

ON THE CONVEX HULL OF PLANAR BROWNIAN SNAKE¹

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The planar Brownian snake is a continuous, strong Markov process taking values in the space of continuous functions in \mathbb{R}^2 that are stopped at some time. For a fixed time the snake is distributed like a planar Brownian motion with a random lifetime. This paper characterizes the convex hull of the trace of the snake paths that exit the half-plane at the origin. It is shown that the convex hull at 0 is roughly a factor of x smoother than the convex hull of a piece of planar Brownian motion at its minimum y -value.

The Brownian snake process of Le Gall is a continuous, strong Markov process taking its values in the space of paths with a random lifetime. For a fixed time t the process is distributed as a Brownian path stopped at a random time, and at a future time s the snake is generated by tracing back on the path at time t until some random point determined by the lifetime process where an independent piece of stopped Brownian path is added on.

For a smooth domain D in dimension 2, it is shown in Abraham and Le Gall [1] that the Brownian snake will, with positive measure, be a path that exits the domain at a given point; that is, it hits points on the boundary of D as it exits D . In particular, for the upper half-plane, H , the origin is a point of exit. Let \tilde{C} denote the convex hull of the trace of those snake paths which exit H at 0 up until the time of exiting H , let f be a function such that $(x, f(x))$ parameterizes $\partial\tilde{C}$ locally near the origin and let T denote the first time that the Brownian snake is a path which exits H at 0. We establish the following two theorems for the convex hull. Here \mathbb{N}_z denotes the excursion measure, to be defined later, under which all paths start at z

THEOREM 0.1. *Let $g(x) = x^2h(x)$ be an even, convex, \mathcal{C}^2 function satisfying $h(e^{-k})/h(e^{-k-1}) \rightarrow 1$. Then, $\mathbb{N}_{(0,1)}$ -almost surely on the event $\{T < \infty\}$,*

$$\liminf_{x \rightarrow 0} \frac{f(x)}{g(x)} = \begin{cases} 0, & \text{if } \sum_k h(e^{-k}) = \infty, \\ \infty, & \text{if } \sum_k h(e^{-k}) < \infty. \end{cases}$$

THEOREM 0.2. *Let $g(x) = x^2h(x)$ be an even, convex function satisfying $h(2^{-k})/h(2^{-k-1}) \rightarrow 1$. Then, $\mathbb{N}_{(0,1)}$ -almost surely on the event $\{T < \infty\}$,*

$$\limsup_{x \rightarrow 0} \frac{f(x)}{g(x)} = \begin{cases} \infty, & \text{if } h(x) \rightarrow 0, \\ 0, & \text{if } h(x) \rightarrow \infty. \end{cases}$$

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REMARK 0.3. It is shown for all finite c that $\limsup_{x \rightarrow 0} f(x)/x^2 \geq c$ with positive $\mathbb{N}_{(0,1)}$ measure, but not almost surely.

REMARK 0.4. The family of functions, for $n \geq 1$, $|\varepsilon| < 2$,

$$h_{n,\varepsilon}(x) = \log(1/x) \log_2(1/x) \cdots (\log_n(1/x))^\varepsilon,$$

satisfy the growth conditions of the theorems.

The convex hull of the two-dimensional Brownian path over $[0, 1]$ at its minimum value has been studied. In Cranston, Hsu and March [4], \liminf results are established which yield an integral test similar to our summation test (cf. [3]). Let f be a function locally giving the convex hull satisfying $f(0) = 0$, and let g be an even, convex function. Then $\liminf f/g = 0(\infty)$ if and only if $\int_{0+} g(x)x^{-2} = \infty (< \infty)$.

A sharp characterization of the \limsup of the convex hull is given by Mountford [11] improving upon the work of Burdzy and San Martin [3]. Mountford establishes that $(1/\pi)x \log_3(1/x)/\log(1/x)$ is an upper function for $f(x)$.

This paper shows that the convex hull in question is a factor of x smoother than that studied for the Brownian motion.

The paper begins with a preliminary section where facts about the Brownian snake and some general results are presented. The proofs of the theorems follow, starting with the \liminf result. The proofs for all four bounds are similar. A geometric criterion, as in [3], is used to bound the probability of a point being in \tilde{C} by the probability that a ray is hit by one of the paths.

To describe this, fix $z = (x, y)$ with $x, y > 0$. Let A be the ray emanating from z on the line connecting z and 0. Fix $0 < x' < x$. Let B be the ray from $(x', 0)$ that intersects z . If a path from $(0, 1)$ to 0 crosses A , then the point z will be in the convex hull generated by this path, since a point on A and the origin will be in the convex hull. Similarly, if no snake path from $(0, 1)$ exiting the half-plane at the origin crosses B , then the point z cannot possibly be in the convex hull generated by such paths. Hence, to characterize if a point z is in the convex hull, we need to understand if these rays are hit by snake paths that exit the half-plane at the origin. This can be done using a known excursion decomposition of the snake process. (See Figure 1).

To make almost sure statements, a type of 0–1 law is proved that makes use of a Poisson description of the process started from an initial starting path.

1. Preliminaries. In this paper a measure with an extra bar, such as \mathbb{N} , will denote a measure for the path-valued process. Measures without the extra bar will represent a Brownian motion which will be denoted generically by a B . As it is convenient, complex variable notation will be used to describe the plane \mathbb{R}^2 . In particular, $re^{i\theta}$ will denote the point $(r \cos \theta, r \sin \theta)$, where we will always assume $r \geq 0$ and $\theta \in [0, 2\pi)$, and if $z = (x, y)$, then $\text{Re}(z) = x$. Finite, positive constants whose values are unimportant will be generically denoted by c_i for some i . For a domain D we denote the Poisson kernel by

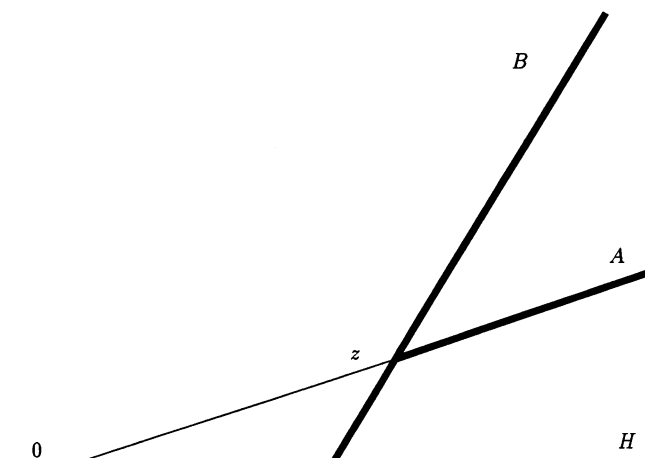


FIG. 1.

$p_D(z, y)$ and the Green function by $G_D(x, y)$. Let $\text{cone}(\gamma) = \{re^{i\alpha}: r > 0, \alpha \in (\pi/2 - \gamma, \pi/2 + \gamma)\}$ denote a cone with opening 2γ , $\gamma < \pi/2$.

The path-valued process, or Brownian snake, is a continuous, strong Markov process on the space of stopped paths from \mathbb{R}_+ to \mathbb{R}^d (cf. [7] and [6] for a construction). The process has as a state space the set of stopped paths. Let \mathscr{W}_z denote those starting from $z \in \mathbb{R}^d$:

$$\mathscr{W}_z = \{(w, \zeta): w \in \mathcal{C}([0, \infty), \mathbb{R}^d), w(0) = z, w(\zeta) = w(\zeta + t) \forall t > 0\}.$$

Let $\mathscr{W} = \bigcup \mathscr{W}_z$. We designate the tip or end of the path by $\hat{w} = w(\zeta)$ and notationally we identify w with W , so that $W_s(t)$ is the value of the path w_s evaluated at t . The notation for the lifetime is usually suppressed.

The path-valued process evolves under the measure $\mathbb{E}_{(w_0, \zeta_0)}$, where the subscript denotes the starting path. The evolution of the process is controlled in some sense by the lifetime process ζ . The process ζ under $\mathbb{E}_{(w_0, \zeta_0)}$ is distributed like a one-dimensional reflecting Brownian motion. The path $W_{s'}(\cdot)$ at $s' > s$ is distributed under $\mathbb{E}_{(w_0, \zeta_0)}$ like a Brownian path in \mathbb{R}^2 stopped at $\zeta_{s'}$. It agrees up until $\eta = \inf\{\zeta_u: s \leq u \leq s'\}$ with the path $W_s(\cdot)$, and its distribution after η is independent of W_s .

Let $\mathbb{P}_{(w_0, \zeta_0)}^*$ be the distribution for W killed when the lifetime process first hits 0. Excursion theory is used to decompose the process into easier-to-understand pieces. The symbol $I = \{(\alpha_i, \beta_i)\}$ will be used for the collection of excursion intervals; the lifetime of an excursion will be denoted by a σ . The excursion measure for excursions of the path-valued process away from the recurrent state $\{\zeta = 0\}$ will be denoted by \mathbb{N}_z , where the point $z \in \mathbb{R}^2$ is identified with the trivial stopped path in \mathscr{W}_z , $(w, 0)$, where $w(0) = z$.

As an example, let $\zeta_0 > 0$, and set $m_s = \inf\{\zeta_u: u \leq s\}$. By the well-known theorem of Lévy, the process $\zeta_s - m_s$ up until $\inf\{s: \zeta_s = 0\}$ is distributed under $\mathbb{P}_{(w_0, \zeta_0)}^*$ like a reflecting Brownian motion started from 0 up until the

time when its local time at the origin becomes $2\zeta_0$. Let I denote the excursion intervals of $\zeta - m$ from 0, and for each excursion interval (α_i, β_i) define

$$W_s^i(t) = W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i} + t).$$

Notice $W_s^i(t) \in \mathscr{W}_{\hat{W}_{\alpha_i}}$. The key tool to understand the process started from a fixed path is the following result (cf. [10], Proposition 2.5).

PROPOSITION 1.1. *The random measure $\sum_{i \in I} \delta_{\zeta_{\alpha_i}, W^i}$ under $\mathbb{P}_{(w_0, \zeta_0)}^*$ is a Poisson random measure with intensity $2 dt \mathbf{1}_{t \leq \zeta_0} \mathbb{N}_{w_0(t)}(\cdot)$.*

That is, the trace of the process has a backbone, $w_0(\cdot)$, and branching off this backbone at a rate $2 dt$ are independent excursions of the process W .

To study the boundary behavior, we use the exit measure, as defined in [10]. Let D be a domain in \mathbb{R}^2 . For a path f we define the exit time from D by

$$\tau_D = \tau_D(f) = \inf\{t: f(t) \notin D\},$$

with the agreement that $\tau_D = \infty$ in the event the set in the infimum is empty. Let I_D denote the excursion intervals from 0 of the process $(\zeta_s - \tau_D(W_s))_+$, that is, the excursions of the path-valued process away from the boundary of D . (These excursions may enter back into the domain; they are simply the excursions relating to the times that the path process, W , is a path that has left D for some duration.) There is a local time on this set denoted by L^D , which may be derived in terms of the local time from 0 of an associated reflecting Brownian motion. The exit measure is a random measure supported on the boundary of D . As defined here, it is a random measure under \mathbb{N}_z , $z \in D$, given by its action on measurable functions ϕ on ∂D as

$$\langle X^D, \phi \rangle = \int_0^\sigma dL_s^D \phi(\hat{W}_s).$$

We will use the following facts about the Brownian snake. Let E_z^D be the distribution of $\{B_{\cdot \wedge \tau_D}, \tau_D < \infty\}$, where B is a Brownian motion in \mathbb{R}^2 . From [10], Proposition 3.3, we have the following formula relating the measures \mathbb{N}_z and E_z^D .

PROPOSITION 1.2. *For every nonnegative, measurable function F on \mathscr{W}_z ,*

$$\mathbb{N}_z \left(\int_0^\sigma dL_s^D F(W_s) \right) = E_z^D(F).$$

In particular, for nonnegative, measurable functions ϕ on ∂D , $\mathbb{N}_z(\langle X^D, \phi \rangle) = E_z^D(\phi(B_{\tau_D}))$.

The exit measure enjoys the following special Markov property (cf. [8], Theorem 2.3). Let

$$\eta_s = \inf \left\{ t: \int_0^t du \mathbf{1}(\zeta_u \leq \tau_D(W_u)) > s \right\},$$

and let \mathcal{E}^D be the \mathbb{N}_z -completion of the σ -field generated by W_{η_s} , a time change of the path-valued process corresponding to the times when the paths have not left D .

THEOREM 1.3 (Special Markov property). *For every nonnegative, measurable function Φ on $C_0(\mathbb{R}_+, \mathcal{W})$,*

$$\mathbb{N}_z\left(\exp - \sum_{i \in I_D} \Phi(W^i) \mid \mathcal{E}^D\right) = \exp - \int X^D(dy) \mathbb{N}_y(1 - \exp - \Phi).$$

Using the special Markov property, we state the following simple corollary.

COROLLARY 1.4. *For Φ as above, by Proposition 1.2,*

$$\begin{aligned} \mathbb{N}_z\left(\sum_{i \in I_D} \Phi(W^i)\right) &= \frac{d}{d\lambda} \mathbb{N}_z\left(\exp - \lambda \sum_{i \in I_D} \Phi(W^i)\right) \Big|_{\lambda=0} \\ &= \frac{d}{d\lambda} \mathbb{N}_z\left(\exp - \int X^D(dy) \mathbb{N}_y(1 - \exp - \lambda\Phi)\right) \Big|_{\lambda=0} \\ &= \mathbb{N}_z\left(\int X^D(dy) \mathbb{N}_y(\Phi)\right) \\ &= E_z(\mathbb{N}_{B_\tau}(\Phi)). \end{aligned}$$

We quote the following relationships between the Brownian snake and the partial differential equation $\Delta u = 4u^2$. Let D be a bounded, Lipschitz domain in \mathbb{R}^d . Let $O \subset \partial D$ be open and $K \subset \partial D$ be compact. Define the stopping times for W :

$$\begin{aligned} T_D &= \inf\{s: \zeta_s = \tau_D(W_s)\}, \\ T_{K,D} &= \inf\{s: \zeta_s = \tau_D(W_s), \hat{W}_s \in K\}. \end{aligned}$$

When $K = \{0\}$ we write $T_{K,D} = T_{0,D}$. The range of the path-valued process killed on ∂D is

$$\mathcal{R}^D = \{W_s(t): \zeta_s = \tau_H(W_s), 0 \leq t \leq \zeta_s \wedge \tau_D(W_s)\}.$$

One has $\{T_{K,D} < \infty\} = \{\mathcal{R}^D \cap K \neq \emptyset\}$. That is, $T_{K,D}$ is finite when the range of the Brownian snake paths killed on ∂D intersects K . From [1], Proposition 1.2, and [9], Proposition 4.4, we have the following result.

PROPOSITION 1.5. *Let $u_1(z) = \mathbb{N}_z(X^D(O) > 0)$ and $u_2(z) = \mathbb{N}_z(T_{K,D} < \infty)$. Then u_1, u_2 satisfy $\Delta u = 4u^2$ in D . The minimal solution satisfying the boundary condition $\lim_{z \rightarrow y} u(z) = \infty$ for all $y \in O$ is u_1 . The maximal solution satisfying the boundary condition $\lim_{z \rightarrow y} u(z) = 0$ for all $y \in K$ is u_2 .*

1.1. Calculations.

LEMMA 1.6 (Scaling). *Let D be some domain such that $0 \in D$. The dilation of D is defined by $\varepsilon D = \{\varepsilon x: x \in D\}$. Let K be a compact subset of the boundary of D . Then*

$$\mathbb{N}_0(T_{K,D} < \infty) = \varepsilon^2 \mathbb{N}_0(T_{\varepsilon K, \varepsilon D} < \infty).$$

PROOF. Set $W_s^{(\varepsilon)}(t) = \varepsilon^{-1} W_{\varepsilon^4 s}(\varepsilon^2 t)$. Then, as remarked in Le Gall [10], Proposition 2.3, due to the scaling properties of Brownian motion and the Itô measure of excursions the distribution of $W_s^{(\varepsilon)}(t)$ under \mathbb{N}_0 is $\varepsilon^{-2} \mathbb{N}_0$. Thus

$$\begin{aligned} \varepsilon^2 \mathbb{N}_0(T_{\varepsilon K, \varepsilon D} < \infty) &= \varepsilon^2 \mathbb{N}_0(\exists s \geq 0: \hat{W}_s \in \varepsilon K, \zeta_s = \tau_{\varepsilon D}) \\ &= \varepsilon^2 \mathbb{N}_0(\exists s \geq 0: \hat{W}_s^{(\varepsilon)} \in K, \zeta_s^{(\varepsilon)} = \tau_D) \\ &= \mathbb{N}_0(\exists s \geq 0: \hat{W}_s \in K, \zeta_s = \tau_D) \\ &= \mathbb{N}_0(T_{K,D} < \infty). \end{aligned} \quad \square$$

By scaling we can calculate $\mathbb{N}_z(T < \infty)$.

LEMMA 1.7. *Let $z = re^{i\theta}$. We have*

$$u(z) = \mathbb{N}_z(T < \infty) = w(\theta)r^{-2},$$

where $w(\theta)$ is the unique positive solution on $(0, \pi)$ to the one-dimensional boundary value problem

$$(1.1) \quad w''(\theta) = 4w(\theta)(1 - w(\theta)), \quad w(0) = w(\pi) = 0.$$

Furthermore, there exist positive, nontrivial constants $c \leq C$ such that, for $0 \leq \theta \leq \pi/2$,

$$(1.2) \quad c\theta \leq w(\theta) = w(\pi - \theta) \leq C\theta.$$

PROOF. First, we show that scaling yields $u(z) = \hat{w}(\theta)r^{-2}$ for some \hat{w} . Set $H(z, \varepsilon) = \{\rho e^{i\gamma}: \rho > \varepsilon, \gamma \in (0, \pi)\} + z$, and let $\bar{B}(x, \varepsilon)$ denote the closed ball of radius ε centered at x . By monotonicity,

$$\begin{aligned} \mathbb{N}_z(T < \infty) &= \mathbb{N}_0(T_{-z, H-z} < \infty) \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{N}_0(T_{\bar{B}(-re^{i\theta}, \varepsilon), H(-re^{i\theta}, \varepsilon)} < \infty) \\ (1.3) \quad &= \lim_{\varepsilon \rightarrow 0} r^{-2} \mathbb{N}_0(T_{(1/r)\bar{B}(re^{i\theta}, \varepsilon), (1/r)H(re^{i\theta}, \varepsilon)} < \infty) \end{aligned}$$

$$\begin{aligned} (1.4) \quad &= r^{-2} \mathbb{N}_0(T_{\{-e^{i\theta}\}, H-e^{i\theta}} < \infty) \\ &= r^{-2} \mathbb{N}_{e^{i\theta}}(T < \infty). \end{aligned}$$

In line (1.3), for a fixed r and θ , the dilation of the sets $(1/r)B(re^{i\theta}, \varepsilon)$ and $(1/r)H(re^{i\theta}, \varepsilon)$ is not degenerate and so the limit in (1.4) follows. Setting $\hat{w}(\theta) = \mathbb{N}_{e^{i\theta}}(T < \infty)$ gives the first result. By symmetry, $\hat{w}(\theta) = \hat{w}(\pi - \theta)$.

The proofs in [9], Proposition 4.4, show that $u(z)$ satisfies $\Delta u = 4u^2$ in H , with boundary condition $u(z) \rightarrow 0$ as $z \rightarrow \partial H \setminus 0$. Thus $\hat{w}(\theta)$ solves (1.1).

In [1] it is shown for $D \subset \mathbb{R}^2$, D smooth, that points on the boundary of D are hit by the path-valued process as it exits the boundary. Thus $\hat{w}(\theta)$ is positive on $(0, \pi)$.

Solutions to $\Delta u = u^2$ are studied in Gmira and Veron [5]. The uniqueness of a positive solution to (1.1) under these boundary conditions is classical and is assured by Proposition 6.5 therein. Hence $\hat{w} = w$.

By a simple calculus trick one has the equivalent differential equation for w :

$$\left(\frac{dw}{d\theta}\right)^2 = \left(\frac{8}{3}\right)w^3 - 4w^2 + (w'(0))^2, \quad w(0) = 0, \quad w(\pi) = 0,$$

where $w'(0)$ is the one-sided derivative. If $w'(0)$ were 0, then the solution would be trivial and it is not; hence $\lim_{\theta \rightarrow 0} w(\theta)/\theta = c_1 \neq 0$. Since $u(e^{i\theta})$ is bounded, for $0 \leq \theta \leq \pi/2$ there exist constants c and C for which

$$c\theta \leq w(\theta) \leq C\theta. \quad \square$$

The next lemma will be used to calculate a lower bound on the event that there exists a path exiting H at 0 that exits a domain through a specified subset of the boundary.

LEMMA 1.8. *For a Greenian domain D , set $A = \sum_{i \in I_D} \Phi(W^i)$, where $\Phi = \Phi^2$. Let $v(z) = \mathbb{N}_z(A)$. Then*

$$\mathbb{N}_z(A > 0) \geq \left(4 \int G_D(z, y) \left(\frac{v(y)}{v(z)}\right)^2 + v(z)^{-1}\right)^{-1}.$$

PROOF. The proof follows that of Proposition 2.2 and Theorem 2.3 in [1]. We have by the Cauchy–Schwarz inequality that

$$\mathbb{N}_z(A > 0) \geq \mathbb{N}_z(A)^2 / \mathbb{N}_z(A^2).$$

It suffices to show that

$$\mathbb{N}_z(A^2) = 4 \int G_D(z, y) v(y)^2 + v(z).$$

First, as in Corollary 1.4,

$$\begin{aligned} \mathbb{N}_z\left(\left(\sum \Phi\right)^2\right) &= \frac{d^2}{d\lambda^2} \mathbb{N}_z\left(\exp -\lambda \sum \Phi(W^i)\right) \Big|_{\lambda=0} \\ &= \frac{d^2}{d\lambda^2} \mathbb{N}_z\left(\exp -2 \int X^D(dy) \mathbb{N}_y(1 - \exp -\lambda\Phi)\right) \Big|_{\lambda=0} \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{N}_z \left(2 \int X^D(dy) \mathbb{N}_y(\Phi^2) \right) + \mathbb{N}_z \left(\left(2 \int X^D(dy) \mathbb{N}_y(\Phi) \right)^2 \right) \\
 &= v(z) + 4 \mathbb{N}_z \left(\left(\int X^D(dy) \mathbb{N}_y(\Phi) \right)^2 \right).
 \end{aligned}$$

Next, with $\tau = \tau_D$,

$$\begin{aligned}
 &\mathbb{N}_z \left(\left(\int X^D(dy) \mathbb{N}_y(\Phi) \right)^2 \right) \\
 &= \mathbb{N}_z \left(\int \int dL_s^D dL_{s'}^D \mathbb{N}_{\hat{W}_s}(\Phi) \mathbb{N}_{\hat{W}_{s'}}(\Phi) \right) \\
 &= 2 \mathbb{N}_z \left(\int \int_{s' > s} dL_s^D dL_{s'}^D \mathbb{N}_{\hat{W}_s}(\Phi) \mathbb{N}_{\hat{W}_{s'}}(\Phi) \right) \\
 (1.5) \quad &= 2 \mathbb{N}_z \left(\int dL_s^D \mathbb{N}_{\hat{W}_s}(\Phi) \mathbb{E}_{W_s}^* \left(\int dL_{s'}^D \mathbb{N}_{\hat{W}_{s'}}(\Phi) \right) \right) \\
 (1.6) \quad &= 2 \mathbb{N}_z \left(\int dL_s^D \mathbb{N}_{\hat{W}_s}(\Phi) \int_0^{\zeta_s} 2 dt \mathbb{N}_{W_s(t)} \left(\int dL_{s'}^D \mathbb{N}_{\hat{W}_{s'}}(\Phi) \right) \right) \\
 &= 4 \mathbb{N}_z \left(\int dL_s^D \mathbb{N}_{\hat{W}_s}(\Phi) \int_0^{\zeta_s} dt \mathbb{N}_{W_s(t)}(\langle X^D, \mathbb{N} \cdot (\Phi) \rangle) \right) \\
 (1.7) \quad &= 4 E_z^D \left(\mathbb{N}_{B_\tau}(\Phi) \int_0^\tau dt E_{B_t}(\mathbb{N}_{B_t}(\Phi)) \right) \\
 (1.8) \quad &= 4 E_z^D \left(\int_0^\tau dt E_{B_t}(\mathbb{N}_{B_t}(\Phi))^2 \right) \\
 &= \int G_D(z, y) v(y)^2.
 \end{aligned}$$

Line (1.5) follows from the strong Markov property of the process W , (1.6) follows from Proposition 1.1, (1.7) is an application of Proposition 1.2 and (1.8) follows by the Markov property of Brownian motion. \square

The following lemma will allow us to make almost sure statements. It states that when an event has a positive probability conditioned on $T < \infty$ for all starting points in a cone, then one of the excursions in the Poisson point process of Proposition 1.1 will have the event happen for it almost surely.

LEMMA 1.9. *Suppose B is an event satisfying:*

- (i) *There exists a constant $c > 0$ depending on γ but not on z such that, when $z \in \text{cone}(\gamma)$, $\mathbb{N}_z(B, T < \infty) = \mathbb{N}_z(B) \geq cu(z)$.*
- (ii) *If Δ is a subinterval of $[0, \sigma]$, then $1_B(W) \geq 1_B(W(\Delta))$, where $W(\Delta)$ is the path process $\{W_t: t \in \Delta\}$.*

Then $\mathbb{N}_z(B, T < \infty) = u(z)$ for all z . That is, B happens almost surely on $\{T < \infty\}$.

PROOF. Let $\Delta_k = [(3/4)2^{-k}, 2^{-k}]$ and $\Omega_k = \{z: |z| \leq A2^{-k/2}, d(z, H) \geq \alpha 2^{-k/2}\}$.

We employ Lemmas 4.2 and 4.3 of [1] as utilized in Theorem 7.1. These assure us that we can find two constants, $\alpha < 1$ and $A > 1$, for which, \mathbb{N}_z -almost surely on $\{T < \infty\}$, the following event happens infinitely often (let $\zeta_T - \Delta_k = [\zeta_t - 2^{-k}, \zeta_t - (3/4)2^{-k}]$):

$$\{\forall t \in \zeta_T - \Delta_k, W_T(t) \in \Omega_k\}.$$

Let κ be the set of k for which this happens $[\#(\kappa) = \infty]$. Then, for $k \in \kappa$, $t \in \zeta_T - \Delta_k$, we have $W_T(t) = re^{i\theta} \in \text{cone}(\cos^{-1}(\alpha/A))$ and $r > \alpha 2^{-k/2}$. By Lemma 1.7 and line (1.2) we have, with $c_4 = c_4(\gamma)$,

$$(1.9) \quad \mathbb{N}_{W_T(t)}(B, T < \infty) \geq c_3 u(W_T(t)) \geq c_4 2^k.$$

Considering the excursions from the minimum of ζ_{T+t} , we have, from Proposition 1.1,

$$\begin{aligned} & \mathbb{N}_z \left(\mathbb{E}_{W_T}^* \left(\sum_{i \in I} \mathbf{1}_B(W^i) = 0 \right), T < \infty \right) \\ &= \lim_{\lambda \rightarrow \infty} \mathbb{N}_z \left(\mathbb{E}_{W_T}^* \left(\exp -\lambda \sum_{i \in I} \mathbf{1}_B(W^i) \right), T < \infty \right) \\ &= \lim_{\lambda \rightarrow \infty} \mathbb{N}_z \left(\exp -2 \int_0^{\zeta_T} dt \mathbb{N}_{W_T(t)}(1 - \exp -\lambda \mathbf{1}_B(W)), T < \infty \right) \\ &= \mathbb{N}_z \left(\exp -2 \int_0^{\zeta_T} dt \mathbb{N}_{W_T(t)}(B), T < \infty \right) \\ &\leq \mathbb{N}_z \left(\exp -2 \sum_{\kappa} \int_{\zeta_T - 2^{-k}}^{\zeta_T - (3/4)2^{-k}} dt \mathbb{N}_{W_T(t)}(B), T < \infty \right) \\ &\leq \mathbb{N}_z(\exp -c_5 \#(\kappa)) = 0. \end{aligned}$$

This gives

$$\begin{aligned} u(z) &= \mathbb{N}_z(T < \infty) \\ &= \mathbb{N}_z \left(1 - \mathbb{E}_{W_T}^* \left(\sum_{i \in I} \mathbf{1}_B(W^i) = 0 \right), T < \infty \right) \\ &= \mathbb{N}_z \left(\mathbb{E}_{W_T}^* \left(\sum_{i \in I} \mathbf{1}_B(W^i) > 0 \right), T < \infty \right) \\ &\leq \mathbb{N}_z(\mathbf{1}_B(W) > 0, T < \infty) \\ &= \mathbb{N}_z(B). \end{aligned}$$

□

We use the following formulas for the Poisson kernel and Green function in the half-plane (cf. [2], Chapter 2):

$$(1.10) \quad \begin{aligned} p_H(z, 0) &= p_H(re^{i\theta}, 0) = \frac{1}{\pi} \frac{\sin \theta}{r}, \\ G_H(x, y) &= \frac{1}{\pi} \log \frac{|\tilde{x} - y|}{|x - y|}, \end{aligned}$$

where \tilde{x} is the reflection of x through the y -axis.

2. Convex hull. Let D_g denote the domain given by the epigraph of the function g , and let I_D denote the excursion intervals from a domain D . Make the following definitions:

$$\begin{aligned} R_{z,\alpha} &= \{z + re^{i\alpha} : r > 0\}, \\ \Gamma_{z,\alpha} &= H \cap \{w : w = z + re^{i\alpha'} : r > 0, \alpha' \in (\alpha, \pi + \alpha)\}, \\ A_{a,\alpha} &= \sum_{i \in I_{\Gamma_{ae^{i\alpha}, \alpha}}} \mathbf{1}_{R_{ae^{i\alpha}, \alpha}}(\hat{W}_0^i) \mathbf{1}_{T < \sigma}(W^i), \\ B_{a,\alpha} &= \sum_{i \in I_{\Gamma_{(a,0), \alpha}}} \mathbf{1}_{R_{(a,0), \alpha}}(\hat{W}_0^i) \mathbf{1}_{T < \sigma}(W^i), \\ C^j &= \sum_{I_{D_g}} \mathbf{1}(\text{Re}(\hat{W}_0^i) \in (e^{-j-1}, e^{-j}]) \mathbf{1}_{T < \sigma}(W^i) \end{aligned}$$

and

$$C_k^n = \sum_k^n C^j, \quad n \geq k \text{ (including } \infty).$$

For $z = ae^{i\alpha} = (b, 0) + e^{i\theta}$ we have $A_{a,\alpha}$ is the number of excursions from the ray emanating from z on the line connecting z to the origin that exit H at the origin at some time, and $B_{b,\theta}$ is the number of excursions from a ray emanating from $(b, 0)$ which intersects the point z that exit H at the origin at some time. As in the Introduction, if $A_{a,\alpha} > 0$, then z is in the convex hull; if $B_{b,\theta} = 0$, then z is not in the convex hull. Thus

$$(2.1) \quad \{A_{a,\alpha} > 0\} \subset \{z \in \tilde{C}\} \subset \{B_{b,\theta} > 0\}.$$

Here C_k^n is the number of excursions that exit D_g with the tip in the interval $(e^{-k}, e^{-n}]$ that at some time exit the half-plane at the origin.

Because of the monotone nature of the problem, we may assume (and do) without loss of generality that for fixed $\alpha > 0$ we have

$$(2.2) \quad x^\alpha \leq h(x) \leq x^{-\alpha}.$$

We need the following two lemmas about B and A .

LEMMA 2.1. *Let $z \in H$, and let ϕ be the conformal map $\phi(z) = ((z - (a, 0))e^{-i\alpha})^\beta$ with $\beta = \pi/(\pi - \alpha)$ which maps a wedge to the half-plane. There exists a constant c_1 independent of z for which*

$$\mathbb{N}_z(B_{a,\alpha}) \leq c_1 w(\alpha) a^{\beta-2} d(\phi(z), H)^{-1}.$$

PROOF. Let τ be the hitting time of $\Gamma_{(a,0),\alpha}$. We use the basic facts about the Brownian snake from the preliminary section to establish

$$\begin{aligned} \mathbb{N}_z(B_{a,\alpha}) &= E^z(B_\tau \in R_{(a,0),\alpha}; u(B_\tau)) \\ &= \int_0^\infty P^z((B_\tau - (a, 0))e^{i\alpha} \in dr) u((a, 0) + re^{i\alpha}) \\ &\leq c_2 \int_0^\infty dr p_H(\phi(z), r^\beta) r^{\beta-1} w(\alpha) r \\ &\quad \times ((a + r \cos \alpha)^2 + (r \sin \alpha)^2)^{-3/2} \\ &= c_2 w(\alpha) a^{\beta-2} \int_0^\infty dr p_H(\phi(z), (ar)^\beta) r^\beta \\ &\quad \times ((1 + r \cos \alpha)^2 + (r \sin \alpha)^2)^{-3/2} \\ &\leq c_3 w(\alpha) a^{\beta-2} d(\phi(z), H)^{-1}. \end{aligned} \quad \square$$

LEMMA 2.2. *Let $z \in H$, $a_k e^{i\alpha_k} = (e^{-k}, g(e^{-k}))$ and $A_k = A_{a_k, \alpha_k}$. Let ϕ be the conformal map $\phi(z) = (ze^{-i\alpha})^\beta$ with $\beta = \pi/(\pi - \alpha)$ which maps a wedge to the half-plane.*

There exist nontrivial constants independent of z for which

$$c_0 h(e^{-k}) \inf_{[1,2]} p(\phi(z), (av)^\beta) \leq \mathbb{N}_z(A_k) \leq c_1 h(e^{-k}) d(\phi(z), H)^{-1}.$$

PROOF. As in the proof of Lemma 2.1, we have

$$\begin{aligned} \mathbb{N}_z(A_k) &= \int_a^\infty p_H(\phi(z), v^\beta) w(\alpha) v^{\beta-3} dv \\ &= w(\alpha_k) a_k^{\beta_k-2} \int_1^\infty p_H(\phi(z), (a_k v)^{\beta_k}) v^{\beta_k-3} dv. \end{aligned}$$

A simple consequence of (1.2) and the assumption (2.2) is that there exist constants c_3, c_4 for which

$$(2.3) \quad c_3 h(e^{-k}) \leq w(\alpha_k) a_k^{\beta_k-2} \leq c_4 h(e^{-k}).$$

With this and the bound $p_H(z, x) \leq d(z, H)^{-1}$, part (i) follows.

Similarly,

$$(2.4) \quad \mathbb{N}_z(A_k) \geq c_5 h_k \inf_{[1,2]} p(\phi(z), (av)^\beta),$$

since

$$\int_1^\infty p_H(\phi(z), (a_k v)^{\beta_k}) v^{\beta_k-3} dv \geq \inf_{[1,2]} p(\phi(z), (a_k v)^{\beta_k}) \int_1^2 v^{\beta-3} dv. \quad \square$$

2.1. Proof of Theorem 0.1.

REMARK 2.3. We first note that it is enough to show that the one-sided limit satisfies the bound almost surely. Because g is assumed to be even, the limit from the left and the limit from the right will share the same bound almost surely.

REMARK 2.4. The assumption that g is \mathcal{C}^2 is made for convenience, but is not essential. One can show that, given g for which $\sum_k h(e^{-k}) < \infty$, there will exist a \mathcal{C}^2 function g_1 satisfying the same condition on the sum and for which $g_2 \geq g_1$. Similarly, if the sum is infinite, one can find a \mathcal{C}^2 function bounding g from below for which the sum is still infinite. The proof of this is to show that the problem is monotone with respect to the domains D_g , and from here notice that any convex function $g(x) = x^2h(x)$ with $h(x) \rightarrow \infty$ has a second derivative of 0 at the origin.

2.1.1. Case $\sum_k h(e^{-k}) < \infty$. Define a_k and α_k by $a_k + b_k e^{i\alpha_k} = (e^{-k}, g(e^{-k})) = z_k$, and set $B_k = B_{a_k, \alpha_k}$. We show first that $\{z_k \notin \tilde{C} \text{ eventually}\}$ happens almost surely. To do this, we show that

$$\mathbb{N}_{(0,1)}(z_k \in \tilde{C} \text{ i.o.}) = 0.$$

This will follow by the Borel–Cantelli argument by showing $\sum_k \mathbb{N}_{(0,1)}(z_k \in \tilde{C}) < \infty$. By (2.1) and Lemma 2.1, we have

$$\begin{aligned} \sum_k \mathbb{N}_{(0,1)}(z_k \in \tilde{C}) &\leq \sum_k \mathbb{N}_{(0,1)}(B_k > 0) \\ &\leq \sum_k c_1 w(\alpha_k) a_k^{\beta_k - 2} d(f(z), H)^{-1} \\ &\leq \sum_k c_2 h(e^{-k}) < \infty. \end{aligned}$$

To finish, we need to interpolate between the points z_k . Suppose $z_k \notin \tilde{C}$. Then, by convexity, we must have that the point $(x, xg(e^{-k})e^k) \notin \tilde{C}$ for $x > e^{-k}$. If we can find a c_3 independent of k for which $xg(e^{-k})e^k \geq c_3g(x)$ for $x \in [e^{-k}, e^{-k+1})$, then on $[e^{-k}, e^{-k+1})$ we have $f(x) \geq c_3g(x)$. By the first part, $z_k \notin \tilde{C}$ happens eventually, so we would conclude that $f(x) \geq c_3g(x)$ eventually or $\liminf f(x)/g(x) \geq c_3$, $\mathbb{N}_{(0,1)}$ -almost surely. Since $cg(x)$ will satisfy the assumptions for any constant c , we conclude that $\liminf f(x)/g(x) = \infty$.

To find c_3 , notice

$$\begin{aligned} g(e^{-k})e^k &= \left(\frac{g(e^{-k})}{g(e^{-k+1})} \frac{e^{-k+1}}{e^{-k}} \right) \frac{g(e^{-k+1})}{e^{-k+1}} \\ &\geq \left(e \frac{g(e^{-k})}{g(e^{-k+1})} \right) \frac{g(x)}{x} \\ &\geq c_3 \frac{g(x)}{x} \end{aligned}$$

for some nontrivial constant c_3 . This follows since $g(x)x^{-1}$ is monotone by convexity and

$$\frac{g(e^{-k})}{g(e^{-k+1})} \rightarrow 1/e^2$$

by our assumptions on g .

2.1.2. *Case $\sum_k h(e^{-k}) = \infty$.* Fix $\gamma \in (0, \pi/2)$. We show that, for k large enough, we have $C_k^\infty > 0$ happens infinitely often \mathbb{N}_z -almost surely on $T < \infty$ for z in $\text{cone}(\gamma)$ and, in particular, for $z = (0, 1)$. Thus there is a sequence of points on the curve $\gamma(x) = (x, g(x))$ that converge to 0 corresponding to excursions leaving from ∂D_g that exit the half-plane at the origin. It is clear that each of these points is in the convex hull of W as it exits 0 and so we conclude that $\liminf f(x)/g(x) \leq 1$. Again, the assumptions on g are also satisfied by cg for any nontrivial constant c and so we conclude that $\liminf f(x)/g(x) = 0$.

Let $h_j = h(e^{-j})$ and $\Delta_j = (e^{-j-1}, e^{-j}]$. The assumption that g is \mathcal{C}^2 assures us that the Poisson kernel, $p_{D_g}(x, y)$, is continuous in $y \in \partial D_g$ and satisfies the following conditions:

1. There exist nontrivial constants for which

$$c_1 d(z, \partial D_g) |z - y|^2 \leq p_{D_g}(z, y) \leq c_2 d(z, \partial D_g) |z - y|^2.$$

We remark that when $z \in \text{cone}(\gamma)$ there exist two nontrivial constants for which

$$(2.5) \quad c_3 |z|^{-1} \leq p_{D_g}(z, y) \leq c_4 |z|^{-1}.$$

2. For all $\varepsilon > 0$ there exists $k_0 = k_0(z)$ such that, for all $k \geq k_0(z)$,

$$(2.6) \quad 1 - \varepsilon \leq \left| \frac{|\Delta_k|^{-1} P^z(B_{\tau_{D_g}} \in \gamma(\Delta_k))}{p_{D_g}(z, 0)} \right| \leq 1 + \varepsilon.$$

This is because p is continuous and $|\gamma(\Delta_k)| |\Delta_k|^{-1} \rightarrow 1$.

We first calculate

$$(2.7) \quad \begin{aligned} \mathbb{N}_z(C^j) &= \int_{\Delta_j} P^z(\text{Re}(B_{\tau_{D_g}}) \in dr) u(\gamma(r)) \\ &\leq c_5 \int_{\Delta_j} P^z(\text{Re}(B_{\tau_{D_g}}) \in dr) g(e^{-j}) e^j e^{-2(j+1)} \\ &\leq c_6 h_j |\Delta_j|^{-1} P^z(B_{\tau_{D_g}} \in \gamma(\Delta_j)). \end{aligned}$$

Fix $\varepsilon > 0$ and z . Find $k_0 = k_0(z)$ as above. Then we have by Lemma 1.8 that

$$(2.8) \quad \mathbb{N}_z(C_k^\infty) \geq \liminf_n \left(\int dy G_{D_g}(z, y) \left(\frac{\mathbb{N}_y(C_k^n)}{\mathbb{N}_z(C_k^n)} \right)^2 + (\mathbb{N}_z(C_k^n))^{-1} \right)^{-1}.$$

For $k \geq k_0$ by (2.6),

$$\begin{aligned} \mathbb{N}_z(C_k^n) &= \sum_k^n \mathbb{N}_z(C^j) \\ &\geq \sum_k^n c_7 h_j |\Delta_j|^{-1} P^z(B_{\tau_{D_g}} \in \gamma(\Delta_j)) \\ &\geq c_8(1 - \varepsilon) p_{D_g}(z, 0) \sum_k^n h_j \rightarrow \infty. \end{aligned}$$

Thus we need only worry about the first term in (2.8):

$$\begin{aligned} &\limsup_n \int dy G_{D_g}(z, y) \left(\frac{\mathbb{N}_y(C_k^n)}{\mathbb{N}_z(C_k^n)} \right)^2 \\ &= \limsup_n \int dy G_{D_g}(z, y) \\ &\quad \times \left(\left(\sum_k^{k_0(y)} \mathbb{N}_y(C^j) \right)^2 + 2 \left(\sum_k^{k_0(y)} \mathbb{N}_y(C^j) \right) \left(\sum_{k_0(y)}^n \mathbb{N}_y(C^j) \right) \right. \\ &\quad \left. + \left(\sum_{k_0(y)}^n \mathbb{N}_y(C^j) \right)^2 \right) \\ &\quad \times \left(c_9 p_{D_g}(z, 0) \sum_k^n h_j \right)^{-2} \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

Since $\sum_k^n h_j \rightarrow \infty$ and the numerator of I does not depend on n , $\text{I} \rightarrow 0$. Using

$$\sum_{k_0(y)}^n \mathbb{N}_y(C^j) \leq c_{10} p_{D_g}(y, 0) \sum_{k_0(y)}^n h_j \leq c_{10} p_{D_g}(y, 0) \sum_k^n h_j$$

yields $\text{II} \rightarrow 0$ and $\text{III} \leq (1 - \varepsilon)^{-2} p_{D_g}^{-2}(z, 0) (1 + \varepsilon)^2 \int dy G_{D_g}(z, y) p_{D_g}^2(y, 0)$.

To show $\text{III} < \infty$, we remark that the monotonicity of the Green function in the domain yields $G_{D_g}(x, y) \leq G_H(x, y)$ on D_g , and the bound $p_{D_g}(z, 0) \leq p_H(z, 0)$ shows that it is enough to show

$$\int dy G_H(z, y) p_H^2(y, 0) < \infty.$$

This follows readily from the formula of (1.10).

Using the bound in (2.5) for $z \in \text{cone}(\pi/4)$, we have shown that

$$\mathbb{N}_z(C_k^\infty > 0) \geq c_{11} u(z).$$

By Lemma 1.9 we conclude that $\mathbb{N}_z(C_k^\infty > 0) \geq u(z)$. Thus $\mathbb{N}_z(C_k^\infty > 0 \text{ i.o.}) = u(z)$. That is, $C_k^\infty > 0$ happens infinitely often almost surely on $T < \infty$. \square

2.2. Proof of Theorem 0.2.

2.2.1. Case $h(x) \rightarrow 0$. Let $g(x) = x^2h(x)$ with $h(x) \rightarrow 0$. For c_0 a nontrivial constant, define $z_k = (e^{-k}, c_0g(e^{-k}))$. Set $a_k = e^{-k}/2$, and define b_k and α_k by $z_k = (a_k, 0) + b_k e^{i\alpha_k}$. Let $B_k = B_{a_k, \alpha_k}$. We have $\{z_k \in \tilde{C}\} \subset \{B_k > 0\}$.

By Lemma 2.1,

$$\begin{aligned}
 \mathbb{N}_{(0,1)}(z_k \notin \tilde{C} \text{ i.o.}, T < \infty) &\geq \lim_k \mathbb{N}_{(0,1)}(z_k \notin \tilde{C}, T < \infty) \\
 &= u((0, 1)) - \lim_k \mathbb{N}_{(0,1)}(z_k \in \tilde{C}, T < \infty) \\
 (2.9) \quad &\geq u((0, 1)) - \lim_k \mathbb{N}_{(0,1)}(B_{a_k, \alpha_k} > 0, T < \infty) \\
 &\geq u((0, 1)) - \lim_k \mathbb{N}_{(0,1)}(B_{a_k, \alpha_k}) \\
 &\geq u((0, 1)) - \lim_k c_1 h(e^{-k}) d(f(z), H)^{-1} \\
 &= u((0, 1)) = \mathbb{N}_{(0,1)}(T < \infty).
 \end{aligned}$$

We note that if the event $z_k \notin \tilde{C}$ i.o. happens $\mathbb{N}_{(0,1)}$ -almost surely on $\{T < \infty\}$, then $\limsup f(x)/g(x) \geq c_0$ happens $\mathbb{N}_{(0,1)}$ -almost surely. Since c_0 is arbitrary we conclude that $\limsup f(x)/g(x) = \infty$, $\mathbb{N}_{(0,1)}$ -almost surely on $\{T < \infty\}$.

REMARK 2.5. From (2.9) one sees that if $h(e^{-k}) \leq c_2$, then we have for $z = r e^{i\theta} \in \text{cone}(\pi/4)$ that there exists a constant for which

$$\mathbb{N}_z(z_k \notin \tilde{C} \text{ i.o.}, T < \infty) \geq \frac{1}{r^2}(c_3 - c_1 c_2 r).$$

Thus we have that for r sufficiently small a lower bound of $cu(z)$ applies. If we could use Lemma 1.9, then we could conclude that $\limsup f(x)/x^2 \geq c_2$, \mathbb{N}_z -almost surely on $T < \infty$. From here we would let $c_2 \rightarrow \infty$ along a countable subsequence to see that $\limsup f(x)/x^2 = \infty$. However, Lemma 1.9 does not apply as it is stated, although one thinks that some such lemma should exist.

2.2.2. Case $h(x) \rightarrow \infty$. Let $z_k = (2^{-k}, g(2^{-k})) = a_k e^{i\alpha_k}$, $A_k = A_{a_k, \alpha_k}$, $\Gamma_k = \Gamma_{0, \alpha_k}$, and let X^k be the exit measure from Γ_k . We show first that, $\mathbb{N}_{(0,1)}$ -almost surely on the event $\{T < \infty\}$, we have $z_k \in \tilde{C}$ eventually. This will follow from the Borel–Cantelli lemma if

$$\sum_k \mathbb{N}_{(0,1)}(z_k \notin \tilde{C}) < \infty.$$

By (2.1) we have $\{z_k \notin \tilde{C}\} \subset \{A_k = 0\}$ on $\{T < \infty\}$; hence it is enough to estimate the latter set. For an excursion from $\partial\Gamma_k$, let $\Psi_k(W) = \mathbf{1}_{R_{z_k, \alpha_k}(\hat{W}_0)} \mathbf{1}_{T < \sigma}(W)$. Let $Y_k = \int 2X^k(dy) \mathbb{N}_y(\Psi_k)$. By the special Markov

property,

$$\begin{aligned}
 \mathbb{N}_z(A_k = 0, T < \infty) &= \lim_{\lambda \rightarrow \infty} \mathbb{N}_z \left(\exp - \left(\lambda \sum_{I_{\Gamma_k}} \Psi_k(W^i) \right), T < \infty \right) \\
 &= \mathbb{N}_z \left(\exp - \left(\int 2X^k(dy) \mathbb{N}_y(\Psi_k > 0) \right), T < \infty \right) \\
 &= \mathbb{N}_z(\exp - Y_k, T < \infty) \\
 &\leq \mathbb{N}_z(\exp - Y_k \circ \Theta_T, T < \infty) \\
 &= \mathbb{N}_z(\mathbb{E}_{W_T}^*(\exp - Y_k), T < \infty) \\
 (2.10) \qquad &= \mathbb{N}_z \left(\exp - \left(\int_0^{\zeta_T} dt \mathbb{N}_{W_T(t)}(1 - \exp - Y_k) \right), T < \infty \right).
 \end{aligned}$$

Line (2.10) follows from the Poisson property of the measure $\sum \delta_{\zeta^i, W^i}$ under $\mathbb{E}_{W_T}^*$ given by Proposition 1.1.

Let Δ_j and Ω_j be as defined in Lemma 1.9. Following [1], define $H_\gamma = \{z \in H: \text{dist}(z, \partial H) \leq \gamma\}$ and

$$\begin{aligned}
 F_{n_0, n}^A(w) &= \frac{1}{n - n_0} \sum_{j=n_0}^{n-1} \mathbf{1} \left(\sup_{0 \leq t \leq 2^{-j}} |w(\zeta - t) - w(\zeta)| > A2^{-j/2} \right), \\
 A_j^\alpha &= \left\{ w \in \mathcal{W}: \zeta \geq 2^{-j}, \left\{ w(\zeta - t); \frac{3}{2}2^{-j-1} \leq t \leq 2^{-j} \right\} \cap H_{\alpha 2^{-j/2}} \neq \emptyset \right\}, \\
 \Phi_{n_0, n}^\alpha(w) &= \frac{1}{n - n_0} \sum_{j=n_0}^{n-1} \mathbf{1}_{A_j^\alpha(w)}.
 \end{aligned}$$

Set

$$\begin{aligned}
 C_{n_0, n}^A &= \{ \zeta_T \in [2^{-n_0}, 2^{n_0}], F_{n_0, n}^A(W_T) > \frac{1}{4} \}, \\
 D_{n_0, n}^\alpha &= \{ \zeta_T \in [4 \cdot 2^{-n_0}, 2^{n_0}], \Phi_{n_0, n}^\alpha(W_T) > \frac{1}{2} \}.
 \end{aligned}$$

Choose λ large enough so that the sequence $\{2^{\sqrt{n}}4^n \exp - \lambda(n - \sqrt{n})\}$ is summable. Then, by [1], Lemmas 4.2 and 4.3, there exist $0 < \alpha < A$ and a constant c_0 for which

$$\begin{aligned}
 \mathbb{N}_{(0,1)}(C_{n_0, n}^A) &\leq \mathbb{N}_{(0,1)}(\exists s \geq 0, \zeta_s \in [2^{-n_0}, 2^{n_0}], F_{n_0, n}^A(W_s) > \frac{1}{4}) \\
 &\leq c_0 2^{n_0} 4^n \exp - \lambda(n - n_0)
 \end{aligned}$$

and

$$\mathbb{N}_{(0,1)}(D_{n_0, n}^\alpha) \leq c_0 2^{n_0} 4^n \exp - \lambda(n - n_0).$$

Let $n \geq 4$, $n_0 = \sqrt{n}$, and fix $A > \alpha$ so that the above holds. We remark that when $\zeta_T \in [4 \cdot 2^{-n_0}, 2^{n_0}]$ holds we have $F_{n_0, n}^A(W_T) \leq \frac{1}{4}$ and $\Phi_{n_0, n}^\alpha(W_T) \leq \frac{1}{2}$ and thus

$$(2.11) \qquad \kappa_n = \{j \leq n: W_T(\zeta_T - \Delta_j) \subset \Omega_j\} \geq \frac{1}{4}(n - \sqrt{n}) \geq \frac{1}{8}n.$$

We will use the following lemma whose proof is postponed.

LEMMA 2.6. *Under the assumptions on g ($h \rightarrow \infty$) for $j \leq k$, the following bound holds:*

$$\inf_{\Omega_j} \mathbb{N}_z(1 - \exp - 2Y_k) \geq c_1 \inf_{\Omega_j} u(z) \geq c_2 2^j.$$

Assuming this lemma, it follows from (2.10) that

$$\begin{aligned} & \mathbb{N}_{(0,1)}(A_k = 0, T < \infty) \\ & \leq \mathbb{N}_{(0,1)}\left(\exp - \left(2 \int_0^{\zeta_T} dt \mathbb{N}_{W_T(t)}(1 - \exp - 2Y_k)\right), T < \infty\right) \\ & \leq \mathbb{N}_{(0,1)}(C_{\sqrt{k},k}^A \cup D_{\sqrt{k},k}^A) \\ & \quad + \mathbb{N}_{(0,1)}(\zeta_T \geq 2^{\sqrt{k}}) + \mathbb{N}_{(0,1)}(\zeta_T \leq 4 \cdot 2^{-\sqrt{k}}) \\ & \quad + \mathbb{N}_{(0,1)}\left(\exp - \left(2 \int_0^{\zeta_T} dt \mathbb{N}_{W_T(t)}(1 - \exp - 2Y_k)\right), \zeta_T \in [4 \cdot 2^{-\sqrt{k}}, 2^{\sqrt{k}}], \right. \\ & \qquad \qquad \qquad \left. F_{n_0,n}^A(W_T) \leq \frac{1}{4}, \Phi_{n_0,n}^\alpha(W_T) \leq \frac{1}{2}, T < \infty\right) \\ & = \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

By a union bound we have

$$\text{I} \leq 2c_0 2^{\sqrt{k}} 4^k \exp - \lambda(k - \sqrt{k}).$$

By the fact that ζ is distributed under $\mathbb{N}_{(0,1)}$ as a Brownian excursion from 0, we have

$$\text{II} \leq \mathbb{N}_{(0,1)}\left(\sup_{[0,\sigma]} \zeta_s > 2^{\sqrt{k}}\right) = 2^{-(\sqrt{k}+1)}.$$

Let $\mathcal{R}_u = \{W_s(t): 0 \leq s \leq \sigma, 0 \leq t \leq u\}$, $\mathcal{R} = \{W_s(t): 0 \leq s \leq \sigma, 0 \leq t \leq \zeta_s\}$. Set $\delta = \text{dist}(z, H)$. In order for ζ_T to be small, it must be that \mathcal{R}_u hits $B(z, \delta)^c$ with a small u . This event can be bounded by the probability that the path process exits distant balls which is given by solving the equation in Proposition 1.5 with an infinite boundary condition, yielding $\mathbb{N}_0(\mathcal{R} \cap B(0, \varepsilon)^c \neq \emptyset) = 2\varepsilon^{-2}$. Thus

$$\begin{aligned} & \text{III} \leq \mathbb{N}_{(0,1)}(T_{(B(z, \delta)^c)} < 4 \cdot 2^{-\sqrt{k}}) \\ & \leq \mathbb{N}_{(0,1)}(\mathcal{R}_{4 \cdot 2^{-\sqrt{k}}} \cap B((0, 1), \delta)^c \neq \emptyset) \\ (2.12) \quad & = (4 \cdot 2^{-\sqrt{k}}) \mathbb{N}_0(\mathcal{R}_1 \cap B(0, \frac{1}{4} \delta 2^{2\sqrt{k}})^c \neq \emptyset) \\ & \leq (4 \cdot 2^{-\sqrt{k}}) \mathbb{N}_0(\mathcal{R} \cap B(0, \frac{1}{4} \delta 2^{2\sqrt{k}})^c \neq \emptyset) \\ & = \frac{128}{\delta^2} 2^{-5\sqrt{k}}. \end{aligned}$$

Line (2.12) follows from the scaling property of the Brownian snake used in the proof of Lemma 1.6.

For IV we have by (2.11) that $\#(\kappa_k) \geq ck$. Thus

$$\begin{aligned} \text{IV} &\leq \mathbb{N}_{(0,1)}\left(\exp - \left(2 \sum_{\kappa_k} \int_{\zeta_T - \Delta_j} dt \inf_{z \in \Omega_j} \mathbb{N}_z(1 - \exp - 2Y_k)\right), \right. \\ &\quad \left. \zeta_T \in [4 \cdot 2^{-\sqrt{k}}, 2^{\sqrt{k}}], F_{n_0, n}^A(W_T) \leq \frac{1}{4}, \Phi_{n_0, n}^\alpha(W_T) \leq \frac{1}{2}, T < \infty\right) \\ &\leq \mathbb{N}_{(0,1)}(\exp - c_3 k, T < \infty) \\ &\leq u((0, 1)) \exp - c_3 k. \end{aligned}$$

Since all these sum in k , we have by the Borel–Cantelli lemma that, $\mathbb{N}_{(0,1)}$ -almost surely on $\{T < \infty\}$, $\{A_k > 0\}$ eventually.

To finish, we need to extrapolate between the points z_k . Following [3], we let $g_1(x) = c_4 g(x)$ with $c_4 > 9$. Choose $k_0 = k_0(\omega)$ so that for $k > k_0$ one has $g_1(2^{-k}) \geq g(2^{-k}) \geq f(2^{-k})$ and $g(2^{-k-1})/g(2^{-k}) \geq 1/3$. Let Q be the point

$$\left(2^{-k}, 2^{-k} c_4 \frac{g(2^{-k})}{2^{-k-1}}\right)$$

and P the point $(2^{-k-1}, c_4 g(2^{-k-1}))$. If the line segment from Q to 0, which goes through P , is contained in \tilde{C} , then $(x, c_4 g(x)) \in \tilde{C}$ for $2^{-k-1} \leq x \leq 2^{-k}$. This will happen if Q lies above $(2^k, g(2^{-k}))$ or

$$2^{-k} \frac{c_4 g(2^{-(k+1)})}{2^{-(k+1)}} \geq g(2^{-k}).$$

This is true by the choice of k_0 and c_4 . \square

Finally, we establish Lemma 2.6.

PROOF OF LEMMA 2.6. Fix $z \in \Omega_j, j < k$. Then $z = re^{i\theta} \in \text{cone}(\cos^{-1}(\alpha/A))$ with $|z| \geq a_k$. For fixed $c_1 > 0$,

$$\mathbb{N}_z(1 - \exp - 2Y_k) \geq \mathbb{N}_z(1 - e^{-2c_1}, Y_k > c_1).$$

By the Cauchy–Schwarz inequality,

$$\mathbb{N}_z(Y_k > c_1) \geq \frac{(\mathbb{N}_z(Y_k) - c_1)^2}{\mathbb{N}_z(Y_k^2)}.$$

We need bounds on $\mathbb{N}_z(Y_k)$ and $\mathbb{N}_z(Y_k^2)$.

By Lemma 2.2,

$$\begin{aligned} \mathbb{N}_z(Y_k) &\geq c_2 h_k \inf_{[1,2]} p_H(r^\beta e^{i\tilde{\theta}}, (av)^\beta) \\ &\geq c_2 h_k r^{-\beta} (p_H(e^{i\tilde{\theta}}, a^\beta) \wedge p_H(e^{i\tilde{\theta}}, 2^\beta)) \\ &\geq c_3 h_k r^{-\beta}. \end{aligned}$$

By Proposition 2.2 of [1] and scaling, we have, with $z = re^{i\theta}$,

$$\begin{aligned} \mathbb{N}_z(Y_k^2) &= 4 \int G_{\Gamma_k}(z, y) \mathbb{N}_z(Y_k)^2 \\ &\leq 4 \int G_H(z, y) \mathbb{N}_z(Y_k)^2 \\ &= 4 \int \rho d\rho d\gamma G_H(re^{i\theta}, \rho e^{i\gamma}) \left(\int_{\alpha_k}^\infty p_H(\rho^\beta e^{i\tilde{\gamma}}, v^\beta) v^{\beta-3} w(\alpha) \right)^2 \\ &= 4r^{2-2\beta} ((w(\alpha_k) \alpha_k^{\beta-2})) \int \rho d\rho d\gamma G_H(e^{i\theta}, \rho e^{i\gamma}) \rho^{-2\beta} \\ &\quad \times \left(\int_1^\infty p_H\left(e^{i\tilde{\gamma}}, \left(\frac{\alpha}{r\rho}\right)^\beta v^\beta\right) v^{\beta-3} \right)^2. \end{aligned}$$

We show first that there exists a constant c_4 independent of x and γ for which

$$(2.13) \quad \int_1^\infty p_H(e^{i\gamma}, (xv)^\beta) v^{\beta-3} < c_4.$$

For $x < 1/2$ we have the left-hand side of (2.13) is equal by a change of variables to

$$\begin{aligned} &x^{-(\beta-2)} \int_x^\infty dv p_H(e^{i\gamma}, v^\beta) v^{\beta-3} \\ &= x^{-(\beta-2)} \left(\int_x^{1/2} + \int_{1/2}^2 + \int_2^\infty \right) dv p_H(e^{i\gamma}, v^\beta) v^{\beta-3} \\ &\leq c_5 x^{-(\beta-2)} \left(\int_x^{1/2} dv v^{\beta-3} + c_6 \int_{1/2}^2 dv p_H(e^{i\gamma}, v^\beta) v^{\beta-1} \right. \\ &\quad \left. + c_7 \int_2^\infty dv v^{-\beta-3} \right) \\ &< c_8. \end{aligned}$$

Similarly, for $x \geq 1/2$ we get the necessary bounds.

Next, by the bounds $G_H(e^{i\theta}, \rho e^{i\gamma}) < c_9 \rho$ for small ρ and $G_H(e^{i\theta}, \rho e^{i\gamma}) < c_{10}(\rho)^{-1}$ for large ρ , it follows that

$$\int \rho d\rho d\gamma G_H(e^{i\theta}, \rho e^{i\gamma}) \rho^{-2\beta} < c_{11}.$$

These combined with (2.3) yield the bound

$$\mathbb{N}_z(Y_k^2) \leq c_{12} h_k r^{2-2\beta}.$$

Thus, for z in $\text{cone}(\cos^{-1}(\alpha/A))$,

$$\begin{aligned} \mathbb{N}_z(Y_k > c_1) &\geq \frac{(c_{13} h_k r^{-\beta} - c_1)^2}{c_{14} h_k^2 r^{2-2\beta}} \\ &= \frac{(c_{13} - c_1 / (h_k r^{-\beta}))^2}{c_{14} r^{-2}} \\ &\geq c_{15} u(z). \end{aligned}$$

The last inequality follows since $h_k \rightarrow 0$ by the assumption on g . This implies Lemma 2.6. \square

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