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REGULARITY AND IRREGULARITY OF $(1+\beta)$-STABLE SUPER-BROWNIAN MOTION

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This paper establishes the continuity of the density of $(1+\beta)$-stable super-Brownian motion $(0<\beta<1)$ for fixed times in $d=1$, and local unboundedness of the density in all higher dimensions where it exists. We also prove local unboundedness of the density in time for a fixed spatial parameter in any dimension where the density exists, and local unboundedness of the occupation density (the local time) in the spatial parameter for dimensions $d\geq 2$ where the local time exists.

1. Introduction. This paper is devoted to the regularity results for the $(1+\beta)$-stable super-Brownian motion. First, we need to introduce the following notation. $\mathcal{M}$ is the space of all Radon measures on $\mathbb{R}^d$ and $\mathcal{M}_F$ is the space of finite measures on $\mathbb{R}^d$ with weak topology ($\Rightarrow$ denotes weak convergence). In general, if F is a set of functions, write F_+ or F^+ for nonnegative functions in F. For any metric space E, let $\mathcal{C}_E[0,\infty)$ [respectively, $\mathcal{D}_E[0,\infty)$] denote the space of continuous (respectively, cadlag) E-valued paths with compact-open (respectively, Skorohod) topology. The integral of a function ϕ with respect to a measure μ is written as $\langle\mu,\phi\rangle$ or $\langle\phi,\mu\rangle$ or $\mu(\phi)$. We use c to denote a positive, finite constant whose value may vary from place to place. A c with some additional notation (as $c_{1,2}$) will denote a specific constant. A constant of the form $c(a,b,\dots)$ means that this constant depends on parameters $a,b,\dots$.

Let $(\Omega,\mathcal{F},\mathcal{F}_\cdot,P)$ be the probability space with filtration, which is sufficiently large to contain all the processes defined below. Let $\mathcal{C}(E)$ denote the space of continuous functions on E and let $\mathcal{C}_{\mathrm{b}}(E)$ be the space of bounded functions in $\mathcal{C}(E)$. Let $\mathcal{C}_{\mathrm{b}}^n=\mathcal{C}_{\mathrm{b}}^n(\mathbb{R}^d)$ denote the subspace of functions in $\mathcal{C}_{\mathrm{b}}=\mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ whose partial derivatives of order n or less are also in $\mathcal{C}_{\mathrm{b}}$. A cadlag adapted measure-valued process X is called a super-Brownian motion with $(1+\beta)$-stable branching if X satisfies the following martingale problem:

$$\begin{aligned}&\exp\{-\langle X_t,\phi\rangle\}-\exp\{-\langle X_0,\phi\rangle\}\\ &\qquad-\int_0^t\exp\{-\langle X_s,\phi\rangle\}\big(-\langle X_s,\tfrac12\Delta\phi\rangle+\gamma\langle X_s,\phi^{1+\beta}\rangle\big)\,ds\end{aligned}\tag{1.1}$$

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is an $\mathcal{F}_t$-martingale for any nonnegative ϕ in $\mathcal{C}_b^2$. In the following, we will assume without loss of generality that $\gamma = 1$ (this can always be done by an appropriate scaling of X).

If $\beta = 1$, $X_{\cdot}$ has continuous sample $\mathcal{M}_F$-valued paths, while, for $0 < \beta < 1$, $X_{\cdot}$ is a.s. discontinuous and has jumps all of the form $\Delta X_t = \delta_{x(t)} m(t)$ at a set of times dense in $[0, \zeta)$, where $\zeta = \inf\{t : X_t(1) = 0\}$ is the lifetime of X (see, e.g., Section 6.2.2 of [3]). For $t > 0$ fixed, X_t is absolutely continuous a.s. if and only if $d < 2/\beta$ (see [7] and Theorem 8.3.1 of [3]). If $\beta = 1$ and $d = 1$, then much more can be said—X_t is absolutely continuous for all $t > 0$ a.s. and has a density $X(t, x)$ which is jointly continuous on $(0, \infty) \times \mathbb{R}$ (see [9] and [11]). In view of the jumps of X (described above) if $0 < \beta < 1$, we see that X_t cannot have a density for a dense set of times a.s. and the regularity properties of the densities that do exist have remained unresolved. In this work, we consider the "stable branching" case of $0 < \beta < 1$ and consider the question:

Is the fixed time density of X_t continuous in space?

The analytic methods used in [7] to prove the existence of a density at a fixed time do not shed any light on its regularity properties. Our main results give a complete answer to this question which we found a bit surprising (see Theorem 1.1): There is a continuous version of the density if and only if $d = 1$. Moreover, when $d > 1$ the density is very badly behaved [see Theorems 1.1(b) and 1.3(b)].

Define a mollifier $J_n(\cdot) = c_d^{-1} \varepsilon_n^{-d/2} \mathbb{1}_{B(0,\sqrt{\varepsilon_n})}(\cdot)$, where $\varepsilon_n = 2^{-n}$, $c_d = \pi^{d/2}/\Gamma(1 + d/2)$ and $B(x, r)$ denotes the open ball with center at x and radius r. As was noted above, if $d < 2/\beta$, then super-Brownian motion X_t with $(1 + \beta)$-stable branching is absolutely continuous in space for fixed times t and so

$$X_t(dx) = \bar{X}(t, x)\, dx,$$

where

$$\bar{X}(t, x) \equiv \begin{cases} \lim_{n\to\infty} J_n * X_t(x), & \text{if it exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Let us introduce some additional notation. Let $\mathcal{C}^{1,2}((0, \infty) \times \mathbb{R}^d)$ denote the set of all real-valued functions ψ on $(0, \infty) \times \mathbb{R}^d$ such that $t \mapsto \psi(t, \cdot)$, $t \mapsto \frac{\partial}{\partial t}\psi(t, \cdot)$ are continuous $\mathcal{C}_b^2$-valued functions on $(0, \infty)$. For each $\mu, \nu \in \mathcal{M}$, $V_t(\mu, \nu)$ denotes the unique (if it exists) nonnegative solution of the nonlinear evolution equation:

$$v_t = S_t\mu - \int_0^t S_{t-s}(v_s^{1+\beta})\, ds + \int_0^t S_{t-s}(\nu)\, ds, \tag{1.3}$$

where S_t is the semigroup of standard Brownian motion in $\mathbb{R}^d$ and p_t is the corresponding transition density. For any measure μ on $\mathbb{R}^d$, we set $S_t\mu(x) \equiv \int_{\mathbb{R}^d} p_t(x - y)\mu(dy)$. In the following, we will identify a nonnegative function, ϕ, integrable on compacts with the associated measure $\phi(x)\, dx$. Let $\mathcal{B}_b$ denote bounded Borel-measurable functions on $\mathbb{R}^d$. It is well known that, for $\phi, \psi \in \mathcal{B}_{b,+}$, $V_t(\phi, \psi)$ exists in any dimension and, moreover, it uniquely characterizes the Laplace transform of X and its weighted occupation time:

$$E\left[\exp\left\{-\langle X_t, \phi\rangle - \int_s^t \langle X_u, \psi\rangle\, du\right\}\Big|\mathcal{F}_s\right] \tag{1.4}$$
$$= \exp\{-\langle X_s,\ V_{t-s}(\phi, \psi)\rangle\}, \qquad P\text{-a.s. } \forall\, x \in \mathbb{R}^d,\ 0 \le s < t.$$

The above equation is derived in Theorem 3.1 of [8] (see also Theorem 7.4.1 of [3]) for any pair (ϕ, ψ) of continuous, nonnegative functions vanishing at ∞. But it is not difficult to get (1.4) for all $\phi, \psi \in \mathcal{B}_{b,+}$, by taking bounded pointwise limits.

By Proposition A.2 of [7] and Lemma 2.1 of [10], for $d < 2/\beta$ and any $\mu \in \mathcal{M}_F$ there exists unique $V_t(\mu, 0) \in \mathcal{C}^{1,2}((0,\infty) \times \mathbb{R}^d)$ that solves (1.3). By Proposition 2.5 of [7], for $d < 2+2/\beta$ and any $\nu \in \mathcal{M}_F$ there exists unique $V_t(0, \nu)$ that solves (1.3).

In the following, we will let (1.2) define a progressively measurable process:

$$\bar{X} : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \mapsto \mathbb{R}_+.$$

It follows from (1.4) and Lemma 2.1(d) of [10] that the Laplace transform of $\bar{X}$ in dimensions $d < 2/\beta$ is given by

$$E\left[\exp\{-a\bar{X}(t,x)\}|\mathcal{F}_s\right]$$
$$= \exp\{-\langle X_s,\ V_{t-s}(a\delta_x, 0)\rangle\}, \qquad P\text{-a.s. } \forall\, x \in \mathbb{R}^d,\ 0 \le s < t,\ a > 0.$$

For any $f : \mathbb{R}^d \mapsto \mathbb{R}$, let $\|f\|_B$ denote the essential supremum (with respect to Lebesgue measure λ) of f on the open set $B \subset \mathbb{R}^d$.

Our first result deals with the properties of $\bar{X}(t,\cdot)$ for fixed t.

THEOREM 1.1 (Regularity and irregularity of density for fixed times). *Fix an arbitrary* $t > 0$.

(a) *For* $d = 1$, *there exists a continuous version of* $\bar{X}(t,\cdot)$ *on* $\mathbb{R}^d$.
(b) *Let* $2 \le d < 2/\beta$. *Then*

$$\|\bar{X}(t,\cdot)\|_{\mathrm{U}} = \infty, \qquad \textit{whenever } X_t(\mathrm{U}) > 0 \textit{ for any open set } \mathrm{U} \subset \mathbb{R}^d,\ P\textit{-a.s.}$$

The proof of part (a) is based on the representation of the density of $(1+\beta)$-stable super-Brownian motion as a solution to a certain SPDE driven by space–time stable noise. This representation was discovered for a more general model in [10] and it allows us to use the Kolmogorov criterion to establish the continuity of the density for fixed times. To prove part (b), we use the cluster representation of super-Brownian motion.

The next result deals with the behavior of the density *in time* for fixed $x \in \mathbb{R}^d$.

THEOREM 1.2 (Unboundedness of density in time). *Let* $0 < \beta < 1$ *and* $d < 2/\beta$. *Fix an arbitrary* $s > 0$. *Then*

$$\|\bar{X}(\cdot, x)\|_{(s,s+\delta)} = \infty \qquad \text{for any } \delta > 0 \text{ and } X_s\text{-a.e. } x,\ P\text{-a.s.} \tag{1.5}$$

The key tool in proving the existence of a density for $(1+\beta)$-stable super-Brownian motion for fixed times is the existence of $V_t(a\delta_x, 0)$ that solves (1.3) (with $\mu = a\delta_x$, $\nu = 0$) in dimensions $d < 2/\beta$. However, as was mentioned earlier, this log-Laplace technique does not help resolve the regularity properties of the density and the continuity of the density at a fixed time remained open.

We would like to mention here a very interesting paper of Saint Loubert Bié [13] that deals with SPDEs driven by Poissonian noise. Although the regularity results obtained in [13] for dimension $d = 1$ do not cover the super-Brownian motion case, they are consistent with our continuity result. Theorem 1.1 gives a complete answer to the problem of continuity of the density of the super-Brownian motion, by stating that the fixed-time density is continuous only in $d = 1$ and is "totally unbounded" in all higher dimensions when it exists. Theorem 1.2 states that in any dimension where the density exists it is "totally unbounded" in time for a fixed point x. The idea of the proof is to establish that X has "too many" small jumps near the point x in the time interval $(s, s+\delta)$ and the huge number of these jumps is the reason for the unboundedness of the density.

Our last result deals with the regularity of the local time in the spatial parameter. Before we present it, recall that the process

$$\mathcal{Y}_t(x) = \mathcal{Y}(t, x) \equiv \begin{cases} \lim_{n\to\infty} \int_0^t J_n * X_s(x)\, ds, & \text{if it exists,} \\ 0, & \text{otherwise} \end{cases} \tag{1.6}$$

is called the local time of X at x. When $d < 2/\beta$, $X_t \ll \lambda$ for a.e. t, P-a.s., and hence the local time is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ for a.e. x, P-a.s. It was derived in [7] that for the super-Brownian motion with Lebesgue initial conditions the local time exists in dimensions $d < 2 + 2/\beta$ and its Laplace transform is given by

$$E\left[\exp\{-a\mathcal{Y}_t(x)\}\right] = \exp\{-\langle X_0, V_t(0, a\delta_x)\rangle\} \qquad \forall x \in \mathbb{R}^d, t \geq 0, a > 0. \tag{1.7}$$

Below we will extend the result of [7] (see also [6] for similar results for the density process) for more general initial conditions—see Lemma 4.5 and Remark 4.6.

We know that, in general, the behavior of $\mathcal{Y}_t(x)$ in x may be more regular than the behavior of the process for a fixed time. For example, Sugitani [14] showed the joint continuity of the local time for the finite-variance super-Brownian motion when $d \leq 3$, whereas the process itself is singular with respect to Lebesgue measure for $d \geq 2$. Surprisingly, in the case of super-Brownian motion with $(1+\beta)$-branching $(0 < \beta < 1)$, we get that the local time is still unbounded in dimensions $2 \leq d < 2 + 2/\beta$.

THEOREM 1.3 (Regularity and irregularity of local time in spatial parameter).
(a) *For* $d = 1$, *there exists a jointly continuous version of* $\mathcal{Y}_t(x)$ *in* $\mathbb{R}_+ \times \mathbb{R}$.
(b) *Assume* $0 < \beta < 1$ *and* $2 \le d < 2 + 2/\beta$. *Then*

$$\|\mathcal{Y}_t(\cdot)\|_{\mathrm{U}} = \infty,$$

whenever $\int_{\mathrm{U}} \mathcal{Y}_t(x)\,dx > 0$ *for any open set* $\mathrm{U} \subset \mathbb{R}^d$ *and any* $t > 0$, P*-a.s.*

Given Theorem 1.1(a), the result of part (a) of the above theorem is not surprising, and its proof is not very difficult. Regarding the proof of part (b), here again, as in the proof of Theorem 1.2, we will exploit the jump structure of $(1+\beta)$-stable super-Brownian motion. We will show that the large number of jumps on any open set of positive $\mathcal{Y}_t(x)\,dx$ measure will ensure the unboundedness of the local time on this set.

The rest of the paper is devoted to the proof of the above theorems.

2. Proofs of Theorem 1.1(a) and Theorem 1.3(a).

We start with a simple lemma.

LEMMA 2.1. *If* $0 < \theta < \beta$ *and* $\phi \ge 0$,

$$E\big[X_t(\phi)^{1+\theta}\big] \le 1 + c(\theta)\bigg[\int_0^t X_0\big(S_{t-r}\big((S_r\phi)^{1+\beta}\big)\big)\,dr + X_0(S_t\phi)^{1+\beta}\bigg].$$

PROOF. The proof follows the first part of the argument used in the proof of Lemma 5.5.4 of [2]. By Lemma 5.5.2(d) and (e) of [2], we have

$$\begin{aligned} &E[X_t(\phi)^{1+\theta}] \\ &\quad\le 1 + c\int_1^\infty r^{1+\theta}\int_0^{2/r} E\big[\exp\{-\langle X_t, u\phi\rangle\} - 1 + \langle X_t,\, u\phi\rangle\big]\,du\,dr \\ &\quad= 1 + c\int_1^\infty r^{1+\theta}\int_0^{2/r} E\big[\exp\{-\langle X_0, V_t(u\phi,0)\rangle\} \\ &\qquad\qquad - 1 + \langle X_0,\, uS_t\phi\rangle\big]\,du\,dr, \end{aligned} \tag{2.1}$$

where the second equality follows by (1.4) and the formula for the mean measure of X_t. By Lemma 5.5.2(c) of [2], the inner integrand is bounded by

$$\begin{aligned} &c\langle X_0,\, uS_t\phi\rangle^{1+\beta} + \big|\exp\{-\langle X_0,\, V_t(u\phi,0)\rangle\} - \exp\{-\langle X_0, uS_t\phi\rangle\}\big| \\ &\quad\le c\langle X_0,\, uS_t\phi\rangle^{1+\beta} + |\langle X_0,\, V_t(u\phi,0)\rangle - \langle X_0,\, uS_t\phi\rangle| \\ &\quad= c\langle X_0,\, uS_t\phi\rangle^{1+\beta} + \int_0^t \langle X_0S_{t-s}\big(V_s(u\phi,0)^{1+\beta}\big)\rangle\,ds \\ &\quad\le cu^{1+\beta}\langle X_0,\, S_t\phi\rangle^{1+\beta} + u^{1+\beta}\int_0^t \langle X_0S_{t-s}\big((S_s\phi)^{1+\beta}\big)\rangle\,ds, \end{aligned} \tag{2.2}$$

where in (2.2) we used the definition (1.3) of $V_t(u\phi, 0)$ and in the last line we used the domination $0 \le V_t(u\phi, 0) \le uS_t\phi$. Now after taking the integral with respect to $du\,dr$ in (2.1), we get the required bound. $\square$

PROOF OF THEOREM 1.1(a). Our goal is to show the continuity of $\bar{X}(t, \cdot)$ [the density of the super-Brownian motion with $(1+\beta)$-stable branching] in $d = 1$. We are going to apply the Kolmogorov criterion of continuity, but first we will show that $\bar{X}$ satisfies a certain SPDE driven by a stable noise. By Proposition 5.1 of [10] (where a more general case is considered), we get that if X satisfies the martingale problem (1.1) then there exists a space–time $(1+\beta)$-stable noise without negative jumps $\dot{\mathrm{L}}$ such that X also satisfies the following SPDE:

$$\begin{aligned}\langle X_t, \phi\rangle &= \langle X_0, \phi\rangle + \int_0^t \langle X_s, \tfrac{1}{2}\Delta\phi\rangle\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \bar{X}(s, y)^{1/(1+\beta)}\phi(y)\mathrm{L}(dy, ds), \qquad t \ge 0,\end{aligned} \tag{2.3}$$

for any ϕ in the Schwartz space of rapidly decreasing infinitely differential functions on $\mathbb{R}$. Note that L is a distribution (generalized function) on $\mathbb{R} \times \mathbb{R}_+$ whose Laplace transform is given by

$$E\biggl[\exp\biggl\{-\int_0^t \int_{\mathbb{R}} \phi(s, x)\mathrm{L}(dx, ds)\biggr\}\biggr] = \exp\biggl\{\int_0^t \int_{\mathbb{R}} \phi(s, x)^{1+\beta}\,dx\,ds\biggr\}$$
$$\forall \phi \ge 0 \text{ such that } \int_0^t \int_{\mathbb{R}^d} \phi(s, x)^{1+\beta}\,dx\,ds < \infty.$$

The stochastic integral with respect to L is defined in [10]. By a standard technique, it is easy to rewrite the SPDE (2.3) in the mild form

$$\begin{aligned}\langle X_t, \phi\rangle &= \langle X_0, S_t\phi\rangle \\ &\quad + \int_0^t \int_{\mathbb{R}} S_{t-s}\phi(y)\bar{X}(s, y)^{1/(1+\beta)}\mathrm{L}(dy, ds), \qquad t \ge 0,\end{aligned} \tag{2.4}$$

for any bounded and measurable ϕ. Now fix $t > 0$ and take $\phi = J_n^x = J_n(x - \cdot)$ in the above. The first term on the right-hand side converges to $S_tX_0(x)$. The left-hand side of (2.4) converges to $\bar{X}(t, x)$ for λ-a.e. x, P-a.s. By Fubini's theorem, we can fix a set $A \subset \mathbb{R}$ of Lebesgue measure 0, such that, for any $x \in A^c$, the left-hand side of (2.4) converges to $\bar{X}(t, x)$, P-a.s.

Now fix arbitrary $x \in \mathbb{R}$ and check the convergence of the stochastic integral in (2.4). To this end, first note that $|S_{t-s}J_n^x(y) - p_{t-s}(y - x)| \le c(t - s)^{-1/2}$ and

$$\int_0^t \int_{\mathbb{R}} (t - s)^{-(1+\beta)/2}\bar{X}(s, y)\,dy\,ds \le \Bigl(\sup_{s\le t} X_s(1)\Bigr)ct^{(1-\beta)/2} < \infty.$$

Therefore, by dominated convergence,

$$\int_0^t \int_{\mathbb{R}} \bigl(|S_{t-s}J_n^x(y) - p_{t-s}(y - x)|\bar{X}(s, y)^{1/(1+\beta)}\bigr)^{1+\beta}\,dy\,ds \to 0,$$

(2.5)

$$P\text{-a.s.}$$

as $n \to \infty$. It is easy to check (see, e.g., the argument after Lemma 5.4 of [10]) that (2.5) implies

$$\int_0^t \int_{\mathbb{R}} S_{t-s} J_n^x(y) \bar{X}(s, y)^{1/(1+\beta)} \mathrm{L}(dy, ds) \tag{2.6}$$

converges in probability to $\int_0^t \int_{\mathbb{R}} p_{t-s}(y - x)\bar{X}(s, y)^{1/(1+\beta)}\mathrm{L}(dy, ds)$, where the existence of the integral is also clear from the above and Section 5 of [10]. Now let $n \to \infty$ in (2.4) with $\phi = J_n^x$ to get

$$\begin{aligned} \bar{X}(t, x) = S_t X_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y)\bar{X}(s, y)^{1/(1+\beta)}\mathrm{L}(dy, ds),& \\ P\text{-a.s., } \lambda\text{-a.e. } x.& \end{aligned} \tag{2.7}$$

The first term on the right-hand side of (2.7) is obviously continuous in x. So, to check the existence of a continuous version of $\bar{X}(t, x)$, it is enough to check the existence of a continuous version of the stochastic integral in x for fixed t. To this end, take arbitrary $x_1, x_2 \in \mathbb{R}$ and define

$$Z_t(x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y)\bar{X}(s, y)^{1/(1+\beta)}\mathrm{L}(dy, ds).$$

Then

$$\begin{aligned} &Z_t(x_1) - Z_t(x_2) \\ &\quad = \int_0^t \int_{\mathbb{R}} \big(p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)\big)\bar{X}(s, y)^{1/(1+\beta)}\mathrm{L}(dy, ds). \end{aligned} \tag{2.8}$$

We may, and will, consider the stable noise without negative jumps as an integral of a compensated Poisson random measure. Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$; that is, for any $A \times B \times C \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, $M(A \times B \times C)$ is distributed according to the Poisson distribution with parameter $\nu(A)\lambda(B \times C)$, where

$$\nu(dr) = \frac{\beta(1+\beta)}{\Gamma(1-\beta)} r^{-2-\beta} \mathbb{1}(r > 0)\, dr$$

and we recall that λ denotes Lebesgue measure. If $\hat{M}$ is the compensator of M, set $\tilde{M} = M - \hat{M}$. Then we have

$$\begin{aligned} \mathrm{L}(ds, dx) &= \int_0^1 r \tilde{M}(dr, ds, dx) + \int_1^\infty r\big(M(dr, ds, dx) - \hat{M}(dr, ds, dx)\big) \\ &\equiv \mathrm{L}^1(ds, dx) + \mathrm{L}^2(ds, dx). \end{aligned}$$

Define

$$Z_t^i(x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\bar{X}(s,y)^{1/(1+\beta)} \mathrm{L}^i(dy,ds), \qquad i = 1, 2.$$

If $M^0(ds,dy) = \int_1^\infty r M(dr,ds,dy)$, then

$$\begin{aligned} Z_t^2(x) &= \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\bar{X}(s,y)^{1/(1+\beta)} M^0(ds,dy) \\ &\quad - \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\bar{X}(s,y)^{1/(1+\beta)} c_\beta \, dy \, ds. \end{aligned}$$

The first integral is a (random) ordinary Riemann–Stieltjes integral as M^0 has finitely many jumps in any compact set. If we can show that, for any $K > 0$,

$$\int_0^t \int_{\mathbb{R}} \sup_{|x| \le K} p_{t-s}(y-x) E\big[\bar{X}(s,y)^{1/(1+\beta)}\big] \, dy \, ds < \infty, \tag{2.9}$$

then

$$\int_0^t \int_{\mathbb{R}} \sup_{|x| \le K} p_{t-s}(y-x)\bar{X}(s,y)^{1/(1+\beta)} M^0(dy,ds) < \infty,$$

as its mean value is a constant multiple of the above expression. The continuity of $Z_t^2(\cdot)$ would therefore follow from (2.9) by dominated convergence. To prove (2.9), first note that

$$\sup_{|x| \le K} p_{t-s}(y-x) \le \mathbb{1}(|y| \le 2K)(t-s)^{-1/2} + \mathbb{1}(|y| > 2K) p_{t-s}(y/2). \tag{2.10}$$

Therefore, the left-hand side of (2.9) is at most

$$\begin{aligned} &\int_0^t (t-s)^{-1/2} \int_{-2K}^{2K} E\big[\bar{X}(s,y)^{1/(1+\beta)}\big] \, dy \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} p_{t-s}(y/2) E\big[\bar{X}(s,y)^{1/(1+\beta)}\big] \, dy \, ds \\ &\quad \le \int_0^t (t-s)^{-1/2} \int_{-2K}^{2K} \big(1 + E[\bar{X}(s,y)]\big) \, dy \, ds \\ &\qquad + \int_0^t \left(\int_{\mathbb{R}} p_{t-s}(y/2)^{1+1/\beta} \, dy \right)^{\beta/(1+\beta)} \\ &\qquad\qquad \times \left(\int_{\mathbb{R}} E[\bar{X}(s,y)] \, dy \right)^{1/(1+\beta)} ds \qquad \text{(by Hölder's inequality)} \\ &\quad \le \int_0^t (t-s)^{-1/2} \big(4K + X_0(1)\big) \, ds \\ &\qquad + c \int_0^t (t-s)^{-1/(2(1+\beta))} X_0(1)^{1/(1+\beta)} \, ds < \infty. \end{aligned}$$

In the last line, we have used (2.7) and a straightforward calculation. This proves (2.9) and hence the continuity of $Z_t^2(\cdot)$.

Now check the continuity of $Z_t^1(x)$. Fix p so that $1+\beta < p < ((1+\beta)^2 \wedge 2)$. By (1.6) of [13] and the Fubini theorem, we get

$$\begin{aligned} &E\big[|Z_t^1(x_1) - Z_t^1(x_2)|^p\big] \\ &\quad \le c\int_0^t \int_{\mathbb{R}} \int_0^1 |p_{t-s}(x_1-y) - p_{t-s}(x_2-y)|^p \\ &\qquad\qquad \times E\big[\bar{X}(s,y)^{p/(1+\beta)}\big] r^{p-2-\beta}\, dr\, dy\, ds. \end{aligned} \tag{2.11}$$

Now recall that by Lemma 2.1, for any $0<\theta<\beta$, there exists a constant $c=c(\theta)$ such that

$$\begin{aligned} &E\big[\langle X_s, \phi\rangle^{1+\theta}\big] \\ &\quad \le 1 + c\bigg[\int_0^s \langle X_0, S_{s-r}\big((S_r\phi)^{1+\beta}\big)\rangle\, dr + \langle X_0, S_s\phi\rangle^{1+\beta}\bigg] \qquad \forall \phi \ge 0. \end{aligned} \tag{2.12}$$

Set $\phi = J_n(y-\cdot)$ and use the above. Use Jensen's inequality to bound the term $S_{s-r}((S_r\phi)^{1+\beta})$ inside the integral by

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} p_r(z-w)^{1+\beta} J_n(w-y)\, dw\, p_{s-r}(z-x)\, dz \\ &\quad \le cr^{-\beta/2}\int_{\mathbb{R}}\int_{\mathbb{R}} p_r(z-w) J_n(w-y)\, dw\, p_{s-r}(z-x)\, dz \\ &\quad = cr^{-\beta/2}\int_{\mathbb{R}} p_s(w-x) J_n(w-y)\, dw \qquad \text{(by Chapman–Kolmogorov)} \\ &\quad \le cr^{-\beta/2}s^{-1/2}. \end{aligned} \tag{2.13}$$

The second term inside the brackets in (2.12) is bounded by

$$\bigg(\int_{\mathbb{R}} S_s\big(J_n(y-\cdot)\big)(x) X_0(dx)\bigg)^{1+\beta} \le cX_0(1)^{1+\beta} s^{-(1+\beta)/2}. \tag{2.14}$$

Let $n\to\infty$. Use (2.12)–(2.14) and Fatou's lemma to get

$$\begin{aligned} E\big[\bar{X}(s,y)^{p/(1+\beta)}\big] &\le \liminf_{n\to\infty} E\big[\langle X_s,\ J_n(y-\cdot)\rangle^{p/(1+\beta)}\big] \\ &\le 1 + c(p)\bigg[\int_0^s X_0(1) r^{-\beta/2}s^{-1/2}\, dr + X_0(1)^{1+\beta}s^{-(1+\beta)/2}\bigg] \\ &\le c(p)\big(s^{(1-\beta)/2}X_0(1) + s^{-(1+\beta)/2}X_0(1)^{1+\beta}\big) \\ &\le c(p, X_0, t)s^{-(1+\beta)/2}, \qquad 0<s\le t, \end{aligned}$$

with $\sup_{t\le T} c(p, X_0, t) < \infty$, $\forall T>0$. Now recall the well-known inequality (see, e.g., (2.4e) of [12]):

$$\begin{aligned}&|p_t(x_1-y)-p_t(x_2-y)|\\&\qquad\le c|x_1-x_2|^{\delta}t^{-\delta/2}\big(p_t(x_1-y)+p_t(x_2-y)\big)\qquad \forall t>0,\ 0\le\delta\le 1.\end{aligned}$$

Next, choose δ as follows:

$$\delta=\begin{cases}1, & \text{if } p<3/2,\\ (3/p)-1-\varepsilon/p, & \text{if } 3/2\le p<2\end{cases}$$

for some $\varepsilon>0$ sufficiently small. Then we can easily derive that

$$\int_{\mathbb{R}}|p_t(x_1-y)-p_t(x_2-y)|^p\,dy\le c|x_1-x_2|^{\xi(p,\varepsilon)}h_{p,\varepsilon}(t)\qquad\forall t>0, \tag{2.15}$$

where $\xi(p,\varepsilon)=p$ for $p<3/2$, $\xi(p,\varepsilon)=3-p-\varepsilon$ for $p\ge 3/2$ and

$$h_{p,\varepsilon}(t)=\begin{cases}t^{1/2-p}, & \text{if } p<3/2,\\ t^{-1+\varepsilon/2}, & \text{if } 3/2\le p<2.\end{cases}$$

Next, use (2.11) and (2.15) to get

$$\begin{aligned}&E\big[|Z_t^1(x_1)-Z_t^1(x_2)|^p\big]\\&\qquad\le c(p,X_0,t)\int_0^t\int_{\mathbb{R}}|p_{t-s}(x_1-y)-p_{t-s}(x_2-y)|^p s^{-(1+\beta)/2}\,dy\,ds\\&\qquad\le c(p,X_0,t)c_{2.1}(t)|x_1-x_2|^{\xi(p)}\qquad\forall x_1,x_2\in\mathbb{R},\end{aligned}\tag{2.16}$$

where

$$c_{2.1}(t)\equiv\int_0^t h_{p,\varepsilon}(t-s)s^{-(1+\beta)/2}\,ds<\infty\qquad\forall t>0,\ \varepsilon>0, \tag{2.17}$$

and for the sake of simplicity we suppressed its dependence on p, ε. By an appropriate choice of p, ε and the Kolmogorov criterion, (2.16) and (2.17) immediately lead to the existence of a continuous version of $Z_t^1(x)$ and hence of $\bar{X}(t,x)$, in x. $\square$

PROOF OF THEOREM 1.3(a). From Theorem 1.1(a), we know that $\liminf_{y\to x,y\in Q}\bar{X}(t,y)$ defines a jointly measurable version of the density $\bar{X}$ which is continuous in x a.s. for each $t>0$. We abuse the notation slightly and write $\bar{X}(t,x)$ for this new version. As in the proof of Theorem 1.1(a), we decompose the integral $\int_0^t\bar{X}(s,x)\,ds$ into the sum of three parts. The first part is trivially jointly continuous, and the second part, $\int_0^t Z^2(s,x)\,ds$, can be shown to be jointly continuous by modifying the ideas used to handle Z_s^2 in the proof of Theorem 1.1(a). It remains to fix $K,T>0$, choose a jointly measurable version of $\int_0^t Z^1(s,x)\,ds$ in $(t,x)\in[0,T]\times[-K,K]$ (which is easy) and show that we can define a jointly continuous version of $\int_0^t Z^1(s,x)\,ds$ in $(t,x)\in[0,T]\times[-K,K]$. From the proof of Theorem 1.1(a), we know that $\liminf_{y\to x,y\in Q}Z_s^1(y)$ defines a version of $Z^1(s,x)$ which is jointly measurable and continuous in x a.s. for each $s>0$. We now write $Z^1(s,x)$ to denote this version. Next, use (2.16) and Corollary 1.2 of [15] to see that

$$\begin{aligned}(2.18)\qquad & E\Big[\sup_{|x|\le K}|Z^1(t,x)|^p\Big]\\ &\quad\le C(T,K)\big(c_{2.1}(t)+E\big[|Z_t^1(0)|^p\big]\big)\qquad\forall\, 0<t\le T,\end{aligned}$$

where $c_{2.1}(t)$ is as in (2.17). It is easy to check that

$$\int_0^T c_{2.1}(t)\,dt<\infty\qquad\forall\, T>0,\ \varepsilon>0.$$

Note that $1+\beta<p<((1+\beta)^2\wedge 2)$ is as in the proof of Theorem 1.1(a) and we have kept track of the constants arising in the proof of Corollary 1.2 of [15] to give an explicit upper bound in (2.18). A calculation similar to that in the proof of Theorem 1.1(a) gives

$$E\big[|Z_t^1(0)|^p\big]\le c'(T)t^{1-(p+\beta)/2}\qquad\forall\, 0<t\le T,$$

which is integrable over $[0,T]$. Therefore,

$$E\Big[\sup_{|x|\le K}|Z^1(t,x)|^p\Big]\le C(T,K)c''(t),$$

where c'' is integrable over $[0,T]$. This implies that

$$E\Big[\sup_{|x|\le K}\int_0^T|Z^1(s,x)|^p\,ds\Big]\le\int_0^T E\Big[\sup_{|x|\le K}|Z^1(s,x)|^p\Big]ds<\infty.$$

Now fix ω outside a null set so that $Z_s^1(x)$ is continuous in x for Lebesgue a.a. $s>0$ and

$$(2.19)\qquad \sup_{|x|\le K}\int_0^T|Z^1(s,x)|^p\,ds<\infty.$$

Now let $(s_n,x_n)\to(s,x)$, $(s_n,x_n)\in[0,T]\times[-K,K]$. To simplify the notation, let us assume that $s_n\ge s$. Then

$$\begin{aligned}&\left|\int_0^{s_n}Z_r^1(x_n)\,dr-\int_0^s Z_r^1(x)\,dr\right|\\ &\quad\le\left|\int_0^{s_n}Z_r^1(x_n)\,dr-\int_0^{s_n}Z_r^1(x)\,dr\right|+\int_s^{s_n}|Z_r^1(x)|\,dr\\ &\quad\le\int_0^T|Z_r^1(x_n)-Z_r^1(x)|\,dr+\int_0^T\mathbb{1}(s\le r\le s_n)|Z_r^1(x)|\,dr.\end{aligned}$$

The second term converges to 0 by dominated convergence [use (2.19) and our choice of ω]. The first term converges to 0 because (2.19) implies uniform integrability on $[0,T]$ with respect to dr. This proves the required joint continuity. $\square$

3. Proof of Theorem 1.1(b). In the proof of Theorem 1.1(b), we will use a cluster representation of super-Brownian motion. First, fix $t > 0$, arbitrary $\varepsilon > 0$ and let $g(\varepsilon) \equiv \beta^{1/\beta}\varepsilon^{1/\beta} \equiv c_\beta \varepsilon^{1/\beta}$. Let $P^*_{\varepsilon,y}$ be the law of a cluster of age ε which is "born" at $y \in \mathbb{R}^d$. More precisely, $P^*_{\varepsilon,y}$ is the suitably normalized canonical measure of the infinitely divisible random measure X_ε starting at δ_y. Apply Proposition 3.3 and formula (3.24) of [4] to see that the Laplace transform of the cluster of age ε, born at $y \in \mathbb{R}^d$, is given by

$$\int e^{-\langle \mu, \phi\rangle} P^*_{\varepsilon,y}(d\mu) = 1 - g_\varepsilon V_\varepsilon(\phi, 0)(y) \qquad \forall \phi \geq 0. \tag{3.1}$$

Use (3.24) of [4] again to get that

$$V_\varepsilon(\theta \mathbf{1}, 0) = \frac{\theta}{(1 + \varepsilon\theta^\beta \beta)^{1/\beta}} \qquad \forall \theta \geq 0. \tag{3.2}$$

Conditionally on $\mathcal{F}_{t-\varepsilon}$, let $\tilde{Z}_{t-\varepsilon}$ be the Poisson random measure on $\mathbb{R}^d$ with intensity $g(\varepsilon)^{-1} X_{t-\varepsilon}(dx)$. By Proposition 3.5 of [4], conditionally on $\mathcal{F}_{t-\varepsilon}$ and $\tilde{Z}_{t-\varepsilon}$, X_t is the sum of independent clusters $\{X_t^{i,\varepsilon,y_i}, i = 1, 2, \ldots, \tilde{Z}_{t-\varepsilon}(1)\}$ with laws P^*_{ε,y_i}, one for each atom y_i of $\tilde{Z}_{t-\varepsilon}$, that is,

$$X_t = \sum_{i=1}^{\tilde{Z}_{t-\varepsilon}(1)} X_t^{i,\varepsilon,y_i}.$$

First, we will prove Theorem 1.1(b) for U a fixed open ball B in $\mathbb{R}^d$. Define a set of open balls $\{\mathrm{B}_k, k = 1, 2, \ldots\}$, such that $\bar{\mathrm{B}}_k \subset \mathrm{B}_{k+1} \subset \mathrm{B}$, for any $k \geq 1$ and $\mathrm{B}_k \uparrow \mathrm{B}$, as $k \to \infty$. Define $N^\varepsilon = \tilde{Z}_{t-\varepsilon}(\mathrm{B})$, $N^{\varepsilon,k} = \tilde{Z}_{t-\varepsilon}(\mathrm{B}_k)$. To simplify the notation, let X_t^ε denote a generic cluster of age ε starting at $\mathbf{0} \in \mathbb{R}^d$ and let P^*_ε be its law. Note that, conditionally on $\mathcal{F}_{t-\varepsilon}$, $N^{\varepsilon,k}$ is a Poisson random variable with intensity $g(\varepsilon)^{-1} X_{t-\varepsilon}(\mathrm{B}_k)$. Fix $q > \sqrt{2/\beta}$ and $p > 0$. Define $r_\varepsilon = q\sqrt{\varepsilon \log(1/\varepsilon)}$ and $R_\varepsilon = r_\varepsilon^d \log(1/\varepsilon)^p$.

The following proposition is the major step in proving unboundedness of the density of X_t conditioned on $X_t(\mathrm{B}_k) > 0$.

PROPOSITION 3.1. *Let* B_k *be as above. Then, for any* $c > 0$,

$$\lim_{\varepsilon \downarrow 0} P\left(\sup_{x \in \mathrm{B}_k} \frac{X_t(B(x, r_\varepsilon))}{r_\varepsilon^d} \leq c \log\left(\frac{1}{\varepsilon}\right)^p\right) \leq P\big(X_t(\mathrm{B}_k) = 0\big).$$

This proposition will be proved via a series of lemmas, and, without loss of generality, we will assume that $c = 1$. First, let us get some trivial estimates on $P(\sup_{x\in \mathrm{B}_k} X_t(B(x,r_\varepsilon))/r_\varepsilon^d \le \log(1/\varepsilon)^p)$.

$$
\begin{aligned}
&P\left(\sup_{x\in\mathrm{B}_k}\frac{X_t(B(x,r_\varepsilon))}{r_\varepsilon^d}\le\log\left(\frac{1}{\varepsilon}\right)^p\right)\\
&\qquad = P\left(\sup_{x\in\mathrm{B}_k}\sum_{i=1}^{\tilde Z_{t-\varepsilon}(1)} X_t^{i,\varepsilon,y_i}(B(x,r_\varepsilon))\le R_\varepsilon\right)\\
&\qquad \le P\left(\sup_{x\in\mathrm{B}_k}\sum_{y_i\in\mathrm{B}_k} X_t^{i,\varepsilon,y_i}(B(x,r_\varepsilon))\le R_\varepsilon\right)\\
&\qquad \le P\left(\sup_{y_i\in\mathrm{B}_k} X_t^{i,\varepsilon,y_i}(B(y_i,r_\varepsilon))\le R_\varepsilon\right)\\
&\qquad = E\Big[E\big[P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon))\le R_\varepsilon\big)^{N^{\varepsilon,k}}\big|\mathcal{F}_{t-\varepsilon}\big]\Big],
\end{aligned}\tag{3.3}
$$

where in the last inequality we used the conditional independence of clusters given $\mathcal{F}_{t-\varepsilon}$, $\tilde Z_{t-\varepsilon}$ and also that the conditional laws of $X_t^{i,\varepsilon,y_i}(B(y_i,r_\varepsilon))$ coincide with the law of $X_t^\varepsilon(B(0,r_\varepsilon))$. Now recall that, conditionally on $\mathcal{F}_{t-\varepsilon}$, $N^{\varepsilon,k}$ is distributed according to the Poisson distribution with intensity $c_\beta^{-1}\varepsilon^{-1/\beta}X_{t-\varepsilon}(\mathrm{B}_k)$. Hence,

$$
\begin{aligned}
&P\left(\sup_{x\in\mathrm{B}_k}\frac{X_t(B(x,r_\varepsilon))}{r_\varepsilon^d}\le\log\left(\frac{1}{\varepsilon}\right)^p\right)\\
&\qquad \le E\big[\exp\big\{-c(\beta)X_{t-\varepsilon}(\mathrm{B}_k)\varepsilon^{-1/\beta}P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon))>R_\varepsilon\big)\big\}\big].
\end{aligned}\tag{3.4}
$$

From (3.4), it is clear that, to prove Proposition 3.1, one needs to establish the limiting behavior of $\varepsilon^{-1/\beta}P_\varepsilon^*(X_t^\varepsilon(B(0,r_\varepsilon))>R_\varepsilon)$ as $\varepsilon\downarrow 0$. Let Ξ_ε be the total mass of the cluster X_t^ε. Then

$$
\begin{aligned}
P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon))>R_\varepsilon\big)&\ge P_\varepsilon^*\big(\Xi_\varepsilon>R_\varepsilon, X_t^\varepsilon\big(B(0,r_\varepsilon)^c\big)=0\big)\\
&\ge P_\varepsilon^*(\Xi_\varepsilon>R_\varepsilon)-P_\varepsilon^*\big(X_t^\varepsilon\big(B(0,r_\varepsilon)^c\big)>0\big).
\end{aligned}\tag{3.5}
$$

Combine (3.1) and (3.2) to get that the Laplace transform of Ξ_ε is given by

$$
E[\exp\{-\theta\,\Xi_\varepsilon\}]=1-\frac{g_\varepsilon\theta}{(1+\varepsilon\theta^\beta\beta)^{1/\beta}}\qquad\forall\,\theta\ge 0.\tag{3.6}
$$

Let Ξ be a nonnegative random variable with the Laplace transform

$$
\begin{aligned}
E[\exp\{-\theta\,\Xi\}]&=1-\frac{\theta}{(1+\theta^\beta)^{1/\beta}}\\
&\equiv h(\theta).
\end{aligned}\tag{3.7}
$$

Then

$$
\begin{aligned}
E\left[\exp\left\{-\theta\frac{\Xi_\varepsilon}{g_\varepsilon}\right\}\right]&=1-\frac{\theta}{(1+\theta^\beta)^{1/\beta}}\\
&=h(\theta),
\end{aligned}
$$

and hence the law of Ξ_ε equals the law of $c_\beta \varepsilon^{1/\beta}\Xi$.

Note that

$$h'(\theta) = -(1+\theta^\beta)^{-(1/\beta)-1}.$$

So

$$h''(\theta) = \big(1+(1/\beta)\big)(1+\theta^\beta)^{-(1/\beta)-2}\beta\theta^{\beta-1} \to \infty$$

as $\theta \downarrow 0$. Therefore, $E[\Xi] = -h'(0) = 1$ and $E[\Xi^2] = \infty$.

LEMMA 3.2. *Ξ is in the domain of attraction of the one-sided $(1+\beta)$-stable law and*

$$P(\Xi > x) = \tilde{L}(x)x^{-1-\beta}, \tag{3.8}$$

where $\tilde{L}$ is slowly varying at ∞.

PROOF. This is standard. One way to show that Ξ is in the domain of attraction of the stable law of index $1+\beta$ is by proving a central limit theorem for $\Xi - 1$ and using a standard result (e.g., Theorem 9.34 of [1]) to see that $P(\Xi > x)$ varies regularly with exponent $1+\beta$. Let Ξ_i be independent copies of Ξ. Then

$$\begin{aligned}
&E\left[\exp\left\{\frac{-\theta\sum_{i=1}^n(\Xi_i - 1)}{n^{1/(\beta+1)}}\right\}\right]\\
&\quad = h(\theta n^{-1/(\beta+1)})^n \exp\{\theta n^{1-1/(\beta+1)}\}\\
&\quad = \left(1 - \frac{\theta n^{-1/(\beta+1)}}{(1+\theta^\beta/n^{\beta/(\beta+1)})^{1/\beta}}\right)^n \exp\{\theta n^{\beta/(\beta+1)}\}\\
&\quad = \left[\left(1 - \frac{\theta n^{-1/(\beta+1)}}{(1+\theta^\beta/n^{\beta/(\beta+1)})^{1/\beta}}\right)\left(1 + \frac{\theta}{n^{1/(\beta+1)}} + O\left(\frac{\theta^2}{n^{2/(\beta+1)}}\right)\right)\right]^n\\
&\quad = \left[1 + \frac{\theta}{n^{1/(\beta+1)}}\left(1 - \frac{1}{(1+\theta^\beta/n^{\beta/(\beta+1)})^{1/\beta}}\right) + O\left(\frac{\theta^2}{n^{2/(\beta+1)}}\right)\right]^n\\
&\quad = \left[1 + \frac{\theta}{n^{1/(\beta+1)}}\left(\frac{(1+\theta^\beta/n^{\beta/(\beta+1)})^{1/\beta} - 1}{(1+\theta^\beta/n^{\beta/(\beta+1)})^{1/\beta}}\right) + O\left(\frac{\theta^2}{n^{2/(\beta+1)}}\right)\right]^n\\
&\quad = \left[1 + \frac{\theta^{\beta+1}}{n\beta} + O\left(\frac{\theta^2}{n^{2/(\beta+1)}}\right) + O\left(\frac{1}{n^{(2\beta+1)/(\beta+1)}}\right)\right]^n\\
&\quad \to \exp\left\{\frac{1}{\beta}\theta^{1+\beta}\right\} \qquad \text{as } n \to \infty\ \forall \theta \ge 0.
\end{aligned}$$

Now the result follows by Theorem 9.34 of [1]. $\square$

The next lemma establishes the limiting behavior of $\varepsilon^{-1/\beta}P_\varepsilon^*(X_t^\varepsilon(B(0, r_\varepsilon)) > R_\varepsilon)$ as $\varepsilon \downarrow 0$.

LEMMA 3.3.

$$\varepsilon^{-1/\beta} P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon)) > R_\varepsilon\big) \to +\infty$$

as $\varepsilon \downarrow 0$.

By (3.5), we need to establish the limiting behavior of $\varepsilon^{-1/\beta} P_\varepsilon^*(\Xi_\varepsilon > R_\varepsilon)$ and $\varepsilon^{-1/\beta} P_\varepsilon^*(X_t^\varepsilon(B(0,r_\varepsilon)^c) > 0)$. By Lemma 3.2, we get

$$\begin{aligned} P_\varepsilon^*(\Xi_\varepsilon > R_\varepsilon) &= P\big(\Xi > R_\varepsilon \varepsilon^{-1/\beta} c_\beta^{-1}\big) \\ &= c(\beta)\tilde{L}(R_\varepsilon \varepsilon^{-1/\beta} c_\beta^{-1})(R_\varepsilon \varepsilon^{-1/\beta})^{-\beta-1}. \end{aligned} \tag{3.9}$$

Next, consider $P_\varepsilon^*(X_t^\varepsilon(B(0,r_\varepsilon)^c) > 0)$. Define the functions $I^\varepsilon(\cdot) \equiv \mathbb{1}_{B(0,r_\varepsilon)^c}(\cdot)$. Then

$$\begin{aligned} P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon)^c) > 0\big) &= 1 - \lim_{\theta\to\infty} E\big[\exp\{-\theta X_t^\varepsilon(I^\varepsilon)\}\big] \\ &= c_\beta \varepsilon^{1/\beta} \lim_{\theta\to\infty} V_\varepsilon(\theta I^\varepsilon, 0)(0), \end{aligned} \tag{3.10}$$

where the last equality follows from (3.1). It is well known (see, e.g., Lemma 6.1.1(c) of [2] and its proof) that

$$\lim_{\theta\to\infty} V_\varepsilon(\theta I^\varepsilon, 0)(0) = -\log\big(P\big(Y_\varepsilon\big(B(0,r_\varepsilon)^c\big) = 0\big)\big),$$

where Y is $(1+\beta)$-stable super-Brownian motion starting at δ_0. Therefore,

$$\begin{aligned} \lim_{\theta\to\infty} V_\varepsilon(\theta I^\varepsilon, 0)(0) &\le -\log\big(P\big(Y_s\big(B(0,r_\varepsilon)^c\big) = 0\ \forall s \le \varepsilon\big)\big) \\ &= \lim_{\theta\to\infty} V_\varepsilon(0, \theta I^\varepsilon)(0), \end{aligned} \tag{3.11}$$

where the second equality follows from (4.2)–(4.4) of [5]. See page 257 of [5] to get that

$$\begin{aligned} \lim_{\theta\to\infty} V_\varepsilon(0,\theta I^\varepsilon)(0) &\le c(d,\beta) r_\varepsilon^{-2/\beta}(r_\varepsilon/\sqrt{\varepsilon})^{d-2+4/\beta}\exp(-r_\varepsilon^2/2\varepsilon) \\ &= c(d,\beta,q)\varepsilon^{-1/\beta}\log(1/\varepsilon)^{(d/2)+(1/\beta)-1}\varepsilon^{q^2/2}. \end{aligned} \tag{3.12}$$

Combine (3.10)–(3.12) to get

$$P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon)^c) > 0\big) \le c(d,\beta,q)\log(1/\varepsilon)^{(d/2)+(1/\beta)-1}\varepsilon^{q^2/2}.$$

This, (3.5) and (3.9) show that

$$\begin{aligned} P_\varepsilon^*\big(X_t^\varepsilon(B(0,r_\varepsilon)) > R_\varepsilon\big) \ge{}& c(\beta)\tilde{L}(R_\varepsilon \varepsilon^{-1/\beta} c_\beta^{-1})(R_\varepsilon \varepsilon^{-1/\beta})^{-\beta-1} \\ &- c(d,\beta,q)\log(1/\varepsilon)^{(d/2)+(1/\beta)-1}\varepsilon^{q^2/2}. \end{aligned}$$

Recall that $R_\varepsilon = r_\varepsilon^d \log(1/\varepsilon)^p = q^d \varepsilon^{d/2} \log(1/\varepsilon)^{p+(d/2)}$. Therefore, some simple algebra gives us that

$$\begin{aligned}
&\varepsilon^{-1/\beta} P_\varepsilon^*\big(X_t^\varepsilon(B(0, r_\varepsilon)) > R_\varepsilon\big) \\
&\quad \geq c(d, p, q)\tilde{L}\big(c(q, \beta)\varepsilon^{(d/2)-(1/\beta)} \log(1/\varepsilon)^{p+(d/2)}\big) \\
&\qquad \times \log(1/\varepsilon)^{-(p+(d/2))(\beta+1)} \varepsilon^{1-d(\beta+1)/2} \\
&\qquad - c(d, \beta, q) \log(1/\varepsilon)^{(d/2)+(1/\beta)-1} \varepsilon^{(q^2/2)-(1/\beta)}.
\end{aligned}$$

The second term goes to 0 as $\varepsilon \to 0$ since $q^2 > 2/\beta$. The first term goes to ∞ since $d \geq 2$, $\beta > 0$ and $\tilde{L}$ is slowly varying at ∞ (recall that $d/2 < 1/\beta$ as we consider the dimensions where the density of the super-Brownian motion exists), and we are done. $\square$

PROOF OF PROPOSITION 3.1. Since t is a fixed time, the process X is continuous at t with probability 1 and hence

$$X_{t-\varepsilon} \Rightarrow X_t \qquad \text{as } \varepsilon \downarrow 0.$$

B_k is an open set and hence, by the standard properties of weak convergence, we get

$$\liminf_{\varepsilon \downarrow 0} X_{t-\varepsilon}(\mathrm{B}_k) \geq X_t(\mathrm{B}_k).$$

Therefore, the right-hand side of (3.4) converges to 0 if $X_t(\mathrm{B}_k) > 0$. Hence, (3.4) implies that

$$\lim_{\varepsilon \downarrow 0} P\left(\sup_{x \in \mathrm{B}_k} \frac{X_t(B(x, r_\varepsilon))}{r_\varepsilon^d} \leq \log\left(\frac{1}{\varepsilon}\right)^p\right) \leq P\big(X_t(\mathrm{B}_k) = 0\big). \qquad \square$$

LEMMA 3.4. *Let* $\{\mathrm{B}_k, k = 1, 2, \ldots\}$, B *be the sets defined at the beginning of the section. Then*

$$P\big(\|\bar{X}(t, \cdot)\|_{\mathrm{B}} = \infty \big| X_t(\mathrm{B}) > 0\big) = 1. \tag{3.13}$$

PROOF. For arbitrary k, we can always fix ε_0 sufficiently small such that

$$\left\{\|\bar{X}(t, \cdot)\|_{\mathrm{B}} \leq \log\left(\frac{1}{\varepsilon}\right)^p\right\} \subset \left\{\sup_{x \in \mathrm{B}_k} \frac{X_t(B(x, r_\varepsilon))}{c_d r_\varepsilon^d} \leq \log\left(\frac{1}{\varepsilon}\right)^p\right\} \qquad \forall \varepsilon \leq \varepsilon_0,$$

and so

$$\begin{aligned}
P\big(\|\bar{X}(t, \cdot)\|_{\mathrm{B}} < \infty\big) &= \lim_{\varepsilon \downarrow 0} P\left(\|\bar{X}(t, \cdot)\|_{\mathrm{B}} \leq \log\left(\frac{1}{\varepsilon}\right)^p\right) \\
&\leq \lim_{\varepsilon \downarrow 0} P\left(\sup_{x \in \mathrm{B}_k} \frac{X_t(B(x, r_\varepsilon))}{r_\varepsilon^d} \leq c_d \log\left(\frac{1}{\varepsilon}\right)^p\right) \\
&\leq P\big(X_t(\mathrm{B}_k) = 0\big) \qquad \forall k \geq 1.
\end{aligned}$$

Letting $k \to \infty$, we get that

$$P\big(\|\bar{X}(t,\cdot)\|_{\mathrm{B}} < \infty\big) \le P\big(X_t(\mathrm{B}) = 0\big).$$

The above implies that $P(\|\bar{X}(t,\cdot)\|_{\mathrm{B}} < \infty | X_t(\mathrm{B}) = 0) = 1$, and we immediately get that

$$P\big(\|\bar{X}(t,\cdot)\|_{\mathrm{B}} < \infty \big| X_t(\mathrm{B}) > 0\big) = 0,$$

and the result follows. □

PROOF OF THEOREM 1.1(b). We may fix ω outside a P-null set so that Lemma 3.4 holds for any rational ball B, that is, for any ball with a rational radius and center. Let U be an arbitrary open set such that $X_t(\mathrm{U}) > 0$. Then, there is always a rational ball $\mathrm{B} \subset \mathrm{U}$ such that $X_t(\mathrm{B}) > 0$, and so the result follows immediately from the previous lemma. □

4. Proof of Theorem 1.2. By the Markov property of super-Brownian motion with $(1+\beta)$-stable branching, we may assume that $s = 0$ and X_0 is absolutely continuous with respect to Lebesgue measure. We will also assume that $X_0 > 0$; otherwise, there is nothing to prove. Define $\tilde{J}_n(x) = \int_0^{1/n} \int_{\mathbb{R}^d} p_s(x - y) J_n(y) n \, dy \, ds$ and note that $\tilde{J}_n$ is also a mollifier. Let us choose x such that

$$\lim_{n\to\infty} \tilde{J}_n * X_0(x) = \lim_{n\to\infty} J_n * X_0(x) > 0 \tag{4.1}$$

and

$$\limsup_{s\downarrow 0} S_s X_0(x) < \infty. \tag{4.2}$$

By standard differentiation theorems, (4.1) and (4.2) hold for X_0-a.e. x. Take x such that (4.1) and (4.2) hold and, without loss of generality, assume that $x = 0$.

We will prove the theorem via a series of lemmas, but first let us introduce some notation. We know (see the proof of Theorem 6.1.3 in [3]) that the jumps of the process X are multipliers of Dirac measures; that is, at the time of a jump, s, $\Delta X_s = r\delta_x$ for some $r > 0, x \in \mathbb{R}^d$. Let $N(dr, ds, dx)$ be the random point measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$ given by $\sum_{(r,s,x):\Delta X_s = r\delta_x} \delta_{r,s,x}$. Then, again by adopting the results in the proof of Theorem 6.1.3 of [3] (see Step 3 there), we get that the compensator measure $\hat{N}(dr, ds, dx)$ of N is given by

$$\hat{N}(dr, ds, dx) = \mathrm{n}(dr)\, X_s(dx)\, ds, \tag{4.3}$$

where

$$\mathrm{n}(dr) = \frac{\beta(\beta+1)}{\Gamma(1-\beta)} r^{-2-\beta}\, dr.$$

Recall our assumptions that $d\beta/2 < 1$ and $\lim_{\varepsilon\downarrow 0} \varepsilon^{-d/2} X_0(B(0,\sqrt{\varepsilon})) > 0$. In the following, fix α such that

$$\max\left(\frac{d(1+\beta)}{2} - 1, \frac{d}{2(\beta+1)}\right) < \alpha < \frac{d}{2}. \tag{4.4}$$

Define the set $B_n = B(0,\sqrt{\varepsilon_n}\,) \setminus B(0,\sqrt{2^{-1}\varepsilon_n}) = B(0,\sqrt{\varepsilon_n}\,) \setminus B(0,\sqrt{\varepsilon_{n+1}}\,)$.

LEMMA 4.1. *Let $\varepsilon_n = 2^{-n}$. There exists a subsequence $\{\varepsilon_{n_k}\}$ such that, for P-a.s. ω, there exists an $N^*(\omega)$ such that*

$$\sup_{0<s<n_k^{-1}} \Delta X_s(B_{n_k})(\omega) > \varepsilon_{n_k}^{\alpha} \qquad \forall\, n_k > N^*(\omega).$$

REMARK 4.2. In other words, for sufficiently small ε_{n_k} there is always a jump in $(0, n_k^{-1}) \times B_{n_k}$ whose height is greater than $\varepsilon_{n_k}^{\alpha}$.

PROOF OF LEMMA 4.1. Define $Z_t^n \equiv N([\varepsilon_n^{\alpha}, \infty] \times [0,t] \times B_n)$. Note that Z_t^n counts the number of jumps of height greater then ε_n^{α} in $[0,t] \times B_n$. Then, for each n, there exists a standard Poisson process A_t^n such that

$$Z_t^n = A^n_{c(\beta)\int_0^t X_s(B_n)\varepsilon_n^{-\alpha(\beta+1)}\,ds}$$

for some $c(\beta) > 0$. Choose $\delta > 0$ sufficiently small such that $\alpha(\beta+1) - \delta > d/2$. Then

$$\begin{aligned}
&P\big(Z_{n^{-1}}^n \le n\big)\\
&\quad= P\left(A^n_{c(\beta)\int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-\alpha(\beta+1)}\,ds} \le n\right)\\
&\quad\le P\left(A^n_{\varepsilon_n^{-\delta}} \le n, c(\beta)\int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-\alpha(\beta+1)}\,ds \ge \varepsilon_n^{-\delta}\right)\\
&\qquad+ P\left(A^n_{c(\beta)\int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-\alpha(\beta+1)}\,ds} \le n,\right.\\
&\qquad\qquad \left. c(\beta)\int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-\alpha(\beta+1)}\,ds \le \varepsilon_n^{-\delta}\right)\\
&\quad\le P\left(A^n_{\varepsilon_n^{-\delta}} \le n\right) + P\left(c(\beta)\int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-d/2} n\,ds\, \varepsilon_n^{d/2-\alpha(\beta+1)+\delta} n^{-1} \le 1\right)\\
&\quad\equiv I^{1,n} + I^{2,n}.
\end{aligned}$$

By a trivial bound on the Poisson probabilities, we get that $I^{1,n}$ is bounded by

$$\begin{aligned}
\sum_{k=0}^{n} P\left(A^n_{\varepsilon_n^{-\delta}} = k\right) &= \sum_{k=0}^{n} \exp\{-2^{n\delta}\}(2^{n\delta})^k/k!\\
&\le \exp\{-2^{n\delta}\}2^{n^2\delta} e,
\end{aligned} \tag{4.5}$$

and the last bound goes to 0 exponentially fast as $n \to \infty$.

Now we will show that

$$I^{2,n} \to 0 \qquad \text{as } n \to \infty. \tag{4.6}$$

First, note that $\varepsilon_n^{d/2-\alpha(\beta+1)+\delta} n^{-1} \to \infty$ as $n \to \infty$, and so it is enough to check that

$$P - \lim_{n\to\infty} n \int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-d/2}\, ds > 0. \tag{4.7}$$

It follows immediately from (1.4) that

$$E\left[\exp\left\{-c_d \int_0^{n^{-1}} X_s(B_n)\varepsilon_n^{-d/2} n\, ds\right\}\right] = \exp\{-\langle V_{n^{-1}}^{(n)}, X_0\rangle\},$$

where $V^{(n)} = V(0, nJ_n)$; that is, it satisfies the following nonlinear evolution equation:

$$\begin{aligned} v_t(x) = &-\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) v_s(y)^{1+\beta}\, dy\, ds \\ &+ \int_0^t \int_{\mathbb{R}^d} p_s(x-y)\big(J_n(y) - 2^{-d/2} J_{n+1}(y)\big) n\, dy\, ds, \end{aligned}$$

where $0 \le t \le n^{-1}, x \in \mathbb{R}^d$. We have to estimate the limiting behavior of $\langle V_{1/n}^{(n)}, X_0\rangle = \langle v_{n^{-1}}, X_0\rangle$ as $n \to \infty$. Note that

$$\begin{aligned} \langle v_{1/n}, X_0\rangle = &-\int_0^{1/n} \int_{\mathbb{R}^d} S_{n^{-1}-s} X_0(y) v_s(y)^{1+\beta}\, dy\, ds \\ &+ (\tilde{J}_n - 2^{-d/2}\tilde{J}_{n+1}) * X_0(0). \end{aligned} \tag{4.8}$$

By our assumption (4.1) $\lim_{n\to\infty} \tilde{J}_n * X_0(0) > 0$ and therefore $\lim_{n\to\infty}(\tilde{J}_n - 2^{-d/2}\tilde{J}_{n+1}) * X_0(0) > 0$. Now it is enough to check that the first term in (4.8) converges to 0 as $n \to \infty$. Since v_t is nonnegative, we have $v_t(x) \le \int_0^t \int_{\mathbb{R}^d} p_s(x-y) J_n(y) n\, dy\, ds$, and so

$$\begin{aligned} &\int_0^{1/n} \int_{\mathbb{R}^d} S_{1/n-s} X_0(x) v_s(y)^{1+\beta}\, dx\, ds \\ &\quad \le \int_0^{1/n} \int_{\mathbb{R}^d} S_{1/n-s} X_0(x) \left(\int_0^s \int_{\mathbb{R}^d} p_u(x-y) J_n(y) n\, dy\, du\right)^{1+\beta} dx\, ds \\ &\quad \le c n^{1+\beta} \int_0^{1/n} \int_{\mathbb{R}^d} S_{n^{-1}-s} X_0(x) \left(\int_0^s p_{u+\sqrt{\varepsilon_n}}(x)\, du\right)^{1+\beta} dx\, ds \\ &\quad \le c n^{1+\beta} \int_0^{1/n} \int_{\mathbb{R}^d} S_{n^{-1}-s} X_0(x) \int_0^s p_{u+\sqrt{\varepsilon_n}}(x)^{1+\beta}\, du\, s^{\beta}\, dx\, ds \\ &\qquad \text{(by Hölder's inequality)} \end{aligned}$$

$$
\begin{aligned}
&\le cn^{1+\beta}\int_0^{1/n}\int_0^s S_{n^{-1}-s+u+\sqrt{\varepsilon_n}}X_0(0)u^{-d\beta/2}\,du\,s^{\beta}\,ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Chapman–Kolmogorov)} \\
&\le c\sup_{0\le s\le 1} S_sX_0(0)n^{1+\beta}\int_0^{1/n} s^{1-d\beta/2}s^{\beta}\,ds \\
&\le c\sup_{0\le s\le 1} S_sX_0(0)n^{1+\beta}n^{-2-\beta+d\beta/2} \\
&= c\sup_{0\le s\le 1} S_sX_0(0)n^{-1+d\beta/2}\to 0 \qquad [\text{by (4.2) and } d<2/\beta]
\end{aligned}
$$

as $n\to\infty$. Combine (4.5) and (4.6) to get that

$$P\big(Z^n_{n^{-1}}\le n\big)\to 0 \qquad \text{as } n\to\infty.$$

Then take subsequence if necessary and apply the Borel–Cantelli lemma to finish the proof. $\square$

REMARK 4.3. In the following, to simplify the notation, we will assume, without loss of generality that, $\{n_k\}=\{n\}$.

Note that

$$
\begin{aligned}
\|\bar X(\cdot,0)\|_{(0,t)} &\ge \sup_{0<s<t-\varepsilon,0<\varepsilon<t}\varepsilon^{-1}\int_s^{s+\varepsilon}\bar X(r,0)\,dr \\
&\ge \lim_{\varepsilon_n\downarrow 0}\varepsilon_n^{-1}\int_{\tau_n}^{\tau_n+\varepsilon_n}\bar X(r,0)\,dr
\end{aligned} \tag{4.9}
$$

for any sequence of stopping times τ_n such that $\tau_n<t-\varepsilon_n$ for all n sufficiently large. Define the time of the first jump of Z^n:

$$\tau_n=\inf\{s:\Delta Z^n_s>0\}$$

and let x_n, r_n be the space location and the height of this jump, that is, $\Delta X_{\tau_n}=r_n\delta_{x_n}$. Recall that, by definition, $(x_n,r_n)\in B_n\times(\varepsilon_n^{\alpha},\infty)$. By Lemma 4.1 and Remark 4.3, it follows that

$$\tau_n<t-\varepsilon_n \tag{4.10}$$

for all n sufficiently large. Therefore, the following lemma, together with (4.9) and (4.10), completes the proof of Theorem 1.2.

LEMMA 4.4.

$$\varepsilon_n^{-1}\int_{\tau_n}^{\tau_n+\varepsilon_n}\bar X(s,0)\,ds\xrightarrow{\mathrm{P}}+\infty \qquad \textit{as } n\to\infty.$$

Before we start the proof of the lemma, we need the following auxiliary result whose proof is deferred to the Appendix. We also have to introduce some additional notation. For any $x \in \mathbb{R}^d$, denote $E_x \equiv (0, \infty) \times (\mathbb{R}^d \setminus \{x\})$ if $d \geq 2$ and $E_x \equiv (0, \infty) \times \mathbb{R}$ if $d = 1$. Define $g_t(\cdot) \equiv \int_0^t p_s(\cdot)\, ds$ for any $t \geq 0$.

LEMMA 4.5. *Let $d < 2 + 2/\beta$. Fix arbitrary $x \in \mathbb{R}^d$ and $a > 0$. Then there exists a unique $v = V(0, a\delta_x)$ in $\mathcal{C}(E_x)_+$ which solves the following nonlinear evolution equation*:

$$v_t(y) = -\int_0^t S_{t-s}(v_s^{1+\beta})\, ds + a \int_0^t p_s(y - x)\, ds \qquad \forall\, (t, y) \in E_x. \tag{4.11}$$

Moreover,

(a) *Note that* $\lim_{n\to\infty} V_t(0, aJ_n(x - \cdot))(y) = V_t(0, a\delta_x)(y)$ *for any* $(t, y) \in E_x$.
(b) *Let $\mu \in \mathcal{M}_F$. Then*

$$\lim_{n\to\infty} \mu\bigl(V_t\bigl(0, aJ_n(x - \cdot)\bigr)\bigr) = \mu\bigl(V_t(0, a\delta_x)\bigr) \qquad \forall\, t > 0$$

if $\mu(g_1(x - \cdot)) < \infty$.
(c) *Let $d \geq 2$ and $\mu \in \mathcal{M}_F$. Then*

$$\lim_{n\to\infty} \mu\bigl(V_t\bigl(0, aJ_n(x - \cdot)\bigr)\bigr) = +\infty \qquad \forall\, t > 0,$$

if $\mu(g_1(x - \cdot)) = \infty$.

REMARK 4.6. It follows from (1.4) and the above that

$$E\bigl[\exp\{-a\mathcal{Y}_t(x)\}\bigr] = \exp\bigl\{-\langle X_0, V_t(0, a\delta_x)\rangle\bigr\}$$
$$\forall\, t, a \geq 0,\ \forall\, x \in \mathbb{R}^d \text{ such that } X_0(g_1(x - \cdot)) < \infty.$$

Now we are ready to present the proof of Lemma 4.4.

PROOF OF LEMMA 4.4. It is clear that we can separate the mass of the process X after time τ_n into two parts: the part that comes from $X_{\tau_n -}$ and the one that comes from $r_n\delta_{x_n}$. Let us denote the second part by $\tilde{X}^n$ and its density by $\tilde{\bar{X}}(s, 0)$. Then, by Remark 4.6 and the strong Markov property, we get

$$\begin{aligned} E\biggl[\exp\biggl\{-\varepsilon_n^{-1} \int_{\tau_n}^{\tau_n+\varepsilon_n} \bar{X}(s, 0)\, ds\biggr\}\biggr] &\leq E\biggl[\exp\biggl\{-\varepsilon_n^{-1} \int_{\tau_n}^{\tau_n+\varepsilon_n} \tilde{\bar{X}}(s, 0)\, ds\biggr\}\biggr] \\ &= E\bigl[\exp\{-\tilde{V}^{(n)}_{\varepsilon_n}(x_n)\varepsilon_n^{\alpha}\}\bigr] + P(\tau_n = \infty), \end{aligned}$$

where $\tilde{V}^{(n)} = V(0, \varepsilon_n^{-1}\delta_0)$ satisfies the following integral equation:

$$\begin{aligned} v_t(x) = -\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) v_s(y)^{1+\beta}\, dy\, ds + \varepsilon_n^{-1} \int_0^t p_s(x)\, ds, \\ 0 \leq t \leq \varepsilon_n, \end{aligned} \tag{4.12}$$

and by (4.10), $P(\tau_n = \infty) \to 0$ as $n \to \infty$. We are interested in the limiting behavior of $v_{\varepsilon_n}(x_n)\varepsilon_n^\alpha$. Define $f_{\varepsilon_n} = \varepsilon_n^{-1}\int_0^{\varepsilon_n} p_s(x_n)\,ds$. Then

$$\begin{aligned} f_{\varepsilon_n} &\ge c\varepsilon_n^{-1}\int_0^{\varepsilon_n} s^{-d/2}e^{-\varepsilon_n/2s}\,ds \qquad (\text{since } |x_n| \le \sqrt{\varepsilon_n}\,) \\ &= c\varepsilon_n^{-d/2}\int_{1/2}^{\infty} w^{-2+d/2}e^{-w}\,dw. \end{aligned}$$

Hence, $\varepsilon_n^\alpha f_{\varepsilon_n} \to \infty$ as $n \to \infty$. Now let us check the limiting behavior of

$$\varepsilon_n^\alpha \int_0^{\varepsilon_n}\int_{\mathbb{R}^d} p_{\varepsilon_n - s}(x_n - y)v_s(y)^{1+\beta}\,dy\,ds.$$

By the domination $v_t(\cdot) \le \varepsilon_n^{-1}\int_0^t p_s(\cdot)\,ds$ [since $v_t(\cdot)$ is nonnegative], we get that

$$\begin{aligned} &\int_0^{\varepsilon_n}\int_{\mathbb{R}^d} p_{\varepsilon_n - s}(x_n - y)v_s(y)^{1+\beta}\,dy\,ds \\ &\quad \le c\int_0^{\varepsilon_n}\int_{\mathbb{R}^d} p_{\varepsilon_n - s}(x_n - y)\biggl(\varepsilon_n^{-1}\int_0^s p_r(y)\,dr\biggr)^{1+\beta}\,dy\,ds \\ &\quad \le c\varepsilon_n^{-1-\beta}\int_0^{\varepsilon_n}\int_{\mathbb{R}^d} p_{\varepsilon_n - s}(x_n - y)\int_0^s p_r(y)^{1+\beta}\,dr\,s^\beta\,dy\,ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by Hölder's inequality}) \\ &\quad \le c\varepsilon_n^{-1-\beta}\int_0^{\varepsilon_n}\int_0^s p_{\varepsilon_n - s + r}(x_n)r^{-d\beta/2}\,dr\,s^\beta\,ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by Chapman–Kolmogorov}) \\ &\quad \le c\varepsilon_n^{-1-\beta}\int_0^{\varepsilon_n}\int_0^s (\varepsilon_n - s + r)^{-d/2}\exp\biggl\{-\frac{\varepsilon_n}{4(\varepsilon_n - s + r)}\biggr\}r^{-d\beta/2}\,dr\,s^\beta\,ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } |x_n| \ge \sqrt{2^{-1}\varepsilon_n}\,) \\ &\quad = c\varepsilon_n^{1-d/2-d\beta/2}\int_0^1\int_0^w (1 - w + u)^{-d/2} \\ &\qquad\qquad\qquad \times \exp\biggl\{-\frac{1}{4(1 - w + u)}\biggr\}u^{-d\beta/2}w^\beta\,du\,dw \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{change of the variables}) \\ &\quad \le c\varepsilon_n^{1-d/2-d\beta/2}\int_0^1\int_0^w u^{-d\beta/2}w^\beta\,du\,dw \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Bigl(\text{since } \sup_{x>0} x^{d/2}\exp\{-x/4\} < \infty\Bigr) \\ &\quad = c\varepsilon_n^{1-d/2-d\beta/2}. \end{aligned}$$

This and (4.4) give

$$\varepsilon_n^\alpha \int_0^{\varepsilon_n} \int_{\mathbb{R}^d} p_{\varepsilon_n - s}(x_n - y) v_s(y)^{1+\beta}\,dy\,ds \le c\varepsilon_n^{1+\alpha-d/2-d\beta/2} \to 0 \qquad \text{as } n\to\infty,$$

by our choice of α and we are done. □

5. Proof of Theorem 1.3(b). As in the proof of Theorem 1.1(b), it is enough to check the claim for a fixed $t>0$ and for U equal to an arbitrary but fixed open ball B. Define a set of open balls $\{\mathrm{B}_k, k=1,2,\dots\}$ such that $\bar{\mathrm{B}}_k \subset \mathrm{B}_{k+1} \subset \mathrm{B}$ for any $k\ge 1$ and $\mathrm{B}_k \uparrow \mathrm{B}$ as $k\to\infty$. As in the proof of Theorem 1.2, we use the jump structure of X and the form of the compensator measure $\hat{N}(dr,ds,dx)$ in (4.3) to verify that

$$\begin{aligned}&\{\Delta X_s = r\delta_x \text{ for some } r>0,\ (s,x)\in[0,t]\times \mathrm{B}_k\}\\ &\quad = \left\{\int_{\mathrm{B}_k} \mathcal{Y}_t(x)\,dx = \int_0^t \int_{\mathrm{B}_k} X_u(dx)\,du > 0\right\} \qquad \forall k\ge 1.\end{aligned} \tag{5.1}$$

Fix an arbitrary $k\ge 1$ and define the stopping times

$$\tau_{l,k} = \inf\{s : \Delta X_s(\{x\}) \ge 2^{-l} \text{ for some } x\in \mathrm{B}_k\} \qquad \forall l\ge 1,$$

$$\tau_{l,k}(t) = \tau_{l,k}\wedge t \qquad \forall l \ge 1.$$

If $\tau_{l,k}\le t$, let $x_{l,k}$ be the point where $\Delta X_{\tau_{l,k}}(\{x_{l,k}\}) \ge 2^{-l}$. If $\tau_l > t$, define $x_{l,k}\equiv \mathbf{0}$. Fix an arbitrary $\varepsilon>0$. By (5.1), there is an $l^* = l^*(\varepsilon,k)$ sufficiently large such that

$$P\left(\tau_{l^*,k} < t \,\Big|\, \int_{\mathrm{B}_k} \mathcal{Y}_t(x)\,dx > 0\right) \ge 1-\varepsilon,$$

and hence

$$P(\tau_{l^*,k} < t) \ge (1-\varepsilon) P\left(\int_{\mathrm{B}_k} \mathcal{Y}_t(x)\,dx > 0\right). \tag{5.2}$$

Since B_k is contained in B, there exists $n_0 = n_0(k)$ such that, for all $n\ge n_0$,

$$\begin{aligned}\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}} &\ge \sup_{x\in \mathrm{B}_k} \int_0^t X_s * J_n(x)\,ds\\ &\ge \int_0^t X_s * J_n(x_{l^*,k})\,ds\\ &\ge \int_{\tau_{l^*,k}(t)}^t X_s * J_n(x_{l^*,k})\,ds.\end{aligned}$$

Consider the Laplace transform of $\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}}$:

$$
\begin{aligned}
E\big[\exp\{-\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}}\}\big] &\le E\bigg[\exp\bigg\{-\int_{\tau_{l^*,k}(t)}^t X_s * J_n(x_{l^*,k})\,ds\bigg\}\bigg] \\
&= E\bigg[E\bigg[\exp\bigg\{-\int_{\tau_{l^*,k}(t)}^t X_s * J_n(x_{l^*,k})\,ds\bigg\}\bigg|\mathcal{F}_{\tau_{l^*,k}}\bigg]\bigg] \\
&= E\bigg[\exp\bigg\{-\big\langle X_{\tau_{l^*,k}(t)}, U^{(n)}_{t-\tau_{l^*,k}(t)}\big\rangle\bigg\}\bigg] \qquad (5.3)\\
&\qquad\qquad\text{(by the strong Markov property)} \\
&\le E\Big[\exp\{-2^{-l}U^{(n)}_{t-\tau_{l^*,k}(t)}(0)\}\mathbb{1}(\tau_{l,k} < t)\Big] + P(\tau_{l^*,k} \ge t) \\
&\qquad\qquad \forall n \ge n_0,
\end{aligned}
$$

where $U^{(n)} = V(0, J_n)$. By Lemma 4.5(c), we have that, for any $t > 0$,

$$
U_t^{(n)}(0) \to +\infty \tag{5.4}
$$

as $n \to \infty$.

Let $n \to \infty$ in (5.3) and use (5.4) to get

$$
\begin{aligned}
E\big[\exp\{-\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}}\}\big] &\le P(\tau_{l^*,k} \ge t) \\
&\le P\bigg(\int_{\mathrm{B}_k} \mathcal{Y}_t(x)\,dx = 0\bigg) + \varepsilon,
\end{aligned}
$$

where the last inequality follows by (5.2). Since ε was arbitrary, we get

$$
E\big[e^{-\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}}}\big] \le P\bigg(\int_{\mathrm{B}_k} \mathcal{Y}_t(x)\,dx = 0\bigg). \tag{5.5}
$$

Inequality (5.5) holds for all $k \ge 1$. Therefore, by letting $k \to \infty$, we get

$$
E\big[e^{-\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}}}\big] \le P\bigg(\int_{\mathrm{B}} \mathcal{Y}_t(x)\,dx = 0\bigg),
$$

and so

$$
P\bigg(\|\mathcal{Y}_t(\cdot)\|_{\mathrm{B}} = \infty \bigg| \int_{\mathrm{B}} \mathcal{Y}_t(x)\,dx > 0\bigg) = 1. \tag{5.6}
$$

□

APPENDIX

Proof of Lemma 4.5. Denote $g_{s,t}(\cdot) \equiv g_t(\cdot) - g_s(\cdot) = \int_s^t p_r(\cdot)\,dr,\ 0 \le s < t$. We start with a simple lemma.

LEMMA A.1. *Let $d < 2 + 2/\beta$. Fix $\eta < 1$ such that $d < 2\eta + 2/\beta$. Then there exists a constant $c(t, \eta)$ such that*

$$
g_{s,t}^{1+\beta}(\cdot) \le c(t, \eta) \int_s^t r^{\beta(\eta - d/2)} p_r(\cdot)\,dr \qquad \forall s \le t.
$$

Moreover, $\sup_{t\le T} c(t,\eta) = c(T,\eta)$ *for any* $T>0$.

PROOF. Let $\alpha = \beta/(1+\beta)$. By Hölder's inequality and simple calculations, we have

$$\begin{aligned} g_{s,t}(y)^{1+\beta} &= \left(\int_s^t p_r(y) r^{\eta\alpha} r^{-\eta\alpha}\,dr\right)^{1+\beta} \\ &\le \int_s^t p_r(y)^{1+\beta} r^{\eta\beta}\,dr \left(\int_s^t r^{-\eta}\,dr\right)^{\beta} \\ &= c(t)\int_s^t r^{\beta(\eta-d/2)} p_r(y)\,dr. \end{aligned}$$

□

PROOF OF LEMMA 4.5. We will assume, without loss of generality, that $x=\mathbf{0}$. Let $\mathcal{L}^{1+\beta} = \mathcal{L}^{1+\beta}(\mathbb{R}^d)$ be the space of equivalent classes of measurable functions on $\mathbb{R}^d$ with finite norm

$$\|f\|_{1+\beta} = \left(\int_{\mathbb{R}^d} |f(y)|^{1+\beta}\,dy\right)^{1/(1+\beta)} < \infty.$$

By Proposition 2.5 of [7], there exists a unique nonnegative solution $\mathbf{V}$ to (4.11) in the space $\mathcal{C}_{\mathcal{L}^{1+\beta}}[0,\infty)$. First, we will show that there exists a solution that belongs to $\mathcal{C}(E_0)$. For this purpose, let us define

$$v(t,y) \equiv -\int_0^t S_{t-s}(\mathbf{V}_s^{1+\beta})\,ds + a\int_0^t p_s(y)\,ds \qquad \forall\,(s,y)\in E_0. \tag{A.1}$$

Then it is clear that v also solves (4.11). To check the continuity of v, note that $\int_0^t p_s(y)\,ds$ is continuous on E_0. Therefore, it is enough to check the continuity of $\int_0^t S_{t-s}(v_s^{1+\beta})(y)\,ds$. Take arbitrary $(t_n,y_n)\to(t,y)\in E_0$ as $n\to\infty$. Then $\mathbb{1}(s<t_n)p_{t_n-s}(y_n-x)\to \mathbb{1}(s<t)p_{t-s}(y-x)$ for $v_s(x)^{1+\beta}\,dx\,ds$-a.e. (s,x). Using the domination $v_s(\cdot)\le a\int_0^s p_r(\cdot)\,dr$, it is enough to check that the sequence of functions $f^n(s,x)\equiv \mathbb{1}(s<t_n)p_{t_n-s}(y_n-x)$ is uniformly integrable with respect to the measure $\hat{\mu}(dx\,ds)\equiv \mathbb{1}(s\le T)(\int_0^s p_r(x)\,dr)^{1+\beta}\,dx\,ds$ for arbitrary $T>t$. Fix $\delta = 1/2d$ and $\eta<1$ such that $d<2\eta+2/\beta$. Then, by Lemma A.1,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} f^n(s,x)^{1+\delta}\hat{\mu}(dx\,ds) \\ &\quad\le c(T,\eta)\int_0^{t_n}\int_{\mathbb{R}^d} p_{t_n-s}(y_n-x)^{1+\delta}\int_0^s r^{\beta(\eta-d/2)}p_r(x)\,dr\,dx\,ds \\ &\quad= c(T,\eta)\int_0^{t_n}(t_n-s)^{-1/4}p_{t_n-s+r}(y_n)\int_0^s r^{\beta(\eta-d/2)}\,dr\,ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Chapman–Kolmogorov)} \\ &\quad\le \begin{cases} c(T,\eta)\displaystyle\int_0^{t_n}(t_n-s)^{-3/4}\,ds, & \text{if } d=1, \\ c(T,\eta)\displaystyle\sup_{r\le T} p_r(y_n), & \text{if } d\ge 2. \end{cases} \end{aligned}$$

In the case of $d \geq 2$ $\sup_{r\leq T} p_r(y_n)$ is bounded uniformly for all n sufficiently large since $y_n \to y \neq \mathbf{0}$. Hence, the uniform integrability of f^n follows and therefore the continuity of v.

To show the uniqueness of the solution to (4.11) in $\mathcal{C}(E_0)_+$, it is enough to show that every such solution is necessarily in $\mathcal{C}_{\mathcal{L}_+^{1+\beta}}[0,\infty)$, since the solution in this space is unique. Let v be an arbitrary solution to (4.11) in $\mathcal{C}(E_0)_+$. Again, use the bound $v_s(\cdot) \leq a\int_0^s p_r(\cdot)\,dr$ and Lemma A.1 to get that $v_s \in \mathcal{L}_+^{1+\beta}$ for all $s \geq 0$. To show the continuity in this space, let $s_n \to t$ as $n \to \infty$. Fix $T > 0$ such that $s_n < T$ for all n. Then $v \in \mathcal{C}(E_0)_+$ and hence $v_{s_n}(x) \to v_t(x)$ $\forall x \in \mathbb{R}^d \setminus \{0\}$. Use the bound $v_{s_n}(\cdot) \leq a\int_0^T p_r(\cdot)\,dr$ $\forall n$, Lemma A.1 and dominated convergence to get that $v_{s_n} \to v_t$ in $\mathcal{L}^{1+\beta}$ and so $v \in \mathcal{C}_{\mathcal{L}_+^{1+\beta}}[0,\infty)$, and uniqueness follows.

Now let us prove properties (a), (b) and (c).

(a) Define $u^{(n)} = V(0, aJ_n)$, $u = V(0, a\delta_0)$. That is,

$$\begin{aligned}u_t^{(n)}(y) = &-\int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-z)u_s^{(n)}(z)^{1+\beta}\,dz\,ds\\ &+ a\int_0^t\int_{\mathbb{R}^d} p_s(y-z)J_n(z)\,dz\,ds\end{aligned} \tag{A.2}$$

for $(t,x) \in E_0$. By Proposition 2.5 of [7], $u^{(n)} \to u$ in $\mathcal{C}_{\mathcal{L}^{1+\beta}}[0,\infty)$. We need to check the pointwise convergence of $u^{(n)}$. Fix $(t,y) \in E_0$. The second term in (A.2) converges trivially to $a\int_0^t p_s(y)\,ds$. Consider the first term. Since $u^{(n)} \to u$ in $\mathcal{C}_{\mathcal{L}^{1+\beta}}[0,\infty)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d} p_{t-s}(y-z)u_s^{(n)}(z)^{1+\beta}\,dz\\ &\qquad \to \int_{\mathbb{R}^d} p_{t-s}(y-z)u_s(z)^{1+\beta}\,dz \qquad \forall\, 0 \leq s < t.\end{aligned}$$

Since $J_n(\cdot) \leq cp_{\sqrt{\varepsilon_n}}(\cdot)$, we use the domination $u_t^{(n)}(x) \leq a\int_0^t\int_{\mathbb{R}^d} p_s(x-y) \times J_n(y)\,dy\,ds$ and Chapman–Kolmogorov equations to get that

$$u_t^{(n)}(x) \leq c\int_0^t p_{s+\sqrt{\varepsilon_n}}(x)\,ds \tag{A.3}$$

$$\leq cg_{t+1}(x) \qquad \forall t > 0, x \in \mathbb{R}^d. \tag{A.4}$$

Note that $\int_{\mathbb{R}^d} p_{t-s}(y-z)g_{t+1}(z)^{1+\beta}\,dz$ is integrable with respect to ds on $[0,t)$ (recall $y \neq 0$ if $d > 1$ and use Lemma A.1). Therefore, by dominated convergence,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-z)u_s^{(n)}(z)^{1+\beta}\,dz\,ds\\ &\qquad \to \int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-z)u_s(z)^{1+\beta}\,dz\,ds \qquad \text{as } n \to \infty,\end{aligned}$$

and (a) follows.

(b) The result follows immediately from (a), (A.4) and dominated convergence.

(c) Define

$$I_t^{1,n} = \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} p_{t-s}(z-y)u_s^{(n)}(y)^{1+\beta}\,dy\,\mu(dz)\,ds,$$

$$I_t^{2,n} = \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} p_s(z-y)J_n(y)\,dy\,\mu(dz)\,ds,$$

that is, $\mu(u_t^{(n)}) = -I_t^{1,n} + aI_t^{2,n}$. First, let us find the rate of convergence of $I_t^{2,n}$ to ∞ as $n\to\infty$.

$$\begin{aligned}
I_t^{2,n} &\ge c\varepsilon_n^{-d/2}\int_{2\varepsilon_n}^t \int_{B(0,\sqrt{\varepsilon_n})}\int_{\mathbb{R}^d} s^{-d/2}\exp\{-|z-y|^2/2s\}\,\mu(dz)\,dy\,ds\\
&\ge c\varepsilon_n^{-d/2}\int_{2\varepsilon_n}^t \int_{B(0,\sqrt{\varepsilon_n})}\int_{\mathbb{R}^d} s^{-d/2}\exp\{-2|z|^2/2s\}\\
&\qquad\qquad \times\exp\{-2|y|^2/2s\}\,\mu(dz)\,dy\,ds\\
&\ge c\varepsilon_n^{-d/2}\int_{2\varepsilon_n}^t S_{s/2}\mu(0)\int_{B(0,\sqrt{\varepsilon_n})}\exp\{-1/2\}\,dy\,ds\\
&= c\int_{\varepsilon_n}^{t/2} S_s\mu(0)\,ds\\
&\to +\infty
\end{aligned}\tag{A.5}$$

as $n\to\infty$.

Now let us bound $I_t^{1,n}$. Recall that $d<2+(2/\beta)$. Choose $\eta<1$ such that $d<2\eta+(2/\beta)$. Then, by the bound (A.3), Lemma A.1 and simple calculations, we get that

$$\begin{aligned}
I_t^{1,n} &\le c(t)\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} p_{t-s}(z-y)\int_{\sqrt{\varepsilon_n}}^{s+\sqrt{\varepsilon_n}} r^{\beta(\eta-d/2)}p_r(y)\,dr\,dy\,\mu(dz)\,ds\\
&= c(t)\int_0^t\int_{\sqrt{\varepsilon_n}}^{s+\sqrt{\varepsilon_n}}\int_{\mathbb{R}^d} p_{t-s+r}(z)\mu(dz)r^{-d\beta/2+\eta\beta}\,dr\,ds\\
&\qquad\qquad\text{(by Chapman–Kolmogorov)}\\
&= c(t)\int_{\sqrt{\varepsilon_n}}^{t+\sqrt{\varepsilon_n}}\int_r^{t+\sqrt{\varepsilon_n}} S_s\mu(0)\,ds\,r^{-d\beta/2+\eta\beta}\,dr\\
&\qquad\qquad\text{(by change of variables)}\\
&\le c(t)\int_{\sqrt{\varepsilon_n}}^{t+\sqrt{\varepsilon_n}} S_r\mu(0)r^{-d\beta/2+\eta\beta+1}\,dr\qquad\text{(by integration by parts).}
\end{aligned}$$

The choice of η implies that $1+\eta\beta-d\beta/2>0$. Hence, by (A.5) and the above, we get that

$$aI_t^{2,n} - I_t^{1,n} \to +\infty$$

as $n \to \infty$. $\square$

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