&\le \gamma_6 \sum_{\substack{z\in\mathscr{D}_0\\ \|z\|\ge A-3Rm}} \exp\{-\beta\gamma_7\|z\|\}\\
&\le \gamma_8 \sum_{k\ge A-3Rm} k^{d-1}\exp(-\beta\gamma_7 k)\\
&\le \gamma_9\int_{A-3Rm}^{+\infty} r^{d-1}\exp(-\beta\gamma_7 r)\,dr\\
&\le \frac{\gamma_{10}}{\beta^d}\exp\{-\beta\gamma_7(A-3mR)\}.
\end{aligned}
\tag{3.16}
$$

Observe also that in view of (3.15), for $z \in \mathbb{R}^d$,

$$
\mathbb{E}\Big[\bar{e}_\lambda(0,z)^\beta\Big] \le \mathbb{E}\Big[\tilde{e}_\lambda(0,z)^\beta\Big] + \gamma_8\exp\{-\beta\gamma_8 u\},
$$

so that

$$
\begin{aligned}
&\sum_{\substack{\|z\|\le A\\ z\in\mathscr{D}_m}} \mathbb{E}\Big[\bar{e}_\lambda(0,z)^\beta\Big]\\
&\quad\le \sum_{\substack{\|z\|\le A\\ z\in\mathscr{D}_m}} \mathbb{E}\Big[\tilde{e}_\lambda(0,z)^\beta\Big] + \gamma_8(2A+1)^{d-1}\exp\{-\beta\gamma_9 u\}.
\end{aligned}
\tag{3.17}
$$

Thanks to (3.13) we can now pick $A = u = \gamma_2 m$ with $\gamma_2 \ge 3R+1$ large enough so that (3.11) holds for suitable constants $\gamma_3, \gamma_4 > 1$.

The promised almost submultiplicative property of $g_\beta(m)$ now comes in the next lemma. The proof bypasses the Van den Berg–Kesten inequality which is at the root of the argument used in Alexander [1] and does not seem easily applicable here. Instead we use a “splitting technique” [see (3.26) and (3.27)] which generates the desired independence property.

LEMMA 3.3. *For $\beta \in (0,1)$ and $m, n \ge 1$,*

$$
g_\beta(m+n) \le \frac{\gamma_{11}}{\beta^d}(m+n)^{3d-2} g_\beta(m) g_\beta(n).
\tag{3.18}
$$

PROOF. We pick $\beta \in (0,1)$, $m, n \ge 1$ and define T as the exit time of $Z_.$ from the box $\Lambda_{\gamma_2(m+n)}$ [see (1.32)]. We also pick a fixed $z \in \mathscr{D}_{m+n} \cap \Lambda_{\gamma_2(m+n)}$ and introduce

$$
L = \sup\{0 < s < H_R(z),\, Z_s \in \mathscr{H}_m^a\},
\tag{3.19}
$$

where we use the convention $\sup\{\emptyset\} = 0$. Observe that when $Z_0 \in B_R(0)$ and $H_R(z) \le \infty$, then $L \in (0, \infty)$. We also consider

$$S = T_{O_m} \circ \theta_L + L \quad \text{and} \quad H = H_{\mathscr{H}_m^a}, \tag{3.20}$$

provided θ_t, $t \ge 0$, stands for the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$. In other words, H is the entrance time in $\mathscr{H}_m^a$ and S is the first exit time of O_m [defined in (3.9)] after L the last visit to $\mathscr{H}_m^a$ before entering $B_R(z)$. Observe that when $y \in \mathbb{R}^d$ is such that $y \cdot e_1 < 3mR + a$, for any path $Z_.$ with $Z_0 = y$ and $H_R(z) < \infty$,

$$S < H_R(z), \quad P_y\text{-a.s.} \tag{3.21}$$

Finally for $x \in \mathbb{R}^d$ we define

$$Q_x = 1\{H_R(z) < \infty\} P_x / P_x[H_R(z) < \infty], \tag{3.22}$$

that is, Wiener measure starting from x conditioned to enter $B_R(z)$. For $x \in B_R(0)$,

$$\begin{aligned}
&E_x\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < T\right] \\
&\quad = E_x\Bigg[\exp\left\{-\int_0^{H} (\lambda + V)(Z_s)\, ds\right\}, H < T, \\
&\qquad\qquad E_{Z_H}\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_2)\, ds\right\}, H_R(z) < T\right]\Bigg].
\end{aligned} \tag{3.23}$$

Now for $y_1 \in \mathscr{H}_m^a \cap \Lambda_{\gamma_2(m+n)}$ (playing the role of Z_H on the event $\{H < T\}$) we have

$$\begin{aligned}
&E_{y_1}\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < T\right] \\
&\quad \le E_{y_1}\left[\exp\left\{-\int_S^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < T\right] \quad \text{[using (3.21)]} \\
&\quad = P_{y_1}[H_R(z) < \infty] \\
&\qquad \times E^{Q_{y_1}}\left[\exp\left\{-\int_S^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < T\right] \\
&\quad \le P_{y_1}[H_R(z) < \infty] \\
&\qquad \times E^{Q_{y_1}}\left[Z_S \in \Lambda_{\gamma_2(m+n)}, \exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\} \circ \theta_S\right].
\end{aligned} \tag{3.24}$$

Observe then that under Q_{y_1}, conditionally on Z_S, the process $Z_{S+.}$ is distributed as Brownian motion starting from Z_S conditioned to enter $B_R(z)$ before $\mathscr{H}_m^a$. Moreover, if we define

$$\partial^+ O_m = \{z \in \partial O_m,\ z \cdot e_1 > 3mR + a\},$$

then $Z_S \in \partial^+ O_m$ Q_{y_1}-a.s. The rightmost member of (3.24) is therefore equal to

$$P_{y_1}[H_R(z) < \infty] \times E^{Q_{y_1}}\left[Z_S \in \Lambda_{\gamma_2(m+n)} \cap \partial^+ O_m, E_{Z_S}\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < H\right] \Big/ P_{Z_S}[H_R(z) < H]\right]$$

$$\leq A(z) \sup_{y \in \Lambda_{\gamma_2(m+n)} \cap \partial^+ O_m} E_y\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < H\right]$$

provided

$$A(z) = \left(\inf_{y \in \Lambda_{\gamma_2}(m+n) \cap \partial + O_m} P_y[H_R(z) < H]\right)^{-1}. \tag{3.25}$$

Inserting the inequality we just derived in (3.23) and taking a supremum over $x \in B_R(0)$, we obtain

$$\tilde{e}_\lambda(0, z) \leq A(z) \sup_{x \in B_R(0)} E_x\left[\exp\left\{-\int_0^H (\lambda + V)(Z_s)\, ds\right\}, H < T\right] \times \sup_{y \in \Lambda_{\gamma_2}(m+n) \cap \partial^+ O_m} E_y\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < H\right].$$

Now on the event $\{H < T\}$, H coincides with one of the $H_R(y)$ for some $y \in \mathscr{D}_m$ with $B_R(y) \cap \Lambda_{\gamma_2(m+n)} \neq \varnothing$. Therefore,

$$\tilde{e}_\lambda(0, z) \leq A(z) \sum_{y, y'} \sup_{x \in B_R(0)} E_x\left[\exp\left\{-\int_0^H (\lambda + V)(Z_s)\, ds\right\}, H_R(y) = H < T\right] \times \sup_{y'' \in B_R(y') \cap \partial^+ O_m} E_{y''}\left[\exp\left\{-\int_0^{H_R(z)} (\lambda + V)(Z_s)\, ds\right\}, H_R(z) < H\right] \tag{3.26}$$

provided y, y' in the summation belong to $\mathscr{D}_m$ and are such that $B_R(y)$ and $B_R(y')$, respectively, intersect $\Lambda_{\gamma_2(m+n)}$. We now raise both members of (3.26) to the power β and integrate over $\mathbb{P}$. The first term in the summation in the right-hand member of (3.26) is measurable with respect to the restriction of the Poisson cloud to $\{z \in \mathbb{R}^d,\ z \cdot e_1 < 3mR\}$, whereas the second term is measurable with respect to the restriction of the Poisson cloud to $\{z \in \mathbb{R}^d, z \cdot e_1 > 3mR\}$. These terms are therefore independent and we find

$$\mathbb{E}\Big[\tilde{e}_\lambda(0,z)^\beta\Big] \le A(z)^\beta \sum_{y,y'} \mathbb{E}\Big[\bar{e}_\lambda(0,y)^\beta\Big]\mathbb{E}\Big[\bar{e}_\lambda(y',z)^\beta\Big]. \tag{3.27}$$

Summing over $z \in \mathscr{D}_{m+n} \cap \Lambda_{\gamma_2(m+n)}$ in view of Lemma 3.2 and of the inequality $A^\beta \le A$, we have thus shown

$$\begin{aligned} &g_\beta(m+n) \\ &\quad \le \frac{\gamma_{12}}{\beta^d} m^{d-1} \sup_{z \in \mathscr{D}_{m+n} \cap \Lambda_{\gamma_2(m+n)}} A(z) g_\beta(m) g_\beta(n)(m+n)^{d-1}. \end{aligned} \tag{3.28}$$

There now remains to give an upper bound on $A(z)$. If $x \in \partial^+ O_m$, we denote by $\bar{x}$ the symmetric of x with respect to the hyperplane $\{z \in \mathbb{R}^d, z \cdot e_1 = 3Rm + a\}$. Now if $z \in \mathscr{D}_{m+n} \cap \Lambda_{\gamma_2(m+n)}$, it follows from the method of images that for $x \in \partial^+ O_m \cap \Lambda_{\gamma_2(m+n)}$,

$$\begin{aligned} &P_x[H_R(z) < H] \\ &\quad \ge c(d,R)\big\{|z-x|^{2-d} - |z-\bar{x}|^{2-d}\big\} \quad (\text{when } d \ge 3) \\ &\quad \ge c(R)\log\left(\frac{|z-\bar{x}|}{|z-x|}\right) \quad (\text{when } d = 2) \\ &\quad = \frac{R-a}{3nR - R - a} \quad (\text{when } d = 1). \end{aligned} \tag{3.29}$$

It is now straightforward to check that for $m, n \ge 1$,

$$\sup\big\{A(z), z \in \mathscr{D}_{m+n} \cap \Lambda_{\gamma_2(m+n)}\big\} \le \gamma_{13}(m+n)^d.$$

Our claim (3.18) now follows.

We define for $\beta \in (0,1)$ and $m \ge 1$,

$$F_\beta(m) = \log g_\beta(m). \tag{3.30}$$

It follows from Lemma 3.3 that for $\beta > 0$ and $m, n \ge 1$,

$$\begin{aligned} &F_\beta(m+n) \le F_\beta(m) + F_\beta(n) + G_\beta(m+n), \\ &\qquad \text{where } G_\beta(k) = \log\left(\frac{\gamma_{11}}{\beta^d}\right) + (3d-2)\log k. \end{aligned} \tag{3.31}$$

The growth of G is "moderate" in the sense that

$$4\sum_{m \ge 2k} \frac{G_\beta(m)}{m(m+1)} \le \frac{\gamma_{12}}{k}\left(1 + \log\frac{1}{\beta} + \log k\right) \quad \text{for } k \ge 1. \tag{3.32}$$

It follows from Hammersley [6] that

$$f(\beta) \overset{\text{def}}{=} \lim_{m\to\infty} \frac{F_\beta(m)}{m} \in [-\infty, \infty) \quad \text{exists}$$

$$(3.33) \qquad \le \frac{F_\beta(m)}{m} + \frac{\gamma_{12}}{m}\left(1 + \log\frac{1}{\beta} + \log m\right) \quad \text{for } m \ge 1.$$

On the other hand, we also have

$$g_\beta(m) \ge \mathbb{E}\left[e_\lambda(0, 3Rme_1)^\beta\right] \ge \exp\{\beta\mathbb{E}[\log e_\lambda(0, 3Rme_1)]\},$$

so that by (1.6),

$$(3.34) \qquad f(\beta) \ge -\beta 3R\alpha_\lambda(e_1).$$

We shall now derive an upper bound on $F_\beta(m)$. In view of (3.11) we have

$$\exp\{F_\beta(m)\} \le \frac{\gamma_3}{\beta^d} \sum_{z\in\mathscr{D}_m\cap\overline{\Lambda}_{\gamma_2 m}} \mathbb{E}\left[\bar{e}_\lambda(0, z)^\beta\right].$$

Observe that each ball of radius R can be covered by a fixed number of balls of radius 1, so that for $z \in \mathscr{D}_m$, $m \ge 1$ and $\beta \in (0, 1)$,

$$\mathbb{E}\left[\bar{e}_\lambda(0, z)^\beta\right]$$
$$\le c(d, R)\mathbb{E}\left[\sup_{x\in B_{2R+1}(0)} E_x\left[\exp\left\{-\int_0^{H(z)}(\lambda + V)(Z_s)\,ds\right\}, H(z) < \infty\right]^\beta\right],$$

thanks to translation invariance. Now $3Rm \le |z| \le \sqrt{d}\,\gamma_2 m$, where R is given in (3.5), and we can apply Harnack's inequality to find

$$\mathbb{E}\left[\bar{e}_\lambda(0, z)^\beta\right] \le \gamma_{13}\mathbb{E}\left[\inf_{B(0)} e_\lambda(x, z)^\beta\right] = \gamma_{13}\mathbb{E}[\exp\{-\beta a_\lambda(0, z)\}]$$
$$= \gamma_{13}\exp\{-\beta\mathbb{E}[a_\lambda(0, z)]\}\mathbb{E}\left[\exp\{\beta(\mathbb{E}[a_\lambda(0, z)] - a_\lambda(0, z))\}\right].$$

We now choose

$$(3.35) \qquad \beta = \kappa\left(\sqrt{d}\,\gamma_2 m\right) \in (0, 1)$$

and conclude from (3.3) that for $m \ge 1$ and $z \in \mathscr{D}_m$,

$$(3.36) \qquad \mathbb{E}\left[\bar{e}_\lambda(0, z)^\beta\right] \le \delta_2\exp\{-\beta\mathbb{E}[a_\lambda(0, z)]\}.$$

Moreover, since $W(\cdot)$ is rotationally invariant,

$$\mathbb{E}[a_\lambda(0, z)] = \mathbb{E}[a_\lambda(0, z')]$$

provided z denotes the image of z by a rotation of axis e_1 with angle π. Translation invariance and the triangle inequality for $a_\lambda(\cdot,\cdot)$ imply

$$\mathbb{E}[a_\lambda(0, 6Rme_1)] \le 2\mathbb{E}[a_\lambda(0, z)]$$

and, therefore,

$$\mathbb{E}\Big[\bar{e}_\lambda(0,z)^\beta\Big] \le \delta_2 \exp\left\{-\frac{\beta}{2}\mathbb{E}[a_\lambda(0,6Rme_1)]\right\}, \tag{3.37}$$

which together with (3.11) shows

$$\exp\{F_\beta(m)\} \le \frac{\delta_3}{\beta^d} m^d \exp\left\{-\frac{\beta}{2}\mathbb{E}[a_\lambda(0,6Rme_1)]\right\} \tag{3.38}$$

with β as in (3.35). Combining this with (3.33) and (3.34), we find that for $m > 1$,

$$-3R\beta\alpha_\lambda(e_1) \le \frac{\delta_4}{m}\left(1+\log\frac{1}{\beta}+\log m\right) - \frac{\beta}{2m}\mathbb{E}[a_\lambda(0,6Rme_1)]$$

and, therefore, for $x = 6Rme_1$,

$$\mathbb{E}[a_\lambda(0,z)] \le \alpha_\lambda(z) + \frac{2\delta_4}{\kappa(\sqrt{d}\,\gamma_2|z|/6R)}\left(1+\log\left(\frac{1}{\kappa(\sqrt{d}\,\gamma_2|z|/6R)}\right)+\log|z|\right).$$

Using the triangle inequality, translation and rotation invariance, we deduce that for any $z \in \mathbb{R}^d$,

$$\mathbb{E}[a_\lambda(0,z)] \le \alpha_\lambda(z) + \frac{\delta_5}{\kappa(\gamma_1|z|)}\left(1+\log\left(\frac{1}{\kappa(\gamma_1|z|)}\right)+\log(1+|z|)\right). \tag{3.39}$$

This together with Lemma 1.2 and (3.2) finishes the proof of Theorem 3.1. □

We can now combine Corollary 2.4, Theorem 2.5 and Theorem 3.1 to find the following corollary:

COROLLARY 3.4. *Assume* (3.1). *When* $d \ge 3$ *or* $\lambda > 0$, *then for* $z \in \mathbb{R}^d$,

$$D_\lambda(0,z) - \gamma_{14}\big(1+|z|^{1/2}\log^+(|z|)\big) \le \alpha_\lambda(z) \le D_\lambda(0,z). \tag{3.40}$$

When $d \le 2$ *and* $\lambda = 0$, *then for* $z \in \mathbb{R}^d$,

$$D_0(0,z) - \gamma_{15}\big(1+|z|^{1/2}(\log^+|z|)^2\big) \le \alpha_0(z) \le D_0(0,z). \tag{3.41}$$

We also have the following corollary:

COROLLARY 3.5. *Assume* (3.1). *When* $d \ge 3$ *or* $\lambda > 3$, *then* $\mathbb{P}$-*a.s. for large* $|z|$,

$$\begin{aligned}&\alpha_\lambda(z) - \gamma_{16}\big(1+|z|^{1/2}\log|z|\big)\\ &\qquad \le d_\lambda(0,z) \le \alpha_\lambda(z) + \gamma_{16}\big(1+|z|^{1/2}\log|z|\big).\end{aligned} \tag{3.42}$$

When $d \leq 2$ *and* $\lambda = 0$, *then* $\mathbb{P}$-*a.s. for large* $|z|$,

$$\alpha_0(z) - \gamma_{17}\big(1 + |z|^{1/2} \log^2 |z|\big) \leq d_0(0, z) \leq \alpha_0(z) + \gamma_{17}\big(1 + |z|^{1/2} \log^2 |z|\big). \tag{3.43}$$

PROOF. Our claim follows from Corollary 3.4 and Theorem 2.1 or 2.5 in the case of d_λ together with a Borel–Cantelli argument. [To this effect, it should be noticed that $\sup_{|x-y| \leq 1} d_\lambda(x, y)$ is uniformly bounded because of (1.35).] □

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