The Annals of Probability
2003, Vol. 31, No. 3, 1494–1532
© Institute of Mathematical Statistics, 2003

SCALING LIMIT OF STOCHASTIC DYNAMICS IN CLASSICAL CONTINUOUS SYSTEMS[1]

BY MARTIN GROTHAUS, YURI G. KONDRATIEV, EUGENE LYTVYNOV AND MICHAEL RÖCKNER

Bielefeld University

We investigate a scaling limit of gradient stochastic dynamics associated with Gibbs states in classical continuous systems on $\mathbb{R}^d$, $d \geq 1$. The aim is to derive macroscopic quantities from a given microscopic or mesoscopic system. The scaling we consider has been investigated by Brox (in 1980), Rost (in 1981), Spohn (in 1986) and Guo and Papanicolaou (in 1985), under the assumption that the underlying potential is in C_0^3 and positive. We prove that the Dirichlet forms of the scaled stochastic dynamics converge on a core of functions to the Dirichlet form of a generalized Ornstein–Uhlenbeck process. The proof is based on the analysis and geometry on the configuration space which was developed by Albeverio, Kondratiev and Röckner (in 1998), and works for general Gibbs measures of Ruelle type. Hence, the underlying potential may have a singularity at the origin, only has to be bounded from below and may not be compactly supported. Therefore, singular interactions of physical interest are covered, as, for example, the one given by the Lennard–Jones potential, which is studied in the theory of fluids. Furthermore, using the Lyons–Zheng decomposition we give a simple proof for the tightness of the scaled processes. We also prove that the corresponding generators, however, do not converge in the L^2-sense. This settles a conjecture formulated by Brox, by Rost and by Spohn.

1. Introduction. The stochastic dynamics $(\mathbf{X}(t))_{t\geq 0}$ of a classical continuous system is an infinite dimensional diffusion process having a Gibbs measure μ (e.g., of the type studied by Ruelle [31]), as an invariant measure. Physically, it describes the stochastic dynamics of Brownian particles which are interacting via the gradient of a pair potential ϕ. Since each particle can move through each position in space, the system is called continuous and is used for modelling gases and fluids. For realistic models which can be described by these stochastic dynamics (e.g., suspensions), we refer to [34].

Since these dynamics are stochastic, they have to be interpreted as mesoscopic processes. The aim of analyzing scaling limits, in general, is to derive from microscopic or mesoscopic systems macroscopic statements and quantities. The type of scaling to study depends on which features of a given system one is interested in (see, e.g., [4, 11, 35]).

Received August 2001; revised July 2002.

[1]Supported by the DFG through SFB 256 Bonn and the DFG-Forschergruppe.

AMS 2000 subject classifications. 60B12, 82C22, 60K35, 60J60, 60H15.

Key words and phrases. Limit theorems, interacting particle systems, diffusion processes.

The scaling we consider in this paper has been investigated in [4, 29]. In his doctoral dissertation Brox [4] has given some heuristic arguments for nonconvergence in law of the scaled process and has conjectured that there is no limiting Markov process. However, assuming the convergence of the generators of the scaled stochastic dynamics averaged over time (cf. Conjecture 6.5 below), Rost [29] has given some heuristic arguments for the existence of a limiting generalized Ornstein–Uhlenbeck process, which, of course, contradicts the statement of Brox. A fundamental and celebrated paper on this problem is due to Spohn [34]. Assuming that the underlying potential is smooth, compactly supported and positive, the author describes a proof of Conjecture 6.5 within the proof of his main theorem (see, however, the remark on page 4 of [34], and Proposition 2 therein, concerning the restriction $d \le 3$). Another approach has been proposed in [8]. The idea of Guo and Papanicolaou has been to prove convergence of the corresponding resolvent. As remarked by themselves, at that time the authors did not have an appropriate infinite dimensional analysis and geometry at their disposal, and therefore their considerations have been on a nonrigorous level.

After these contributions, for a long time there was no progress in this problem. Recently, however, some new techniques have been introduced. In [1, 2] an infinite dimensional analysis and geometry on the configuration space was developed. In this paper we shall make use of these concepts to tackle once more the problem described above.

The stochastic dynamics $(\mathbf{X}(t))_{t\ge 0}$ of a classical continuous system takes values in the configuration space

$$\Gamma := \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}$$

and informally solves the following infinite system of stochastic differential equations:

$$\begin{aligned} dx(t) &= -\beta \sum_{\substack{y(t)\in\mathbf{X}(t),\\ y(t)\neq x(t)}} \nabla\phi\big(x(t)-y(t)\big)\,dt + \sqrt{2}\,dB^x(t), \qquad x(t)\in\mathbf{X}(t), \\ \mathbf{X}(0) &= \gamma, \qquad \gamma\in\Gamma, \end{aligned} \tag{1}$$

where $(B^x)_{x\in\gamma}$ is a sequence of independent Brownian motions. The study of such diffusions has been initiated by Lang [14] (see also [33]), who considered the case $\phi \in C_0^3(\mathbb{R}^d)$ using finite-dimensional approximations and stochastic differential equations. More singular ϕ, which are of particular interest in physics (as, e.g., the Lennard–Jones potential), have been treated by Osada [25] and Yoshida [37] (see also [5, 36] for the hard core case). Osada and Yoshida were the first to use Dirichlet forms for the construction of such processes. However, they could not write down the corresponding generators or martingale problems explicitly; hence they could not prove that their processes actually solve (1) weakly. This, however, was proved in [2] by showing an integration-by-parts formula for the respective

Gibbs measures. Thus the latter work became the starting point of this paper. In [2], Dirichlet forms also were used and all constructions were designed to work particularly for singular potentials of the above-mentioned type; see Theorem 3.2. Additionally, and this is essential for our considerations, an explicit expression for the corresponding generator and martingale problem was provided, which shows that the process in [2] indeed solves (1) in the weak sense.

The scaled process $(\mathbf{X}_\varepsilon(t))_{t\geq 0}$ studied in this paper is defined by

$$\mathbf{X}_\varepsilon(t) := S_{\mathrm{out},\varepsilon}\big(S_{\mathrm{in},\varepsilon}(\mathbf{X}(\varepsilon^{-2}t))\big), \qquad t \geq 0,\ \varepsilon > 0,$$

and we are interested in the scaling limit for $\varepsilon \to 0$. The first scaling $S_{\mathrm{in},\varepsilon}$ scales the position of the particles inside the configuration space as follows:

$$\Gamma \ni \gamma \mapsto S_{\mathrm{in},\varepsilon}(\gamma) := \{\varepsilon x \mid x \in \gamma\} \in \Gamma, \qquad \varepsilon > 0.$$

Hence, for small $\varepsilon > 0$, this scaling concentrates the particles toward the origin. The second scaling $S_{\mathrm{out},\varepsilon}$ leads us out of the configuration space and is given by

$$\Gamma \ni \gamma \mapsto S_{\mathrm{out},\varepsilon}(\gamma) := \varepsilon^{d/2}\big(\gamma - \rho^{(1)}_{\tilde{\mu}_\varepsilon}\, dx\big) \in \mathcal{D}',$$

where $\mathcal{D}'$ is the dual space of $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$. In the second scaling we first center the configuration γ by subtracting the first correlation measure $\rho^{(1)}_{\tilde{\mu}_\varepsilon}\, dx$ of the Gibbs measure $\tilde{\mu}_\varepsilon := S^*_{\mathrm{in},\varepsilon}\mu$. Furthermore, we scale the mass of the particles by $\varepsilon^{d/2}$ to avoid divergence of the total mass at the origin as $\varepsilon \to 0$.

We start with constructing the Dirichlet form $\mathcal{E}_\varepsilon$, the generator H_ε and the semigroup $(T_{\varepsilon,t})_{t\geq 0}$ associated with $(\mathbf{X}_\varepsilon(t))_{t\geq 0}$. These objects are images of the Dirichlet form, generator and semigroup, respectively, which are associated with the original stochastic dynamics $(\mathbf{X}(t))_{t\geq 0}$; see Theorem 4.1.

The first convergence we show is the following (see Theorem 5.3). We prove that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(F, G) = \mathcal{E}_{\nu_\mu}(F, G), \tag{2}$$

for all smooth cylinder functions $F, G \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \mathcal{D}')$. The limit Dirichlet form $\mathcal{E}_{\nu_\mu}$ is defined on $L^2(\mathcal{D}', \nu_\mu)$ with ν_μ being white noise, and associated with a generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ solving the stochastic differential equation

$$d\mathbf{X}(t, x) = \frac{\rho^{(1)}_\phi(\beta, 1)}{\chi_\phi(\beta)} \Delta \mathbf{X}(t, x)\, dt + \sqrt{2\rho^{(1)}_\phi(\beta, 1)}\, d\mathbf{W}(t, x), \tag{3}$$

where $(\mathbf{W}(t))_{t\geq 0}$ is a Brownian motion in $\mathcal{D}'$ with covariance operator $-\Delta$. The coefficient $\rho^{(1)}_\phi(\beta, 1)/\chi_\phi(\beta)$ is called the bulk diffusion coefficient and β is the inverse temperature. The convergence (2) determines the limit process uniquely [see Remark 5.4(i)] and requires only very weak assumptions. The interaction potential ϕ only has to be stable (S) and we have to assume the low activity high temperature regime (see below for precise definitions). A basic ingredient in the proof is the convergence of the image measures $\mu_\varepsilon := S^*_{\mathrm{out},\varepsilon} S^*_{\mathrm{in},\varepsilon}\mu$ to the Gaussian white noise measure ν_μ as $\varepsilon \to 0$; see Theorem 5.1. The latter fact has been proved by Brox [4].

The convergence in terms of the Dirichlet forms, however, up to this point has no probabilistic interpretation. Hence, we also study convergence in law of the scaled processes. By $\mathbf{P}^\varepsilon$ we denote the law of the scaled equilibrium processes, that is, the law of the scaled process starting with a distribution equal to the equilibrium measure μ_ε. Then, in Theorem 6.1, we prove that the family $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ is tight. This has been shown before by Brox [4] and Spohn [34], for smooth compactly supported potentials. Our proof, again, works under quite weak assumptions on the potential. We only need conditions which ensure the existence of the original stochastic process and have to assume the low activity high temperature regime. In the proof we use the well-known Lyons–Zheng decomposition [18, 19] of the scaled process and the Burkholder–Davies–Gundy inequalities to establish the required estimate of the increments. Since the state space of the scaled process is a space of distributions, we first prove tightness in a weak sense. Then, via some Hilbert–Schmidt embeddings, we find a negative, weighted Sobolev spaces $\mathcal{H}_{-m}$ as state space such that the family $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ is tight on $C([0,\infty),\mathcal{H}_{-m})$.

It remains to prove that all accumulation points coincide with the generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ above. A well-known method to identify the limit is based on considering the associated martingale problem. More precisely, if we could prove that all accumulation points of $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ satisfy the martingale problem for the generator H associated with (3) with initial condition ν_μ, then a (slight modification of a) uniqueness result of Holley and Stroock [9] implies that all these accumulation points coincide.

The obvious first idea to prove that all limit points solve the martingale problem for H is to try to prove strong convergence of $H_\varepsilon \to H$ as $\varepsilon \to 0$. In [4, 29, 34] it has, however, been conjectured that, in general, the difference

$$\|(H - H_\varepsilon)F\|_{L^2(\mu_\varepsilon)}, \qquad F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\mathcal{D}'),$$

does not tend to zero as $\varepsilon \to 0$. In Theorem 6.3 we prove that this conjecture is indeed true. The proof is quite an elaborate task and is done via a (mathematically rigorous) high temperature expansion. A basic tool for this is provided by Theorem A.4, where we derive explicit formulas for the derivative of the correlation functions with respect to the inverse temperature β using the so-called K-transform from [12], and by Theorem B.1, where we prove a coercivity identity for Gibbs measures.

It turns out that for the above-described identification of the accumulation points of $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$, however, a weaker convergence of the generators is sufficient. In Theorem 6.7 we prove convergence in law under the assumption that

Conjecture 6.5 is true, that is, under the assumption that the generators converge in time average.

To complete the program also from a purely probabilistic point of view, it remains to prove Conjecture 6.5 in physically relevant models. This will be the subject of future work.

The progress achieved in this paper may be summarized by the following core results:

1. Convergence of Dirichlet forms is shown; see Remark 5.4.
2. The tightness result as in [4, 34] is generalized; see Remark 6.2.
3. Conjecture on nonconvergence of generators is proved.
4. A mathematically rigorous high temperature expansion of all correlation functionals is developed (up to second order in $\beta = 1/T$).
5. All above results apply to physically relevant potentials; in particular, singularities at the origin, nontrivial negative part and infinite range are allowed.

Hypotheses on the potential are weakened not for the sake of generality, but in order to cover the physically relevant potentials (as, e.g., Lennard–Jones potential).

2. Gibbs states of classical continuous systems.

2.1. *Configuration space and Poisson measure.* Let $\mathbb{R}^d$, $d \geq 1$, be equipped with the norm $|\cdot|_{\mathbb{R}^d}$ given by the Euclidean scalar product $(\cdot,\cdot)_{\mathbb{R}^d}$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the corresponding Borel σ-algebra. $\mathcal{O}_c(\mathbb{R}^d)$ denotes the system of all open sets in $\mathbb{R}^d$, which have compact closure. The Lebesgue measure on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ we denote by dx.

The *configuration space* Γ over $\mathbb{R}^d$ is defined by

$$\Gamma := \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

Here $|A|$ denotes the cardinality of a set A. Via the identification of $\gamma \in \Gamma$ with $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}_p(\mathbb{R}^d)$, where ε_x denotes the Dirac measure in $x \in \mathbb{R}^d$, Γ can be considered as a subset of the set $\mathcal{M}_p(\mathbb{R}^d)$ of all positive Radon measures on $\mathbb{R}^d$. Hence Γ can be topologized by the vague topology, that is, the topology generated by maps

$$\gamma \mapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f(x)\, d\gamma(x) = \sum_{x\in\gamma} f(x),$$

where $f \in C_0(\mathbb{R}^d)$, the set of continuous functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-algebra.

For a given $z > 0$ (activity parameter), let π_z denote the Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity measure $z\,dx$. This measure is characterized via its Fourier transform

$$\int_\Gamma \exp(i\langle f,\gamma\rangle)\, d\pi_z(\gamma) = \exp\left(z \int_{\mathbb{R}^d} \left(\exp(if(x)) - 1\right) dx\right), \qquad f \in \mathcal{D},$$

where $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$, the set of smooth functions on $\mathbb{R}^d$ with compact support.

2.2. *Gibbs measures in the low activity high temperature regime.* Let ϕ be a symmetric pair potential, that is, a measurable function $\phi:\mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\phi(x) = \phi(-x)$. For $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ the conditional energy $E_\Lambda^\phi:\Gamma \to \mathbb{R} \cup \{\infty\}$ with empty boundary condition is defined by

$$E_\Lambda^\phi(\gamma) := \sum_{\{x,y\}\subset\gamma_\Lambda} \phi(x-y) = E_\Lambda^\phi(\gamma_\Lambda),$$

where $\gamma_\Lambda := \gamma \cap \Lambda$ and the sum over the empty set is defined to be zero.

For every $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ we define a cube

$$Q_r = \{x \in \mathbb{R}^d \mid r_i - 1/2 \le x_i < r_i + 1/2\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$ we set $\gamma_r := \gamma_{Q_r}$, $r \in \mathbb{Z}^d$. Additionally, we introduce for $n \in \mathbb{N}$ a cube Λ_n with side length $2n-1$ centered at the origin in $\mathbb{R}^d$.

Let us recall some standard assumptions from statistical mechanics. For our results we have to require some of the following conditions:

(SS) (Superstability). There exist $A(\phi) > 0$, $B(\phi) \ge 0$ such that, if $\gamma = \gamma_{\Lambda_n}$ for some $n \in \mathbb{N}$, then

$$E_{\Lambda_n}^\phi(\gamma) \ge \sum_{r\in\mathbb{Z}^d} \bigl(A(\phi)|\gamma_r|^2 - B(\phi)|\gamma_r|\bigr).$$

(SS) obviously implies:

(S) (Stability). For any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma$, we have

$$E_\Lambda^\phi(\gamma) \ge -B(\phi)|\gamma_\Lambda|.$$

A consequence of (S), in turn, is, of course, that ϕ is bounded from below. For $\beta \ge 0$, $z > 0$, let us define

$$C(\beta\phi, z) := \exp\bigl(2\beta B(\phi)\bigr) \int_{\mathbb{R}^d} \bigl|\exp\bigl(-\beta\phi(x)\bigr) - 1\bigr| z\,dx.$$

We also need the following:

(UI) (Uniform integrability). We have

$$C(\beta\phi, z) < \exp(-1).$$

For a given potential ϕ the set of pairs (β, z) such that condition (UI) holds is called the *low activity high temperature* (LA–HT) *regime*; see [23, 30]; (UI) is stronger than *integrability* (I) [i.e., $C(\beta\phi, z) < \infty$], which is also called *regularity*; see, for example, [31].

(LR) (Lower regularity). There exists a decreasing positive function $a:\mathbb{N}\to\mathbb{R}_+$ such that

$$\sum_{r\in\mathbb{Z}^d} a(\|r\|)<\infty$$

and, for any Λ', Λ'' which are finite unions of cubes of the form Q_r and disjoint,

$$W^\phi(\gamma'\mid\gamma'')\ge-\sum_{r',r''\in\mathbb{Z}^d} a(\|r'-r''\|)|\gamma'_{r'}||\gamma''_{r''}|,$$

provided $\gamma'=\gamma'_{\Lambda'}$, $\gamma''=\gamma''_{\Lambda''}$. Here

$$W^\phi(\gamma'\mid\gamma''):=\sum_{x\in\gamma',y\in\gamma''}\phi(x-y)$$

is the interaction energy and $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$.

On $(\Gamma,\mathcal{B}(\Gamma))$ we consider the finite volume Gibbs measures μ_Λ in $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$ with empty boundary condition:

$$d\mu_\Lambda(\gamma):=\frac{1}{Z_\Lambda}\exp\bigl(-\beta E_\Lambda^\phi(\gamma_\Lambda)\bigr)\,d\pi_z(\gamma),$$

where $\beta\ge0$ is the inverse temperature and

$$Z_\Lambda=\int_\Gamma\exp\bigl(-\beta E_\Lambda^\phi(\gamma_\Lambda)\bigr)\,d\pi_z(\gamma)$$

is the partition function. Using (S) one easily proves that it is finite. In, for example, [22, 23] it has been proved that in the LA–HT regime the weak limit

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\mu_\Lambda=\mu \tag{4}$$

exists. Furthermore, it can be shown that μ is a Gibbs measure; see [13, 32]. The measure μ in (4) we call the Gibbs measure corresponding to (ϕ,β,z) and the construction with empty boundary condition.

2.3. *K-transform and correlation functions.* Next, we recall the definition of correlation functions using the concept of the so-called K-transform (see, e.g., [12, 15–17]).

Denote by Γ_0 the space of finite configurations over $\mathbb{R}^d$:

$$\Gamma_0:=\bigsqcup_{n=0}^\infty\Gamma_0^{(n)},\qquad\Gamma_0^{(0)}:=\{\varnothing\},\qquad\Gamma_0^{(n)}:=\{\eta\subset\mathbb{R}^d\mid|\eta|=n\},\qquad n\in\mathbb{N}.$$

Let $\widetilde{\mathbb{R}^{d\times n}}=\{(x_1,\dots,x_n)\in\mathbb{R}^{d\times n}\mid x_i\ne x_j\text{ for }i\ne j\}$ and let S^n denote the group of all permutations of $\{1,\dots,n\}$. Through the natural bijection

$$\widetilde{\mathbb{R}^{d\times n}}/S^n\leftrightarrow\Gamma_0^{(n)} \tag{5}$$

one defines a topology on $\Gamma_0^{(n)}$. The space Γ_0 is equipped then with the topology of disjoint union. Let $\mathcal{B}(\Gamma_0)$ denote the Borel σ-algebra on Γ_0.

A $\mathcal{B}(\Gamma_0)$-measurable function $G:\Gamma_0 \to \mathbb{R}$ is said to have bounded support if there exist $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $\operatorname{supp}(G) \subset \bigsqcup_{n=0}^{N} \Gamma_{0,\Lambda}^{(n)}$, where $\Gamma_{0,\Lambda}^{(n)} = \{\eta \subset \Lambda \mid |\eta| = n\}$.

For any $\gamma \in \Gamma$ let $\sum_{\eta \Subset \gamma}$ denote the summation over all $\eta \subset \gamma$ such that $|\eta| < \infty$. For a function $G:\Gamma_0 \to \mathbb{R}$, the K-transform of G is defined by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) \tag{6}$$

for each $\gamma \in \Gamma$ such that at least one of the series $\sum_{\eta \Subset \gamma} G^+(\eta)$ or $\sum_{\eta \Subset \gamma} G^-(\eta)$ converges, where $G^+ := \max\{0, G\}$ and $G^- := -\min\{0, G\}$.

Let μ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. The correlation measure corresponding to μ is defined by

$$\rho_\mu(A) := \int_\Gamma (K\mathbb{1}_A)(\gamma)\, d\mu(\gamma), \qquad A \in \mathcal{B}(\Gamma_0);$$

ρ_μ is a measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ (see [12] for details, in particular, measurability issues).

Let $G \in L^1(\Gamma_0, \mathcal{B}(\Gamma_0), \rho_\mu)$. Then $\|KG\|_{L^1(\mu)} \le \|K|G|\|_{L^1(\mu)} = \|G\|_{L^1(\rho_\mu)}$; hence $KG \in L^1(\Gamma, \mathcal{B}(\Gamma), \mu)$ and $KG(\gamma)$ is for μ-a.e. $\gamma \in \Gamma$ absolutely convergent. Moreover, then obviously

$$\int_{\Gamma_0} G(\eta)\, d\rho_\mu(\eta) = \int_\Gamma (KG)(\gamma)\, d\mu(\gamma); \tag{7}$$

see [12, 16, 17].

The Lebesgue–Poisson measure λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ with activity parameter $z > 0$ is defined by

$$\lambda_z := \delta_\varnothing + \sum_{n=1}^{\infty} \frac{z^n}{n!}\, dx^{\otimes n},$$

where $dx^{\otimes n}$ is defined via the bijection (5).

For the Gibbs measure μ in the LA–HT regime corresponding to ϕ satisfying (S) and the construction with empty boundary condition, the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure (see, e.g., [23, 30]). Its Radon–Nikodym derivative

$$\rho_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_z}(\eta), \qquad \eta \in \Gamma_0,$$

w.r.t. λ_z we denote by the same symbol, and the functions

$$\rho_\mu^{(n)}(x_1, \dots, x_n) := \rho_\mu(\{x_1, \dots, x_n\}), \qquad x_1, \dots, x_n \in \mathbb{R}^d,\ x_i \ne x_j \text{ if } i \ne j, \tag{8}$$

are called the nth order correlation functions of the measure μ. Furthermore, the correlation functions can be expressed as functions of the underlying potential ϕ, inverse temperature β and activity z, that is, $\rho_\mu = \rho_\phi(\beta, z)$ (see, e.g., [23, 30]). Hence, due to the translation invariance of the pair interaction, the correlation functions as well as the Gibbs measure μ are translation invariant. In particular, $\rho_\phi^{(1)}(\beta, z)$ does not depend on $x_1 \in \mathbb{R}^d$.

Additionally, for these functions the so-called Ruelle bound holds: for fixed $\beta \geq 0$, $z > 0$, there exists a constant $\xi > 0$ such that for all n and $x_1, \dots, x_n \in \mathbb{R}^d$, $x_i \neq x_j$ for $i \neq j$, we have

$$\rho_\phi^{(n)}(\beta, z, x_1, \dots, x_n) \leq \xi^n \tag{9}$$

(see [30]). Using this bound one gets, in particular, that all local moments of μ are finite:

$$\int_\Gamma |\gamma_\Lambda|^n \, d\mu(\gamma) < \infty \qquad \forall n \in \mathbb{N},\ \Lambda \in \mathcal{O}_c(\mathbb{R}^d). \tag{10}$$

3. Dirichlet forms, their generators and corresponding stochastic dynamics. Here we recall the analysis and geometry on configuration space developed in [1, 2].

Let $T_x(\mathbb{R}^d) = \mathbb{R}^d$ denote the tangent space to $\mathbb{R}^d$ at a point $x \in \mathbb{R}^d$. The tangent space to Γ at a point $\gamma \in \Gamma$ is defined as the Hilbert space

$$T_\gamma(\Gamma) := L^2\big(\mathbb{R}^d \to T(\mathbb{R}^d), \gamma\big) = \bigoplus_{x \in \gamma} T_x(\mathbb{R}^d).$$

Thus, each $V(\gamma) \in T_\gamma(\Gamma)$ has the form $V(\gamma) = (V(\gamma, x))_{x \in \gamma}$, where $V(\gamma, x) \in T_x(\mathbb{R}^d)$, and

$$\|V(\gamma)\|^2_{T_\gamma(\Gamma)} = \sum_{x \in \gamma} \|V(\gamma, x)\|^2_{T_x(\mathbb{R}^d)} = \sum_{x \in \gamma} \|V(\gamma, x)\|^2_{\mathbb{R}^d}.$$

Let $\gamma \in \Gamma$ and $x \in \gamma$. We denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \varnothing$. Now, for a function $F : \Gamma \to \mathbb{R}$, $\gamma \in \Gamma$ and $x \in \gamma$, we define a function $F_x(\gamma, \cdot) : \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by

$$\mathcal{O}_{\gamma,x} \ni y \mapsto F_x(\gamma, y) := F(\gamma - \varepsilon_x + \varepsilon_y) \in \mathbb{R}.$$

We say that a function $F : \Gamma \to \mathbb{R}$ is differentiable at $\gamma \in \Gamma$ if, for each $x \in \gamma$, the function $F_x(\gamma, \cdot)$ is differentiable at x and

$$\nabla^\Gamma F(\gamma) := \big(\nabla_x F(\gamma)\big)_{x \in \gamma} \in T_\gamma(\Gamma),$$

where

$$\nabla_x F(\gamma) := \nabla_y F_x(\gamma, y)\big|_{y=x}.$$

Evidently, this definition is independent of the choice of the set $\mathcal{O}_{\gamma,x}$. We call $\nabla^\Gamma F(\gamma)$ the gradient of F at $\gamma \in \Gamma$.

We define a set of smooth cylinder functions $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ as the set of all functions on Γ of the form

$$\gamma \mapsto F(\gamma) = g_F(\langle f_1,\gamma\rangle,\ldots,\langle f_N,\gamma\rangle), \tag{11}$$

where $f_1,\ldots,f_N \in \mathcal{D}$ and $g_F \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$. Clearly, $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ is dense in $L^p(\mu)$, $p \geq 1$. Any function F of the form (11) is differentiable at each point $\gamma \in \Gamma$, and its gradient is given by

$$(\nabla^{\Gamma} F)(\gamma,x) = \sum_{j=1}^{N} \partial_j g_F(\langle f_1,\gamma\rangle,\ldots,\langle f_N,\gamma\rangle)\nabla f_j(x), \qquad \gamma \in \Gamma,\ x \in \gamma, \tag{12}$$

where ∂_j denotes the partial derivative w.r.t. the jth variable. For $F, G \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ we define

$$\mathcal{E}_{\mu}^{\Gamma}(F,G) := \int_{\Gamma} \big(\nabla^{\Gamma} F(\gamma), \nabla^{\Gamma} G(\gamma)\big)_{T_{\gamma}(\Gamma)}\, d\mu(\gamma).$$

Gibbs measures μ in the LA–HT regime corresponding to stable potentials and the construction with empty boundary condition have all local moments finite; see (10). Thus, for such measures, with the help of (12) we have $(\nabla^{\Gamma} F(\gamma), \nabla^{\Gamma} G(\gamma))_{T_{\gamma}(\Gamma)} \in L^1(\mu)$. Furthermore, the gradient respects μ-classes $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)^{\mu}$ determined by $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ (see, e.g., [21, 27]. Hence, $(\mathcal{E}_{\mu}^{\Gamma}$, $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma))$ is a densely defined, positive definite, symmetric bilinear form on $L^2(\mu)$.

To ensure closability of this bilinear form we have to assume further properties of the potential ϕ:

(D) (Differentiability). The function $\exp(-\phi)$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d \setminus \{0\}$ and the weak gradient $\nabla\phi$ (which is a locally dx-integrable function on $\mathbb{R}^d \setminus \{0\}$), considered as a dx-a.e. defined function on $\mathbb{R}^d$, satisfies

$$\nabla\phi \in L^1\big(\mathbb{R}^d, \exp(-\phi)\,dx\big) \cap L^2\big(\mathbb{R}^d, \exp(-\phi)\,dx\big).$$

Note that, for many typical potentials in statistical physics, we have $\phi \in C^{\infty}(\mathbb{R}^d \setminus \{0\})$. For such "outside the origin regular" potentials, condition (D) nevertheless does not exclude a singularity at the point $0 \in \mathbb{R}^d$.

Before we can formulate the next assumption we need to define the set S_{∞} of tempered configurations:

$$S_{\infty} := \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n := \left\{\gamma \in \Gamma \,\middle|\, \sum_{r \in \Lambda_N \cap \mathbb{Z}^d} |\gamma_r|^2 \leq n^2 |\Lambda_N \cap \mathbb{Z}^d|\ \forall N \in \mathbb{N}\right\}.$$

(LS) (Local summability). For all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and all $\gamma \in S_\infty$,

$$\lim_{n\to\infty} \sum_{y\in\gamma_{\Lambda_n\setminus\Lambda}} \nabla\phi(\cdot - y)$$

exists in $L^1_{\mathrm{loc}}(\Lambda, dx)$.

Assuming (ϕ, β, z) satisfies (SS), (UI), (LR), (D) and (LS), and that μ is the corresponding Gibbs measure constructed with empty boundary condition, one can prove an integration-by-parts formula for the gradient ∇^Γ (see [2], Theorem 4.3). Utilizing this formula we obtain, for $F, G \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma)$,

$$\mathcal{E}_\mu^\Gamma(F, G) = \int_\Gamma H_\mu^\Gamma F\, G\, d\mu, \tag{13}$$

where

$$\begin{aligned} H_\mu^\Gamma F(\gamma) &= -\sum_{i,j=1}^N \partial_i\partial_j g_F(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle)\langle(\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \gamma\rangle \\ &\quad - \sum_{j=1}^N \partial_j g_F(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle) \\ &\qquad \times\left(\langle \Delta f_j, \gamma\rangle - \beta \sum_{\{x,y\}\subset\gamma} \big(\nabla\phi(x-y), \nabla f_j(x) - \nabla f_j(y)\big)_{\mathbb{R}^d}\right) \end{aligned} \tag{14}$$

for μ-a.e., $\gamma \in \Gamma$ and $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma)$ as in (11). Moreover, $H_\mu^\Gamma F \in L^2(\mu)$ for each $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma)$ (see [2], Lemma 4.1). Utilizing (13), in [2], Proposition 5.1, the following statement has been proven.

PROPOSITION 3.1. *Assume that* (ϕ, β, z) *fulfill conditions* (SS), (UI), (LR), (D) *and* (LS), *and let* μ *be the corresponding Gibbs measure constructed with empty boundary condition. Then the bilinear form* $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma))$ *is closable on* $L^2(\mu)$ *and its closure* $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ *is a symmetric Dirichlet form which is conservative. Its generator is the Friedrichs extension of* H_μ^Γ, *which will be denoted by the same symbol.*

Of course, H_μ^Γ generates a strongly continuous contraction semigroup

$$T_t^\mu := \exp(-tH_\mu^\Gamma), \qquad t \ge 0.$$

The existence of the diffusion process corresponding to $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ was shown in [2], Theorem 5.2, and [21], Theorem 4.13. For all $d \ge 1$, it lives on the bigger state space $\ddot{\Gamma}$ consisting of all integer-valued Radon measures on $\mathbb{R}^d$ (see, e.g., [10]). For $d \geq 2$ in [28], Corollary 1, the authors have proven that the set $\ddot{\Gamma}/\Gamma$ is $\mathcal{E}_\mu^\Gamma$-exceptional. Thus, the associated diffusion process can be restricted to a process on Γ. For simplicity of notation, we exclude the case $d = 1$ in what follows. However, all our further considerations also work in that case.

THEOREM 3.2. *Let (ϕ, β, z) satisfy the same conditions as in Proposition* 3.1 *and let μ be the corresponding Gibbs measure constructed with empty boundary condition. Then*:

(i) *There exists a conservative diffusion process* (*i.e.*, *a conservative strong Markov process with continuous sample paths*)

$$\mathbf{M} = \big(\mathbf{\Omega}, \hat{\mathbf{F}}, (\hat{\mathbf{F}}_t)_{t \geq 0}, (\Theta_t)_{t \geq 0}, (\mathbf{X}(t))_{t \geq 0}, (\mathbf{P}_\gamma)_{\gamma \in \Gamma}\big)$$

on Γ which is properly associated with $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$; that is, for all (*$\mu$-versions*) *of $F \in L^2(\Gamma, \mu)$ and all $t > 0$, the function*

$$\gamma \mapsto p(t, F)(\gamma) := \int_\Omega F(\mathbf{X}(t))\, d\mathbf{P}_\gamma, \qquad \gamma \in \Gamma,$$

is an $\mathcal{E}_\mu^\Gamma$-quasicontinuous version of $T_t^\mu F$. The process $\mathbf{M}$ *is up to μ-equivalence unique*, *has μ as an invariant measure and is called microscopic stochastic dynamics*.

(ii) *The diffusion process* $\mathbf{M}$ *is up to μ-equivalence the unique diffusion process having μ as invariant measure and solving the martingale problem for $(-H_\mu^\Gamma, D(H_\mu^\Gamma))$; that is, for all $G \in D(H_\mu^\Gamma)$*,

$$G(\mathbf{X}(t)) - G(\mathbf{X}(0)) + \int_0^t H_\mu^\Gamma G(\mathbf{X}(s))\, ds, \qquad t \geq 0,$$

is an $\hat{\mathbf{F}}_t$-martingale under $\mathbf{P}_\gamma$ (*hence starting in γ*) *for $\mathcal{E}_\mu^\Gamma$-q.a. $\gamma \in \Gamma$.*

In the above theorem $\mathbf{M}$ is canonical, that is, $\Omega = C([0, \infty) \to \Gamma)$, $\mathbf{X}(t)(\xi) = \xi(t)$, $\xi \in \Omega$. The filtration $(\hat{\mathbf{F}}_t)_{t \geq 0}$ is the natural "minimum completed admissible filtration" (cf. [6], Chapter. A.2, or [20], Chapter IV) obtained from $\sigma\{\langle f, \mathbf{X}(s)\rangle \mid 0 \leq s \leq t,\ f \in \mathcal{D}\}$, $t \geq 0$. $\hat{\mathbf{F}} := \hat{\mathbf{F}}_\infty := \bigvee_{t \in [0,\infty)} \hat{\mathbf{F}}_t$ is the smallest σ-algebra containing all $\hat{\mathbf{F}}_t$ and $(\Theta_t)_{t \geq 0}$ are the corresponding natural time shifts. For a detailed discussion of these objects and the notion of quasicontinuity we refer to [20]. The second part of the above theorem was proved in [2], Theorem 5.3.

REMARK 3.3. Let us consider the diffusion process $(\mathbf{X}(t))_{t \geq 0}$ provided by Theorem 3.2. In (14) we have an explicit formula for the action of the associated generator $-H_\mu^\Gamma$ on smooth cylinder functions. Utilizing an extension of Itô's formula to this infinite dimensional situation on a heuristic level we find the associated infinite system of stochastic differential equations

$$
(15) \qquad \begin{aligned} dx(t) &= -\beta \sum_{\substack{y(t)\in \mathbf{X}(t),\\ y(t)\neq x(t)}} \nabla\phi\big(x(t)-y(t)\big)\,dt + \sqrt{2}\,dB^x(t), \qquad x(t)\in \mathbf{X}(t),\\ \mathbf{X}(0) &= \gamma, \qquad \gamma\in\Gamma, \end{aligned}
$$

where $(B^x)_{x=x(0)\in\mathbf{X}(0)}$ is a sequence of independent Brownian motions. Theorem 3.2(ii) implies that the process $((\mathbf{X}(t))_{t\geq 0}, \mathbf{P}_\gamma)$ solves the infinite system (15) in the sense of the associated martingale problem for $\mathcal{E}_\mu^\Gamma$-q.a. $\gamma\in\Gamma$ as a starting point.

4. Scaling of stochastic dynamics and associated Dirichlet form. We perform the scaling of the process $(\mathbf{X}(t))_{t\geq 0}$ in two steps.

First scaling. We scale the position of the particles inside the configuration space as follows:

$$
\Gamma \ni \gamma \mapsto S_{\mathrm{in},\varepsilon}(\gamma) := \{\varepsilon x \mid x\in\gamma\}\in\Gamma, \qquad \varepsilon>0;
$$

that is, for $f\in\mathcal{D}$, the scaling is given through $\langle f, S_{\mathrm{in},\varepsilon}(\gamma)\rangle = \sum_{x\in\gamma} f(\varepsilon x)$. Obviously, $S_{\mathrm{in},\varepsilon}$ is a homeomorphism on Γ. From now on we assume that μ corresponds to $(\phi,\beta,1)$, $\beta\geq 0$, and the construction with empty boundary condition. Let us define the image measure $\tilde{\mu}_\varepsilon := S_{\mathrm{in},\varepsilon}^*\mu$. This measure is also defined on $(\Gamma,\mathcal{B}(\Gamma))$ and it is easy to check that it is the Gibbs measure corresponding to $(\phi_\varepsilon,\beta,\varepsilon^{-d})$ and the construction with empty boundary condition, where $\phi_\varepsilon := \phi(\varepsilon^{-1}\cdot)$. Furthermore, since $C(\beta\phi_\varepsilon,\varepsilon^{-d}) = C(\beta\phi,1)$, recall (UI), the measure $\tilde{\mu}_\varepsilon$ is in the LA–HT regime if and only if this is true for μ.

Second scaling. This scaling leads us out of the configuration space and is given by

$$
\Gamma\ni\gamma\mapsto S_{\mathrm{out},\varepsilon}(\gamma) := \varepsilon^{d/2}\big(\gamma - \rho_{\phi_\varepsilon}^{(1)}(\beta,\varepsilon^{-d})\varepsilon^{-d}\,dx\big)\in\Gamma_\varepsilon,
$$

where $\Gamma_\varepsilon := S_{\mathrm{out},\varepsilon}(\Gamma)\subset\mathcal{D}'$, $\varepsilon>0$, $\mathcal{D}'$ is the topological dual of $\mathcal{D}$ (where both $\mathcal{D}$ and $\mathcal{D}'$ are equipped with their respective usual locally convex topology). We consider Γ_ε as a topological subspace of $\mathcal{D}'$, thus Γ_ε is equipped with the corresponding Borel σ-algebra. Obviously, $S_{\mathrm{out},\varepsilon}:\Gamma\to\Gamma_\varepsilon$ is continuous, hence Borel-measurable. Since it is also one-to-one and since both Γ and $\mathcal{D}'$ are standard measurable spaces, it follows by [26], Chapter V, Theorem 2.4, that Γ_ε is a Borel subset of $\mathcal{D}'$ and that $S_{\mathrm{out},\varepsilon}^{-1}:\Gamma_\varepsilon\to\Gamma$ is also Borel-measurable. The function $\rho_{\phi_\varepsilon}^{(1)}(\beta,\varepsilon^{-d})$ is the first correlation function corresponding to the Gibbs measure $\tilde{\mu}_\varepsilon$, that is,

$$
\int_{\mathbb{R}^d} f(x)\rho_{\phi_\varepsilon}^{(1)}(\beta,\varepsilon^{-d})\varepsilon^{-d}\,dx = \int_\Gamma \langle f,\gamma\rangle\,d\tilde{\mu}_\varepsilon(\gamma) \qquad \forall f\in C_0(\mathbb{R}^d).
$$

Applied to a test function $f \in \mathcal{D}$, the second scaling gives

$$\langle f, S_{\mathrm{out},\varepsilon}(\gamma)\rangle = \varepsilon^{d/2}\Biggl(\sum_{x\in\gamma} f(x) - \rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\varepsilon^{-d}\int f(x)\,dx\Biggr), \tag{16}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual paring between $\mathcal{D}$ and $\mathcal{D}'$. Here we assume the LA–HT regime. So, as mentioned before, $\rho^{(1)}_{\phi}(\beta,1)$ is a constant, and thus by definition of $\tilde{\mu}_\varepsilon$ $\rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})$ is also a constant; see Section 2.3. Obviously, the random variable (16) is centered w.r.t. the measure $\tilde{\mu}_\varepsilon$.

Scaled process. The scaled process of interest is

$$\mathbf{X}_\varepsilon(t) := S_{\mathrm{out},\varepsilon}\bigl(S_{\mathrm{in},\varepsilon}(\mathbf{X}(\varepsilon^{-2}t))\bigr), \qquad t\ge 0,\ \varepsilon>0.$$

Associated Dirichlet form. Next for each $\varepsilon>0$ we construct a Dirichlet form $\mathcal{E}_\varepsilon$ such that $(\mathbf{X}_\varepsilon(t))_{t\ge0}$ is the unique process which is properly associated with $\mathcal{E}_\varepsilon$.

Let $\mu_\varepsilon := S^*_{\mathrm{out},\varepsilon}S^*_{\mathrm{in},\varepsilon}\mu = S^*_{\mathrm{out},\varepsilon}\tilde{\mu}_\varepsilon$. Then we define a unitary mapping $\mathcal{S}_{\mathrm{out},\varepsilon}: L^2(\Gamma_\varepsilon,\mu_\varepsilon)\to L^2(\Gamma,\tilde{\mu}_\varepsilon)$ by defining $\mathcal{S}_{\mathrm{out},\varepsilon}F$ to be the $\tilde{\mu}_\varepsilon$-class represented by $\tilde{F}\circ S_{\mathrm{out},\varepsilon}$ for any μ_ε-version $\tilde{F}$ of $F\in L^2(\Gamma_\varepsilon,\mu_\varepsilon)$. Using this mapping we define a bilinear form $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$ as the image bilinear form of $(\mathcal{E}^\Gamma_{\tilde{\mu}_\varepsilon}, D(\mathcal{E}^\Gamma_{\tilde{\mu}_\varepsilon}))$ under the mapping $\mathcal{S}_{\mathrm{out},\varepsilon}$:

$$\mathcal{E}_\varepsilon(F,G) := \mathcal{E}^\Gamma_{\tilde{\mu}_\varepsilon}(\mathcal{S}_{\mathrm{out},\varepsilon}F, \mathcal{S}_{\mathrm{out},\varepsilon}G), \qquad F,G\in D(\mathcal{E}_\varepsilon), \tag{17}$$

where $D(\mathcal{E}_\varepsilon) := \mathcal{S}^{-1}_{\mathrm{out},\varepsilon}D(\mathcal{E}^\Gamma_{\tilde{\mu}_\varepsilon})$. Let $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma_\varepsilon)$ be defined analogously to the space $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma)$. Then, obviously, $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma_\varepsilon)\subset D(\mathcal{E}_\varepsilon)$; hence $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$ is densely defined. It follows by [20], Chapter VI, Exercise 1.1, that $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$ is a Dirichlet form. It is called the image Dirichlet form of $(\mathcal{E}^\Gamma_{\tilde{\mu}_\varepsilon}, D(\mathcal{E}^\Gamma_{\tilde{\mu}_\varepsilon}))$ under the mapping $S_{\mathrm{out},\varepsilon}$. Its generator $(-H_\varepsilon, D(H_\varepsilon))$ is given by

$$H_\varepsilon = \mathcal{S}^{-1}_{\mathrm{out},\varepsilon}H^\Gamma_{\tilde{\mu}_\varepsilon}\mathcal{S}_{\mathrm{out},\varepsilon}, \qquad D(H_\varepsilon) = \mathcal{S}^{-1}_{\mathrm{out},\varepsilon}D\bigl(H^\Gamma_{\tilde{\mu}_\varepsilon}\bigr). \tag{18}$$

Then for $F\in\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D},\Gamma_\varepsilon)\subset D(H_\varepsilon)$, we have

$$\begin{aligned}
H_\varepsilon F(\omega) = &-\sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i\partial s_j}(\langle f_1,\omega\rangle,\ldots,\langle f_N,\omega\rangle)\\
&\qquad\times\bigl\langle(\nabla f_i,\nabla f_j)_{\mathbb{R}^d}, \varepsilon^{d/2}\omega + \rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\,dx\bigr\rangle\\
&-\sum_{j=1}^N \partial_j g_F(\langle f_1,\omega\rangle,\ldots,\langle f_N,\omega\rangle)\\
&\qquad\times\Biggl(\langle\Delta f_j,\omega\rangle - \varepsilon^{d/2}\beta\sum_{\{x,y\}\subset S^{-1}_{\mathrm{out},\varepsilon}\omega}(\nabla\phi_\varepsilon(x-y),\\
&\qquad\qquad\nabla f_j(x)-\nabla f_j(y))_{\mathbb{R}^d}\Biggr),
\end{aligned} \tag{19}$$

where F is of the form (11) and the variable ω is running through Γ_ε. Note that the last term is well defined for μ_ε-a.e. $\omega \in \Gamma_\varepsilon$.

THEOREM 4.1. *Let* $(\phi, \beta, 1)$ *fulfill conditions* (SS), (UI), (LR), (D) *and* (LS), *and let* μ *be the corresponding Gibbs measure constructed with empty boundary condition. Then for all* (μ_ε*-versions*) *of* $F \in L^2(\Gamma_\varepsilon, \mu_\varepsilon)$ *and all* $t > 0$ *the function*

$$\omega \mapsto p_\varepsilon(t, F)(\omega) := \int_\Omega F(\mathbf{X}_\varepsilon(t))\, d\mathbf{P}_{S_{\mathrm{in},\varepsilon}^{-1} S_{\mathrm{out},\varepsilon}^{-1}\omega}, \qquad \omega \in \Gamma_\varepsilon,$$

is a μ_ε*-version of* $T_{\varepsilon,t}F := \exp(-tH_\varepsilon)F$. *For*

$$\mathbf{Q}_\omega := \mathbf{P}_{S_{\mathrm{in},\varepsilon}^{-1} S_{\mathrm{out},\varepsilon}^{-1}\omega}, \qquad \omega \in \Gamma_\varepsilon,$$

the process $\mathbf{M}^\varepsilon = (\Omega, \hat{\mathbf{F}}, (\hat{\mathbf{F}}_{t/\varepsilon^2})_{t\ge 0}, (\Theta_{t/\varepsilon^2})_{t\ge 0}, (\mathbf{X}_\varepsilon(t))_{t\ge 0}, (\mathbf{Q}_\omega)_{\omega\in\Gamma_\varepsilon})$ *is a diffusion process and thus up to* μ_ε*-equivalence the unique process in this class which is properly associated with* $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$ *and has* μ_ε *as an invariant measure.*

PROOF. For $F \in L^2(\Gamma_\varepsilon, \mu_\varepsilon)$, we have $F(\mathbf{X}_\varepsilon(t)) = (\mathcal{S}_{\mathrm{in},\varepsilon}\mathcal{S}_{\mathrm{out},\varepsilon}F)(\mathbf{X}(\varepsilon^{-2}t))$, $t \ge 0$, where $\mathcal{S}_{\mathrm{in},\varepsilon}F := F \circ S_{\mathrm{in},\varepsilon}$. By Theorem 3.2, we have

$$\begin{aligned}
&\big(\mathcal{S}_{\mathrm{out},\varepsilon}^{-1}\mathcal{S}_{\mathrm{in},\varepsilon}^{-1}\exp(-t\varepsilon^{-2}H_\mu^\Gamma)\mathcal{S}_{\mathrm{in},\varepsilon}\mathcal{S}_{\mathrm{out},\varepsilon}F\big)(\omega)\\
&\quad= \int_\Omega \mathcal{S}_{\mathrm{out},\varepsilon}\mathcal{S}_{\mathrm{in},\varepsilon}F(\mathbf{X}(\varepsilon^{-2}t))\, d\mathbf{P}_{S_{\mathrm{in},\varepsilon}^{-1} S_{\mathrm{out},\varepsilon}^{-1}\omega}\\
&\quad= \int_\Omega F(\mathbf{X}_\varepsilon(t))\, d\mathbf{P}_{S_{\mathrm{in},\varepsilon}^{-1} S_{\mathrm{out},\varepsilon}^{-1}\omega}
\end{aligned} \tag{20}$$

for μ_ε almost all $\omega \in \Gamma_\varepsilon$. We note that $(\mathcal{E}_{\tilde{\mu}_\varepsilon}^\Gamma, D(\mathcal{E}_{\tilde{\mu}_\varepsilon}^\Gamma))$ is obviously the image Dirichlet form under the map $\mathcal{S}_{\mathrm{in},\varepsilon}$ of $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ times ε^{-2}. Hence we have, for the corresponding generator $(H_{\tilde{\mu}_\varepsilon}^\Gamma, D(H_{\tilde{\mu}_\varepsilon}^\Gamma))$,

$$H_{\tilde{\mu}_\varepsilon}^\Gamma = \mathcal{S}_{\mathrm{in},\varepsilon}^{-1}\varepsilon^{-2}H_\mu^\Gamma\mathcal{S}_{\mathrm{in},\varepsilon}, \qquad D\big(H_{\tilde{\mu}_\varepsilon}^\Gamma\big) = \mathcal{S}_{\mathrm{in},\varepsilon}^{-1}\big(D(H_\mu^\Gamma)\big). \tag{21}$$

Using the Hille–Yosida theorem (via resolvent), (18) and (21), we can conclude that

$$\mathcal{S}_{\mathrm{out},\varepsilon}^{-1}\mathcal{S}_{\mathrm{in},\varepsilon}^{-1}\exp(-t\varepsilon^{-2}H_\mu^\Gamma)\mathcal{S}_{\mathrm{in},\varepsilon}\mathcal{S}_{\mathrm{out},\varepsilon} = \exp(-tH_\varepsilon) \tag{22}$$

on $L^2(\Gamma_\varepsilon, \mu_\varepsilon)$. Thus, by (20) and (22) the first statement of the theorem is proved. The fact that $\mathbf{M}^\varepsilon$ is a diffusion is straightforward to check. In particular, it then follows by [20], Chapter IV, Theorem 3.5, that $\mathbf{M}^\varepsilon$ is properly associated with $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$. □

5. Convergence of Dirichlet forms. Our aim is to show convergence of $(\mathbf{X}_\varepsilon(t))_{t\geq 0}$ to a generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ as $\varepsilon \to 0$. In this section we prove this in terms of the corresponding Dirichlet forms.

It will turn out that the limit Dirichlet form is defined in $L^2(\mathcal{D}', \nu_\mu)$, where ν_μ is the Gaussian white noise measure on $\mathcal{D}'$ with covariance operator $\chi_\phi(\beta)\,\mathrm{Id}$ and

$$\chi_\phi(\beta) := \rho_\phi^{(1)}(\beta, 1) + \int_{\mathbb{R}^d} u_\phi^{(2)}(\beta, 1, x, 0)\, dx$$

is the *compressibility* of the Gibbs state μ; see (35) below for the definition of the Ursell function $u_\phi^{(2)}$ and Proposition A.3 for the existence of the integral. The measure ν_μ exists due to the Bochner–Minlos theorem via its characteristic function given by

$$\int_{\mathcal{D}'} \exp(i\langle f, \omega\rangle)\, d\nu_\mu(\omega) = \exp\biggl(-\frac{\chi_\phi(\beta)}{2}\int_{\mathbb{R}^d} (f(x))^2\, dx\biggr), \qquad f \in \mathcal{D}.$$

For $n \in \mathbb{Z}$, we define a weighted Sobolev space $\mathcal{H}_n$ as the closure of $\mathcal{D}$ w.r.t. the Hilbert norm

$$\|f\|_n^2 = \langle f, f\rangle_n := \int_{\mathbb{R}^d} A^n f(x) f(x)\, dx, \qquad f \in \mathcal{D},$$

where $Af(x) = -\Delta f(x) + |x|^2 f(x)$, $x \in \mathbb{R}^d$, that is, A is the Hamilton operator of the harmonic oscillator with ground state eigenvalue d. We identify $\mathcal{H}_0 = L^2(\mathbb{R}^d, dx)$ with its dual and obtain

$$\mathcal{D} \subset S(\mathbb{R}^d) \subset \mathcal{H}_n \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{H}_{-n} \subset S'(\mathbb{R}^d) \subset \mathcal{D}', \qquad n \in \mathbb{N}.$$

Here, as usual, $S'(\mathbb{R}^d)$ denotes the space of tempered distributions which is the topological dual of $S(\mathbb{R}^d)$, the Schwartz space of smooth functions on $\mathbb{R}^d$ decaying faster than any polynomial. Of course, $\mathcal{H}_{-n}$ is the topological dual of $\mathcal{H}_n$ w.r.t. $\mathcal{H}_0$. The dual paring between these spaces we denote by $\langle\cdot,\cdot\rangle$. Since the embeddings $\mathcal{H}_n \subset \mathcal{H}_{n-d}$ are Hilbert–Schmidt for all $n \in \mathbb{Z}$, it follows by the Bochner–Minlos theorem that $\nu_\mu(\mathcal{H}_{-d}) = 1$.

The first part of the following theorem is an easy generalization of Proposition 3.9 in [4]. The second and third parts have been proved in [4], Proposition 5.4 and Theorem 6.5, respectively.

THEOREM 5.1. *Let us assume that* $(\phi, \beta, 1)$ *fulfill* (S) *and* (UI), *and let* μ *be the corresponding Gibbs measure constructed with empty boundary condition. Then*:

(i) *There exists* $C^{(1)} \in (0, \infty)$ *such that*

$$\int_{\mathcal{D}'} \|\omega\|_{-(d+1)}^2\, d\mu_\varepsilon(\omega) \leq C^{(1)}$$

uniformly in $\varepsilon \in (0, 1]$ *and, in particular,* $\mu_\varepsilon(\mathcal{H}_{-(d+1)}) = 1$.

(ii) *For each* $f \in \mathcal{D}$, *we have* $\lim_{\varepsilon\to 0} \mathbb{E}_{\mu_\varepsilon}[\langle f, \cdot\rangle^2] = \mathbb{E}_{\nu_\mu}[\langle f, \cdot\rangle^2]$.

(iii) *The family of measures* $(\mu_\varepsilon)_{\varepsilon>0}$ *converges weakly on* $\mathcal{H}_{-(d+1)}$ *to the Gaussian measure* ν_μ *as* $\varepsilon \to 0$.

We shall also use the following lemma, which is easy to derive by using the properties of correlation functions (see Section 2.3) and recalling that $\tilde{\mu}_\varepsilon = S^*_{\mathrm{in},\varepsilon}\mu$ is the Gibbs measure corresponding to $(\phi_\varepsilon, \beta, \varepsilon^{-d})$ and the construction with empty boundary condition.

LEMMA 5.2. *Let the conditions of Theorem* 5.1 *hold*. *Then we have*

$$\rho^{(1)}_{\phi_\varepsilon}(\beta, \varepsilon^{-d}) = \rho^{(1)}_{\phi}(\beta, 1), \qquad \rho^{(2)}_{\phi_\varepsilon}(\beta, \varepsilon^{-d}, x, y) = \rho^{(2)}_{\phi}\left(\beta, 1, \frac{x-y}{\varepsilon}, 0\right).$$

We define the Dirichlet form $(\mathcal{E}_{\nu_\mu}, D(\mathcal{E}_{\nu_\mu}))$ as the closure of the bilinear form

$$\begin{aligned}\mathcal{E}_{\nu_\mu}(F, G) &= -\rho^{(1)}_{\phi}(\beta, 1) \int_{\mathcal{D}'} \int_{\mathbb{R}^d} \partial_x F(\omega) \Delta \partial_x G(\omega)\, dx\, d\nu_\mu(\omega)\\ &= \rho^{(1)}_{\phi}(\beta, 1) \int_{\mathcal{D}'} \int_{\mathbb{R}^d} \big(\nabla \partial_x F(\omega), \nabla \partial_x G(\omega)\big)_{\mathbb{R}^d}\, dx\, d\nu_\mu(\omega),\end{aligned}$$

where $F, G \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D}, \mathcal{D}')$, and the space $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D}, \mathcal{D}')$ is defined analogously to $\mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D}, \Gamma)$. Here $\partial_x F$ denotes the derivative of $F = g_F(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle) \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D}, \mathcal{D}')$ in direction ε_x, $x \in \mathbb{R}^d$, that is,

$$\partial_x F(\omega) = \frac{d}{dt} F(\omega + t\varepsilon_x)\bigg|_{t=0} = \sum_{j=1}^{N} \partial_j g_F(\langle f_1, \omega\rangle, \dots, \langle f_N, \omega\rangle) f_j(x), \qquad \omega \in \mathcal{D}',$$

where $N \in \mathbb{N}$ and $f_1, \dots, f_N \in \mathcal{D}$.

Integrating by parts in the Gaussian space (see, e.g., [3], Theorem 6.1.2 and 6.1.3), we obtain

$$\mathcal{E}_{\nu_\mu}(F, G) = \int_{\mathcal{D}'} HF(\omega) G(\omega)\, d\nu_\mu(\omega), \qquad F, G \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D}, \mathcal{D}'),$$

where

$$\begin{aligned} HF = {}& -\rho^{(1)}_{\phi}(\beta, 1) \sum_{i,j=1}^{N} \partial_i \partial_j g_F(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle)\\ & \times \int_{\mathbb{R}^d} \big(\nabla f_i(x), \nabla f_j(x)\big)_{\mathbb{R}^d}\, dx \\ & - \frac{\rho^{(1)}_{\phi}(\beta, 1)}{\chi_\phi(\beta)} \sum_{j=1}^{N} \partial_j g_F(\langle f_1, \cdot\rangle, \dots, \langle f_N, \cdot\rangle) \langle \Delta f_j, \cdot\rangle. \end{aligned} \tag{23}$$

It is well known (see, e.g., [3], Theorem 6.1.4) that the operator H is essentially self-adjoint on $\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$. We preserve the same notation for its closure. The operator H generates an infinite dimensional Ornstein–Uhlenbeck semigroup

$$T_t := \exp(-tH), \qquad t \geq 0,$$

in $L^2(\nu_\mu)$. This semigroup is associated with a generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ on $\mathcal{D}'$ (see [3], Chapter 6, Section 1.5).

THEOREM 5.3. *Suppose that* $(\phi,\beta,1)$ *satisfy conditions* (S) *and* (UI), *and let* μ *be the corresponding Gibbs measure construction with empty boundary condition. Then for all* $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$, *we have*

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(F,G) = \mathcal{E}_{\nu_\mu}(F,G). \tag{24}$$

REMARK 5.4. (i) The $(\mathbf{X}(t))_{t\geq 0}$ is the unique process associated with the closure of the pre-Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}'))$ on $L^2(\mathcal{D}',\nu_\mu)$. In this sense the convergence of bilinear forms proven in Theorem 5.3 uniquely determines the limiting process $(\mathbf{X}(t))_{t\geq 0}$.

(ii) The generator H corresponds to the following stochastic differential equation:

$$d\mathbf{X}(t,x) = \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)}\Delta\mathbf{X}(t,x)\,dt + \sqrt{2\rho_\phi^{(1)}(\beta,1)}\,d\mathbf{W}(t,x),$$

where $(\mathbf{W}(t))_{t\geq 0}$ is a Brownian motion in $\mathcal{D}'$ with covariance operator—Δ, and the coefficient $\rho_\phi^{(1)}(\beta,1)/\chi_\phi(\beta)$ is called *bulk diffusion coefficient*.

(iii) The generality of the class of admissible potentials is very important from the physical point of view. Before one could only treat smooth, compactly supported, positive potentials. However, any physically realistic potential has a singularity at the origin. Furthermore, it is of physical interest to study potentials which also have a negative part.

(iv) The proof of Theorem 5.3 is straightforward. However, it identifies the bulk diffusion coefficient for very general potentials. This coefficient is, from the physical point of view, the most interesting quantity.

PROOF OF THEOREM 5.3. We first note that each function $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \mathcal{D}')$, when restricted to Γ_ε, belongs to $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\varepsilon) \subset D(\mathcal{E}_\varepsilon)$. Furthermore, since $\mathcal{B}(\mathcal{D}') \cap \Gamma_\varepsilon = \mathcal{B}(\Gamma_\varepsilon)$, the measure μ_ε can be considered as a measure on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$. By the polarization identity, it is sufficient to prove (24) for the case $G = F = g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)$. Evaluating (17) and applying Lemma 5.2

we obtain

$$
\begin{aligned}
\mathcal{E}_\varepsilon(F,F) &= \varepsilon^d \sum_{i,j=1}^N \int_\Gamma \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \gamma\rangle \\
&\qquad \times \partial_i g_F\Big(\big\langle f_1, \varepsilon^{d/2}\big(\gamma - \varepsilon^{-d}\rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\,dx\big)\big\rangle, \\
&\qquad\qquad \dots, \big\langle f_N, \varepsilon^{d/2}\big(\gamma - \varepsilon^{-d}\rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\,dx\big)\big\rangle\Big) \\
&\qquad \times \partial_j g_F\Big(\big\langle f_1, \varepsilon^{d/2}\big(\gamma - \varepsilon^{-d}\rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\,dx\big)\big\rangle, \\
&\qquad\qquad \dots, \big\langle f_N, \varepsilon^{d/2}\big(\gamma - \varepsilon^{-d}\rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\,dx\big)\big\rangle\Big)\,d\tilde{\mu}_\varepsilon(\gamma) \\
&= \varepsilon^{d/2} \sum_{i,j=1}^N \int_{\mathcal{D}'} \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \omega\rangle \partial_i g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle) \\
&\qquad \times \partial_j g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle)\,d\mu_\varepsilon(\omega) \\
&\quad + \rho^{(1)}_\phi(\beta,1) \sum_{i,j=1}^N \int_{\mathbb{R}^d} \big(\nabla f_i(x), \nabla f_j(x)\big)_{\mathbb{R}^d}\,dx \\
&\qquad \times \int_{\mathcal{D}'} \partial_i g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle) \\
&\qquad\qquad \times \partial_j g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle)\,d\mu_\varepsilon(\omega).
\end{aligned}
\tag{25}
$$

By Theorem 5.1(iii) we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\mathcal{D}'} \partial_i g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle)\partial_j g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle)\,d\mu_\varepsilon(\omega) \\
&\qquad = \int_{\mathcal{D}'} \partial_i g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle)\partial_j g_F(\langle f_1,\omega\rangle, \dots, \langle f_N,\omega\rangle)\nu_\mu(\omega);
\end{aligned}
$$

hence, the second term in (25) converges to $\mathcal{E}_{\nu_\mu}(F,F)$ and it only remains to show that first term in (25) converges to zero as $\varepsilon \to 0$. However, this is obvious from Theorem 5.1(i), because $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\mathcal{D}')$. $\square$

6. Convergence in law. The convergence in terms of the Dirichlet forms admits no probabilistic interpretation. Hence, next we study convergence in law of the scaled processes.

The laws of the scaled equilibrium processes $\mathbf{P}^\varepsilon := \mathbf{Q}_{\mu_\varepsilon} \circ \mathbf{X}_\varepsilon^{-1} (= \mathbf{P}_\mu \circ \mathbf{X}_\varepsilon^{-1})$ are probability measures on $C([0,\infty),\Gamma_\varepsilon)$, where $\mathbf{Q}_{\mu_\varepsilon} := \int_{\Gamma_\varepsilon} \mathbf{Q}_\omega\,d\mu_\varepsilon(\omega)$ and $\mathbf{P}_\mu := \int_\Gamma \mathbf{P}_\gamma\,d\mu(\gamma)$ (cf. Theorem 4.1). Since $C([0,\infty),\Gamma_\varepsilon)$ is a Borel subset of $C([0,\infty),\mathcal{D}')$ (under the natural embedding) with compatible measurable structures we can consider $\mathbf{P}^\varepsilon$ as a measure on $C([0,\infty),\mathcal{D}')$ and by using Theorem 3.2(ii) we find that the process $(\mathbf{X}(t))_{t\ge 0}$ corresponding to $\mathbf{P}^\varepsilon$, that is, the realization of $(\mathbf{X}_\varepsilon(t))_{t\ge 0}$ as a coordinate process in $C([0,\infty),\mathcal{D}')$, solves the martingale problem for $(-H_\varepsilon, D(H_\mu^\Gamma))$ w.r.t. the corresponding minimum completed admissible filtration $(\mathbf{F}_t)_{t\geq 0}$ for all $\varepsilon > 0$.

6.1. *Tightness.*

THEOREM 6.1. *Let* $(\phi, \beta, 1)$ *fulfill conditions* (SS), (UI), (LR), (D) *and* (LS), *and let* μ *be the corresponding Gibbs measure constructed with empty boundary condition. Then there exists* $m \in \mathbb{N}$, $m \geq d+1$, *such that the family of probability measures* $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ *can be restricted to the space* $C([0,\infty), \mathcal{H}_{-m})$. *Furthermore,* $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ *is tight on* $C([0,\infty), \mathcal{H}_{-m})$.

REMARK 6.2. Theorem 6.1 has been proved before by Brox [4] and Spohn [34], for smooth, compactly supported potentials only. Their proof can be generalized to a more general class of potentials. However, their technique requires that $\partial_j \phi x^i$ (here $\partial_j \phi$ is the partial derivative of the potential in direction j and x^i the ith component of the identity) is locally integrable w.r.t. the Lebesgue measure. From the physical point of view this is a very restrictive assumption on the singularity of the potential at the origin.

PROOF OF THEOREM 6.1. Let $f \in \mathcal{D}$. By Theorem 5.1(i) we know, in particular, that the functions $\langle f, \cdot\rangle$, $\langle \nabla f, \cdot\rangle \in L^2(\mu_\varepsilon)$. Hence it is easy to show by approximation that $\langle f, \cdot\rangle \in D(\mathcal{E}_\varepsilon)$. Consider the conservative diffusion process $\mathbf{M}^\varepsilon$ on Γ_ε associated with $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$ according to Theorem 4.1. We may regard $\mathbf{M}^\varepsilon$ on the state space $\mathcal{D}'$ (common to all $\mathbf{M}^\varepsilon$, $\varepsilon > 0$). Considering its distribution on $C([0,\infty), \mathcal{D}')$ we may regard its canonical realization $\mathbf{M}^\varepsilon = (\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{Q}^\varepsilon_\omega)_{\omega\in\mathcal{D}'})$. So, in particular, $\Omega = C([0,\infty), \mathcal{D}')$, $X(t)(\omega) = \omega(t), t \geq 0$, $\theta_t(\omega) = \omega(t+\cdot)$ and $\mathbf{P}^\varepsilon = \int_{\Gamma_\varepsilon} \mathbf{Q}^\varepsilon_\omega \, d\mu_\varepsilon(\omega)$. Fix $T > 0$. Below we canonically project the process onto $\Omega_T := C([0,T], \mathcal{D}')$ without expressing this explicitly. We define the time reversal $r_T(\omega) := \omega(T - \cdot)$, $\omega \in \Omega_T$. Now, by the well-known Lyons–Zheng decomposition (cf. [6, 19] and also [18] for its infinite dimensional variant), we have, for all $0 \leq t \leq T$,

$$\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(0)\rangle = \tfrac{1}{2}\mathbf{M}_t(\varepsilon, f) + \tfrac{1}{2}\big(\mathbf{M}_{T-t}(\varepsilon, f)(r_T) - \mathbf{M}_T(\varepsilon, f)(r_T)\big),$$

$\mathbf{P}^\varepsilon$-a.e., where $(\mathbf{M}_t(\varepsilon, f))_{0\leq t\leq T}$ is a continuous $(\mathbf{P}^\varepsilon, (\mathbf{F}_t)_{0\leq t\leq T})$-martingale and $(\mathbf{M}_t(\varepsilon,\ f)(r_T))_{0\leq t\leq T}$ is a continuous $(\mathbf{P}^\varepsilon, (r_T^{-1}(\mathbf{F}_t))_{0\leq t\leq T})$-martingale. [We note that $\mathbf{P}^\varepsilon \circ r_T^{-1} = \mathbf{P}^\varepsilon$ because $(T_{\varepsilon,t})_{t\geq 0}$ is symmetric on $L^2(\mu_\varepsilon)$.] Moreover, by (25), the bracket of $\mathbf{M}(\varepsilon, f)$ is given by

$$\langle \mathbf{M}(\varepsilon, f)\rangle_t = 2\int_0^t \varepsilon^{d/2}\big\langle |\nabla f|^2_{\mathbb{R}^d}, \mathbf{X}(u)\big\rangle + \rho_\phi^{(1)}(\beta, 1)\int_{\mathbb{R}^d} |\nabla f(x)|^2_{\mathbb{R}^d}\, dx\, du,$$

as, for example, directly follows from [6], Theorem 5.2.3 and Theorem 5.1.3(i). We note here that both theorems in [6] are formulated and proved for locally compact separable metric spaces, while $\mathcal{D}'$ is not of this type. However, both theorems carry over to general state spaces by virtue of the local compactification and regularization procedure developed in [20], Chapter VI.2, which is easily seen to be applicable in our case (see, e.g., [20], Chapter VI, Theorem 2.4, in regard to [6], Theorem 5.1.3(i)). Hence by the Burkholder–Davies–Gundy inequalities and since $\mathbf{P}^\varepsilon \circ r_T^{-1} = \mathbf{P}^\varepsilon$, we can find $C^{(3)} \in (0,\infty)$ such that, for all $f \in \mathcal{D}$, $0 < \varepsilon \le 1$, $0 \le s \le t \le T$,

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{P}^\varepsilon}\big[|\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(s)\rangle|^4\big]\\
&\quad \le \mathbb{E}_{\mathbf{P}^\varepsilon}\big[|\mathbf{M}_t(\varepsilon, f) - \mathbf{M}_s(\varepsilon, f)|^4\big]\\
&\qquad + \mathbb{E}_{\mathbf{P}^\varepsilon}\big[|\mathbf{M}_{T-t}(\varepsilon, f)(r_T) - \mathbf{M}_{T-s}(\varepsilon, f)(r_T)|^4\big]\\
&\quad \le C^{(3)}\Bigg(\mathbb{E}_{\mathbf{P}^\varepsilon}\Bigg[\bigg(\int_s^t \bigg(\varepsilon^{d/2}\big\langle|\nabla f|^2_{\mathbb{R}^d}, \mathbf{X}(u)\big\rangle\\
&\qquad\qquad + \rho_\phi^{(1)}(\beta,1)\int_{\mathbb{R}^d}|\nabla f(x)|^2_{\mathbb{R}^d}\,dx\bigg)\,du\bigg)^2\Bigg]\\
&\qquad + \mathbb{E}_{\mathbf{P}^\varepsilon}\Bigg[\bigg(\int_{T-t}^{T-s}\bigg(\varepsilon^{d/2}\big\langle|\nabla f|^2_{\mathbb{R}^d}, \mathbf{X}(T-u)\big\rangle\\
&\qquad\qquad + \rho_\phi^{(1)}(\beta,1)\int_{\mathbb{R}^d}|\nabla f(x)|^2_{\mathbb{R}^d}\,dx\bigg)\,du\bigg)^2\Bigg]\Bigg)\\
&\quad \le 4C^{(3)}(t-s)^2\bigg(\varepsilon^d\int_{\Gamma_\varepsilon}\big\langle|\nabla f|^2_{\mathbb{R}^d}, \omega\big\rangle^2\,d\mu_\varepsilon(\omega)\\
&\qquad\qquad + \rho_\phi^{(1)}(\beta,1)^2\bigg(\int_{\mathbb{R}^d}|\nabla f(x)|^2_{\mathbb{R}^d}\,dx\bigg)^2\bigg)\\
&\quad \le C^{(4)}(t-s)^2\big(\||\nabla f|^2_{\mathbb{R}^d}\|^2_{d+1} + \||\nabla f|_{\mathbb{R}^d}\|^4_0\big),
\end{aligned}
\tag{26}
$$

where $C^{(4)} := 4C^{(3)}\max(C^{(1)}), \rho_\phi^{(1)}(\beta,1)^2$ and $C^{(1)}$ as in Theorem 5.1(i).

Now we can use (26) to define $\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(s)\rangle$ for $f \in S(\mathbb{R}^d)$ via an approximation as an element in $L^4(\Omega, \mathbf{P}^\varepsilon)$. Then, of course, the estimate (26) is also true for $f \in S(\mathbb{R}^d)$.

Let $m \in \mathbb{N}$ and let $(e_i)_{i\in\mathbb{N}}$ be the sequence of Hermite functions, forming an orthonormal system in $\mathcal{H}_{m-2d}$. Then $(a_i^{m-d}e_i)_{i\in\mathbb{N}}$, where $(a_i)_{i\in\mathbb{N}}$ are the eigenvalues of A w.r.t. the Hermite functions, forms an orthonormal system in $\mathcal{H}_{-m}$. Since the mappings $f \mapsto \||\nabla f|^2_{\mathbb{R}^d}\|^2_{d+1}$ and $f \mapsto \||\nabla f|_{\mathbb{R}^d}\|^4_0$ are continuous on $S(\mathbb{R}^d)$, we can choose $\alpha > 0$ and $m \in \mathbb{N}$ large enough that

$$\| |\nabla f|^2_{\mathbb{R}^d} \|^2_{d+1} + \| |\nabla f|_{\mathbb{R}^d} \|^4_0 \leq \alpha \|f\|^4_{m-2d} \qquad \forall f \in S(\mathbb{R}^d).$$

In particular, for all $i \in \mathbb{N}$, we have $\| |\nabla e_i|^2_{\mathbb{R}^d} \|^2_{d+1} + \| |\nabla e_i|_{\mathbb{R}^d} \|^4_0 \leq \alpha$. Hence, by the above, we can estimate

$$\begin{aligned} &\big(\mathbb{E}_{\mathbf{P}^\varepsilon}\big[\|\mathbf{X}(t) - \mathbf{X}(s)\|^4_{-m}\big]\big)^{1/2} \\ &\leq \sum_{i=0}^{\infty} a_i^{-2d} \big(\mathbb{E}_{\mathbf{P}^\varepsilon}\big[|\langle e_i, \mathbf{X}(t)\rangle - \langle e_i, \mathbf{X}(s)\rangle|^4\big]\big)^{1/2} \qquad (27) \\ &\leq C^{(5)}(t-s), \end{aligned}$$

where the constant $C^{(5)} := (\alpha C^{(4)})^{1/2} \sum_{i=0}^{\infty} a_i^{-2d}$ is finite, because A^{-d} is a Hilbert–Schmidt operator. Since by Theorem 5.1(iii) $\mu_\varepsilon \to \nu_\mu$ as $\varepsilon \to 0$, now the tightness of $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ on $C([0,\infty), \mathcal{H}_{-m})$ follows by standard arguments. □

6.2. *Identification of the limit via the associated martingale problem.* To identify the limit by Theorem 6.7 below it would be sufficient to show that each accumulation point $\mathbf{P}$ of $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ solves the martingale problem for $(-H, D_0)$, where $D_0 := \{G(\langle f, \cdot\rangle) \mid G \in C^2_{\mathrm{b}}(\mathbb{R}),\ f \in S(\mathbb{R}^d)\}$, with initial distribution ν_μ, that is,

$$G\big(\langle f, \mathbf{X}(t)\rangle\big) - G\big(\langle f, \mathbf{X}(0)\rangle\big) + \int_0^t HG(\langle f, \cdot\rangle)(\mathbf{X}(s))\,ds, \qquad t \geq 0,$$

is an $\mathbf{F}_t$-martingale under $\mathbf{P}$ and $\mathbf{P} \circ \mathbf{X}(0)^{-1} = \nu_\mu$. One well-known way to establish this property is to prove convergence of the generators H_ε to the generator H as $\varepsilon \to 0$. Thus, first we study the difference $\|(H - H_\varepsilon)G(\langle f, \cdot\rangle)\|_{L^2(\mu_\varepsilon)}$ for $\varepsilon \to 0$. To see that $HG(\langle f, \cdot\rangle) \in L^2(\mu_\varepsilon)$ we use representation (23).

6.2.1. *Nonconvergence of generators.* Using (19) and (23) again, by an approximation argument it is easy to show that $\langle f, \cdot\rangle$, $f \in \mathcal{D}$, is an element of $D(H_\varepsilon)$ and $D(H)$. As we shall prove now at least on such functions the above convergence does not hold if we have nontrivial interactions. For the proof of the following theorem we refer to Appendix C.

THEOREM 6.3. *Let the potential ϕ be isotropic, that is, $\phi(x) = V(r)$, $r = |x|_{\mathbb{R}^d}$, $x \in \mathbb{R}^d$. Furthermore, let $x^k x^l \partial_i \partial_j \phi \in L^1(\mathbb{R}^d, dx)$ and $x^i \partial_i \phi \in L^2(\mathbb{R}^d, dx)$. Additionally, let the assumptions required in Theorems* A.4 *and* B.1 *hold and let μ be the Gibbs measure associated with $(\phi, \beta, 1)$ and the construction with empty boundary condition, where $\beta \in [0, \beta_0]$ and $\beta_0 > 0$ is as in Theorem* A.4. *Then there exists a function $[0, \beta_0] \ni \beta \mapsto R_\phi(\beta) \in \mathbb{R}_+$ such that*

$$\lim_{\varepsilon \to 0} \|(H - H_\varepsilon)\langle f, \cdot\rangle\|_{L^2(\mu_\varepsilon)} = R_\phi(\beta)\|\Delta f\|^2_{L^2(dx)} \qquad \forall f \in \mathcal{D}.$$

Furthermore, if $\mu \neq \pi_1$, *then there exist* $\beta_1(\phi) \in (0, \beta_0]$ *such that* $R_\phi(\beta) > 0$ *for all* $\beta \in (0, \beta_1]$.

REMARK 6.4. Theorem 6.3 states that for high temperatures (small inverse temperature) and sufficiently smooth isotropic potentials the generators do not converge in the L^2-sense. It applies obviously to compactly supported potentials $\phi \in C_0^2(\mathbb{R}^d)$ and has been conjectured in [4, 29, 34].

6.2.2. *A conditional theorem on convergence in law.* To identify the limit the following weaker type of convergence is sufficient.

CONJECTURE 6.5. *Let* $(\phi, \beta, 1)$ *fulfill conditions* (SS), (UI), (LR), (D) *and* (LS), *and let* μ *be the corresponding Gibbs measure constructed with empty boundary condition. Furthermore, for* $G \in C_{\mathrm{b}}^2(\mathbb{R})$, $f \in \mathcal{D}$ *and* $t, s \geq 0$, *define*

$$V_\varepsilon(f, t, s) := \int_t^{t+s} G'\big(\langle f, \mathbf{X}(u)\rangle\big)(H - H_\varepsilon)\langle f, \cdot\rangle(\mathbf{X}(u))\, du.$$

Then

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mathbf{P}^\varepsilon}[|V_\varepsilon(f, t, s)|] = 0.$$

REMARK 6.6. Conjecture 6.5 states that the scaled generators converge in time average, whereas Theorem 6.3 concerns convergence of the scaled generators at an arbitrary fixed time. Conjecture 6.5 was first formulated in [29]. In [34] the author describes a proof of Conjecture 6.5 for positive, smooth, compactly supported potentials and $d \leq 3$, but with $G(x) = x$ (see [34], Assumption (C), page 10). It is easy to show that, if Conjecture 6.5 holds for $G(x) = x$, then it also holds for all $G \in C_{\mathrm{b}}^2(\mathbb{R})$.

THEOREM 6.7. *Let* $(\phi, \beta, 1)$ *fulfill conditions* (SS), (UI), (LR), (D) *and* (LS), *and let* μ *be the corresponding Gibbs measure constructed with empty boundary condition. Assume Conjecture* 6.5. *Additionally, let* $\mathbf{P}$ *be an accumulation point of* $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ *on* $C([0,\infty), \mathcal{H}_{-m})$ *with* $m \in \mathbb{N}$ *as in Theorem* 6.1. *Then* $\mathbf{P}$ *solves the martingale problem for* $(-H,\ D_0)$ *with initial distribution* ν_μ, *that is, for all* $G \in C_{\mathrm{b}}^2(\mathbb{R})$, $f \in S(\mathbb{R}^d)$,

$$G\big(\langle f, \mathbf{X}(t)\rangle\big) - G\big(\langle f, \mathbf{X}(0)\rangle\big) + \int_0^t HG(\langle f, \cdot\rangle)(\mathbf{X}(s))\, ds, \qquad t \geq 0, \tag{28}$$

is an $\mathbf{F}_t$-*martingale under* $\mathbf{P}$ *and* $\mathbf{P} \circ \mathbf{X}(0)^{-1} = \nu_\mu$. *The measure* $\mathbf{P}$ *is uniquely determined by these properties, in particular, all such* $\mathbf{P}$ *coincide. Hence* $\mathbf{P}_\varepsilon \to \mathbf{P}$ *weakly as* $\varepsilon \to 0$.

PROOF. Let $f \in \mathcal{D}$, $t, s \geq 0$, and define the following random variables on $C([0,\infty), \mathcal{H}_{-m})$:

$$U_\varepsilon(f,t,s) := G\bigl(\langle f, \mathbf{X}(t)\rangle\bigr) - G\bigl(\langle f, \mathbf{X}(s)\rangle\bigr) + \int_t^{t+s} H_\varepsilon G(\langle f, \cdot\rangle)(\mathbf{X}(u))\,du,$$

$$U(f,t,s) := G\bigl(\langle f, \mathbf{X}(t)\rangle\bigr) - G\bigl(\langle f, \mathbf{X}(s)\rangle\bigr) + \int_t^{t+s} HG(\langle f, \cdot\rangle)(\mathbf{X}(u))\,du,$$

$$S_\varepsilon(f,t,s) := \varepsilon^{d/2}\int_t^{t+s} G''\bigl(\langle f, \mathbf{X}(u)\rangle\bigr)\mathbf{X}\bigl(|\nabla f|^2_{\mathbb{R}^d}\bigr)(u)\,du.$$

Utilizing Theorem 5.1(i) it follows that

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mathbf{P}^\varepsilon}[|S_\varepsilon(f,t,s)|] = 0. \tag{29}$$

The trace filtration obtained by restricting $(\mathbf{F}_t)_{t\geq 0}$ to $C([0,\infty), \mathcal{H}_{-m})$ coincides with the natural filtration of $C([0,\infty), \mathcal{H}_{-m})$, which we also denote by $(\mathbf{F}_t)_{t\geq 0}$. Since $\mathbf{P}^\varepsilon$ solves the martingale problem for $(-H_\varepsilon, D_0)$ w.r.t. $(\mathbf{F}_t)_{t\geq 0}$, we have for all $\mathbf{F}_t$-measurable bounded, continuous, $F_t : C([0,\infty), \mathcal{H}_{-m}) \to \mathbb{R}$ and $\varepsilon > 0$ that $\mathbb{E}_{\mathbf{P}^\varepsilon}[F_t U_\varepsilon(f,t,s)] = 0$. Thus, together with Conjecture 6.5 and (29), it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}&\mathbb{E}_{\mathbf{P}^\varepsilon}[F_t U(f,t,s)]\\ &= \lim_{\varepsilon\to 0}\mathbb{E}_{\mathbf{P}^\varepsilon}\bigl[F_t\bigl(U_\varepsilon(f,t,s) + V_\varepsilon(f,t,s) + S_\varepsilon(f,t,s)\bigr)\bigr] = 0.\end{aligned} \tag{30}$$

Let $\mathbf{P}$ be an accumulation point of $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ on $C([0,\infty), \mathcal{H}_{-m})$, that is, $\mathbf{P}^{\varepsilon_n} \to \mathbf{P}$ weakly for some subsequence $\varepsilon_n \to 0$ for $n \to \infty$. Obviously, by Theorem 5.1(iii), we have $\mathbf{P}\circ\mathbf{X}(t)^{-1} = \nu_\mu$ for all $t \geq 0$, in particular, $\mathbf{P}\circ\mathbf{X}(0)^{-1} = \nu_\mu$. By (30), it remains to show that

$$\lim_{n\to\infty}\mathbb{E}_{\mathbf{P}^{\varepsilon_n}}[F_t U(f,t,s)] = \mathbb{E}_{\mathbf{P}}[F_t U(f,t,s)]. \tag{31}$$

Obviously, we only have to prove (31) with $U(f,t,s)$ replaced by the last summand in its definition, because for the first two summands convergence is clear. To do this we set

$$\begin{aligned}h &:= HG(\langle f, \cdot\rangle)\\ &= -\rho_\phi^{(1)}(\beta,1)G''(\langle f,\cdot\rangle)\||\nabla f|_{\mathbb{R}^d}\|_0^2 - \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)}G'(\langle f,\cdot\rangle)\langle\Delta f,\cdot\rangle.\end{aligned}$$

Then

$$\begin{aligned}&\left|\mathbb{E}_{\mathbf{P}}\left[F_t\int_t^{t+s} HG(\langle f,\cdot\rangle)(\mathbf{X}(u))\,du\right] - \mathbb{E}_{\mathbf{P}^{\varepsilon_n}}\left[F_t\int_t^{t+s} HG(\langle f,\cdot\rangle)(\mathbf{X}(u))\,du\right]\right|\\ &\quad\leq \int_t^{t+s}\bigl|\mathbb{E}_{\mathbf{P}}[F_t h(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\varepsilon_n}}[F_t h(\mathbf{X}(u))]\bigr|\,du\end{aligned}$$

and for $K_r := \{\omega \in \mathcal{H}_{-m} \mid \|\omega\|_{-m} \le r\}$, $r > 0$, we have both for the positive and negative parts h^+, h^- of h and $u \in [t, t+s]$, setting $h_r^\pm := h^\pm \wedge \sup_{K_r} |h|$,

$$\begin{aligned}
&\bigl|\mathbb{E}_{\mathbf{P}}[F_t h^\pm(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\varepsilon_n}}[F_t h^\pm(\mathbf{X}(u))]\bigr| \\
&\quad \le \left| \int_{\{\mathbf{X}(u)\in K_r\}} |F_t| h_r^\pm(\mathbf{X}(u))\, d\mathbf{P} - \int_{\{\mathbf{X}(u)\in K_r\}} |F_t| h_r^\pm(\mathbf{X}(u))\, d\mathbf{P}^{\varepsilon_n} \right| \\
&\qquad + \int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t|\,|h|(\mathbf{X}(u))\, d\mathbf{P} \\
&\qquad + \int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t|\,|h|(\mathbf{X}(u))\, d\mathbf{P}^{\varepsilon_n} \\
&\quad \le \bigl|\mathbb{E}_{\mathbf{P}}[|F_t| h_r^\pm(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\varepsilon_n}}[|F_t| h_r^\pm(\mathbf{X}(u))]\bigr| \\
&\qquad + 2\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t|\,|h|(\mathbf{X}(u))\, d\mathbf{P} \\
&\qquad + 2\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t|\,|h|(\mathbf{X}(u))\, d\mathbf{P}^{\varepsilon_n}.
\end{aligned}$$

However, for all $r > 0$,

$$\begin{aligned}
&\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t|\,|h|(\mathbf{X}(u))\, d\mathbf{P}^{\varepsilon_n} \\
&\quad \le \|F_t\|_\infty \int_{\mathcal{H}_{-m}\setminus K_r} |h|\, d\mu_\varepsilon \\
&\quad \le \frac{1}{r}\rho_\phi^{(1)}(\beta,1) C^{(1)} \|F_t\|_\infty \biggl(\frac{\|G''\|_\infty}{r} \||\nabla f|_{\mathbb{R}^d}\|_0^2 + \frac{\|G'\|_\infty}{\chi_\phi(\beta)} \|\Delta f\|_m \biggr),
\end{aligned}$$

where we used $|\langle \Delta f, \omega\rangle| \le \|\Delta f\|_m \|\omega\|_{-m}$ and $1 \le \|\omega\|_{-m}/r$ on $\mathcal{H}_{-m} \setminus K_r$. The constant $C^{(1)}$ is as in Theorem 5.1(i). Similarly,

$$\begin{aligned}
&\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t|\,|h|(\mathbf{X}(u))\, d\mathbf{P} \\
&\qquad \le \rho_\phi^{(1)}(\beta,1) \|F_t\|_\infty \|G''\|_\infty \||\nabla f|_{\mathbb{R}^d}\|_0^2 \frac{1}{r^2} \int_{\mathcal{H}_{-m}} \|\omega\|_{-m}^2\, d\nu_\mu(\omega) \\
&\qquad\quad + \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)} \|F_t\|_\infty \|G'\|_\infty \|\Delta f\|_m \frac{1}{r} \int_{\mathcal{H}_{-m}} \|\omega\|_{-m}^2\, d\nu_\mu(\omega),
\end{aligned}$$

and since the Gaussian measure ν_μ has measure 1 on $\mathcal{H}_{-m}$, there exists a constant $C^{(6)} \in (0,\infty)$ such that $\int_{\mathcal{H}_{-m}} \|\omega\|_{-m}^2\, d\nu_\mu(\omega) \le C^{(6)}$. Hence by the weak convergence of $\mathbf{P}^{\varepsilon_n} \to \mathbf{P}$ as $n \to \infty$ and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\limsup_{n\to\infty}\int_t^{t+s}&\big|\mathbb{E}_{\mathbf{P}}[F_t h^{\pm}(\mathbf{X}(u))]-\mathbb{E}_{\mathbf{P}^{\varepsilon_n}}[F_t h^{\pm}(\mathbf{X}(u))]\big|\,du\\&\le\frac{2s}{r}\rho_\phi^{(1)}(\beta,1)\max\{C^{(1)},C^{(6)}\}\|F_t\|_\infty\\&\quad\times\left(\frac{\|G''\|_\infty}{r}\||\nabla f|_{\mathbb{R}^d}\|_0^2+\frac{\|G'\|_\infty}{\chi_\phi(\beta)}\|\Delta f\|_m\right)\end{aligned}$$

for all $r>0$. Letting $r\to\infty$, equality (31) follows and therefore

$$\mathbb{E}_{\mathbf{P}}[F_t U(f,t,s)]=0\qquad\forall f\in\mathcal{D}. \tag{32}$$

However, by an approximation, (32) is also true for all $f\in S(\mathbb{R}^d)$.

Now it remains to show that $\mathbf{P}$ is uniquely determined by (28), but this follows by an easy generalization of Theorem 1.4 in [9]. All the assumptions required there are fulfilled in our situation except for the assumption on the operator B. This operator B in our case is $\sqrt{-\Delta}$, which is not bounded as required in [9]. Analyzing the proof, however, one finds that continuity and boundedness of the function

$$[0,\infty)\ni t\mapsto\langle B\exp(t\Delta)f,B\exp(t\Delta)f\rangle\in[0,\infty)$$

for a fixed $f\in S(\mathbb{R}^d)$ is sufficient, which in our case is obviously true. □

APPENDIX A

Inverse temperature derivative of correlation functions. First, we have to define the finite volume correlation functions

$$\begin{aligned}\rho_{\phi,\Lambda}(\beta,z,\eta)&:=Z_{\phi,\Lambda}^{-1}(\beta,z)\int_{\Gamma_{0,\Lambda}}\exp\big(-\beta E_\Lambda^\phi(\eta\cup\xi)\big)\,d\lambda_z(\xi),\qquad\beta\ge0,\ z>0,\\Z_{\phi,\Lambda}(\beta,z)&:=\int_{\Gamma_{0,\Lambda}}\exp\big(-\beta E_\Lambda^\phi(\xi)\big)\,d\lambda_z(\xi),\qquad\eta\in\Gamma_{0,\Lambda},\ \Lambda\in\mathcal{O}_c(\mathbb{R}^d),\end{aligned}$$

where we restricted the Lebesgue–Poisson measure to $\Gamma_{0,\Lambda}:=\bigsqcup_{n=0}^\infty\Gamma_{0,\Lambda}^{(n)}$; see Section 2.3.

The proof of the following lemma is an easy generalization of [13], Theorem 3.3.18.

LEMMA A.1. *Let* (ϕ,β_0,z) *satisfy conditions* (S) *and* (UI). *Furthermore, let* ϕ *fulfill the condition*

$$0<\int_{\mathbb{R}^d\setminus\Lambda_0}\big(\exp(\beta_0|\phi(x)|)-1\big)\,dx<\infty \tag{33}$$

for some $\Lambda_0 \in \mathcal{O}_c(\mathbb{R}^d)$. *Then*

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \rho^{(n)}_{\phi,\Lambda}(\beta, z, x_1, \dots, x_n) = \rho^{(n)}_{\phi}(\beta, z, x_1, \dots, x_n) \tag{34}$$

for all $z > 0$ *and uniformly in* $\beta, x_1, \dots, x_n$ *on any set* $[0, \beta_0] \times (\Lambda')^n$, *where* $\Lambda' \in \mathcal{O}_c(\mathbb{R}^d)$.

REMARK A.2. Condition (33) is obviously fulfilled for smooth, compactly supported potentials ϕ. Or, if $\phi \in L^1(\mathbb{R}^d \setminus \Lambda_0)$ and bounded on $\mathbb{R}^d \setminus \Lambda_0$ for some $\Lambda_0 \in \mathcal{O}_c(\mathbb{R}^d)$, and not dx-a.e. zero on $\mathbb{R}^d \setminus \Lambda_0$, then condition (33) is also fulfilled.

Via a recursion formula one can transform the correlation functions $\rho^{(n)}_{\phi,\Lambda}$ into the so-called *Ursell functions* $u^{(n)}_{\phi,\Lambda}$ and vice versa (see, e.g., [22, 31]). Their relation is given by

$$\rho_{\phi,\Lambda}(\beta, z, \eta) = \sum_{\substack{\eta_1 \cup \cdots \cup \eta_j = \eta, \\ \eta_k \cap \eta_l = \varnothing,\, k \neq l,\, j \in \mathbb{N}}} u_{\phi,\Lambda}(\beta, z, \eta_1) \cdots u_{\phi,\Lambda}(\beta, z, \eta_j), \qquad \eta \in \Gamma_0, \tag{35}$$

where $u^{(n)}_{\phi,\Lambda}$ is related to $u_{\phi,\Lambda}$ analogously to (8). Correspondingly, $u^{(n)}_{\phi}$ and u_ϕ are defined with ρ_ϕ replacing $\rho_{\phi,\Lambda}$. Due to the translation invariance of the correlation functions, Ursell functions are also translation invariant. Furthermore, by an easy generalization of Theorem 4.5 in [4] (see also [31], Chapter 4), we obtain the following integrability property.

PROPOSITION A.3. *Let* (ϕ, β, z) *satisfy conditions* (S) *and* (UI). *Then*, *for each* $n \geq 1$, *there exists a nonnegative measurable function* $U^{(n+1)}_{\phi,\beta,z} : \mathbb{R}^{d \times n} \to \mathbb{R}_+$, *such that*

$$\left|u^{(n+1)}_{\phi,\Lambda}(\beta, z, \cdot, 0)\right| \leq U^{(n+1)}_{\phi,\beta,z} \qquad \forall\, \Lambda \in \mathcal{O}_c(\mathbb{R}^d),$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{d\times n}} \left|U^{(n+1)}_{\phi,\beta,z}(x_1, \dots, x_n) f(x_1, \dots, x_n)\right| dx_1 \cdots dx_n \\
&\quad \leq \exp\bigl(2n\beta B(\phi)\bigr) \left(\sum_{m=0}^{\infty} \frac{1}{m!} (n + m + 1)^{n+m-1} C(\beta\phi, z)^m \right) \\
&\qquad \times \sup_{x_n \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| \exp\bigl(-\beta\phi(x_n - y_n)\bigr) - 1 \right| \\
&\qquad \times \sup_{x_{n-1} \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| \exp\bigl(-\beta\phi(x_{n-1} - y_{n-1})\bigr) - 1 \right| \\
&\qquad\ \cdots \sup_{x_1 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| \exp\bigl(-\beta\phi(x_1 - y_1)\bigr) - 1 \right| \left| f(y_1, \dots, y_n) \right| dy_1 \cdots dy_n,
\end{aligned}$$

for all measurable functions $f:\mathbb{R}^{d\times n}\to\mathbb{R}$.

THEOREM A.4. *Let* (ϕ,β_0,z) *satisfy conditions* (S) *and* (UI), *and let either* $\phi\equiv 0$ *or* $\phi\in L^1(\mathbb{R}^d,dx)\cap L^2(\mathbb{R}^d,dx)$ *and condition* (33) *hold. Then* $\rho_\phi\in C^2([0,\beta_0])$ *and for* λ_z*-a.e.* $\eta\in\Gamma_0$, *we have*

$$\begin{aligned}\frac{\partial\rho_\phi}{\partial\beta}(\beta,z,\eta)={}&-E^\phi(\eta)\rho_\phi(\beta,z,\eta)\\&-\int_{\mathbb{R}^d}W^\phi(\eta\mid x)\rho_\phi(\beta,z,\eta\cup\{x\})z\,dx\\&-\frac{1}{2}\int_{\mathbb{R}^{d\times 2}}\phi(x-y)\Big(\rho_\phi(\beta,z,\eta\cup\{x,y\})\\&\qquad-\rho_\phi(\beta,z,\eta)\rho_\phi^{(2)}(\beta,z,x,y)\Big)z^2\,dx\,dy,\end{aligned}\tag{36}$$

where $E^\phi(\eta):=\lim_{\Lambda\nearrow\mathbb{R}^d}E^\phi_\Lambda(\eta)$.

PROOF. First, we note that the expression on the right-hand side of (36) is well defined and finite. Indeed, since $\phi\in L^1(\mathbb{R}^d)$ and the correlation functions are bounded [see (9)], the first integral in this expression is finite. Using (35) and Proposition A.3, one finds that the second integral is also finite.

Analyzing the properties of the Lebesgue–Poisson measure we find, for $\eta\in\Gamma_{0,\Lambda}$,

$$\begin{aligned}\frac{\partial\rho_{\phi,\Lambda}}{\partial\beta}(\beta,z,\eta)={}&-E^\phi_\Lambda(\eta)\rho_{\phi,\Lambda}(\beta,z,\eta)-\int_\Lambda W^\phi(\eta\mid x)\rho_{\phi,\Lambda}(\beta,z,\eta\cup\{x\})z\,dx\\&-\frac{1}{2}\int_{\Lambda^2}\phi(x-y)\Big(\rho_{\phi,\Lambda}(\beta,z,\eta\cup\{x,y\})\\&\qquad-\rho_{\phi,\Lambda}(\beta,z,\eta)\rho^{(2)}_{\phi,\Lambda}(\beta,z,x,y)\Big)z^2\,dx\,dy.\end{aligned}$$

Using (34), Proposition A.3 and that bound (9) also holds for finite volume correlation functions, uniformly in $\Lambda\subset\mathbb{R}^d$, twice applying the dominated convergence theorem shows

$$\begin{aligned}&\lim_{\Lambda\nearrow\mathbb{R}^d}\left(\frac{\partial\rho_{\phi,\Lambda}}{\partial\beta}\right)(\beta,z,\eta)\\&\quad=-E^\phi(\eta)\rho_\phi(\beta,z,\eta)-\int_{\mathbb{R}^d}W^\phi(\eta\mid x)\rho_\phi(\beta,z,\eta\cup\{x\})z\,dx\\&\qquad-\frac{1}{2}\int_{\mathbb{R}^{d\times 2}}\phi(x-y)\Big(\rho_\phi(\beta,z,\eta\cup\{x,y\})\\&\qquad\qquad-\rho_\phi(\beta,z,\eta)\rho_\phi^{(2)}(\beta,z,x,y)\Big)z^2\,dx\,dy\end{aligned}$$

for λ_z-a.e. $\eta\in\Gamma_0$. It remains to show that the derivative and the infinite volume limit can be interchanged. We evidently have to show this only for potentials which are not identically equal to zero. By using Lemma A.1 and Proposition A.3, we see that for $z > 0$, $\eta \in \Gamma_0$, fixed the function $\frac{\partial \rho_{\phi,\Lambda}}{\partial \beta}(\beta, z, \eta)$ converges uniformly on $[0, \beta_0]$ as $\Lambda \nearrow \mathbb{R}^d$ and

$$\frac{\partial \rho_\phi}{\partial \beta}(\beta, z, \eta) = \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\partial \rho_{\phi,\Lambda}}{\partial \beta}(\beta, z, \eta).$$

The second order derivative can be derived analogously. The only difference is that in the second order derivative the potential ϕ appears in its second power. Hence, for $\phi \in L^1(\mathbb{R}^d, dx) \cap L^2(\mathbb{R}^d, dx)$ we obtain that $\rho_\phi \in C^2([0, \beta_0])$. A more detailed proof can be found in [7]. □

APPENDIX B

Coercivity identity for Gibbs measures. Here we derive an analog of the usual coercivity identity on $L^2(\mathbb{R}^d, g\,dx)$ for $L^2(\Gamma, \mu)$, where μ is a Ruelle measure on Γ, whose potential satisfies some weak additional conditions.

First we have to develop a little further the analysis and geometry as in Section 3. For each $\gamma \in \Gamma$, consider the triple

$$T_{\gamma,\infty}(\Gamma) \supset T_\gamma(\Gamma) \supset T_{\gamma,0}(\Gamma). \tag{37}$$

Here, $T_{\gamma,0}(\Gamma)$ consists of all finite sequences from $T_\gamma(\Gamma)$, and $T_{\gamma,\infty}(\Gamma) := (T_{\gamma,0}(\Gamma))'$ is the dual space consisting of all sequences $V(\gamma) = (V(\gamma, x))_{x\in\gamma}$, where $V(\gamma, x) \in T_x(\mathbb{R}^d)$. The pairing between any $V(\gamma) \in T_{\gamma,\infty}(\Gamma)$ and $v(\gamma) \in T_{\gamma,0}(\Gamma)$ with respect to the zero space $T_\gamma(\Gamma)$ is given by

$$\big(V(\gamma), v(\gamma)\big)_{T_\gamma(\Gamma)} := \sum_{x\in\gamma} \big(V(\gamma, x), v(\gamma, x)\big)_{T_x(\mathbb{R}^d)}.$$

This series is, in fact, finite.

For $\gamma \in \Gamma$, we define $B_\mu(\gamma) = (B_\mu(\gamma, x))_{x\in\gamma} \in T_{\gamma,\infty}(\Gamma)$ by

$$B_\mu(\gamma, x) := -\beta \sum_{y\in\gamma\setminus\{x\}} \nabla\phi(x - y), \qquad x \in \gamma. \tag{38}$$

As follows from the proof of Lemma 4.1 in [2], for μ-a.e. $\gamma \in \Gamma$ the series on the right-hand side of (38) converges absolutely in $\mathbb{R}^d$, provided (ϕ, β, z) satisfies (SS), (UI), (LR) and (D), and that μ is the corresponding Gibbs measure constructed with empty boundary condition. Observe that

$$\begin{aligned} H_\mu^\Gamma F(\gamma) &:= -\Delta^\Gamma F(\gamma) - \big(B_\mu(\gamma), \nabla^\Gamma F(\gamma)\big)_{T_\gamma(\Gamma)}, \\ \Delta^\Gamma F(\gamma) &:= \sum_{x\in\gamma} \Delta_x F(\gamma), \\ \Delta_x F(\gamma) &:= \Delta_y F_x(x, y)|_{y=x}, \end{aligned} \tag{39}$$

where H_μ^Γ is the generator as in (14) and $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma)$ as in (11). Of course, Δ acting on differentiable functions defined on $\mathbb{R}^d$ is denoting the Laplacian on $\mathbb{R}^d$. We call B_μ the logarithmic derivative of the measure μ.

Let $A(\gamma) \in (T_{\gamma,\infty}(\Gamma))^{\otimes 2}$ [cf. (37)], so that $A(\gamma) = (A(\gamma, x, y))_{x,y\in\gamma}$, where $A(\gamma, x, y) \in T_y(\mathbb{R}^d) \otimes T_x(\mathbb{R}^d)$. We realize $A(\gamma)$ as a linear operator acting from $T_{\gamma,0}(\Gamma)$ into $T_{\gamma,\infty}(\Gamma)$ setting

$$\begin{aligned} T_{\gamma,0}(\Gamma) \ni V(\gamma) &\mapsto A(\gamma)V(\gamma) \\ &:= \left(\sum_{x\in\gamma} \big(A(\gamma, x, y), V(\gamma, x)\big)_{T_x(\mathbb{R}^d)} \right)_{y\in\gamma} \in T_{\gamma,\infty}(\Gamma). \end{aligned}$$

Evidently, if $A(\gamma) \in (T_{\gamma,0}(\Gamma))^{\otimes 2}$, then $A(\gamma)$ defines a linear continuous operator in $T_\gamma(\Gamma)$. We denote by $A(\gamma)^*$ its adjoint operator.

For a vector field $\Gamma \ni \gamma \mapsto W(\gamma) \in T_{\gamma,\infty}(\Gamma)$, we define its derivative $\nabla^\Gamma W(\gamma)$ as a mapping

$$\Gamma \ni \gamma \mapsto \nabla^\Gamma W(\gamma) = \big(\nabla^\Gamma W(\gamma, x, y)\big)_{x,y\in\gamma} \in \big(T_{\gamma,\infty}(\Gamma)\big)^{\otimes 2}$$

such that

$$\nabla^\Gamma W(\gamma, x, y) := \nabla_y W(\gamma, x) = \begin{cases} \nabla_z W(\gamma - \varepsilon_y + \varepsilon_z, x)|_{z=y}, & \text{if } x \neq y, \\ \nabla_z W(\gamma - \varepsilon_y + \varepsilon_z, z)|_{z=y}, & \text{if } x = y, \end{cases}$$

if all derivatives $\nabla_y W(\gamma, x)$, $x, y \in \gamma$, exist. For a function $F : \Gamma \to \mathbb{R}^d$, we write $F'' := \nabla^\Gamma \nabla^\Gamma F$, if it exists.

THEOREM B.1 (Coercivity identity). *Let the potential* ϕ *satisfy* (SS), (I) *and* (LR), *and the three following conditions*:

(i) $\phi \in C^2(\mathbb{R}^d \setminus \{0\})$, $e^{-\phi}$ *is continuous on* $\mathbb{R}^d$ *and* $e^{-\phi}\nabla\phi$ *can be extended to a continuous, vector-valued function on* $\mathbb{R}^d$;

(ii) *for each* $\gamma \in S_\infty$, *the three series* $\sum_{x\in\gamma} \phi(\cdot - x)$, $\sum_{x\in\gamma} \nabla\phi(\cdot - x)$ *and* $\sum_{x\in\gamma} \nabla^2\phi(\cdot - x)$ *converge locally uniformly on* $X \setminus \gamma$;

(iii) *we have*

$$\begin{aligned} &\nabla\phi \in L^1\big(\mathbb{R}^d, \exp(-\phi(x))\, dx\big) \cap L^2\big(\mathbb{R}^d, \exp(-\phi(x))\, dx\big), \\ &\nabla^2\phi \in L^1\big(\mathbb{R}^d, \exp(-\phi(x))\, dx\big). \end{aligned}$$

Furthermore, let μ be the Gibbs measure corresponding to (ϕ, β, z) and the construction with empty boundary condition. Then, for any $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma)$,

$$\begin{aligned}
&\|H_\mu^\Gamma F\|_{L^2(\mu)}^2 \\
&\quad = \int_\Gamma \mathrm{Tr}_{T_\gamma(\Gamma)} F''(\gamma) F''(\gamma)^* \, d\mu(\gamma) \\
&\qquad - \int_\Gamma \big(\nabla^\Gamma F(\gamma), \nabla^\Gamma B_\mu(\gamma) \nabla^\Gamma F(\gamma)\big)_{T_\gamma(\Gamma)} \, d\mu(\gamma) \\
&\quad = \int_\Gamma \mathrm{Tr}_{T_\gamma(\Gamma)} F''(\gamma) F''(\gamma)^* \, d\mu(\gamma) \\
&\qquad + \beta \int_\Gamma \sum_{\{x,y\} \subset \gamma} \Big(\big(\nabla^\Gamma F(\gamma, x) - \nabla^\Gamma F(\gamma, y)\big), \\
&\qquad\qquad \nabla^2 \phi(x-y) \times \big(\nabla^\Gamma F(\gamma, x) - \nabla^\Gamma F(\gamma, y)\big)\Big)_{\mathbb{R}^d} \, d\mu(\gamma).
\end{aligned} \tag{40}$$

REMARK B.2. As easily seen, conditions (i)–(iii) of the above theorem imply (D) and (LS).

REMARK B.3. As will be seen from the proof of Theorem B.1, the coercivity identity (40) holds for each monomial $F = \langle f, \cdot \rangle^n$, where $f \in \mathcal{D}$ and $n \in \mathbb{N}$.

PROOF OF THEOREM B.1. Let $G : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$ be measurable, then, by [24], we have the following, due to condition (ii):

$$\begin{aligned}
&\int_\Gamma \sum_{x \in \gamma} G(\gamma, x) \, d\mu(\gamma) \\
&\qquad = \int_\Gamma \int_{\mathbb{R}^d} z \exp\Big(-\beta \sum_{y \in \gamma} \phi(x-y)\Big) G(\gamma + \varepsilon_x, x) \, dx \, d\mu(\gamma).
\end{aligned} \tag{41}$$

Let $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma)$. By (39) and (41), we get

$$\begin{aligned}
\|H_\mu^\Gamma F\|_{L^2(\mu)}^2 &= \int_\Gamma \sum_{x \in \gamma} \Big(\Delta_x F(\gamma) + \big(B_\mu(\gamma, x), \nabla^\Gamma F(\gamma, x)\big)_{T_x(\mathbb{R}^d)}\Big)^2 \, d\mu(\gamma) \\
&\quad + \int_\Gamma \sum_{x, y \in \gamma, \, x \neq y} \Big(\Delta_x F(\gamma) + \big(B_\mu(\gamma, x), \nabla^\Gamma F(\gamma, x)\big)_{T_x(\mathbb{R}^d)}\Big) \\
&\qquad \times \Big(\Delta_y F(\gamma) + \big(B_\mu(\gamma, y), \nabla^\Gamma F(\gamma, y)\big)_{T_y(\mathbb{R}^d)}\Big) \, d\mu(\gamma)
\end{aligned} \tag{42}$$

$$
\begin{aligned}
&= \int_\Gamma \int_{\mathbb{R}^d} z \exp\biggl(-\beta \sum_{y\in\gamma} \phi(x-y)\biggr) \\
&\qquad \times \Bigl(\Delta_x F(\gamma+\varepsilon_x) \\
&\qquad\qquad + \bigl(B_\mu(\gamma+\varepsilon_x, x), \nabla_x F(\gamma+\varepsilon_x)\bigr)_{T_x(\mathbb{R}^d)}\Bigr)^2 \, dx \, d\mu(\gamma) \\
&\quad + \int_\Gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} z^2 \exp\biggl(-\beta \biggl(\sum_{y_1\in\gamma} \phi(x_1-y_1) \\
&\qquad\qquad\qquad + \sum_{y_2\in\gamma\cup\{x_1\}} \phi(x_2-y_2)\biggr)\biggr) \\
&\qquad \times \Bigl(\Delta_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}) \\
&\qquad\qquad + \bigl(B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}, x_1), \\
&\qquad\qquad\qquad \nabla_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\bigr)_{T_{x_1}(\mathbb{R}^d)}\Bigr) \\
&\qquad \times \Bigl(\Delta_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}) \\
&\qquad\qquad + \bigl(B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}, x_2), \\
&\qquad\qquad\qquad \nabla_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\bigr)_{T_{x_2}(\mathbb{R}^d)}\Bigr) \, dx_1 \, dx_2 \, d\mu(\gamma).
\end{aligned}
$$

By conditions (i) and (ii), we conclude that, for each fixed $\gamma \in S_\infty$, the function

$$
g_\gamma(x) := \exp\biggl(-\beta \sum_{y\in\gamma} \phi(x-y)\biggr)
$$

is continuous on $\mathbb{R}^d$, two times continuously differentiable on $\mathbb{R}^d \setminus \gamma$, and ∇g_γ extends to a continuous function on $\mathbb{R}^d$. Moreover, by (38), $B_\mu(\gamma+\varepsilon_x, x)$ is the logarithmic derivative of the measure $\nu_\gamma := g_\gamma \, dx$. Finally, it is easy to see from (i)–(iii) that the function

$$
g_\gamma(x)\bigl(\log g_\gamma(x)\bigr)'' = \exp\biggl(-\beta \sum_{y\in\gamma} \phi(x-y)\biggr) \beta \sum_{y\in\gamma} \nabla^2 \phi(x-y)
$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Thus, the usual coercivity identity on the space of square-integrable functions $L^2(\mathbb{R}^d, d\nu_\gamma)$ implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \exp\biggl(-\beta \sum_{y\in\gamma} \phi(x-y)\biggr) \\
&\quad \times \Bigl(\Delta_x F(\gamma+\varepsilon_x) + \bigl(B_\mu(\gamma+\varepsilon_x, x), \nabla_x F(\gamma+\varepsilon_x)\bigr)_{T_x(\mathbb{R}^d)}\Bigr)^2 \, dx \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \exp\biggl(-\beta \sum_{y\in\gamma} \phi(x-y)\biggr) \\
&\qquad \times \Bigl(\operatorname{Tr}_{T_x(\mathbb{R}^d)} \nabla_x \nabla_x F(\gamma+\varepsilon_x)\bigl(\nabla_x \nabla_x F(\gamma+\varepsilon_x)\bigr)^* \\
&\qquad\qquad - \bigl(\nabla_x F(\gamma+\varepsilon_x), \nabla_x B_\mu(\gamma+\varepsilon_x, x)\nabla_x F(\gamma+\varepsilon_x)\bigr)_{T_x(\mathbb{R}^d)}\Bigr)\, dx.
\end{aligned}
$$

Absolute analogously, a slight modification of the proof of the coercivity identity on $\mathbb{R}^d$ implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp\biggl(-\beta \sum_{y_1\in\gamma} \phi(x_1-y_1) - \beta \sum_{y_2\in\gamma}\phi(x_2-y_2) - \beta\phi(x_1-x_2)\biggr) \\
&\qquad \times \Bigl(\Delta_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}) \\
&\qquad\qquad + \bigl(B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}, x_1), \nabla_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\bigr)_{T_{x_1}(\mathbb{R}^d)}\Bigr) \\
&\qquad \times \Bigl(\Delta_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}) \\
&\qquad\qquad + \bigl(B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}, x_2), \nabla_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\bigr)_{T_{x_2}(\mathbb{R}^d)}\Bigr)\, dx_1\, dx_2 \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp\biggl(-\beta \sum_{y_1\in\gamma} \phi(x_1-y_1) - \beta \sum_{y_2\in\gamma}\phi(x_2-y_2) - \beta\phi(x_1-x_2)\biggr) \\
&\qquad \times \Bigl(\|\nabla_{x_2}\nabla_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\|^2_{T_{x_2}(\mathbb{R}^d)\otimes T_{x_1}(\mathbb{R}^d)} \\
&\qquad\quad - \bigl(\nabla_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}), \nabla_{x_2} B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}, x_1) \\
&\qquad\qquad\qquad \times \nabla_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\bigr)_{T_{x_2}(\mathbb{R}^d)}\Bigr)\, dx_1\, dx_2.
\end{aligned}
\tag{44}
$$

Next, by (38), (i) and (ii), we get, for any $\gamma\in S_\infty$,

$$
\nabla_y B_\mu(\gamma, x) = \begin{cases} -\beta \displaystyle\sum_{z\in\gamma\setminus\{x\}} \nabla^2\phi(x-z), & \text{if } x = y, \\ \beta\nabla^2\phi(x-y), & \text{otherwise.}\end{cases}
\tag{45}
$$

By (41), (45), condition (iii) and estimate (4.29) in [2], we have, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$,

$$
\begin{aligned}
&\int_\Gamma \int_\Lambda z \exp\biggl(-\beta \sum_{y\in\gamma}\phi(x-y)\biggr) \\
&\qquad \times \bigl(1 + \|\nabla_x B_\mu(\gamma+\varepsilon_x, x)\|_{T_x(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)}\bigr)\, dx\, d\mu(\gamma) \\
&\quad = \int_\Gamma \sum_{x\in\gamma_\Lambda} \bigl(1+\|\nabla_x B_\mu(\gamma, x)\|_{T_x(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)}\bigr)\, d\mu(\gamma)
\end{aligned}
\tag{46}
$$

$$\leq \int_\Gamma \sum_{x\in\gamma_\Lambda} \Bigl(1+\beta \sum_{y\in\gamma\setminus\{x\}} \|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\Bigr)\, d\mu(\gamma)$$

$$= \int_\Lambda \rho_\mu^{(1)}(x)\,dx + \int_\Lambda\int_{\mathbb{R}^d} \rho_\mu^{(2)}(x,y)\beta\|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\,dy\,dx$$

$$\leq \int_\Lambda \rho_\mu^{(1)}(x)\,dx + C^{(8)}\int_\Lambda\int_{\mathbb{R}^d} \beta\|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d} e^{-\beta\phi(x-y)}\,dy\,dx$$

$$< \infty,$$

where $C^{(8)} \in (0,\infty)$ is a constant, and analogously

$$\int_\Gamma\int_\Lambda\int_\Lambda z^2 \exp\Bigl(-\beta\sum_{y_1\in\gamma}\phi(x_1-y_1) - \beta\sum_{y_2\in\gamma}\phi(x_2-y_2) - \beta\phi(x_1-x_2)\Bigr)$$

$$\times \bigl(1+\|\nabla_{x_2} B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_1)\|_{T_{x_2}(\mathbb{R}^d)\otimes T_{x_1}(\mathbb{R}^d)}\bigr)\,dx_1\,dx_2\,d\mu(\gamma)$$

$$= \int_\Gamma \sum_{x,y\in\gamma_\Lambda,\, x\neq y} \bigl(1+\|\nabla_y B_\mu(\gamma,x)\|_{T_y(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)}\bigr)\,d\mu(\gamma) \tag{47}$$

$$= \int_\Gamma \sum_{x,y\in\gamma_\Lambda,\, x\neq y} \bigl(1+\beta\,\|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\bigr)\,d\mu(\gamma)$$

$$< \infty.$$

Now, by (41)–(44), (46) and (47),

$$\begin{aligned}\|H_\mu^\Gamma F\|^2_{L^2(\mu)} = \int_\Gamma \sum_{x\in\gamma} \Bigl(&\mathrm{Tr}_{T_x(\mathbb{R}^d)}\, \nabla_x\nabla_x F(\gamma)\bigl(\nabla_x\nabla_x F(\gamma)\bigr)^* \\ &- \bigl(\nabla_x F(\gamma),\, \nabla_x B_\mu(\gamma,x)\nabla_x F(\gamma)\bigr)_{T_x(\mathbb{R}^d)}\Bigr) \\ &+ \sum_{x,y\in\gamma,\, x\neq y} \Bigl(\|\nabla_y\nabla_x F(\gamma)\|^2_{T_y(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)} \\ &- \bigl(\nabla_y F(\gamma),\, \nabla_y B_\mu(\gamma,x)\nabla_x F(\gamma)\bigr)_{T_y(\mathbb{R}^d)}\Bigr)\,d\mu(\gamma) \\ = &\int_\Gamma \mathrm{Tr}_{T_\gamma(\Gamma)}\, F''(\gamma)F''(\gamma)^*\,d\mu(\gamma) \\ &- \int_\Gamma \bigl(\nabla^\Gamma F(\gamma),\, \nabla^\Gamma B_\mu(\gamma)\nabla^\Gamma F(\gamma)\bigr)_{T_\gamma(\Gamma)}\,d\mu(\gamma).\end{aligned} \tag{48}$$

Finally, from (45) and (48), we get the second equality in (40). □

APPENDIX C

Proof for nonconvergence of generators.

PROOF OF THEOREM 6.3. We have

$$\begin{aligned}\|(H-H_\varepsilon)\langle f,\cdot\rangle\|^2_{L^2(\mu_\varepsilon)} = \int_{\Gamma_\varepsilon} & H\langle f,\omega\rangle H\langle f,\omega\rangle\, d\mu_\varepsilon(\omega)\\ & - 2\int_{\Gamma_\varepsilon} H\langle f,\omega\rangle H_\varepsilon\langle f,\omega\rangle\, d\mu_\varepsilon(\omega)\\ & + \int_{\Gamma_\varepsilon} H_\varepsilon\langle f,\omega\rangle H_\varepsilon\langle f,\omega\rangle\, d\mu_\varepsilon(\omega).\end{aligned} \tag{49}$$

A direct consequence of Theorem 5.1(ii) is that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} H\langle f,\omega\rangle H\langle f,\omega\rangle\, d\mu_\varepsilon(\omega) = \frac{(\rho^{(1)}_\phi(\beta,1))^2}{\chi_\phi(\beta)}\|\Delta f\|^2_{L^2(dx)}.$$

Furthermore, we have

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} H\langle f,\omega\rangle H_\varepsilon\langle f,\omega\rangle\, d\mu_\varepsilon(\omega) = \frac{(\rho^{(1)}_\phi(\beta,1))^2}{\chi_\phi(\beta)}\|\Delta f\|^2_{L^2(dx)}, \tag{50}$$

where (50) can be shown in the same way as the convergence of Dirichlet forms in Theorem 5.3, the argument is even more simple. Showing convergence of the third term in (49) is a quite elaborate task. Using (18), the coercivity identity provided in Theorem B.1, (7) and Lemma 5.2, we obtain

$$\begin{aligned}&\int_{\Gamma_\varepsilon} H_\varepsilon\langle f,\omega\rangle H_\varepsilon\langle f,\omega\rangle\, d\mu_\varepsilon(\omega)\\ &\quad= \varepsilon^d \int_\Gamma H_{\tilde\mu_\varepsilon}\langle f,\gamma\rangle H_{\tilde\mu_\varepsilon}\langle f,\gamma\rangle\, d\tilde\mu_\varepsilon(\gamma)\\ &\quad= \rho^{(1)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d})\|\Delta f\|^2_{L^2(dx)}\\ &\qquad + \frac{\varepsilon^{-d}}{2}\beta\int_{\mathbb{R}^{2d}}\big(\nabla f(x)-\nabla f(y),\\ &\qquad\qquad \nabla^2\phi_\varepsilon(x-y)(\nabla f(x)-\nabla f(y))\big)_{\mathbb{R}^d}\rho^{(2)}_{\phi_\varepsilon}(\beta,\varepsilon^{-d},x,y)\,dx\,dy\\ &\quad= \rho^{(1)}_\phi(\beta,1)\|\Delta f\|^2_{L^2(dx)}\\ &\qquad + \frac{\varepsilon^{-(d+2)}}{2}\beta\int_{\mathbb{R}^{2d}}\bigg(\nabla f(x)-\nabla f(y),\nabla^2\phi\Big(\frac{x-y}{\varepsilon}\Big)(\nabla f(x)-\nabla f(y))\bigg)_{\mathbb{R}^d}\\ &\qquad\qquad \times\rho^{(2)}_\phi\Big(\beta,1,\frac{x-y}{\varepsilon},0\Big)\,dx\,dy.\end{aligned}$$

By the mean value theorem, we get

$$\frac{\varepsilon^{-(d+2)}}{2}\Big(\nabla f(x)-\nabla f(y), \nabla^2\phi\Big(\frac{x-y}{\varepsilon}\Big)\big(\nabla f(x)-\nabla f(y)\big)\Big)_{\mathbb{R}^d}$$
$$=\frac{\varepsilon^{-d}}{2}\int_0^1\int_0^1\Big(\nabla^2 f(y+q_1(x-y))\frac{x-y}{\varepsilon},$$
$$\nabla^2\phi\Big(\frac{x-y}{\varepsilon}\Big)\nabla^2 f(y+q_2(x-y))\frac{x-y}{\varepsilon}\Big)_{\mathbb{R}^d} dq_1\, dq_2.$$

Thus, we obtain an approximate identity and

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} H_\varepsilon\langle f,\omega\rangle H_\varepsilon\langle f,\omega\rangle\, d\mu_\varepsilon(\omega)$$
$$=\rho_\phi^{(1)}(\beta,1)\|\Delta f\|^2_{L^2(dx)}$$
$$+\tfrac{1}{2}\sum_{i,j,k,l=1}^{d}\int_{\mathbb{R}^d}\beta x^k x^l \partial_i\partial_j\phi(x)\rho_\phi^{(2)}(\beta,1,x,0)\,dx$$
$$\times\int_{\mathbb{R}^d}\partial_i\partial_k f(y)\partial_j\partial_l f(y)\,dy,$$

where x^i is the ith component of $x\in\mathbb{R}^d$. Set

$$D_\phi(\beta,i,j,k,l)$$
(51)
$$:=\int_{\mathbb{R}^d}\beta x^k x^l\partial_i\partial_j\phi(x)\rho_\phi^{(2)}(\beta,1,x,0)\,dx.$$

For isotropic potentials the corresponding second correlation function is also isotropic and the coefficient (51) turns out to be

$$\int_{\mathbb{R}^d}\beta\Big(\frac{x^i x^j}{r^3}\big(rV''(r)-V'(r)\big)+\frac{V'(r)}{r}\delta_{i,j}\Big)x^k x^l\tilde{\rho}_\phi^{(2)}(\beta,1,r)\,dx,$$

here $\delta_{i,j}$ is the Kronecker delta. Hence, for isotropic potentials the coefficient $D_\phi(\beta,i,j,k,l)$ is only different from zero if each index in the set $\{i,j,k,l\}$ at least occurs twice. Utilizing polar coordinates, the identity $\int_0^{2\pi}\sin^4(\theta)\,d\theta = 3\int_0^{2\pi}\sin^2(\theta)\cos^2(\theta)\,d\theta$ and the symmetry of $\int_{\mathbb{R}^d}\partial_i\partial_k f(y)\,\partial_j\partial_l f(y)\,dy$ in all its indexes, we find

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} H_\varepsilon\langle f,\omega\rangle H_\varepsilon\langle f,\omega\rangle\, d\mu_\varepsilon(\omega)=D_\phi(\beta)\|\Delta f\|^2_{L^2(dx)},$$

where

$$D_\phi(\beta)=\rho_\phi^{(1)}(\beta,1)+\tfrac{1}{2}\int_{\mathbb{R}^d}\beta x^1 x^1\partial_1\partial_1\phi(x)\rho_\phi^{(2)}(\beta,1,x,0)\,dx.$$

Next, we compare the coefficients $D_\phi(\beta)$ and $(\rho_\phi^{(1)}(\beta,1))^2/\chi_\phi(\beta)$ in terms of a high temperature expansion. The latter coefficient is the *isothermal compressibility* of the fluid or gas characterized by μ. Applying Theorem A.4, we obtain

$$D_\phi(\beta) = 1 + \beta^2 \frac{\partial^2}{\partial\beta^2} D_\phi(0) + o(\beta^2),$$

$$\frac{(\rho_\phi^{(1)}(\beta,1))^2}{\chi_\phi(\beta)} = 1 + \beta^2 \frac{\partial^2}{\partial\beta^2} \frac{(\rho_\phi^{(1)}(\cdot,1))^2}{\chi_\phi}(0) + o(\beta^2), \qquad \beta \in [0,\beta_0],$$

where

$$\frac{\partial^2}{\partial\beta^2} D_\phi(0) = -\biggl(\int_{\mathbb{R}^d} \phi(x)\,dx\biggr)^2 + \int_{\mathbb{R}^d} \bigl(x^1\partial_1\phi(x)\bigr)^2\,dx,$$

$$\frac{\partial^2}{\partial\beta^2} \frac{(\rho_\phi^{(1)}(\cdot,1))^2}{\chi_\phi}(0) = -\biggl(\int_{\mathbb{R}^d} \phi(x)\,dx\biggr)^2.$$

Thus, the remainder function is given by

$$\begin{aligned} R_\phi(\beta) &= D_\phi(\beta) - \frac{(\rho_\phi^{(1)}(\beta,1))^2}{\chi_\phi(\beta)} \\ &= \beta^2 \int_{\mathbb{R}^d} \bigl(x^1\partial_1\phi(x)\bigr)^2\,dx + o(\beta^2), \qquad \beta \in [0,\beta_0]. \end{aligned} \tag{52}$$

Hence, $R_\phi \equiv 0$ on $[0,\beta_0]$ is equivalent to $\int_{\mathbb{R}^d}(x^1\partial_1\phi(x))^2\,dx = 0$. This, in turn, is equivalent to

$$\partial_1\phi(x) = 0 \qquad \text{for } dx\text{-a.e., } x \in \mathbb{R}^d. \tag{53}$$

Since the potential is isotropic, the only potential in consideration which fulfills (53) is $\phi \equiv 0$. Hence, by (52) for $\mu \neq \pi_1$ there exists $\beta_1 \in (0,\beta_0]$ such that $R_\phi(\beta) > 0$ for all $\beta \in (0,\beta_1]$. □

Acknowledgments. We thank Sergio Albeverio, Hans Otto Georgii, Tobias Kuna and Herbert Spohn for discussions and helpful comments. We also thank a very conscientious referee for a careful reading of the manuscript and making very useful comments and suggestions.

REFERENCES

[1] ALBEVERIO, S., KONDRATIEV, YU. G. and RÖCKNER, M. (1998). Analysis and geometry on configuration spaces. *J. Funct. Anal.* **154** 444–500.

[2] ALBEVERIO, S., KONDRATIEV, YU. G. and RÖCKNER, M. (1998). Analysis and geometry on configuration spaces: The Gibbsian case. *J. Funct. Anal.* **157** 242–291.

[3] BEREZANSKY, YU. M. and KONDRATIEV, YU. G. (1995). *Spectral Methods in Infinite-Dimensional Analysis*. Kluwer, Dordrecht. [Originally published (1988) in Russian, Naukova Dumka, Kiev.]

[4] BROX, T. (1980). Gibbsgleichgewichtsfluktuationen für einige Potentiallimiten. Ph.D. dissertation, Univ. Heidelberg.

[5] FRADON, M., ROELLY, S. and TANEMURA, H. (2000). An infinite system of Brownian balls with infinite range interaction. *Stochastic Process. Appl.* **90** 43–66.

[6] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). *Dirichlet Forms and Symmetric Markov Processes*. de Gruyter, Berlin.

[7] GROTHAUS, M., KONDRATIEV, YU. G., LYTVYNOV, E. and RÖCKNER, M. (2001). Scaling limit of stochastic dynamics in classical continuous systems. Bibos Preprint 01-08-051, Univ. Bielefeld.

[8] GUO, M. Z. and PAPANICOLAOU, G. (1985). Bulk diffusion for interacting Brownian particles. In *Statistical Physics and Dynamical Systems* 41–48. Birkhäuser, Boston.

[9] HOLLEY, R. A. and STROOCK, D. W. (1978). Generalized Ornstein–Uhlenbeck processes and infinite particle branching Brownian motion. *Publ. Res. Inst. Sci.* **14** 741–788.

[10] KALLENBERG, O. (1975). *Random Measures*. Academic Press, San Diego.

[11] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. Springer, Berlin.

[12] KONDRATIEV, YU. G. and KUNA, T. (2002). Harmonic analysis on configuration space. I. General theory. *Infinite Dimens. Anal. Quantum Probab. Relat.* **5** 201–233.

[13] KUNA, T. (1999). Studies in configuration space analysis and applications. Ph.D. dissertation, Univ. Bonn.

[14] LANG, R. (1977). Unendlichdimensionale Wienerprozesse mit Wechselwirkung I, II. *Z. Wahrsch. Verw. Gebiete* **38** 55–72; **39** 277–299.

[15] LENARD, A. (1973). Correlation functions and the uniqueness of the state in classical statistical mechanics. *Comm. Math. Phys.* **30** 35–44.

[16] LENARD, A. (1975). States of classical statistical mechanical systems of infinitely many particles I. *Arch. Rational Mech. Anal.* **59** 219–239.

[17] LENARD, A. (1975). States of classical statistical mechanical systems of infinitely many particles II. *Arch. Rational Mech. Anal.* **59** 241–256.

[18] LYONS, T. J. and ZHANG, T. S. (1994). Decomposition of Dirichlet processes and its applications. *Ann. Probab.* **22** 494–524.

[19] LYONS, T. J. and ZHENG, W. A. (1988). A crossing estimate for the canonical process on a Dirichlet space and a tightness result. *Colloque Paul Lévy sur les processes stochastique. Astérisque* **157/158** 249–272.

[20] MA, Z.-M. and RÖCKNER, M. (1992). *Introduction to the Theory of* (*Non-Symmetric*) *Dirichlet Forms*. Springer, Berlin.

[21] MA, Z.-M. and RÖCKNER, M. (2000). Construction of diffusions on configuration spaces. *Osaka J. Math.* **37** 273–314.

[22] MALYSHEV, V. A. and MINLOS, R. A. (1991). *Gibbs Random Fields*: *Cluster Expansion*. Kluwer, Dordrecht.

[23] MINLOS, R. A. (1967). Limiting Gibbs distribution. *Funktsional. Anal. Prilozhen.* **1** 60–73.

[24] NGUYEN, X. X. and ZESSIN, H. (1979). Integral and differentiable characterizations of the Gibbs process. *Math. Nachr.* **88** 105–115.

[25] OSADA, H. (1996). Dirichlet form approach to infinite-dimensional Wiener process with singular interactions. *Comm. Math. Phys.* **176** 117–131.

[26] PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces*. Academic Press, New York.

[27] RÖCKNER, M. (1998). Stochastic analysis on configuration spaces: Basic ideas and recent results. In *New Directions in Dirichlet Forms* (J. Jost, W. Kendall, U. Mosco, M. Röckner and K.-T. Sturm, eds.) 157–232. Amer. Math. Soc., Providence, RI.
[28] RÖCKNER, M. and SCHMULAND, B. (1998). A support property for infinite-dimensional interacting diffusion processes. *C. R. Acad. Sci. Paris Sér. I Math.* **326** 359–364.
[29] ROST, H. (1981). Hydrodynamik gekoppelter Diffusionen: Fluktuationen im Gleichgewicht. *Dynamics and Processes*. *Lecture Notes in Math.* **1031** 97–107. Springer, Berlin.
[30] RUELLE, D. (1963). Correlation functions of classical gases. *Ann. Phys.* **25** 109–120.
[31] RUELLE, D. (1969). *Statistical Mechanics. Rigorous Results.* Benjamins, Amsterdam.
[32] RUELLE, D. (1970). Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.* **18** 127–159.
[33] SHIGA, T. (1979). A remark on infinite-dimensional Wiener processes with interactions. *Z. Wahrsch. Verw. Gebiete* **47** 299–304.
[34] SPOHN, H. (1986). Equilibrium fluctuations for interacting Brownian particles. *Comm. Math. Phys.* **103** 1–33.
[35] SPOHN, H. (1991). *Large Scale Dynamics of Interacting Particles*. Springer, Berlin.
[36] TANEMURA, H. (1997). Uniqueness of Dirichlet forms associated with systems of infinitely many Brownian balls in $\mathbb{R}^d$. *Probab. Theory Related Fields* **109** 275–299.
[37] YOSHIDA, M. W. (1996). Construction of infinite-dimensional interacting diffusion process through Dirichlet forms. *Probab. Theory Related Fields* **106** 265–297.

M. GROTHAUS
BIBOS AND MATHEMATICS DEPARTMENT
BIELEFELD UNIVERSITY
33615 BIELEFELD
GERMANY
AND
IAM
BONN UNIVERSITY
53115 BONN
GERMANY
E-MAIL: grothaus@wiener.iam.uni-bonn.de
URL: http://wiener.iam.uni-bonn.de/~grothaus/

Y. G. KONDRATIEV
BIBOS AND MATHEMATICS DEPARTMENT
BIELEFELD UNIVERSITY
33615 BIELEFELD
GERMANY
AND
INSTITUTE FOR MATHEMATICS
NASU
252601 KIEV
UKRAINE
E-MAIL: kondrat@mathematik.uni-bielefeld.de

E. LYTVYNOV
BIBOS
BIELEFELD UNIVERSITY
33615 BIELEFELD
GERMANY
AND
IAM
BONN UNIVERSITY
53115 BONN
GERMANY
E-MAIL: lytvynov@wiener.iam.uni-bonn.de

M. RÖCKNER
BIBOS AND MATHEMATICS DEPARTMENT
BIELEFELD UNIVERSITY
33615 BIELEFELD
GERMANY
E-MAIL: roeckner@mathematik.uni-bielefeld.de