The Annals of Probability
2003, Vol. 31, No. 3, 1338–1376

ON THE SHARP MARKOV PROPERTY FOR GAUSSIAN RANDOM FIELDS AND SPECTRAL SYNTHESIS IN SPACES OF BESSEL POTENTIALS

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Let $\Phi = \{\phi(x) : x \in \mathbb{R}^2\}$ be a Gaussian random field on the plane. For $A \subset \mathbb{R}^2$, we investigate the relationship between the σ-field $\mathcal{F}(\Phi, A) = \sigma\{\phi(x) : x \in A\}$ and the infinitesimal or germ σ-field $\bigcap_{\varepsilon>0} \mathcal{F}(\Phi, A_\varepsilon)$, where A_ε is an ε-neighborhood of A. General analytic conditions are developed giving necessary and sufficient conditions for the equality of these two σ-fields. These conditions are potential theoretic in nature and are formulated in terms of the reproducing kernel Hilbert space associated with Φ. The Bessel fields Φ_β satisfying the pseudo-partial differential equation $(I - \Delta)^{\beta/2}\phi(x) = \dot{W}(x)$, $\beta > 1$, for which the reproducing kernel Hilbert spaces are identified as spaces of Bessel potentials $\mathcal{L}^{\beta,2}$, are studied in detail and the conditions for equality are conditions for spectral synthesis in $\mathcal{L}^{\beta,2}$. The case $\beta = 2$ is of special interest, and we deduce sharp conditions for the sharp Markov property to hold here, complementing the work of Dalang and Walsh on the Brownian sheet.

Introduction. We consider a stochastically continuous mean-zero real-valued random field $\Phi = \{\phi(x) : x \in \mathbb{R}^n\}$. Our focus in this paper is the relationship between the *sharp σ-field*

$$\mathcal{F}(\Phi, \Gamma) \stackrel{\text{def}}{=} \sigma\{\phi(x) : x \in \Gamma\}$$

and the *germ σ-field*

$$\overline{\mathcal{F}}(\Phi, \Gamma) \stackrel{\text{def}}{=} \bigcap_{\varepsilon>0} \mathcal{F}(\Phi, \Gamma_\varepsilon),$$

with Γ_ε denoting the uniform neighborhood $\{x : \operatorname{dist}(x, \Gamma) < \varepsilon\}$ of Γ. Writing $\bar{\Gamma}$ for the closure of Γ and equating σ-fields that differ only by null sets,

$$\mathcal{F}(\Phi, \Gamma) \subseteq \overline{\mathcal{F}}(\Phi, \Gamma) \quad \text{and} \quad \mathcal{F}(\Phi, \Gamma) = \mathcal{F}(\Phi, \bar{\Gamma}).$$

If $\Gamma \subset \mathbb{R}^n$ is a closed set separating $\mathbb{R}^n$ into complementary open sets D_+ and D_-, and if $\overline{\mathcal{F}}(\Phi, D_+)$ and $\overline{\mathcal{F}}(\Phi, D_-)$ are conditionally independent given $\overline{\mathcal{F}}(\Phi, \Gamma)$, we will say that Φ satisfies the *germ field Markov property* at Γ. If Φ satisfies the more restrictive condition with $\overline{\mathcal{F}}(\Phi, D_+)$ and $\overline{\mathcal{F}}(\Phi, D_-)$

Received October 2001; revised July 2002.
AMS 2000 subject classifications. Primary 60G15, 60G60; secondary 31B15, 31B25, 60H15.
Key words and phrases. Gaussian fields, germ fields, sharp Markov property, spectral synthesis.

conditionally independent given $\mathcal{F}(\Phi, \Gamma)$, we will say that Φ satisfies the *sharp Markov property at* Γ.

There is no universal agreement in the literature on terminology concerning the Markov properties of random fields; see Dalang and Walsh (1992), Pitt (1971) and Rozanov (1982). In particular, Dalang and Walsh (1992) use slightly different definitions for both Markov properties in which the two germ σ-fields $\overline{\mathcal{F}}(\Phi, D_-)$ and $\overline{\mathcal{F}}(\Phi, D_+)$ are replaced with the sharp fields $\mathcal{F}(\Phi, D_-)$ and $\mathcal{F}(\Phi, D_+)$. For our purposes, our definition has the advantage since, in the typical cases of interest when Γ is the boundary of both sets D_+ and D_-, the relationship between the two Markov properties is relatively direct and elementary. Namely, in this case,

$$\overline{\mathcal{F}}(\Phi, \Gamma) \subseteq \overline{\mathcal{F}}(\Phi, D_+) \cap \overline{\mathcal{F}}(\Phi, D_-),$$

and, modulo null sets, $\overline{\mathcal{F}}(\Phi, D_+)$ and $\overline{\mathcal{F}}(\Phi, D_-)$ are not conditionally independent over any proper sub-σ-field of $\overline{\mathcal{F}}(\Phi, \Gamma)$. We can thus state the following result.

PROPOSITION A. *Suppose that* Γ *is a closed set separating* $\mathbb{R}^n$ *into complementary open sets* D_+ *and* D_-, *with* $\Gamma = \overline{D}_- \cap \overline{D}_+$. *Then* $\Phi = \{\phi(x) : x \in \mathbb{R}^n\}$ *satisfies the sharp Markov property at* Γ *iff* Φ *satisfies the germ field Markov property at* Γ *and the identity*

$$\overline{\mathcal{F}}(\Phi, \Gamma) = \mathcal{F}(\Phi, \Gamma) \tag{1}$$

holds.

These considerations raise some fundamental questions.

QUESTION 1. What conditions on the closed set $\Gamma \subset \mathbb{R}^n$ imply that (1) holds?

Similarly, the seminal results of Dalang and Walsh on the sharp Markov property and their use of the fields $\mathcal{F}(\Phi, D_-)$ and $\mathcal{F}(\Phi, D_+)$, rather than $\overline{\mathcal{F}}(\Phi, D_-)$ and $\overline{\mathcal{F}}(\Phi, D_+)$, raise the same questions for open sets.

QUESTION 2. If $O \subset \mathbb{R}^n$ is open, what conditions on O imply that

$$\mathcal{F}(\Phi, O) = \overline{\mathcal{F}}(\Phi, O) \tag{2}$$

holds?

In this paper these two questions are studied in detail for the case of Gaussian random fields. Our approach is essentially based on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Phi)$ (defined in Section 2) associated with Φ. It converts these problems into spectral synthesis problems in the space $\mathcal{H}$ and is, in principle, very general. However, in practice, the approach requires detailed structural knowledge of the space $\mathcal{H} = \mathcal{H}(\Phi)$, and this is not generally available. For this reason, we present the conceptual formulation of this approach restricted to the family of continuous, mean-zero, stationary Gaussian fields $\Phi_\beta = \Phi = \{\phi(x) : x \in \mathbb{R}^2\}$ on the plane which satisfy the stochastic pseudo-differential equations

$$(I - \Delta)^{\beta/2}\phi(x) = \dot{W}(x), \qquad x \in \mathbb{R}^2, \tag{3}$$

where $\beta > 1$ is constant and $\dot{W}(x)$ is a stationary Gaussian white noise on $\mathbb{R}^2$.

REMARK. For reasons explained below, we call these Bessel fields of index β. A Fourier calculation (see Section 3) shows that Φ_β has spectral density given by

$$\Delta_\beta(\lambda) = \Delta(\lambda) = \frac{1}{(2\pi)^2}\frac{1}{(1+|\lambda|^2)^\beta}, \qquad \lambda \in \mathbb{R}^2,$$

and covariance function

$$\rho(x, y) = E\phi(x)\phi(y) = G_{2\beta}(x - y),$$

where, for any $\beta > 0$, G_β is defined to be the inverse Fourier transform of $(1 + |\lambda|^2)^{-\beta/2}$. Note that G_β is the *Bessel kernel of index* β and is essentially a Bessel function of the third kind [see Adams and Hedberg (1996), page 11].

The reproducing kernel spaces $\mathcal{H}(\Phi_\beta)$ are easily identified: each u in $\mathcal{H}(\Phi_\beta)$ is square integrable, and the $\mathcal{H}(\Phi_\beta)$ norm of u is given by

$$\|u\|^2_{\beta,2} = \int_{\mathbb{R}^2} (1 + |\lambda|^2)^\beta\, |\hat{u}(\lambda)|^2\, d\lambda,$$

where $\hat{u}(\lambda) = \int_{\mathbb{R}^2} e^{-ix\cdot\lambda} u(x)\, dx$ is the Fourier transform of u. When $\beta = m$ is an integer, this is the Sobolev space of functions with L^2-derivatives of order not greater than m. For noninteger β, the norm is given by the same expression as above but now the space is identified as the space of Bessel potentials $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$ of order β. The spaces $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$ have been thoroughly studied [see Stein (1970) and Adams and Hedberg (1996) for details].

The paper is organized as follows: For the reader's convenience, we have gathered together references, notation and essential facts on Bessel potential spaces in Section 1. Readers familiar with this material can skip the section or use it as a guide for notation. The general formulation of the problems in the setting of the reproducing kernel Hilbert spaces is given in Section 2. The case of $\beta = 2$,

$$(I - \Delta)\phi(x) = \dot{W}(x), \qquad x \in \mathbb{R}^2,$$

deserves special attention. Because of its original occurrence in Whittle (1954), it is called the Whittle field. It is the simplest, most familiar and most interesting of the Bessel fields. In addition, nearly all key ideas used in treating the general case occur in the Whittle case. For this reason, Section 3 presents the Whittle field in complete detail. The discussion is extended to general $\beta > 1$ in Section 4.

RESULTS ON THE WHITTLE FIELD. We now state our principal results for the Whittle field. Except for Theorem 3.11 and related material in Section 3.4, these results were announced in Pitt and Robeva (1994) and are proved in the dissertation of Robeva (1997). When $\Phi = \{\phi(x) : x \in \mathbb{R}^2\}$ is the Whittle field, the associated reproducing kernel space $\mathcal{H}$ is identified as the Sobolev space $\mathcal{L}^{2,2}(\mathbb{R}^2)$ of continuous square-integrable functions on the plane that have weak L^2-derivatives of first and second orders. Let $C_{\log}(A)$ denote logarithmic capacity on $\mathbb{R}^2$. It is known that each function $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$ is, off a set A with $C_{\log}(A) = 0$, differentiable. For $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$, we write $u|_S = 0$ when the restriction of u to a set $S \subseteq \mathbb{R}^2$ vanishes and $\nabla u|_S = 0$ to indicate that, off a set of zero logarithmic capacity, the restriction of ∇u to S vanishes.

THEOREM 1 (3.1). *For the Whittle field Φ and a set $S \subseteq \mathbb{R}^2$, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ iff $u \in \mathcal{L}^{2,2}$ and $u|_S = 0$ implies $\nabla u|_S = 0$ except on a set of logarithmic capacity* 0.

This result, which follows from a theorem on spectral synthesis in $\mathcal{L}^{2,2}(\mathbb{R}^2)$ of Hedberg (1981) (presented below as Theorem 1.5), is the basis for our subsequent analysis. It has an immediate corollary that $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ holds for any sufficiently small set S satisfying $C_{\log}(S) = 0$. As we shall see, this also holds for any sufficiently irregular set Γ.

Following Saks (1937), page 262, we define a contingent of a set. For points $x \neq y \in \mathbb{R}^2$, we let $l(x, y)$ denote the line in $\mathbb{R}^2$ that contains x and y. If x is an accumulation point of a set $S \subseteq \mathbb{R}^2$, and if l is a line through x, we will say that l is a *contingent of S at x* provided there is a sequence of points $y_n \neq x$ in S that converges to x with $\lim_{n\to\infty} l(x, y_n) = l$. We let $\mathrm{Contg}(S, x)$ denote the set of all contingents to S at x. For a subset $S \subseteq \mathbb{R}^2$, we say that S has a tangent at $x \in S$ provided that $\mathrm{Contg}(S, x)$ contains a unique line. We write $\mathcal{T}(S)$ for the set of points in S at which S has a tangent. We have the following result.

THEOREM 2 (3.2). *Let $S \subseteq \mathbb{R}^2$ be closed. If $C_{\log}(\mathcal{T}(S)) = 0$, then $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.*

The problem of identifying S for which $\mathcal{F}(\Phi, S) \neq \overline{\mathcal{F}}(\Phi, S)$ is essentially that of identifying level sets S of functions in $\mathcal{L}^{2,2}(\mathbb{R}^2)$ with gradients that do not vanish identically on S. Using a change-of-variable argument, we prove the following result.

THEOREM 3 (3.6). *If $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ is a smooth curve for which $\gamma'(s) \neq 0$ is Hölder continuous with exponent $\alpha > 1/2$, then $\mathcal{F}(\Phi, \Gamma) \neq \overline{\mathcal{F}}(\Phi, \Gamma)$.*

It is possible to strengthen Theorem 3.6 in a quantitative manner by considering the sharp field to be the tangential part of $\overline{\mathcal{F}}(\Phi, \Gamma)$ and introducing the normal component $\mathcal{F}_n(\Phi, \Gamma)$ generated by generalized normal derivatives of the form

$$\int_\Gamma f(x)\,\partial_n\phi(x)\,d\sigma(x) \equiv \lim_{h\to 0}\int_\Gamma f(x)\frac{\phi(x+hn(x))-\phi(x)}{h}\,d\sigma(x). \tag{4}$$

Here σ denotes the arc length, $n(x)$ is a continuous unit normal vector to Γ and $f(x)$ is a continuous function defined on Γ.

Associated with the σ-fields $\mathcal{F}(\Phi, \Gamma)$ and $\mathcal{F}_n(\Phi, \Gamma)$ are two subspaces of the Hilbert space $L^2(\Omega, \Sigma, P)$, defined as the closed linear spans $H(\Phi, \Gamma) = \overline{\mathrm{sp}}\{\phi(x), x \in S\}_{L^2}$ and $H_n(\Phi, \Gamma) = \overline{\mathrm{sp}}\{\int_\Gamma f(x)\,\partial_n\phi(x)\,d\sigma(x) : f \in C(\Gamma)\}_{L^2}$. The assumption that Φ is Gaussian implies [see, e.g., Rozanov (1982), page 41] that $\mathcal{F}(\Phi, S) = \sigma(H(\Phi, S))$ and $\mathcal{F}_n(\Phi, S) = \sigma(H_n(\Phi, S))$, and if $\overline{H}(\Phi, S) \overset{\text{def}}{=} \bigcap H(\Phi, O)$, with the intersection taken over all neighborhoods O of $\overline{S}$, $\overline{\mathcal{F}}(\Phi, S) = \sigma(\overline{H}(\Phi, S))$. Thus, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ if and only if $H(\Phi, S) = \overline{H}(\Phi, S)$. Moreover, we have the following result.

THEOREM 4 (3.11). *Let $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ be a smooth C^1-curve with $\gamma'(s) \neq 0$.*

(i) *For any $f \in C(\Gamma)$, the quantity $\int_\Gamma f(x)\,\partial_n\phi(x)\,d\sigma(x)$ defined by* (4) *exists. Moreover, $H(\Phi, \Gamma) + H_n(\Phi, \Gamma)$ is dense in $\overline{H}(\Phi, \Gamma)$;*

(ii) *If $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ is a smooth simple curve and $\gamma'(s) \neq 0$ is Hölder continuous with exponent $\alpha > 1/2$, then there is a positive angle between the two spaces $H(\Phi, \Gamma)$ and $H_n(\Phi, \Gamma)$. Thus, the sum $H(\Phi, \Gamma) + H_n(\Phi, \Gamma)$ is closed in $L^2(P)$ and*

$$\overline{H}(\Phi, \Gamma) = H(\Phi, \Gamma) + H_n(\Phi, \Gamma).$$

(iii) *If Γ is straight* (*i.e., a line segment*), *then $H(\Phi, \Gamma)$ and $H_n(\Phi, \Gamma)$ are orthogonal.*

REMARK. The significance of part (ii) derives from Gebelein's inequality [see, e.g., Dym and McKean (1976), page 66], which implies that the degree of dependence between the two σ-fields $\mathcal{F}(\Phi, \Gamma) = \sigma(H(\Phi, \Gamma))$ and $\sigma(H_n(\Phi, \Gamma))$ is comparable to the cosine of the angle between $H(\Phi, \Gamma)$ and $H_n(\Phi, \Gamma)$. It is also of note that this degree of dependence can increase as the curve Γ becomes less smooth. In fact, we use a lacunary trigonometric series construction to show that the condition $\alpha > 1/2$ in this theorem is essentially the best possible.

THEOREM 5 (3.7). *If $0 < \alpha < 1/2$, there is a smooth curve $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ with $\gamma'(s) \neq 0$ being Hölder continuous with exponent α and $\mathcal{F}(\Phi, \Gamma) = \overline{\mathcal{F}}(\Phi, \Gamma)$.*

The following results concern open sets.

THEOREM 6 (3.4). *$\mathcal{F}_D(\Phi, D) = \overline{\mathcal{F}}_D(\Phi, D)$ holds for any bounded connected open set D.*

EXAMPLE 1 (3.5). *There exists a bounded open set $D \subset \mathbb{R}^2$ for which $\mathcal{F}(\Phi, D) \neq \overline{\mathcal{F}}(\Phi, D)$.*

A corollary of Theorem 3.4, concerning the germ field Markov property for the Whittle field Φ, is that for closed separating sets Γ for which the complementary open sets D_+ and D_- are connected and satisfy $\overline{D}_+ \cap \overline{D}_- = \Gamma$, the germ field Markov property of Φ at Γ, as defined in this paper, coincides with the germ field Markov property as defined in Dalang and Walsh (1992). In fact, this follows since $\mathcal{F}(\Phi, D_-) = \overline{\mathcal{F}}(\Phi, D_-)$ and $\mathcal{F}(\Phi, D_+) = \overline{\mathcal{F}}(\Phi, D_+)$.

NOTATION CONVENTIONS. $\mathbb{R}^n$ is the n-dimensional Euclidean space, a point $x \in \mathbb{R}^n$ is denoted by $x = (x_1, \dots, x_n)$ the Euclidean inner product is $x \cdot y = x_1 y_1 + \cdots + x_n y_n$, and the norm is $|x| = (x \cdot x)^{1/2}$. In $\mathbb{R}^2$, $(x, y) \in \mathbb{R}^2$ is sometimes used to avoid subindices. For an arbitrary set $S \subset \mathbb{R}^n$, $\bar{S}$ denotes its closure, S^o denotes its interior and $S^c = \mathbb{R}^n \setminus S$ is its complement. The n-dimensional Lebesgue measure of S is denoted by $|S|$. The indicator function of the set S is denoted by $\mathbb{1}_S$. If $\{[a_i, b_i]\}$ is a finite set of intervals, linear combinations of their indicator functions are called simple functions. The open ball with center x and radius r is denoted by $B(x, r)$.

We write $(\partial/\partial x_j)u = \partial_j u$ and the gradient $\nabla u = (\partial_1 u, \dots, \partial_n u)$. Higher derivatives are denoted by $D^\kappa u = \partial_1^{\kappa_1} \cdots \partial_n^{\kappa_n}$, where $\kappa = (\kappa_1, \dots, \kappa_n)$ is a multiindex of positive integers. We denote $|\kappa| = \kappa_1 + \cdots + \kappa_n$. We also write $\nabla^m u$ for the vector $(D^\kappa u)_{|\kappa|=m}$ of all derivatives of order m and $|\nabla^m u|$ for its Euclidean norm.

The class of infinitely differentiable functions on Ω is denoted by $C^\infty(\Omega)$, and $C_0^\infty(\Omega)$ is the subset of functions with compact support on Ω. For an integer $m \geq 0$ and $0 < \alpha \leq 1$, $C^{m,\alpha}(\Omega)$ is the Hölder class of functions on Ω with m continuous derivatives such that the derivatives of highest order are Hölder continuous of order α. $C_0^{m,\alpha}(\Omega)$ is the class of all $C^{m,\alpha}(\Omega)$-functions with compact support on Ω. The Schwartz class of rapidly decreasing C^∞-functions is denoted by $\mathcal{S}$.

We use c to denote a constant whose value may change from one line to another.

1. Bessel potential function spaces. We collect here basic facts on Bessel potential and Sobolev spaces. We will use these results only on $\mathbb{R}^2$ and with $p = 2$, but because there is no added difficulty in discussing the general case of $\mathbb{R}^n$, we do so here. Let G_β, $\beta \in \mathbb{R}$, be the Bessel kernel of order β defined as the inverse Fourier transform of $\hat{G}_\beta(\lambda) = (1 + |\lambda|^2)^{-\beta/2}$. For $\beta > 0$, G_β is an

L^1-function and is continuous for $\beta > n$. For a general exponent β, $\hat{G}_\beta(\lambda)$, $\lambda \in \mathbb{R}^n$, is identified as a tempered distribution. If f is in the Schwartz class $\mathcal{S}$ of rapidly decreasing C^∞-functions, then $\hat{G}_\beta \hat{f} \in \mathcal{S}$. The pseudo-differential operators $\mathcal{G}_\beta = (I - \Delta)^{-\beta/2}$, $\beta \in \mathbb{R}$, have *Bessel potential* representations

$$f = \mathcal{G}_\beta(g) = G_\beta * g, \qquad \beta \in \mathbb{R},$$

with

$$g = \mathcal{G}_{-\beta}(f) = (I - \Delta)^{\beta/2} f.$$

From the definition of $\mathcal{G}_\beta$,

$$\mathcal{G}_\beta \mathcal{G}_\gamma = \mathcal{G}_{\beta+\gamma} \quad \text{and} \quad \mathcal{G}_\beta^{-1} = \mathcal{G}_{-\beta}.$$

When $\beta > 0$, $G_\beta(x)$ decays exponentially at ∞. At 0, for $0 < \beta < n$, the asymptotics are

$$G_\beta(x) = O(|x|^{-n+\beta}) \qquad \text{as } |x| \to 0$$

and, for $\beta = n$,

$$G_n(x) = O\left(\log \frac{1}{|x|}\right) \qquad \text{as } |x| \to 0$$

[see, e.g., Stein (1970), page 135, or Adams and Hedberg (1996), page 13]. For $\beta \in \mathbb{R}$ and $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, the *space of Bessel potentials* is defined by

$$\mathcal{L}^{\beta,p} = \mathcal{L}^{\beta,p}(\mathbb{R}^n) = \{u : u = \mathcal{G}_\beta(g), g \in L^p(\mathbb{R}^n)\},$$

with norm given by $\|u\|_{\beta,p} = \|G_\beta * g\|_{\beta,p} = \|g\|_p$, where $\|\cdot\|_p$ denotes the L^p-norm [see, e.g., Adams and Hedberg (1996), page 11].

The C_0^∞ is dense in $\mathcal{L}^{\beta,p}$ and the dual space of $\mathcal{L}^{\beta,p}$ is $\mathcal{L}^{-\beta,p'}$, with $pp' = p + p'$ for all $\beta \in \mathbb{R}$ and $1 < p < \infty$. For $\alpha < \beta$, the inclusion $\mathcal{L}^{\alpha,p} \subset \mathcal{L}^{\beta,p}$ holds and $\|u\|_{\alpha,p} \le \|u\|_{\beta,p}$. When $\beta \ge 0$ is an integer, the space $\mathcal{L}^{\beta,p}$ can be identified with the Sobolev space $W^{\beta,p}$ of weakly differentiable functions of order β:

$$W^{\beta,p}(\mathbb{R}^n) \overset{\text{def}}{=} \left\{u \in L^p : \int_{\mathbb{R}^n} \sum_{0 \le |k| \le \beta} |\nabla^k u|^p < \infty\right\}$$

and the norm $\|u\|_{W^{\beta,p}} = (\int_{\mathbb{R}^n} \sum_{0 \le |k| \le \beta} |\nabla^k u|^p)^{1/p}$ is equivalent to the norm of $\mathcal{L}^{\beta,p}$ [Stein (1970), page 135].

The following interpolation result [Triebel (1984), page 69] shows how the Bessel potential spaces extend the chain of Sobolev spaces.

THEOREM 1.1. *If $0 < \theta < 1$ and $\beta = (1 - \theta)\beta_0 + \theta\beta_1$, then the interpolation spaces of exponent θ between the Bessel spaces $\mathcal{L}^{\beta_0,p_0}$ and $\mathcal{L}^{\beta_1,p_1}$ is $\mathcal{L}^{\beta,p}$, where $1/p = (1 - \theta)/p_0 + \theta/p_1$.*

It is shown in Stein (1970), page 136, that a function $u \in \mathcal{L}^{\beta,p}$ if and only if $u \in \mathcal{L}^{\beta-1,p}$ and, for each j, $\partial_j u \in \mathcal{L}^{\beta-1,p}$. Moreover, the norms $\|u\|_{\beta,p}$ and $\|u\|_{\beta-1,p} + \|\nabla u\|_{\beta-1,p}$ are equivalent. When $\beta p > n$, the functions in $\mathcal{L}^{\beta,p}(\mathbb{R}^n)$ are continuous but not for $\beta p \leq n$ [Adams and Hedberg (1996), Chapter 6]. A useful way of measuring the deviation from continuity is given by the (β, p)-*capacity* [see Adams and Hedberg (1996), Chapter 2]. The capacity of a set $S \subseteq \mathbb{R}^n$ can be defined as

$$C_{\beta,p}(S) = \inf\{\|g\|_{L^p}^p : g \geq 0,\ G_\beta * g \geq 1 \text{ on } S\}. \tag{1.1}$$

A property of points will be said to hold (β, p)-*quasieverywhere* [(β, p)-q.e.] if it holds for all points except those belonging to a set of (β, p)-capacity 0. A function u, defined (β, p)-q.e. on $\mathbb{R}^n$, is called (β, p)-*quasicontinuous* if for each $\varepsilon > 0$ there is an open set D with $C_{\alpha,p}(D) < \varepsilon$ such that f is continuous on $\mathbb{R}^n \setminus D$.

THEOREM 1.2 [Adams and Hedberg (1996), Chapter 6]. *Let $u \in \mathcal{L}^{\beta,p}(\mathbb{R}^n)$ and $\beta p \leq n$. After possible redefinition on a set of measure* 0, *u is (β, p)-quasicontinuous. Moreover, if u and v are two (β, p)-quasicontinuous functions such that $u(x) = v(x)$ a.e., then $u(x) = v(x)$ (β, p)-q.e.*

If $u \in \mathcal{L}^{\beta,p}$, $1 < p \leq n/\beta$, and $S \subset \mathbb{R}^n$ are arbitrary, then the *trace of u on S*, denoted $u|_S$, is the restriction to S of any (β, p)-quasicontinuous representative of u. In particular, $u|_S = 0$ or $\nabla u|_S = 0$ means these statements hold q.e.

THEOREM 1.3 [Stoke (1984)]. *Let $u \in \mathcal{L}^{\beta,p}$, where $\beta \geq 1$ and $1 \leq p < \infty$ are such that $\beta p > n$ but $(\beta - 1)p \leq n$. Then u is differentiable $(\beta - 1, p)$-q.e.*

It is useful to recast the quasicontinuity of functions in $\mathcal{L}^{\beta,p}$ as *fine continuity*. If $1 < p \leq n/\beta$, a set $S \subset \mathbb{R}^n$ is called (β, p)-*thin* at a point $x \in \mathbb{R}^n$ if

$$\int_0^1 \left(\frac{C_{\beta,p}(S \cap B(x,r))}{r^{n-\beta p}}\right)^{p'-1} \frac{dr}{r} < \infty,$$

where $pp' = p + p'$. Otherwise, S is said to be (β, p)-*thick* at x. If $x \in \mathbb{R}^n$, the set $O \subset \mathbb{R}^n$, $x \in O$, is a (β, p)-*fine neighborhood* of x if the set O^c is (β, p)-*thin* at x. A set $S \subset \mathbb{R}^n$ is (β, p)-*finely open* if it is a (β, p)-fine neighborhood of all of its points. A function u, defined for $x \in S \subset \mathbb{R}^n$, is (β, p)-*finely continuous* at x if the set $\{y \in S : |f(y) - f(x)| \geq \varepsilon\}$ is (β, p)-*thin* at x for all $\varepsilon > 0$.

THEOREM 1.4 [Adams and Hedberg (1996), page 177]. *A function f is (β, p)-quasicontinuous iff f is (β, p)-finely continuous (β, p)-q.e.*

For a set $S \subset \mathbb{R}^n$, denote by $\mathcal{L}_{00}^{\beta,p}(S)$ the closure in $\mathcal{L}^{\beta,p}(\mathbb{R}^n)$ of the functions $u \in \mathcal{L}^{\beta,p}$ with compact support contained in S. If S is open, then $\mathcal{L}_{00}^{\beta,p}(S)$ is the closure in $\mathcal{L}^{\beta,p}(\mathbb{R}^n)$ of $C_0^\infty(S)$. The following result is due to Hedberg (1981) for integer order Bessel spaces (Sobolev spaces) and generalized by Netrusov [Adams and Hedberg (1996), page 281] for $\beta > 0$.

THEOREM 1.5. *Let $\beta > 0$, $1 < p < \infty$, $u \in \mathcal{L}^{\beta,p}(\mathbb{R}^n)$ and $S \subset \mathbb{R}^n$ be arbitrary. Then the following statements are equivalent*:

(i) $D^\kappa u|_S = 0$ *for all multiindices* κ, $0 \le |\kappa| < \beta$;
(ii) $u \in \mathcal{L}_{00}^{\beta,p}(S^c)$;

REMARKS. (1) Only the $p = 2$ case occurs in what follows. For $n - 2\beta > 0$, the capacity $C_{\beta,2}$ defined by (1.1) is equivalent to the capacity defined through the power kernel $k(x) = 1/|x|^{n-2\beta}$ from classical potential theory [Adams and Hedberg (1996), Chapter 5]. More precisely, if $k(x) \ge 0$ is a decreasing continuous extended real-valued function on $[0, \infty]$, the k-energy of a positive measure μ is $\mathcal{E}(k, \mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k(|x - y|)\, d\mu(x)\, d\mu(y)$. Then, for compact sets $A \subseteq \mathbb{R}^n$, define the k-capacity of A as

$$k - C(A) = \sup_{\mu} \left\{ \frac{1}{\mathcal{E}(k, \mu)} : \operatorname{supp} \mu \subseteq A,\ \mu(A) = 1 \right\}, \tag{1.2}$$

and, for an arbitrary $S \subseteq \mathbb{R}^n$,

$$k - C(S) = \sup_{A} \{ k - C(A) : A \text{ is compact },\ A \subseteq S \}. \tag{1.3}$$

When $n - 2\beta = 0$, the capacity (1.1) is equivalent to the classical logarithmic capacity $C_{\log}(S)$.

(2) If Λ_t is the t-dimensional Hausdorff measure, $\Lambda_t(S) < \infty$ implies $k - C(S) = 0$ for $k(x) = 1/|x|^t$ [Falconer (1985), Theorem 6.4], and therefore $\Lambda_{n-2\beta}(S) < \infty$ implies $C_{\beta,2}(S) = 0$.

(3) When $\beta \le 1/2$ and S is a line segment, the energy integral for the power kernel $k(x) = 1/|x|^{n-2\beta}$ diverges, and therefore $C_{\beta,2}(S) = 0$. For $\beta > 1/2$, the line segments have positive $C_{\beta,2}$-capacity.

(4) When $p = 2$ and $0 < \beta < 2$, the spaces $\mathcal{L}^{\beta,2}$ are characterized in terms of the L^2-modulus of continuity and the L^2-modulus of smoothness,

$$\omega(t) = \left(\int_{\mathbb{R}^n} |u(x + t) - u(x)|^2\, dx \right)^{1/2}$$

and

$$\tilde{\omega}(t) = \left(\int_{\mathbb{R}^n} |u(x + t) + u(x - t) - 2u(x)|^2\, dx \right)^{1/2}.$$

THEOREM 1.6 [See, e.g., Stein (1970), page 141]. *Let* $u \in L^2(\mathbb{R}^n)$.

(i) *Suppose* $0 < \beta < 1$. *Then* $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^n)$ *if and only if*

$$\int_{\mathbb{R}^n} \frac{(\omega(t))^2}{|t|^{n+2\beta}}\, dt < \infty.$$

(ii) *Suppose* $0 < \beta < 2$. *Then* $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^n)$ *if and only if*

$$\int_{\mathbb{R}^n} \frac{(\tilde{\omega}(t))^2}{|t|^{n+2\beta}}\, dt < \infty.$$

Theorem 1.6 extends to Bessel spaces of general indices $\gamma > 0$: if $\gamma = m + \beta$, m-integer, and $0 < \beta < 1$ or $0 < \beta < 2$, then $u \in \mathcal{L}^{\gamma,2}$ iff $u \in \mathcal{L}^{m,2}$ and the proper integrability condition (i) or (ii) is satisfied.

2. Gaussian random fields. Here we show that for a stochastically continuous real-valued Gaussian random field Φ the equality $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ is equivalent to an approximation condition in the reproducing kernel Hilbert space associated with Φ (Theorem 2.2).

Let $\Phi = \{\phi(x) : x \in \mathbb{R}^2\}$ be a real-valued mean-zero Gaussian random field defined over a complete probability space (Ω, Σ, P) and let $\mathcal{F}(\Phi, S)$ and $\overline{\mathcal{F}}(\Phi, S)$ be the sharp and the germ σ-fields of Φ for a set S, as defined in the Introduction. Associated with these σ-fields, as described in the Introduction, are two subspaces of the Hilbert space $L^2(P) = L^2(\Omega, \Sigma, P)$:

1. $H(\Phi, S) \stackrel{\text{def}}{=} \overline{\text{sp}}\{\phi(x), x \in S\}_{L^2}$—the closed linear subspace of $L^2(P)$ obtained as the closed linear span of $\{\phi(x), x \in S\}$ in $L^2(P)$;
2. $\overline{H}(\Phi, S) \stackrel{\text{def}}{=} \bigcap H(\Phi, O)$, with the intersection taken over all neighborhoods O of $\overline{S}$.

The assumption that Φ is Gaussian implies [see, e.g., Rozanov (1982), page 41] that

$$\mathcal{F}(\Phi, S) = \sigma(H(\Phi, S)), \qquad \overline{\mathcal{F}}(\Phi, S) = \sigma\big(\overline{H}(\Phi, S)\big),$$

and thus

$$\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S) \qquad \text{if and only if } H(\Phi, S) = \overline{H}(\Phi, S). \tag{2.1}$$

We rephrase this in terms of the reproducing kernel Hilbert space associated with Φ as follows.

Let $\mathcal{H}(\Phi, S)$ be the space of functions on $\mathbb{R}^2$ given by

$$\mathcal{H}(\Phi, S) = \big\{u(x) \stackrel{\text{def}}{=} EX\phi(x) : X \in H(\Phi, S)\big\},$$

with the inner product

$$\langle u_1, u_2\rangle_{\mathcal{H}} = \langle u_1, u_2\rangle = EX_1X_2,$$

where $u_1(x) = EX_1\phi(x)$ and $u_2(x) = EX_2\phi(x)$.

It is clear that each function $\rho(x,\cdot)$, $x \in S$, determined by the correspondence $y \mapsto \rho(x, y)$, belongs to $\mathcal{H}(\Phi, S)$ and that:

(i) the map $J : X \mapsto EX\phi(x)$ determines an isometry between $H(\Phi, S)$ and $\mathcal{H}(\Phi, S)$;

(ii) $\mathcal{H}(\Phi, S)$ is spanned by the functions $\{\rho(x,\cdot), x \in S\}$;

(iii) for each $u \in \mathcal{H}(\Phi, S)$, the reproducing property $u(x) = \langle u, \rho(x,\cdot)\rangle$ holds.

Here $\mathcal{H}(\Phi) = \mathcal{H}(\Phi, \mathbb{R}^2)$ is the *reproducing kernel Hilbert space* of Φ and ρ is the *reproducing kernel* of $\mathcal{H}(\Phi)$.

Setting

$$\overline{\mathcal{H}}(\Phi, S) = \bigcap_{O \supset \bar{S}} \mathcal{H}(\Phi, O),$$

the isometry (i), maps the spaces $H(\Phi, S)$ and $\overline{H}(\Phi, S)$ of random variables onto the function subspaces $\mathcal{H}(\Phi, S)$ and $\overline{\mathcal{H}}(\Phi, S)$ of $\mathcal{H}(\Phi)$, and from (2.1) it follows that

$$\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S) \quad \text{if and only if } \mathcal{H}(\Phi, S) = \overline{\mathcal{H}}(\Phi, S). \tag{2.2}$$

The following elementary result identifies the orthogonal complement of $\mathcal{H}(\Phi, S)$ in $\mathcal{H}(\Phi)$ and is fundamental for our discussion.

PROPOSITION 2.1. *For $u \in \mathcal{H}(\Phi)$, $u \in \mathcal{H}(\Phi, S)^{\perp}$ iff $u(x) = 0$ holds for all $x \in S$.*

PROOF. Observe that $u \perp \mathcal{H}(\Phi, S)$ iff $\langle u, \rho(x,\cdot)\rangle = 0$ holds for all $x \in S$. But the reproducing property (iii) of ρ gives $u(x) = \langle u, \rho(x,\cdot)\rangle$ and the result follows. □

This result prompts the following notation. Let

$$\mathcal{H}_0(\Phi, S) \stackrel{\text{def}}{=} \{u \in \mathcal{H}(\Phi) : u(x) = 0 \text{ for } x \in S\} \tag{2.3}$$

and

$$\mathcal{H}_{00}(\Phi, S) \stackrel{\text{def}}{=} \overline{\cup \mathcal{H}_0(\Phi, O)}, \tag{2.4}$$

where the union is over all neighborhoods O of $\bar{S}$ and the closure is taken in the norm of $\mathcal{H}(\Phi)$.

Our principal result in this section is the following criterion.

THEOREM 2.2. *For a continuous Gaussian random field Φ and a set S, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ iff $\mathcal{H}_0(\Phi, S) = \mathcal{H}_{00}(\Phi, S)$, that is, iff each function in $\mathcal{H}_0(\Phi, S)$ is a limit in the norm of $\mathcal{H}(\Phi)$ of a sequence of functions that vanish on neighborhoods of $\bar{S}$.*

PROOF. By (2.2), $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ holds iff $\mathcal{H}(\Phi, S) = \overline{\mathcal{H}}(\Phi, S)$ and thus the spaces $\mathcal{H}(\Phi, S)$ and $\overline{\mathcal{H}}(\Phi, S)$ are equal if and only if their orthogonal complements are equal. It only remains to identify the orthogonal complement of $\overline{\mathcal{H}}(\Phi, S)$ as $\mathcal{H}_{00}(\Phi, S)$, which follows directly from the definition of $\overline{\mathcal{H}}(\Phi, S)$ and Proposition 2.1. □

REMARKS. (1) In analogy with the classical results of Beurling (1948), approximation results such as Theorem 2.2 are called *spectral synthesis* theorems. If $\mathcal{H}_0(\Phi, S) = \mathcal{H}_{00}(\Phi, S)$, the set S is said to *admit spectral synthesis* in the function space $\mathcal{H}(\Phi)$.

(2) The geometric conditions on S under which the approximation in Theorem 2.2 is possible depend on the structure of the reproducing kernel Hilbert space. For example, for the Bessel fields of order β, $\mathcal{H}(\Phi) = \mathcal{L}^{\beta,2}(\mathbb{R}^2)$. We identify $\mathcal{H}_{00}(\Phi, S) = \mathcal{L}_{00}^{\beta,2}(S^c)$ (see Section 1 and the comment preceding Theorem 1.5), and therefore the necessary and sufficient conditions for spectral synthesis $\mathcal{H}_0(\Phi, S) = \mathcal{H}_{00}(\Phi, S)$ follow from Theorem 1.5.

3. The Whittle field. We present a detailed analysis of the Whittle field $\Phi = \{\Phi(x), x \in \mathbb{R}^2\}$ of conditions when the sharp and germ σ-fields are equal. Note that Φ satisfies the stochastic differential equation

$$(I - \Delta)\phi(x) = \dot{W}(x), \qquad x \in \mathbb{R}^2. \tag{3.1}$$

Here $\dot{W}$ is a Gaussian white noise with $E\dot{W}(A)\dot{W}(B) = |A \cap B|$. The major results from this section were listed in the Introduction.

The operator $I - \Delta$ has an inverse given by convolution with the Bessel kernel $G_2(x)$, and we can write

$$\phi(x) = \int_{\mathbb{R}^2} G_2(x - y)\dot{W}(y)\,dy, \tag{3.2}$$

and so $E\phi(x)\phi(y) = G_2 * G_2(x - y)$. Taking Fourier transforms on both sides, we obtain the spectral representation of the covariance function

$$\begin{aligned} \rho(x, y) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\lambda}(1 + |\lambda|^2)^{-2}\,d\lambda \\ &= \int_{\mathbb{R}^2} e^{i(x-y)\cdot\lambda}\,\Delta(\lambda)\,d\lambda, \end{aligned} \tag{3.3}$$

where $\Delta(\lambda) = (2\pi(1 + |\lambda|^2))^{-2}$. The family $\{e^{ix\cdot\lambda} : x \in \mathbb{R}^2\}$ spans $L^2(\mathbb{R}^2, \Delta)$ and the functions $u \in \mathcal{H}(\Phi)$ are given by

$$u(x) = \int_{\mathbb{R}^2} e^{ix\cdot\lambda} f(\lambda)\,\Delta(\lambda)\,d\lambda = G_2 * g(x), \tag{3.4}$$

where f satisfies $\int_{\mathbb{R}^2} |f(\lambda)|^2(1 + |\lambda|^2)^{-2}\,d\lambda < \infty$ and $\hat{g}(\lambda) = f(\lambda)(1 + |\lambda|^2)^{-1} \in L^2$. Therefore, we can identify the space $\mathcal{H}(\Phi)$ as $\mathcal{L}^{2,2}$ and $\|u\|_{2,2} = \|g\|_2$.

Theorems 2.2 and 1.5 lead to the following criterion.

THEOREM 3.1. *For the Whittle field Φ and a set $S \subseteq \mathbb{R}^2$, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ iff $u \in \mathcal{L}^{2,2}$ and $u|_S = 0$ implies $\nabla u|_S = 0$ except on a set of logarithmic capacity* 0.

We study separately the cases of general closed sets, general open sets and graphs of curves in $\mathbb{R}^2$.

3.1. *The sharp and germ fields for closed sets.* A corollary of Theorem 3.1 is that $C_{\log}(S) = 0$ implies $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$; that is, if the set S is small enough, the sharp and the germ σ-fields are equal. However, $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ can also occur for large sets.

THEOREM 3.2. *Let S be a closed set and let $\mathcal{T}(S)$ be the set of all $x \in S$ for which S has a tangent line. Then $C_{\log}(\mathcal{T}(S)) = 0$ implies $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$.*

That this result is not exact is shown in Theorem 3.7.

Theorem 3.2 appeared in Pitt and Robeva (1994), but the proof is short and we include it here for completeness. We begin with a lemma.

LEMMA 3.3. *Let $S \subseteq \mathbb{R}^2$ be closed and $x_0 \in S \setminus \mathcal{T}(S)$. Let u be differentiable at x_0 and vanishing on S. Then $\nabla u(x_0) = 0$.*

PROOF. Since u is differentiable at x_0,

$$u(y) - u(x_0) = (y - x_0) \cdot \nabla u(x_0) + o(|y - x_0|), \qquad \text{as } |y - x_0| \to 0.$$

Thus, for each unit vector v which is a cluster point of the vectors $(y - x_0)/|y - x_0|$, $y \in S$, $y \to x_0$, will imply $v \cdot \nabla u(x_0) = 0$. For $x_0 \notin \mathcal{T}(S)$, there are two linearly independent vectors of this type, and hence $\nabla u(x_0) = 0$. □

PROOF OF THEOREM 3.2. According to Theorem 1.3, for each function $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$, there exists a set E with $C_{\log}(E) = 0$ and such that u is differentiable at all $x \in E^c$. By Lemma 3.3, we have that $\nabla u(x) = 0$ for each $x \in S \setminus (E \cup \mathcal{T}(S))$. Since $C_{\log}(E \cup \mathcal{T}(S)) = 0$, Theorem 3.1 implies the result. □

Theorem 3.2 implies $\mathcal{F}(\Phi, \Gamma) = \overline{\mathcal{F}}(\Phi, \Gamma)$ for all nowhere differentiable curves $\Gamma \subset \mathbb{R}^2$, for example, the von Koch snowflake curve. Other examples of closed sets $S \in \mathbb{R}^2$ with $\mathcal{F}(\Phi, S) = \overline{\mathcal{F}}(\Phi, S)$ are two-dimensional Cantor sets and the Sierpinski gasket and carpet.

3.2. *The sharp and germ fields for open sets.* We consider open sets $D \subseteq \mathbb{R}^2$ and seek conditions that imply $\mathcal{F}(\Phi, D) = \overline{\mathcal{F}}(\Phi, D)$. Our positive result is as follows.

THEOREM 3.4. *$\mathcal{F}_D(\Phi, D) = \overline{\mathcal{F}}_D(\Phi, D)$ holds for any bounded connected open set D.*

PROOF. By Theorem 3.1, it suffices to show that each u in $\mathcal{L}^{2,2}(\mathbb{R}^2)$, with $u|_D = 0$, satisfies $\nabla u(x) = 0$ on $\overline{D}$ except for a set of points x of logarithmic capacity 0. But D is open, so $\nabla u(x) = 0$ on D, and, off a set of logarithmic capacity 0, $\nabla u(x)$ is $(2, 2)$-finely continuous (see Theorem 1.4). Thus, it suffices to show that D is thick at each point in ∂D. The Beurling criterion for regular points of the Dirichlet problem [see, e.g., Tsuji (1975), page 105] implies that D is thick at each point x in the boundary of D and the proof is complete. □

The next example gives an open set D for which $\mathcal{F}_D(\Phi, D) \neq \overline{\mathcal{F}}_D(\Phi, D)$. This is related to Example 3.5 in Dalang and Walsh (1992) for the Brownian sheet B and is given, in a different context, by Hedberg (1980).

EXAMPLE 3.5. *There exists a bounded open set $D \subset \mathbb{R}^2$ for which $\mathcal{F}(\Phi, D) \neq \overline{\mathcal{F}}(\Phi, D)$.*

OUTLINE OF PROOF. Added details may be found in Hedberg (1980). Using Theorem 1.5, it suffices to construct an open set D and a function $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$ such that u vanishes identically on D but $\nabla u \neq 0$ on a subset of ∂D of positive logarithmic capacity. To construct the set D, let $\{\bar{B}(a_n, R_n)\}_{n=1}^{\infty}$ be a sequence of closed disjoint disks in $\mathbb{R}^2$ with centers a_n in $[-1/2, 1/2] \times \{0\}$, such that $\bigcup_{n=1}^{\infty} \bar{B}(a_n, R_n)$ is dense in $[-1/2, 1/2] \times \{0\}$. Choose R_n so that $\sum_{n=1}^{\infty} R_n < 1/2$. Further, let $0 < r_n < R_n$, $n \geq 1$, be such that

$$\sum_{n=1}^{\infty} \left(\log \frac{R_n}{r_n} \right)^{-1} < \infty.$$

Set $D = \bigcup_{n=1}^{\infty} B(a_n, r_n)$. Notice that every $x \in [-1/2, 1/2] \times \{0\} \setminus (\bigcup_{n=1}^{\infty} \bar{B}(a_n, R_n))$ belongs to ∂D and since $\sum_{n=1}^{\infty} R_n < 1/2$, the set of such points has positive one-dimensional Lebesgue measure and thus positive logarithmic capacity.

The desired function $u \in \mathcal{L}^{2,2}$ is constructed by starting with a sequence of decreasing functions $\{v_n(r)\}_{n=1}^{\infty}$, with $v_n \in C_0^{\infty}[0, \infty)$, and $v_n(r) = 1$ for $r \leq r_n$ and $v_n(r) = 0$ for $r \geq R_n$, and satisfying

$$|v_n'(r)| \leq \frac{c}{r} \left(\log \frac{R_n}{r_n} \right)^{-1}, \qquad |v_n''(r)| \leq \frac{c}{r^2} \left(\log \frac{R_n}{r_n} \right)^{-1},$$

where c is a constant.

Let $g(x) \in C_0^\infty(\mathbb{R}^1)$ be such that $g(x) = g(x_1, x_2) = x_2$ in a neighborhood of $[-1/2, 1/2] \times \{0\}$ and set $u(x) = g(x)(1 - \sum_{n=1}^\infty v_n(|x - a_n|))$. One can show that $u \in \mathcal{L}^{2,2}$, and it is immediate that $u|_D = 0$. On the other hand, if $(x_1, 0) \in [-1/2, 1/2] \times \{0\} \setminus (\bigcup_{n=1}^\infty \bar{B}(a_n, R_n))$ is a point on ∂D, then for $x = (x_1, x_2)$, $u(x) = g(x) = x_2$ for small x_2. Thus, $\partial_2 u = 1$ on a set of positive logarithmic capacity. □

3.3. *The sharp and germ fields for smooth curves.* In this section, we prove the following result.

THEOREM 3.6. *If* $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ *is a smooth curve for which* $\gamma'(s) \neq 0$ *is Hölder continuous with exponent* $\alpha > 1/2$, *then* $\mathcal{F}(\Phi, \Gamma) \neq \overline{\mathcal{F}}(\Phi, \Gamma)$.

That Theorem 3.6 is best possible is shown by the next theorem.

THEOREM 3.7. *For each* $\alpha < 1/2$, *there exists a smooth curve* $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ *with* $\gamma'(s)$ *being Hölder continuous with exponent* α, *and* $\mathcal{F}(\Phi, \Gamma) = \overline{\mathcal{F}}(\Phi, \Gamma)$.

PROOF OF THEOREM 3.6. We need to build a function $u \in \mathcal{L}^{2,2}$ such that $u|_\Gamma = 0$ but $\nabla u|_\Gamma \neq 0$ on a set of positive logarithmic capacity.

If $\Gamma = \{(s, t) : s = c\}$ is a vertical line, the result is obvious. In all other cases, it is possible to find a smooth function $t = \tilde{\gamma}(s)$, $s \in \mathbb{R}$, with $\|\tilde{\gamma}\|_\infty + \|\tilde{\gamma}'\|_\infty < \infty$ and such that $\tilde{\gamma}'(s)$ is Hölder continuous of order α and Γ intersects the graph $\tilde{\Gamma}$ of $\tilde{\gamma}(s)$ on a simple arc L of positive length. We will construct a function v that is locally in $\mathcal{L}^{2,2}$ so that $v|_{\tilde{\Gamma}} = 0$ but $\nabla v|_{\tilde{\Gamma}} \neq 0$. Then, for sufficiently small $r > 0$, choosing a ball $B(a, 2r) \subset L$ centered at $a \in L$ and a smooth $\phi \in C_0^\infty$ with $\phi(s, t) = 1$ on $B(a, r)$ and $\phi(s, t) = 0$ on $\mathbb{R}^n \setminus B(a, 2r)$, setting $u = v\phi$ will complete the proof. To construct the function v, define

$$w(s, t) = \begin{cases} P_t * \tilde{\gamma}(s), & \text{when } t \geq 0, \\ 3P_{-t} * \tilde{\gamma}(s) - 2P_{-2t} * \tilde{\gamma}(s), & \text{when } t < 0, \end{cases}$$

where P_t is the Poisson kernel.

The function $w(s, t)$ is a C^1-function with bounded derivatives and $w(s, 0) = \tilde{\gamma}(s)$. The rate of increase of the higher order derivatives of w is given by

$$\|D^\kappa w\|_\infty \leq c \frac{1}{t^{m-1-\alpha}} \qquad \text{for } |\kappa| = m \text{ as } |t| \to 0 \tag{3.5}$$

[see Stein (1970), page 62]. Consider now the change of variables $k : \mathbb{R}^2 \to \mathbb{R}^2$, $(s, t) \mapsto (x, y)$, given by

$$x = s, \qquad y = ct + w(s, t), \tag{3.6}$$

where $c > 0$ is chosen so that $\partial y/\partial t > 0$ for all s and t. Thus, k takes the upper half plane $\mathbb{R}_+^2$ onto the domain above the graph of the curve $t = \tilde{\gamma}(s)$.

Further, since the Jacobian of k is given by

$$J = \begin{pmatrix} 1 & 0 \\ \partial_1 w & c + \partial_2 w \end{pmatrix}, \tag{3.7}$$

$|J| = c + \partial_2 w > 0$.

Set $h(x, y) = (s, t)$ to be the inverse of k and define a function v as the t-coordinate of h, $t(x, y)$. It is now clear that $v|_{\tilde{\Gamma}} = 0$ and since $\partial t/\partial y = 1/(c + \partial_2 w) > 0$, $\nabla v|_{\tilde{\Gamma}} \neq 0$. Thus, we only have to check whether the second-order derivatives of v are locally in L^2. But since the Jacobian of k is bounded from above and below away from 0, a direct calculation shows that $D^\kappa v$, $|\kappa| = 2$, are locally square integrable if and only if $D^\kappa w$, $|\kappa| = 2$, are locally in L^2. Thus, the estimates (3.5), and the requirement $\alpha > 1/2$, imply that

$$\int_B |D^\kappa w(s, t)|^2 \, ds \, dt \le c \int_B \left| \frac{1}{t^{1-\alpha}} \right|^2 ds \, dt < \infty \qquad \text{for } |\kappa| = 2.$$

Thus, v is locally in $\mathcal{L}^{2,2}$ and the proof of Theorem 3.6 is complete. □

PROOF OF THEOREM 3.7. For a given $\alpha < 1/2$, we will construct a curve $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ with $\gamma'(s)$ of Hölder class C^α such that, for each $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$, $u|_\Gamma = 0$ implies $\nabla u|_\Gamma = 0$. The curve Γ will be the graph of the function $g(x)$ given by the lacunary series

$$g(x) = \sum_{k=0}^{\infty} a_k \bigl(1 - \cos(2\pi n_k x)\bigr), \qquad x \in [0, 1], \tag{3.8}$$

with $n_k = 2^{k!}$ and $a_k = (n_k)^{-(1+\alpha)}$. That $\gamma' \in C^{0,\alpha}$ follows from the Weierstrass–Hardy theorem [Zygmund (1959), page 48].

Since ∇u is (1, 2)-quasicontinuous, to prove that $\nabla u|_\Gamma = 0$, it is enough to show that $\partial_1 u = 0$ and $\partial_2 u = 0$ a.e. on Γ. But since $u|_\Gamma = 0$, the function $F(x) = u(x, g(x))$ vanishes on $[0, 1]$ and $\partial_1 u + g' \partial_2 u = 0$ almost everywhere. Thus, it suffices to show that

$$\int_\Gamma f(x) \, \partial_2 u(x, \tilde{y}) \, dx = 0 \tag{3.9}$$

holds for each simple function f on $[0, 1]$; here $\tilde{y} = g(x)$.

We make three approximations.

1. The integral in (3.9) is approximated by line integrals $\int_{\Gamma_N} f(x) \, \partial_2 u(x, y) \, dx$, where Γ_N are the graphs of the partial sums $y = g_N(x)$ of the lacunary series (3.8).
2. The integral $\int_{\Gamma_N} f(x) \, \partial_2 u(x, y) \, dx$ is approximated by integrals of the form $h^{-1} \int_{\Gamma_N} f(x)[u(x, y + h) - u(x, y)] \, dx$.

3. A numerical approximation of the integrals

$$h^{-1}\int_{\Gamma_N} f(x)u(x,y)\,dx \quad \text{and} \quad h^{-1}\int_{\Gamma_N} f(x)u(x,y+h)\,dx$$

is invoked to show that, for special values of h, these integrals are $o(h)$.

Together these steps imply (3.9). The steps are listed as a sequence of lemmas.

LEMMA 3.8. *Let $g_N(x)$ be the Nth partial sum of* (3.8)

$$g_N(x) = \sum_{k=0}^{N} a_k\big(1-\cos(2\pi 2^{k!}x)\big).$$

Then, for each integer $N \geq 0$, we have

(i) $g_N(x) + 2a_{N+1} = g_{N+1}(x)$ *at the points* $x = (j-\frac{1}{2})/n_{N+1}$, $j = 1, 2, \dots, n_{N+1}$;

(ii) $g_N(x) = g_{N+1}(x)$ *at the points* $x = j/n_{N+1}$, $j = 0, 1, \dots, n_{N+1}$;

(iii) *the points* $\{(j/n_{N+1}, g_N(j/n_{N+1})), j = 0, 1, \dots, n_{N+1}\}$ *lie on the graph* Γ *of the limit function* $g(x) = \lim_{N\to\infty} g_N(x)$.

PROOF. Verification is routine. □

LEMMA 3.9. *Let Γ_N be the graph of $y = g_N(x)$. Given $\varepsilon > 0$ and a constant $c > 0$, there exists an N_0 such that, for any simple function f and for any $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$ with $\|u\|_{2,2} \leq 1$,*

$$\left|\int_{\Gamma_N} f(x)\,\partial_2 u(x,y)\,dx - \frac{1}{2a_{N+1}}\int_{\Gamma_N} f(x)[u(x,y+2a_{N+1}) - u(x,y)]\,dx\right| < \varepsilon\|f\|_2$$

for all $N > N_0$.

PROOF. The proof splits into two steps.

Step 1. For a simple function f, define the linear functionals $\tau_h(u)$ on $\mathcal{L}^{2,2}(\mathbb{R}^2)$ by

$$\tau_h(u) = \int_0^1 f(x)\,\partial_2 u(x,0)\,dx - \frac{1}{h}\int_0^1 f(x)[u(x,h) - u(x,0)]\,dx \tag{3.10}$$

on $\mathcal{L}^{2,2}$. Then the functional norm of τ_h satisfies $\|\tau_h\|_{-2,2} = O(\sqrt{h})$ for $h \to 0$.

To see this, write $u(x,y) = \int_{\mathbb{R}^2} e^{i(\lambda x+\mu y)} g(\lambda,\mu)\hat{G}_2(\lambda,\mu)\,d\lambda\,d\mu$, where $g \in L^2(\mathbb{R}^2)$. Then

$$\tau_h(u) = \iint g(\lambda,\mu)\left[i\mu - \frac{e^{i\mu h}-1}{h}\right]\hat{G}_2(\lambda,\mu)\hat{f}(\lambda)\,d\lambda\,d\mu$$

and, by Schwartz's inequality,

$$\|\tau_h\|_{-2,2}^2 \le \iint |\hat{f}(\lambda)|^2 \left[i\mu - \frac{e^{i\mu h} - 1}{h} \right]^2 \Delta(\lambda, \mu)\, d\lambda\, d\mu,$$

where $\Delta(\lambda, \mu) = |\hat{G}_2(\lambda, \mu)|^2 = 1/(1 + \lambda^2 + \mu^2)^2$ is the spectral density of Φ. Writing this integral as a sum $I_1 + I_2$ with

$$I_1 = \iint_{|\mu|h<1} |\hat{f}(\lambda)|^2 \left[i\mu - \frac{e^{i\mu h} - 1}{h} \right]^2 \Delta(\lambda, \mu)\, d\lambda\, d\mu,$$

we have

$$I_1 = O\left(\iint_{|\mu|h<1} |\hat{f}(\lambda)|^2 h^2 \mu^4\, \Delta(\lambda, \mu)\, d\lambda\, d\mu \right) = O(h).$$

Similarly, with

$$I_2 = \iint_{|\mu|h>1} |\hat{f}(\lambda)|^2 \left[i\mu - \frac{e^{i\mu h} - 1}{h} \right]^2 \Delta(\lambda, \mu)\, d\lambda\, d\mu,$$

we have

$$I_2 = O\left(\iint_{|\mu|h>1} |\hat{f}(\lambda)|^2 \mu^2\, \Delta(\lambda, \mu)\, d\lambda\, d\mu \right) = O(h),$$

and thus

$$\|\tau_h\|_{-2,2} = O(\sqrt{h}) \qquad \text{for } h \to 0, \tag{3.11}$$

as advertised.

Step 2. We define a second functional on $\mathcal{L}^{2,2}(\mathbb{R}^2)$ by the formula

$$\begin{aligned} T_N(u) = & \int_{\Gamma_N} f(x)\, \partial_2 u(x, y)\, dx \\ & - \frac{1}{2a_{N+1}} \int_{\Gamma_N} f(x)[u(x, y + 2a_{N+1}) - u(x, y)]\, dx, \end{aligned} \tag{3.12}$$

and show that $\|T_N\|_{-2,2} \to 0$ when $N \to \infty$.

The expression (3.12) can be rewritten in the form of (3.10) if we change variables with $\omega : \mathbb{R}^2 \to \mathbb{R}^2$, given by $(s, t) \mapsto (x, y)$, with $x = s$, $y = t - g_N(s)$. Next, define the composition operator $W(f) = f \circ \omega$. Then $T_N = \tau_h \circ \omega$ with $h = 2a_{N+1}$. Thus, $\|T_N\|_{-2,2} \le \|\tau_h\|_{-2,2} \|W\|_{-2,2}$. We can estimate the norm $\|W\|_{-2,2}$ of the functional W by estimating the $\mathcal{L}^{2,2}$-norm of $u(x, y + g_N(x))$. But

$$\|u(x, y + g_N(x))\|_{2,2} = \left(\sum_{|\kappa| \le 2} \sum_{m=0}^{2} \|\nabla^m u(x, y + g_N(x))\|_2^2 \right)^{1/2},$$

and direct calculations give

$$\iint \left|u\big(x, y + g_N(x)\big)\right|^2 dx\,dy = \|g_N\|_\infty^2 \|u\|_{2,2}^2,$$

$$\iint \left|\nabla u\big(x, y + g_N(x)\big)\right|^2 dx\,dy \le c\big(1 + \|g_N'\|_\infty^2\big)\|u\|_{2,2}^2,$$

$$\iint \left|\nabla^2 u\big(x, y + g_N(x)\big)\right|^2 dx\,dy \le c\big(1 + \|g_N'\|_\infty^2 + \|g_N''\|_\infty^2\big)\|u\|_{2,2}^2.$$

Thus, $\|W\|_{-2,2} \le c\|g_N''\|_\infty \le c4\pi^2 N(n_N)^{1/2+\delta}$, and since $\|\tau_h\|_{-2,2} = O(\sqrt{a_{N+1}})$ for $h = 2a_{N+1}$, we can estimate the norm of T_N as

$$\|T_N\|_{-2,2} \le cN(n_N)^{1/2+\delta}\sqrt{a_{N+1}} \le c\frac{N(n_N)^{1/2+\delta}}{(n_{N+1})^{3/4-\delta/2}} \to 0 \qquad \text{as } N \to \infty. \quad \square$$

The next lemma [Theorem 2.3 in Pitt, Robeva and Wang (1995)] bounds the error in the midpoint approximation for integrals of the form $\int_0^1 f(x)u(x,0)\,dx$, where f is a smooth function on [0, 1] and $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$.

LEMMA 3.10. *For every smooth function f on* [0, 1],

$$\sup_{\|u\|_{2,2}\le 1}\left|\int_0^1 f(x)u(x,0)\,dx - \frac{1}{M}\sum_{j=0}^{M} f\left(\frac{j+\frac{1}{2}}{M}\right)u\left(\frac{j+\frac{1}{2}}{M}, 0\right)\right| = O(M^{-3/2}).$$

The proof of Theorem 3.7 is completed by estimating (3.9).

Let f be a simple function on (0, 1). It is elementary to show that, for each $u \in \mathcal{L}^{2,2}$,

$$\lim_{N\to\infty}\int_{\Gamma_N} f(x)\,\partial_2 u(x,y)\,dx = \int_\Gamma f(x)\,\partial_2 u(x,\tilde{y})\,dx$$

[where, as before, $\tilde{y} = g(x)$] and hence it suffices to show that

$$\lim_{N\to\infty}\int_{\Gamma_N} f(x)\,\partial_2 u(x,y)\,dx = 0 \tag{3.13}$$

for each $u \in \mathcal{L}^{2,2}$ with $u|_\Gamma = 0$. Fixing $\varepsilon > 0$, by Lemma 3.9 there exists an $N_0 = N_0(\varepsilon)$ such that, for all $N > N_0$,

$$\left|\int_{\Gamma_N} f(x)\,\partial_2 u(x,y)\,dx - \frac{1}{2a_{N+1}}\int_{\Gamma_N} f(x)[u(x, y+2a_{N+1}) - u(x,y)]\,dx\right| < \varepsilon. \tag{3.14}$$

Making the change of variables $x = x$, $y = g_N(x)$, we get

$$\int_{\Gamma_N} f(x)u(x,y)\,dx = \int_0^1 f(x)u\big(x, g_N(x)\big)\,dx.$$

This change of variables brings a change in the $\mathscr{L}^{2,2}$-norm of the function u by a factor no greater than $\|g_N''\|_\infty$. Applying Lemma 3.10 with $M = n_{N+1}$ gives

$$
\begin{aligned}
\int_{\Gamma_N} f(x)u(x,y)\,dx &= \int_0^1 f(x)u\big(x, g_N(x)\big)\,dx \\
&= \frac{1}{n_{N+1}} \sum_{k=1}^{n_{N+1}-1} f\left(\frac{k}{n_{N+1}}\right) u\left(\frac{k}{n_{N+1}}, g_N\left(\frac{k}{n_{N+1}}\right)\right) \\
&\quad + O\big(n_{N+1}^{-3/2}\big)\|g_N''\|_\infty.
\end{aligned} \tag{3.15}
$$

Since $u|_\Gamma = 0$, Lemma 3.8(iii) implies that $u(k/n_{N+1}, g_N(k/n_{N+1})) = 0$ for the values of k under consideration. From (3.15), we therefore have

$$
\int_{\Gamma_N} f(x)u(x,y)\,dx = O\big(n_{N+1}^{-3/2}\big)\|g_N''\|_\infty. \tag{3.16}
$$

For the second integral in (3.14), we have

$$
\begin{aligned}
&\int_{\Gamma_N} f(x)u(x, y + 2a_{N+1})\,dx \\
&\quad = \int_0^1 f(x)u(x, g_N(x) + 2a_{N+1})\,dx \\
&\quad = \frac{1}{n_{N+1}} \sum_{k=1}^{n_{N+1}} f\left(\frac{k-\frac{1}{2}}{n_{N+1}}\right) u\left(\frac{k-\frac{1}{2}}{n_{N+1}}, g_{N+1}\left(\frac{k-\frac{1}{2}}{n_{N+1}}\right)\right) \\
&\qquad + O\big(n_{N+1}^{-3/2}\big)\|g_N''\|_\infty.
\end{aligned} \tag{3.17}
$$

Now apply Lemma 3.8 to the effect that $g_{N+1}(x) = g_N(x) + 2a_{N+1}$ at the points $x = (k - \frac{1}{2})/n_{N+1}$, $k = 1, 2, \ldots, n_{N+1}$, and notice that, for these values, the points $(x, g_{N+1}(x))$ belong to the graph of Γ. Thus, u vanishes at these points and (3.17) gives

$$
\int_{\Gamma_N} f(x)u(x, y + 2a_{N+1})\,dx = O\big(n_{N+1}^{-3/2}\big)\|g_N''\|_\infty. \tag{3.18}
$$

Combining (3.14), (3.16) and (3.18) gives

$$
\begin{aligned}
\left|\int_{\Gamma_N} f(x)\,\partial_2 u(x,y)\,dx\right| &\le \varepsilon + \frac{\|g_N''\|_\infty}{2a_{N+1}} O\big(n_{N+1}^{-3/2}\big) \\
&= \varepsilon + \frac{1}{2}\big(n_{N+1}^{-\delta}\big)\|g_N''\|_\infty,
\end{aligned} \tag{3.19}
$$

since $a_{N+1} = (n_{N+1})^{-3/2+\delta}$. Now substituting $\|g_N''\|_\infty \le 4\pi^2 N(n_N)^{1/2+\delta}$ into (3.19) gives

$$
\left|\int_{\Gamma_N} f(x)\,\partial_2 u(x,y)\,dx\right| \le \varepsilon + c\frac{N n_N^{1/2+\delta}}{n_{N+1}^{\delta}} < 2\varepsilon
$$

for sufficiently large N. Therefore, (3.13) follows and we have thus $\int_\Gamma f(x) \times \partial_2 u(x, \tilde{y})\, dx = 0$. The proof is complete. □

REMARKS. (1) It is known that if $\partial\Omega = \Gamma$ is sufficiently smooth, then the ground-state eigenfunction ν for the Laplace equation on Ω is smooth [see, e.g., Agmon (1965), Theorem 14.6] and, in particular, $\nu \in \mathcal{L}^{2,2}(\Omega)$ and $\nu|_\Gamma = 0$ but $\partial\nu/\partial n|_\Gamma < 0$. Therefore, for smooth domains, the eigenfunction $\nu(x, y)$ provides an example of a function in $\mathcal{L}^{2,2}$ that vanishes on $\partial\Omega$ but $\nabla\nu(x, y)|_{\partial\Omega} \neq 0$. A theorem of Kondrat'ev and Eidel'man (1979) gives the following condition on $\partial\Omega$ for this to be the case.

Let $C^{1,\omega}$ be the class of all continuously differentiable functions for which the modulus of continuity of the derivatives does not exceed $\omega(t)$ and let $J_\omega = \int_0 (\omega(t)t^{-1})^2\, dt$.

(i) If $\partial\Omega \in C^{1,\omega}$ and $J_\omega < \infty$, then $\nu \in \mathcal{L}^{2,2}(\Omega)$.

(ii) If ω is such that $J_\omega = \infty$, then there exists a domain Ω with $\partial\Omega \in C^{1,\omega}$ such that $\nu \notin \mathcal{L}^{2,2}(\Omega)$.

(2) Comparing Theorems 3.6 and 3.7, it is clear that the question of whether $\mathcal{F}(\Phi, \Gamma) = \overline{\mathcal{F}}(\Phi, \Gamma)$ or not when $\Gamma = \{\gamma(t) : t \in [0, 1]\}$ with $\gamma'(t)$ being Hölder continuous with exponent $1/2$ remains open. Both proofs appear not to generalize for $\alpha = 1/2$.

3.4. *A closer look at the generators of* $\overline{\mathcal{F}}(\Phi, \Gamma)$. We present a description of $\overline{\mathcal{F}}(\Phi, \Gamma)$ as generated by the random field ϕ and its "normal derivatives" $\partial_n\phi$ on Γ. Since the Whittle field is not classically differentiable, these derivatives require a weak interpretation. Following McKean (1963), Pitt (1971) and Piterbarg (1983), define the generalized normal derivatives $\partial_n\phi$ as

$$\int_\Gamma f(x)\, \partial_n\phi(x)\, d\sigma(x) = \lim_{h\to 0} \frac{1}{h} \int_\Gamma f(x)\big[\phi\big(x + hn(x)\big) - \phi(x)\big]\, d\sigma(x), \tag{3.20}$$

provided that the limit exists as a weak limit in $L^2(P)$. Denote the subspace generated by the normal derivatives:

$$H_n(\Phi, \Gamma) = \overline{\mathrm{sp}}\left\{ \int_\Gamma f(x)\, \partial_n\phi(x)\, d\sigma(x) : f \in C(\Gamma) \right\}_{L^2}.$$

The next result decomposes $\overline{H}(\Phi, \Gamma)$ into the sharp piece $H(\Phi, \Gamma)$ and the normal piece $H_n(\Phi, \Gamma)$ and bounds the degree of dependence between these parts based on the smoothness of Γ.

THEOREM 3.11. *Let* $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ *be a smooth* C^1*-curve with* $\gamma'(s) \neq 0$.

(i) *For any* $f \in C(\Gamma)$, *the quantity* $\int_\Gamma f(x)\, \partial_n\phi(x)\, d\sigma(x)$ *defined by* (3.20) *exists. Moreover*, $H(\Phi, \Gamma) + H_n(\Phi, \Gamma)$ *is dense in* $\overline{H}(\Phi, \Gamma)$;

(ii) *If* $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ *is a smooth simple curve and* $\gamma'(s) \neq 0$ *is Hölder continuous with exponent* $\alpha > 1/2$, *then there is a positive angle between the two spaces* $H(\Phi, \Gamma)$ *and* $H_n(\Phi, \Gamma)$. *Thus, the sum* $H(\Phi, \Gamma) + H_n(\Phi, \Gamma)$ *is closed in* $L^2(P)$ *and*

$$\overline{H}(\Phi, \Gamma) = H(\Phi, \Gamma) + H_n(\Phi, \Gamma). \tag{3.21}$$

(iii) *If* Γ *is straight* (*i.e.*, *a line segment*), *then* $H(\Phi, \Gamma)$ *and* $H_n(\Phi, \Gamma)$ *are orthogonal.*

PROOF. (i) Using the isometry $\phi(x) \mapsto \rho(x, \cdot)$ between $H(\Phi)$ and $\mathcal{H}(\Phi) = \mathcal{L}^{2,2}(\mathbb{R}^2)$, the existence of the weak limit (3.20) is seen to be equivalent to the existence of the limit

$$\lim_{h \to 0} \frac{1}{h} \int_\Gamma f(x)\big[u\big(x + hn(x)\big) - u(x)\big]\, d\sigma(x) \tag{3.22}$$

for each $u \in \mathcal{L}^{2,2}(\mathbb{R}^2)$. But $C_0^\infty(\mathbb{R}^2)$ is dense in $\mathcal{L}^{2,2}(\mathbb{R}^2)$, and the limit (3.22) exists for $u \in C_0^\infty(\mathbb{R}^2)$, so it suffices to show that the linear functionals $T_h(u)$ defined by

$$T_h(u) = \frac{1}{h} \int_\Gamma f(x)\big[u\big(x + hn(x)\big) - u(x)\big]\, d\sigma(x)$$

are uniformly bounded on $\mathcal{L}^{2,2}$.

Write $T_h(u) = \int_\Gamma \nabla u \cdot d\bar{\mu}_h(x)$, where $\bar{\mu}_h(A)$ is the vector measure $\bar{\mu}_h(A) = (1/h) \int_\Gamma \int_0^h n(x) \mathbb{1}_A(x + tn(x))\, dt\, d\sigma(x)$. We will show that

$$\|T_h\|_{-2,2} \le c \iint G_2(x - y)\, d|\mu_h|(x)\, d|\mu_h|(y) \tag{3.23}$$

and prove that the integrals $\iint G_2(x - y)\, d|\mu_h|(x)\, d|\mu_h|(y)$ are uniformly bounded. Since the functions ∂u_i are in $\mathcal{L}^{1,2}$, with $\|\partial u_i\|_{1,2} \le \|u\|_{2,2}$, it will be enough to prove that, for each linear functional defined on C_0^∞ by an integral of the form $l(f) = \int f\, d\mu$, where μ is a positive finite measure with compact support,

$$\|l\|_{-1,2}^2 = \iint G_2(x - y)\, d\mu(x)\, d\mu(y). \tag{3.24}$$

This result is well known, but we sketch the proof for completeness. By standard approximation arguments, it suffices to treat the case when $\mu(dx) = m(x)\, dx$, where $m \in C_0^\infty$. Then

$$l(f) = \int f m(x)\, dx = \int (I - \Delta) f(x) (I - \Delta)^{-1} m(x)\, dx$$

or

$$l(f) = \int (I - \Delta)^{1/2} f(x) (I - \Delta)^{1/2} (I - \Delta)^{-1} m(x)\, dx.$$

Written in this form, we recognize the norm $\|l\|_{-1,2}$ as the $\mathcal{L}^{1,2}$-norm of the function $g(x) = (I - \Delta)^{-1}m(x)$. But $\|g(x)\|_{1,2} = \|(I - \Delta)^{1/2}g\|_2 = \|(I - \Delta)^{-1/2}m\|_2$. Since $(I - \Delta)^{-1/2}$ is the convolution operator with kernel G_1 and since $G_1 * G_1 = G_2$, we see that

$$\|l\|_{-1,2} = \|g\|_{1,2} = \int (I - \Delta)^{-1}m^2(x)\,dx = \iint G_2(x-y)m(x)m(y)\,dx\,dy.$$

Replacing $m(x)\,dx$ with $d\mu(x)$ completes the proof of (3.24).

Now since the hypothesis that $\gamma'(s) \neq 0$ is continuous implies that the family $\{\mu_h\}$ satisfies the uniform Lipschitz condition $|\mu_h|(B(x,r)) \le cr$ for all $h < 1$, all $r > 0$ and all $x \in \mathbb{R}^2$, and since $G_2(x)$, $x \in \mathbb{R}^2$, has a logarithmic singularity at the origin, we may combine (3.23) and this Lipschitz condition with the elementary identity

$$\int_{|x-y|<\delta} \log\frac{1}{|x-y|}\,d\mu(y) = \int_0^\delta \mu\big(B(x,r)\big)\frac{dr}{r} + \mu\big(B(x,\delta)\big)\log\frac{1}{\delta}$$

to complete the proof. Namely, if $E_h = \{\tilde{x} : \tilde{x} = x + tn(x), x \in \Gamma, t \in [0,h]\}$ then, for sufficiently small δ,

$$\begin{aligned}\sup_{x\in E_h}\int_{|y-x|<\delta} d\mu_h(y) &\le \sup_{x\in E_h}\left(\int_0^\delta \mu_h\big(B(x,r)\big)\frac{dr}{r} + \mu_h\big(B(x,\delta)\big)\log\frac{1}{\delta}\right)\\ &\le c\delta + c\delta\log\frac{1}{\delta} \le c,\end{aligned}$$

where c is independent of h.

We show that $H(\Phi,\Gamma)$ and $H_n(\Phi,\Gamma)$ span $\overline{H}(\Phi,\Gamma)$ by checking that each $X \in H(\Phi)$, which is perpendicular to both $H(\Phi,\Gamma)$ and $H_n(\Phi,\Gamma)$, is also perpendicular to $\overline{H}(\Phi,\Gamma)$. Setting $u(x) = EX\phi(x) \in \mathcal{L}^{2,2}$, we note that $X \in H(\Phi,\Gamma)^\perp$ implies $u = 0$ on Γ. Also, for $X \in H_n(\Phi,\Gamma)^\perp$,

$$0 = E\left(\int_\Gamma f(x)\,\partial_n\phi(x)\,d\sigma(x)X\right) = \int_\Gamma f(x)n(x)\cdot\nabla u(x)\,d\sigma(x) \tag{3.25}$$

for each $f \in C(\Gamma)$.

By the Sobolev trace theorem, $\nabla u(x)$ is square integrable on Γ with respect to σ and (3.25) implies that $\nabla u(x) = 0$, σ-a.e. on Γ. Since $\nabla u(x)$ is quasicontinuous, $\nabla u(x) = 0$ q.e. on Γ, and, by Theorem 1.5, $u \in \mathcal{L}^{2,2}_{00}(\mathbb{R}^2 \setminus \Gamma)$. Thus, X is perpendicular to $\overline{H}(\Phi,\Gamma)$ and the proof of (i) is complete.

(ii) The proof is based on a change-of-variables argument similar to that used in the proof of Theorem 3.6 but requiring more precision. We begin with the observation that the cosine of the angle θ between the spaces $H(\Phi,\Gamma)$ and $H_n(\Phi,\Gamma)$ is given by

$$\begin{aligned}\cos(\theta) = \sup\big\{&EXY : X \in H(\Phi,\Gamma), Y \in H_n(\Phi,\Gamma),\\ &\text{and } \|X\|_2 = \|Y\|_2 = 1\big\},\end{aligned} \tag{3.26}$$

and that the assertion $\theta > 0$ is equivalent to:

There exists an $\varepsilon > 0$ *so that* $\|Y - X\|_2 \geq \varepsilon$ *holds for all* $X \in H(\Phi, \Gamma)$ *and*

$$Y \in H_n(\Phi, \Gamma), \text{ provided only that } \|Y\|_2 \geq 1. \tag{3.27}$$

Since $X = \int_\Gamma f(x)\phi(x)\,d\sigma$ and $Y = \int_\Gamma g(x)\,\partial_n\phi(x)\,d\sigma$ are dense in $H(\Phi, \Gamma)$ and $H_n(\Phi, \Gamma)$, respectively, (3.26) will follow if (3.27) holds for X and Y of this form.

We rephrase this in terms of the reproducing kernel Hilbert space $\mathscr{H}(\Phi)$ and the linear isometry between $H(\Phi)$ and $\mathscr{H}(\Phi)$ determined by the correspondence $\phi \mapsto \rho(x, \cdot)$. We interpret the norms of the random variables $\|Y\|_2$ and $\|Y - X\|_2$ as norms of linear functionals

$$\|Y\|_2 = \sup\left\{\int_\Gamma g(x)\,\partial_n u(x)\,d\sigma : u \in \mathscr{H}(\Phi), \|u\|_{2,2} \leq 1\right\},$$

$$\|Y - X\|_2 = \sup\left\{\int_\Gamma \big(g(x)\,\partial_n u(x) - f(x)u(x)\big)\,d\sigma : u \in \mathscr{H}(\Phi), \|u\|_{2,2} \leq 1\right\}.$$

Next, since by using a partition-of-unity argument the general case of a smooth simple curve Γ may be reduced to the case when Γ is the graph of a function $t = \gamma(s)$, $s \in [0, 1]$, of class $C^{1,\alpha}$, (3.27) will follow if we prove the following proposition.

PROPOSITION 3.12. *Let* $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ *be the graph of a smooth* C^1*-function and let* $\gamma'(s) \neq 0$ *be Hölder continuous with exponent* $\alpha > 1/2$. *Let* f *and* g *denote functions in* $C[0, 1]$ *and, for notational convenience, introduce the linear functionals on* $\mathcal{L}^{2,2}(\mathbb{R}^2)$:

$$T_f(u) \stackrel{\text{def}}{=} \int f(x)u(x)\,d\sigma(x),$$

$$N_g(u) \stackrel{\text{def}}{=} \int g(x)\,\partial_n u(x)\,d\sigma(x).$$

There exists a constant $c > 0$ *such that* $\|N_g - T_f\|_{-2,2} \geq c\|N_g\|_{-2,2}$.

PROOF. We again use the change of variables $k : (s, t) \mapsto (x, y)$ from the proof of Theorem 3.6:

$$x = s, \qquad y = ct + w(s, t), \tag{3.28}$$

let $h : (x, y) \mapsto (s, t)$ be its inverse and set $v = u \circ k$. The proof involves three steps. We will show that there are constants $c_1, c_2, c_3, \dots,$ independent of g, so that:

1. Using the change of variables (3.28), we can write $N_g(u) = \tilde{N}_g(v)$, where $\tilde{N}_g(v)$ is a linear functional supported on $[0, 1] \times \{0\} \subseteq \mathbb{R}^2$ and

$$\|\tilde{N}_g\|_{-2,2} \geq c_1\|N_g\|_{-2,2}. \tag{3.29}$$

2. We will decompose $\tilde{N}_g$ into its normal and tangential parts $\tilde{N}_g = N^1 - T^1$ and show that

$$\|T^1\|_{-2,2} \le c_2 \|N^1\|_{-2,2}. \tag{3.30}$$

Thus, $\|\tilde{N}_g\|_{-2,2} \le (c_2+1)\|N^1\|_{-2,2}$ and $\|N^1\|_{-2,2} \ge c_3 \|N_g\|_{-2,2}$, where $c_3 = c_1/(c_2+1)$.

3. The proof will be completed by showing that, for all f,

$$\|N_g - T_f\|_{-2,2} \ge c_4 \|N^1\|_{-2,2}. \tag{3.31}$$

The proof of (3.29) is immediate, for when we write $u = v \circ h$, we have $c^{-1}\|u\|_{2,2} \le \|v\|_{2,2} \le c\|v\|_{2,2}$ and $h(\Gamma) = [0, 1]$.

For (3.31), a direct calculation shows that, for any f, we can write

$$\tilde{N}_g(u) = N^1(v) - T^1(v) - \tilde{T}_f(v),$$

where

$$\begin{aligned} N^1(v) &= \int_0^1 n(s)\,\partial_t v(s,0)\,ds, \\ T^1(v) &= \int_0^1 n(s)m(s)\,\partial_s v(s,0)\,ds, \\ \tilde{T}_f(v) &= \int_0^1 \sqrt{1+(\gamma'(s))^2}\,v(s,0)\,ds \end{aligned} \tag{3.32}$$

and

$$n(s) = g(s)/\partial_t y(s,0), \qquad m(s) = \big(1+\partial_t y(s,0)\big)\big(1+\gamma'(s)\big). \tag{3.33}$$

Introducing the subspaces $\mathcal{L}_{\mathrm{e}}^{2,2}$ and $\mathcal{L}_{\mathrm{o}}^{2,2}$ of $\mathcal{L}^{2,2}$ of functions $v(s,t)$ that are even (respectively odd) in the variable t, we observe that $\mathcal{L}^{2,2} = \mathcal{L}_{\mathrm{e}}^{2,2} \oplus \mathcal{L}_{\mathrm{o}}^{2,2}$ and $\mathcal{L}_{\mathrm{e}}^{2,2} \subseteq \mathrm{Ker}(N^1)$, $\mathcal{L}_{\mathrm{o}}^{2,2} \subseteq \mathrm{Ker}(T^1 + \tilde{T}_f)$. It follows that $\|N^1 - (T^1 + \tilde{T}_f)\|_{-2,2} \ge \|N^1\|_{-2,2}$, thus implying (3.31).

The proof of (3.30) depends on a lemma.

LEMMA 3.13. *The function $m(s) = (1+\partial_t y(s,0))(1+\gamma'(s))$ is in the Hölder class C^α and multiplication by m defines a bounded operator on the Bessel space $\mathcal{L}^{1/2,2}(\mathbb{R}^1)$.*

PROOF. For $t > 0$, $y(s,t) = ct + P_t * \gamma(s)$, where P_t is the Poisson kernel on the upper half plane. Thus,

$$\partial_t y(s,0) = c + \frac{1}{\pi}\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{\gamma(s-s_1)-\gamma(s)}{s_1^2}\,ds_1 = c + \frac{1}{\pi}\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{\gamma'(s_1)}{s_1 - s}\,ds_1,$$

where P.V. denotes the principal-value integral. But this is the Hilbert transform of γ' and the Hilbert transform is a bounded operator on the space C^α [see, e.g., Torchinski (1986), page 214, for the periodic case]. So $s \mapsto \partial_t y(s,0)$ is in $C^{0,\alpha}$, and since $C^{0,\alpha}$ is an algebra, $m(s) \in C^{0,\alpha}$. Multiplication by $m \in C^{0,\alpha}$ with $\alpha > 1/2$ is easily seen to be a bounded operator on $\mathcal{L}^{1/2,2}(\mathbb{R}^1)$ and the proof of the lemma is complete. $\square$

To complete the proof of the proposition, we compute the norms of N^1 and T^1 in terms of the Fourier transforms of $n(s)$ and $n(s)m(s)$. Using (3.32) and writing $v(s,t) = \int_{\mathbb{R}^2} e^{i(s\lambda_1 + t\lambda_2)} V(\lambda)\,\Delta(\lambda)\,d\lambda$, with $\Delta(\lambda) = (2\pi(1+|\lambda|^2))^{-2}$ and $\|v\|_{2,2}^2 = \int_{\mathbb{R}^2} |V(\lambda)|^2\,\Delta(\lambda)\,d\lambda$, we have $N^1(v) = i\int_{\mathbb{R}^2} \hat{n}(\lambda_1)\lambda_2 V(\lambda)\,\Delta(\lambda)\,d\lambda$. By Schwartz's inequality,

$$\|N^1\|_{-2,2}^2 = \int_{\mathbb{R}^2} |\hat{n}(\lambda_1)|^2 \lambda_2^2\,\Delta(\lambda)\,d\lambda$$
$$= c_4^2 \int_{-\infty}^{\infty} |\hat{n}(\lambda_1)|^2 \frac{d\lambda_1}{\sqrt{1+\lambda_1^2}} = c_4^2 \|n\|_{-1/2,2}^2,$$

where $c_4 = 2\int_0^\infty p^2/(1+p^2)^2\,dp < \infty$. Similarly, one can show that

$$\|T^1\|_{-2,2}^2 = c_5^2 \int_{-\infty}^{\infty} |\widehat{nm}(\lambda_1)|^2 \frac{\lambda_1^2}{(1+\lambda_1^2)^{3/2}}\,d\lambda_1, \qquad \text{with } c_5^2 = 2\int_0^\infty \frac{dp}{(1+p^2)^2}.$$

Thus, $\|T^1\|_{-2,2}^2 \le c_5^2 \|nm\|_{-1/2,2}^2$. But multiplication by m defines a bounded linear transformation on $\mathcal{L}^{1/2,2}(\mathbb{R}^1)$ and, by duality, on its dual space $\mathcal{L}^{-1/2,2}(\mathbb{R}^1)$. Therefore,

$$\|T^1\|_{-2,2} \le c_5 \|nm\|_{-1/2,2} \le c_6 \|n\|_{-1/2,2} \le c_7 \|N^1\|_{-2,2},$$

and the proof of (ii) is complete.

(iii) Let Γ be a straight line. Without loss of generality, we may assume that $\Gamma = (-\infty,\infty) \times \{0\} \subset \mathbb{R}^2$. Then the two spaces $H(\Phi,\Gamma)$ and $\bar{H}(\Phi,\Gamma)$ are easily identified in the Fourier picture [Pitt (1973)]. Precisely, the map $\phi(x) \to e^{ix\cdot\lambda}$ determines an isometry from $H(\Phi,\mathbb{R}^2)$ onto $L^2(\mathbb{R}^2,\Delta)$, and the images of the subspaces $H(\Phi,\Gamma)$ and $H_n(\Phi,\Gamma)$ are given by $\{u(\lambda_1,0) : u \in L^2(\mathbb{R}^2,\Delta)$ and $\int_{\mathbb{R}^2} |u(\lambda_1,0)|^2\,\Delta(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2 < \infty\}$ and $\{\lambda_2 v(\lambda_1,0) : v \in L^2(\mathbb{R}^2,\Delta)$ and $\int_{\mathbb{R}^2} |\lambda_2 v(\lambda_1,0)|^2\,\Delta(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2 < \infty\}$, respectively. Thus, since Δ is even, $\langle u(\lambda_1,0), \lambda_2 v(\lambda_1,0)\rangle = 0$ and $H(\Phi,\Gamma)$ and $H_n(\Phi,\Gamma)$ are orthogonal. $\square$

4. The Bessel fields on $\mathbb{R}^2$. In this section, we extend the discussion of the Whittle field given in Section 3 to include the general Bessel fields on $\mathbb{R}^2$, that is, the continuous, stationary, mean-zero Gaussian fields Φ_β satisfying the pseudo-differential equation

$$(I-\Delta)^{\beta/2}\phi(x) = \dot{W}(x), \qquad x \in \mathbb{R}^2, \tag{4.1}$$

for $\beta > 1$.

The general framework remains essentially unchanged, but details and techniques vary depending on the particular range of β. In particular, the results for open sets D on when $\mathcal{F}(\Phi_\beta, D) = \overline{\mathcal{F}}(\Phi_\beta, D)$ are found to depend on β, and we show that this equality always holds for $k < \beta \le k + 1/2$. Also, as β increases, the fields become smoother, and higher order normal derivatives occur in the description of the boundary spaces $\overline{H}(\Phi, \Gamma)$.

The field Φ_β may be represented in the form

$$\phi(x) = \int_{\mathbb{R}^2} G_\beta(x - y)\dot{W}(y)\,dy,$$

which implies the formula $\rho(x, y) = E\phi(x)\phi(y) = G_\beta * G_\beta(x - y) = G_{2\beta}(x - y)$ for the covariance function $\rho(x, y)$ as well as the Fourier representation

$$\rho(x, y) = \int_{\mathbb{R}^2} e^{i(x-y)\cdot\lambda}\,\Delta_\beta(\lambda)\,d\lambda,$$

with $\Delta_\beta(\lambda) = (2\pi)^{-2}(1 + |\lambda|^2)^{-\beta}$. It follows that the reproducing kernel Hilbert space $\mathcal{H}(\Phi_\beta)$ is the Bessel potential space $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$. As described in Section 1, this implies that each $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^2)$ is k-times differentiable off of an exceptional set A with $C_{\beta-k,2}(A) = 0$. Combining this with Theorems 1.5 and 2.2 gives the following result.

THEOREM 4.1. *Let Φ_β with $k < \beta \le k + 1$, for some integer $k \ge 1$, be the solution of* (4.1). *Let $S \subseteq \mathbb{R}^2$. Then $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$ iff, for any $u \in \mathcal{L}^{\beta,2}$, $u|_S = 0$ implies $\nabla^m u|_S = 0$ for all integers m with $0 < m \le k$.*

In the range $1 < \beta \le 2$, therefore, $C_{\beta-1,2}(S) = 0$ implies that $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$. For $\beta > 2$, however, $C_{\beta-1,2}(S) > 0$ for $S \ne \varnothing$ (see Remark 1 following Theorem 1.5), and the condition $C_{\beta-1,2}(S) = 0$ is never applicable. Observe also that if $C_{\beta-m,2}(S) = 0$ for some integer m, $0 < m < k$, the conditions $\nabla^m u|_S = 0, \dots, \nabla^k u|_S = 0$ in Theorem 4.1 are vacuous. In addition, when β moves from one of the ranges $k < \beta \le k + 1$, k-positive integer, to the next, the order of the derivatives involved in Theorem 4.1 increases by 1.

4.1. *The sharp and germ fields for closed sets.* The exact analogue of Theorem 3.2 is the following theorem.

THEOREM 4.2. *Let S be a closed set and let $\mathcal{T}(S)$ be the set of all $x \in S$ for which S has a tangent line. Let $k < \beta \le k + 1$, where $k \ge 1$ is an integer. Then $C_{\beta-k,2}(\mathcal{T}(S)) = 0$ implies $\mathcal{F}(\Phi_\beta, S) = \overline{\mathcal{F}}(\Phi_\beta, S)$.*

PROOF. Notice that, for the range $k < \beta \le k + 1$, the functions $\nabla^m u$ are continuous for all integers $0 \le m \le k - 1$ and thus vanish on S. It follows from Theorem 1.3 that $\nabla^k u$ is classically differentiable $(\beta - k, 2)$-q.e., and we can show (as in Lemma 2.3) that if x_0 is a point of differentiability for $\nabla^k u$ and S does not have a tangent at x_0, then $\nabla^k u(x_0) = 0$. Therefore,

$$C_{\beta-k,2}\big(S \setminus \{x \in S : \nabla^k u(x) \neq 0\}\big) \leq C_{\beta-k,2}(\mathcal{T}(S)) = 0,$$

and the proof is complete. □

4.2. *The sharp and germ fields for open sets.* Results on the equality of the two σ-fields $\mathcal{F}(\Phi, D)$ and $\overline{\mathcal{F}}(\Phi, D)$ for open sets D are of interest for arbitrary fields Φ, and it is remarkable how few results are available in the literature. To our knowledge, the only positive result concerning stationary Gaussian processes occurs as an exercise on page 245 in the book of Dym and McKean (1976): *If there is a constant $k < \infty$ such that the spectral density $\Delta(\lambda)$ satisfies, for all $a \geq 1$, $\Delta(a\lambda) \leq c\Delta(\lambda)$, then Φ satisfies $\mathcal{F}(\Phi, I) = \overline{\mathcal{F}}(\Phi, I)$ for each time interval I.* In Pitt (1973), this elementary result and argument are extended for stationary Gaussian fields on $\mathbb{R}^d$ (see the comment on page 350) in a form valid for bounded open star-shaped sets D. Examples are also given in Dym and McKean (1976) of spectral densities for which the corresponding process Φ satisfies $\mathcal{F}(\Phi, I) \neq \overline{\mathcal{F}}(\Phi, I)$ for specific time intervals I.

THEOREM 4.3. *For each $k \geq 1$, if $k < \beta \leq k + 1/2$ and if $D \subseteq \mathbb{R}^2$ is open, then $\mathcal{F}_D(\Phi_\beta, D) = \overline{\mathcal{F}}_D(\Phi_\beta, D)$.*

PROOF. We study the case $k = 1$ first. Let $1 < \beta \leq 3/2$ and let $u|_D = 0$. Then, since u is continuous, $u|_{\overline{D}} = 0$. Now let $E \subseteq \mathbb{R}^2$ be the set of x at which u fails to be classically differentiable. Then, by Theorem 1.3, $C_{\beta-1,2}(E) = 0$. The set $\mathcal{T}(\overline{D})$ is contained in a countable union of rectifiable curves [Saks (1937), page 264] and hence has σ-finite linear Hausdorff measure. This implies (see Remark 2 following Theorem 1.5) that $C_{\beta-1,2}(\mathcal{T}(\overline{D})) = 0$. Finally, for each point $x \in \overline{D} \setminus \{E \cup \mathcal{T}(\overline{D})\}$, $u(x)$ is classically differentiable, and just as in the proof of Lemma 3.3, $\nabla u(x) = 0$.

Turning to the general case $k \geq 1$, when $k < \beta \leq k + 1/2$ and $u|_D = 0$, $\nabla^{k-1} u(x)$ is continuous and vanishes identically on $\overline{D}$. Pick any $(k-1)$-st-order partial derivative of u and call it v. We must show that $C_{\beta-k,2}(\{x \in \overline{D} : \nabla v(x) \neq 0\}) = 0$. We let E be the set where v is nondifferentiable. Then, as before, $C_{\beta-k,2}(E) = C_{\beta-k,2}(\mathcal{T}(\overline{D})) = 0$ and $\nabla v(x) = 0$ for each $x \in \overline{D} \setminus \{E \cup \mathcal{T}(\overline{D})\}$. □

A generalization of Theorem 3.4 gives the following result.

THEOREM 4.4. *For each $k \geq 1$, if $k + 1/2 < \beta \leq k + 1$, $\mathcal{F}_D(\Phi_\beta, D) = \overline{\mathcal{F}}_D(\Phi_\beta, D)$ holds for any bounded connected open set D.*

PROOF. The proof is based on an elementary capacity inequality for radially decreasing kernels $k(x)$ [Landkof (1972), page 158].

LEMMA 4.5. *Let $k(x)$ be a radially decreasing kernel on $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $|g(x) - g(y)| \le |x - y|$ for all $x, y \in \mathbb{R}^n$. If μ is a finite measure on $\mathbb{R}^n$, for a set $S \subseteq \mathbb{R}^n$, define the image measure $\nu(S) = \mu_g(S) = \mu(g^{-1}(S))$. Then, for each compact set $A \subseteq \mathbb{R}^n$, $k - C(A) \ge k - C(g(A))$.*

We first prove Theorem 4.4 for the range $3/2 < \beta < 2$. The idea is the same as in Theorem 3.4. More precisely, since $\nabla u(x) = 0$ on D and since ∇u is $(\beta - 1, 2)$-finely continuous $(\beta - 1, 2)$-q.e., it is enough to show that for an open connected set $D \subseteq \mathbb{R}^2$, the set D is $(\beta - 1, 2)$-thick at each x on the boundary of D. But, as stated in Remark 1 following Theorem 1.5, the capacity $C_{\beta-1,2}$ is equivalent to the k-capacity with $k(x) = |x|^{-(2-2(\beta-1))}$. It follows from Lemma 4.5 and from the fact that line segments have positive capacities for $3/2 < \beta \le 2$ that D is $(\beta - 1, 2)$-thick in its boundary, $(\beta - 1, 2)$-q.e.

For general $k + 1/2 < \beta < k + 1$, notice that, if $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^2)$, then $\nabla^m u$ are continuous for all integers $0 \le m \le k - 1$ and thus vanish on ∂D. Since $\nabla^k u$ is in the Bessel space $\mathcal{L}^{\beta-k,2}(\mathbb{R}^2)$ and thus finely continuous $(\beta - k, 2)$-q.e., the claim that D is $(\beta - k, 2)$-thick in its boundary, $(\beta - k, 2)$-q.e., completes the proof in this case.

Finally, when $\beta = k + 1$, the result follows as in Theorem 3.4. □

The construction of Adams and Hedberg (1996), page 321, leads to a generalization of Example 3.5.

THEOREM 4.6. *For each integer $k \ge 1$ and $k + 1/2 < \beta \le k + 1$, there exists a disconnected open set D for which $\mathcal{F}_D(\Phi_\beta, D) \ne \overline{\mathcal{F}}_D(\Phi_\beta, D)$.*

PROOF. It will be enough to construct a set D and a function $u \in \mathcal{L}^{\beta,2}$ such that $u|_{\bar{D}} = 0$ and $\partial^k u(x) \ne 0$ on a subset of ∂D with positive $C_{\beta-k,2}$-capacity. We outline the proof where it differs from that of Example 3.5.

The balls $B(a_n, R_n)$, $B(a_n, r_n)$ and the set D are constructed in the same way. The decreasing $C_0^\infty[0, \infty)$-functions v_n are chosen to satisfy $v_n(r) = 1$ for $r \le r_n$, $v_n(r) = 0$ for $r \ge R_n$ and $|v_n^{(j)}(r)| \le cr^{-j}(\log R_n/r_n)^{-1}$ for $1 \le j \le k$ [see Adams and Hedberg (1996), page 321]. As in Example 3.5, all points $x \in [-1/2, 1/2] \times \{0\} \setminus (\bigcup_{n=1}^\infty \bar{B}(a_n, R_n))$ belong to ∂D and since $\sum_{n=1}^\infty R_n < 1/2$, the set of such points has positive one-dimensional Lebesgue measure.

Now set $g(x) = g(x_1, x_2) = x_2^k$ in a neighborhood of $[-1/2, 1/2] \times \{0\}$ and define u as in Example 3.5. It is routine to verify that $u \in \mathcal{L}^{k+1,2}$ $[\sum_{|\sigma|=k+1} \int |D^\sigma(g(x)v_n(|x - a_n|))|^2\,dx < c\log(R_n/r_n)^{-1}$, when R_n is small enough] and therefore $u \in \mathcal{L}^{\beta,2}$. Moreover, $u|_{\bar{D}} = 0$ and $\partial^k u(x) = k!$ for any $x \in [-1/2, 1/2] \times \{0\} \cap \partial D$. Since line segments have positive $C_{\beta-k,2}$-capacity for $k + 1/2 < \beta \le k + 1$, this implies that $\nabla^k u|_{\partial D} \ne 0$. □

4.3. *The sharp and germ fields for smooth curves.* Our next result shows that, for β in the range $k < \beta \le k + 1/2$ and for a smooth curve Γ of Hölder class $C^{k,\alpha}$ with $\alpha > \beta - (k + 1/2)$, no kth-order derivatives arise in the spectral synthesis problem.

LEMMA 4.7. *Let $\Gamma = \{\gamma(s) : s \in [0, 1]\}$ be a $C^{k,\alpha}$-curve and let β satisfy $k < \beta \le k + 1/2$ for some integer $k \ge 1$. Define $t = (2 - 2(\beta - k))^{-1}$ and let $\alpha \ge 1/t = 2(k + 1 - \beta)$. Then, for t-dimensional Hausdorff measure Λ_t, $\Lambda_t(\Gamma) < \infty$ and $C_{\beta-k,2}(\Gamma) = 0$.*

PROOF. If $\Lambda_t(S) < \infty$, it follows that the capacity of S, corresponding to the power kernel $1/|x|^t$, is 0 [Falconer (1985), Theorem 6.4]. By Remark 1 following Theorem 1.5 for $t = 2(k + 1 - \beta)$, it follows that $C_{(2-t)/2,2}(S) = 0$, that is, $C_{\beta-k,2}(S) = 0$. Since a Hölder-continuous curve Γ of order α has finite $1/\alpha$-dimensional Hausdorff measure, the proof is complete. □

For a smooth curve Γ, the germ space $\overline{H}(\Phi_\beta, \Gamma)$ can be decomposed into a sum of the tangential (sharp) space $H(\Phi_\beta, \Gamma)$ and the spaces $H_{\partial_n^m}(\Phi_\beta, \Gamma)$ generated by the normal derivatives $\partial_n^m \phi$, $1 \le m < \beta$. The normal derivatives $\partial_n^m \phi$ are defined as generalized functions given by the values of the derivatives $F^{(m)}(0)$ where

$$F(h) = \int_\Gamma f(x)\phi\big(x + hn(x)\big)\, d\sigma(x), \tag{4.2}$$

provided the derivatives exist in a weak sense in $L^2(P)$.

For $1 < m < \beta$, the subspace of normal derivatives of order m is denoted by

$$H_{\partial_n^m}(\Phi_\beta, \Gamma) = \overline{\mathrm{sp}}\bigg\{ \int_\Gamma f(x)\, \partial_n^m \phi(x)\, d\sigma(x) : f \in C(\Gamma)\bigg\}_{L^2}, \tag{4.3}$$

with this notation extended to include the sharp field $H(\Phi_\beta, \Gamma)$ for $m = 0$.

THEOREM 4.8. *Let $k \ge 1$ be an integer, $k + 1/2 < \beta \le k + 3/2$, and let $\Gamma = \{\gamma(t), t \in [0, 1]\}$ be a smooth curve with continuous derivative $\gamma'(t) \ne 0$.*

(i) *Then, for each $f \in C^1(\Gamma)$ and each integer value m, $0 < m \le k$, the derivatives $F^{(m)}(0)$ exist and the space $\sum_{m=0}^k H_{\partial_n^m}(\Phi_\beta, \Gamma)$ is dense in $\overline{H}(\Phi_\beta, \Gamma)$.*

(ii) *If, furthermore, $0 < \beta - (k + 1/2) < \alpha \le 1$ and Γ is a smooth $C^{k,\alpha}$-curve, there is a positive angle between the subspace $H_{\partial_n^i}(\Phi_\beta, \Gamma)$ and the linear span of the $H_{\partial_n^j}(\Phi_\beta, \Gamma)$ for $j \ne i$, $0 \le i, j \le k$.*

PROOF. (i) The $\beta = 2$ case of the theorem is covered in Theorem 3.11. We next consider the case $2 < \beta < 5/2$. In this case, Φ_β is continuously differentiable and it is clear that $H(\Phi_\beta, \Gamma) = \overline{\mathrm{sp}}\{\phi(x) : x \in \Gamma\}$ while $H_{\partial_n}(\Phi_\beta, \Gamma) = \overline{\mathrm{sp}}\{\partial_n \phi(x) : x \in \Gamma\}$. We only need to show that $H(\Phi_\beta, \Gamma) + H_{\partial_n}(\Phi_\beta, \Gamma)$ is dense in $\overline{H}(\Phi_\beta, \Gamma)$

and we do this by identifying their orthogonal complements. Thus, we let $X \in H(\Phi_\beta)$ be perpendicular to both $H(\Phi_\beta, \Gamma)$ and $H_{\partial_n}(\Phi_\beta, \Gamma)$. The function $u(x) = EX\phi(x)$ is then in $\mathcal{L}^{\beta,2}$ and both $u(x)$ and $\nabla u(x)$ vanish identically on Γ. In addition, $\nabla^2 u|_\Gamma = 0$ $(\beta - 2, 2)$-q.e. since $C_{\beta-2,2}(\Gamma) = 0$. By Netrusov's theorem (Theorem 1.5), $u \in \mathcal{L}^{\beta,2}_{00}$ so $X \in \overline{H}(\Phi_\beta, \Gamma)^\perp$, and the proof is complete.

The general case with $k < \beta < k + 1/2$ is proved similarly and the general case with $k = \beta$ is proved by modifying the $\beta = 2$ proof.

Next, we treat the case with $3/2 < \beta < 2$. The proof of the existence of the boundary integral follows exactly the proof given in Theorem 3.11. The functionals $\{T_h\}$ and the vector measures $\{\mu_h\}$ are defined exactly as before, and the estimate

$$\|T_h\|^2_{-\beta,2} \le 2 \iint G_{2\beta-2}(x - y)\, d|\mu_h|(x)\, d|\mu_h|(y),$$

together with the uniform Lipschitz condition $|\mu_h(B(x, r))| \le cr$, holds as before. But $G_{2\beta-2}(x)$ has a power singularity with index $4 - 2\beta$ at $x = 0$, and integration by parts shows that $\int G_{2\beta-2}(x - y)\, d|\mu_h(y)|$ is uniformly bounded in x and h, which proves the existence of the boundary integrals.

To show that $H(\Phi_\beta, \Gamma)$ and $H_{\partial_n}(\Phi_\beta, \Gamma)$ span $\overline{H}(\Phi_\beta, \Gamma)$, we proceed as in Theorem 3.11. It suffices to show that each X in $H(\Phi_\beta, \Gamma)^\perp \cap H_{\partial_n}(\Phi_\beta, \Gamma)^\perp$ is in $\overline{H}(\Phi_\beta, \Gamma)^\perp$, or, what is the same, that the function $u(x) = EX\phi(x)$ which is in $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$ is also in $\mathcal{L}^{\beta,2}_{00}(\Gamma)$. But $u|_\Gamma = 0$, so by Netrusov's theorem it suffices to show that $\nabla u|_\Gamma = 0$ $(\beta - 1, 2)$-q.e. By quasicontinuity, we only need to show that $\int_\Gamma f(x)\, \partial_n u(x)\, d\sigma(x) = 0$ for each simple function f on Γ. For a fixed f, both $\int_\Gamma f(x)\, \partial_n u(x)\, d\sigma(x)$ and $\lim_{h\to 0}(1/h) \int_\Gamma f(x)[u(x + hn(x) - u(x)]\, d\sigma(x) = EX \int_\Gamma f(x)\, \partial_n \phi(x)\, d\sigma(x)$ define continuous linear functionals on $\mathcal{L}^{\beta,2}$. The functionals agree on the dense subspace C_0^2 of $\mathcal{L}^{\beta,2}$ and thus they are equal. But $X \in \overline{H}(\Phi_\beta, \Gamma)^\perp$, so $0 = EX \int_\Gamma f(x)\, \partial_n \phi(x)\, d\sigma(x)$ for each f, and the result follows.

For general β satisfying $k + 1/2 < \beta < k + 1$, the result may be proved analogously.

(ii) This part of the proof is also adapted from the corresponding argument in Theorem 3.11. The goal is to reduce the general case to the case when $\Gamma \subseteq \mathbb{R}^1 \subset \mathbb{R}^2$. The reduction is done in two parts. First, as in Theorem 3.11, we reduce the general case to the case when Γ is the graph of a real-valued function $y = g(x)$ of Hölder class $C^{k,\alpha}$ with compact support. Second, the case when Γ is a graph of a function is reduced, via a change of variables, to the case $\Gamma \subseteq \mathbb{R}^1$.

To treat the case $\Gamma = \mathbb{R}^1 = \{(x, 0) : -\infty < x < \infty\}$, it will suffice to show that, for each integer m with $0 \le m \le k$, there is a positive angle between the spaces $K_m = \sum_{j=0}^{m-1} H_{\partial_y^j}(\Phi_\beta, \mathbb{R}^1)$ and $L_m = H_{\partial_y^m}(\Phi_\beta, \mathbb{R}^1)$. Dense sets of variables in these two spaces are given by X and Y of the form

$$X = \sum_{j=0}^{m-1} \int_{-\infty}^{\infty} a_j(x)\, \partial_y^j \phi(x, 0)\, dx \quad \text{and} \quad Y = \int_{-\infty}^{\infty} b(x)\, \partial_y^m \phi(x, 0)\, dx.$$

Passing to the spectral representation, we have

$$EXY = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \sum_{j=0}^{m-1} \hat{a}_j(s)(it)^j \overline{\hat{b}(s)(it)^m} \frac{ds\,dt}{(1+s^2+t^2)^\beta}.$$

Setting $c(s) = \sqrt{1+s^2}$ and making the change of variables $t = c(s)\tau$ gives

$$EXY = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} c(s)^{1-2\beta} \left\{ \sum_{j=0}^{m-1} \bigl(a_j(s) c^j(s)(i\tau)^j\bigr) \times \overline{\hat{b}(s) c^m(s)(i\tau)^m} \frac{d\tau}{(1+\tau^2)^\beta} \right\} \frac{ds}{(1+s^2)^\beta}.$$

From this, is is clear that the cosine of the angle θ_m between K_m and L_m is given by

$$\cos(\theta_m) = \sup\{EXY : X \in K_m, Y \in L_m, \text{ and } \|X\|_2 = \|Y\|_2 = 1\} = \|P_m\|,$$

where $\|P_m\|$ is the norm of the orthogonal projection in $L^2(\mathbb{R}^1, d\tau/(1+\tau^2)^\beta)$ of the one-dimensional space $\overline{\mathrm{sp}}\{\tau^m\}$ onto the space $\mathcal{P}_m = \overline{\mathrm{sp}}\{\tau^j : 0 \le j < m\}$. Since $\mathcal{P}_m$ is finite dimensional and $\tau^m \notin \mathcal{P}_m$, $\|\mathcal{P}_m\| < 1$ and the proof is complete in this case.

We now consider the case when $\Gamma = \{(s,t) : t = g(s) : s \in \mathbb{R}^1\}$ and $g \in C_0^{k,\alpha}$ and γ and its kth-order derivatives are bounded. Here, as before, we consider a change of variables $(s,t) \to (x,y)$ given by a diffeomorphism $(x,y) = k(s,t)$, where $x = s$ and $y = ct + w(s,t)$ with $w(s,t) = P_t * \gamma(s)$ for $t > 0$. Here P_t is the Poisson kernel for the upper half plane and $c > 0$ is chosen so that, at $t = 0$, $\partial y/\partial t$ is positive and bounded away from 0 and both k and k^{-1} are Lipschitz. We extended $w(s,t)$ into the lower half plane $t < 0$ by, for example, setting $w(s,-t) = \sum_{j=1}^{m+1} a_j P_{jt} * \gamma(s)$ with $\sum_{j=1}^{m+1} j^l a_j = (-1)^l$, $0 \le l \le m$. With this definition, the reader may check that $w(s,t) \in C^{k,\alpha}$ and $\partial_t^{k+1} w(s,t) = O(1/|t|^{k+1-\alpha})$ [Stein (1970), page 62].

The next step is to show that the operator $W(u) \stackrel{\text{def}}{=} u \circ k$ is bounded and invertible on $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$. Since the case $\beta = 2$ has been covered in Theorem 3.11, we begin with the range $3/2 < \beta < 5/2$ with $\beta \ne 2$.

To show that, for any $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^2)$, $W(u) \in \mathcal{L}^{\beta,2}(\mathbb{R}^2)$, it will be enough to show [Stein (1970), page 139] that $\nabla(u \circ k) \in \mathcal{L}^{\beta-1,2}$. Since $\nabla(u \circ k) = \nabla u(k) \cdot \nabla k$, it will be enough to check that:

1. $u \mapsto u \circ k$ is a bounded operator from $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$ to L^2;
2. $u \mapsto (\nabla u) \circ k$ is a bounded operator from $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$ to $\mathcal{L}^{\beta-1,2}$;
3. ∇k is a (bounded) multiplier on $\mathcal{L}^{\beta-1,2}$; that is, for any $v \in \mathcal{L}^{\beta-1,2}$, $v \cdot \nabla k \in \mathcal{L}^{\beta-1,2}$.

Since k and k^{-1} are both Lipschitz, (1) is immediate. To prove (2), notice that, if $1/2 < \beta - 1 < 1$, a C^1 change of variables preserves L^2 and $\mathcal{L}^{1,2}$. The result for the range $3/2 < \beta < 2$ follows by interpolation. For the range $2 < \beta < 5/2$, we can again use interpolation. If k is $C^{1,\alpha}$ with $\alpha > 1/2$, the change of variables k preserves $\mathcal{L}^{1,2}$, and we already proved, in Theorem 3.11, that k also preserves $\mathcal{L}^{2,2}$. Thus, (2) follows for $2 < \beta \le 5/2$, even with the minimal smoothness of $C^{1,\alpha}$, $\alpha > 1/2$, on k.

To prove (3), we study the cases $3/2 < \beta < 2$ and $2 < \beta \le 5/2$ separately. Denote $z = (s,t)$ and $m(z) = \nabla k(z)$.

Case 1 ($3/2 < \beta < 2$). Because of Theorem 1.6(i), it is enough to show that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(z+h)m(z+h) - v(z)m(z)|^2/|h|^{2\beta}\,dz\,dh < \infty$. Rewriting $m(z+h)v(z+h) - m(z)v(z) = m(z+h)[v(z+h) - v(z)] + v(z)[m(z+h) - m(z)]$, the integral above will be finite if we show that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |m(z+h)[v(z+h) - v(z)]|^2/(|h|^{2\beta})\,dz\,dh < \infty$ and $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(z)[m(z+h) - m(z)]|^2/(|h|^{2\beta})\,dz\,dh < \infty$. The first integral is easily seen to be finite, since $m(z)$ is continuous and bounded, and $v \in \mathcal{L}^{\beta-1,2}(\mathbb{R}^2)$.

For the second integral, it is straightforward to verify that it converges over the regions $\{|z| > 1, h \in \mathbb{R}^2\}$ and $\{|z| < 1, |h| > 1\}$. For the region $\{|z| < 1, |h| < 1\}$, we write $z = (s,t)$ and $h = (h_1, h_2)$ and break the integral into two pieces over the regions $\{|h| < 1, |s| < 1, |t| < |h|\}$ and $\{|h| < 1, |s| < 1, |h| \le |t| < 1\}$. We use that $v \in \mathcal{L}^{\beta-1,2}$ with $3/2 < \beta < 2$ implies, by the trace theorem, that there is a constant c with $\int_{|s|<1} |v(s,t)|^2\,ds \le c\|v\|^2_{\beta-1,2}$, t-a.e. For the first integral, we thus have

$$\begin{aligned} I_1 &= \int_{|h|<1}\int_{|s|<1}\int_{|t|<|h|} \frac{|v(s,t)|^2|m(s+h_1,t+h_2) - m(s,t)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\ &\le c\int_{|h|<1}\int_{|s|<1}\int_{|t|<|h|} \frac{|v(s,t)|^2|h|^{2\alpha}}{|h|^{2\beta}}\,ds\,dt\,dh \\ &\le c\|v\|^2_{\beta-1,2}\int_{|h|<1}\int_{|t|<|h|} \frac{|h|^{2\alpha}}{|h|^{2\beta}}\,dt\,dh \le c\|v\|^2_{\beta-1,2}\int_{|h|<1} |h|^{2\alpha-2\beta+1}\,dh, \end{aligned}$$

which is bounded by $c\|v\|^2_{\beta-1,2}$ for $\alpha > \beta - 3/2$. To estimate the second integral, we use $|\nabla m(s,t)| \le ct^{\alpha-1}$ near $t = 0$ [Stein (1970), page 62] to get

$$\begin{aligned} I_2 &= \int_{|h|<1}\int_{|s|<1}\int_{1>|t|\ge|h|} \frac{|v(s,t)|^2|m(s+h_1,t+h_2) - m(s,t)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\ &\le c\int_{|h|<1}\int_{|s|<1}\int_{1>|t|\ge|h|} \frac{|v(s,t)|^2|h|^2|\nabla m(t,s)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\ &\le c\int_{|h|<1} |h|^{2-2\beta}\int_{|s|<1}\int_{1>|t|\ge|h|} |v(s,t)|^2 t^{2\alpha-2}\,dt\,ds\,dh \\ &\le c\|v\|^2_{\beta-1,2}\int_{|h|<1} |h|^{2-2\beta}\int_{1>|t|\ge|h|} t^{2\alpha-2}\,dt\,dh, \end{aligned}$$

which is also bounded by $c\|v\|^2_{\beta-1,2}$ for $\alpha > \beta - 3/2$, thus completing the proof for the range $3/2 < \beta < 2$.

Case 2 ($2 < \beta < 5/2$). We invoke Strichartz's theorem [Strichartz (1967)] that each space $\mathcal{L}^{\beta,2}(\mathbb{R}^2)$ with $\beta > 1$ is an algebra under pointwise multiplication. Therefore, since $v = \nabla u(k) \in \mathcal{L}^{\beta-1,2}$, (3) will follow if we know that $m = \nabla k \in \mathcal{L}^{\beta-1,2}$, which, by Theorem 1.6(ii), is the same as $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |m(z+h) + m(z-h) - 2m(z)|^2/|h|^{2\beta}\,dz\,dh < \infty$. As for the range $3/2 < \beta < 2$, it is immediate to verify that this integral converges over the regions $\{|z| > 1, h \in \mathbb{R}^2\}$ and $\{|z| < 1, |h| > 1\}$. For the region $\{|z| < 1, |h| < 1\}$, using $z = (s,t)$ and $h = (h_1, h_2)$, we again break the integral into two pieces over the regions $\{|h| < 1, |s| < 1, |t| < |h|\}$ and $\{|h| < 1, |s| < 1, |h| \le |t| < 1\}$, respectively:

$$
\begin{aligned}
I_1 &= \int_{|h|<1}\int_{|s|<1}\int_{|t|<|h|} \frac{|m(z+h) + m(z-h) - 2m(z)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\
&\le c\int_{|h|<1}\int_{|s|<1}\int_{|t|<|h|} \frac{|m(z+h) - m(z)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\
&\le c\int_{|h|<1}\int_{|t|<|h|} \frac{|h|^{2\alpha}}{|h|^{2\beta}}\,dt\,dh \\
&\le c\int_{|h|<1} |h|^{2\alpha-2\beta+1}\,dh,
\end{aligned}
$$

which is finite for $\alpha > \beta - 3/2$.

For the second integral, we use $f(x+h) + f(x-h) - 2f(x) = \int_{-h}^{h}(h - |s|)f''(x+s)\,ds$ and $|\nabla^3 w(s,t)| = |\nabla^2 m(s,t) \le ct^{\alpha-2}$ near $t = 0$ [Stein (1970), page 62] to get

$$
\begin{aligned}
I_2 &= \int_{|h|<1}\int_{|s|<1}\int_{1>|t|\ge|h|} \frac{|m(z+h) + m(z-h) - 2m(z)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\
&\le c\int_{|h|<1}\int_{|s|<1}\int_{1>t\ge|h|} \frac{|h|^4|\nabla^2 m(t,s)|^2}{|h|^{2\beta}}\,ds\,dt\,dh \\
&\le c\int_{|h|<1} |h|^{4-2\beta}\int_{1>t\ge|h|} t^{2(\alpha-2)}\,dt\,dh,
\end{aligned}
$$

which is also finite for $\alpha > \beta - 3/2$, thus completing the proof of Theorem 4.8(ii) for $k = 1$.

For spaces of higher order $\mathcal{L}^{\beta,2}$ with $\beta \ge 5/2$, the proof may be reduced to the case just discussed by consideting the higher order gradient ∇^{k-1} and invoking the result that $u \in \mathcal{L}^{\beta,2}$ iff $u \in \mathcal{L}^{k-1,2}$ and $\nabla^{k-1}u$ is in the space $\mathcal{L}^{\beta-k+1,2}$.

Finally, we comment that, for $\Gamma = \{(x,y) : y = g(s), s \in [0,1]\}$ with $g \in C_0^{1,\alpha}$, the adjoint W^* of the change-of-variables operator W maps the normal derivative operators ∂_n^m along Γ onto operators $b_m(s)\partial_t^m$ + *terms of lower order in* ∂_t that satisfy $\inf\{b_m(s)\} > 0$. Using this observation and the case when $\Gamma = \mathbb{R}^1$, it is routine to establish that the angles are positive in the general case. The proof of Theorem 4.8 is complete. $\square$

We now determine the critical smoothness condition on the curve Γ that guarantees that the sharp and the germ σ-fields of Φ_β, $k+1/2 < \beta \le k+3/2$, will be different. Because of Theorem 4.8, it is clear that we have to consider curves that are no smoother than Hölder of class $C^{k,\alpha}$ with $0 < \alpha \le \beta - (k+1/2)$. We show that there exist curves Γ of class $C^{k,\alpha}$ for which $u|_\Gamma = 0$ implies $\nabla u|_\Gamma = 0$ for all functions $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^2)$. As the smoothness of Γ decreases, $u|_\Gamma = 0$ may also imply that higher order gradients vanish on Γ. The critical smoothness on Γ that allows $u|_\Gamma = 0$ to imply $\nabla^m u|_\Gamma = 0$ for all $m = 1, 2, \dots, k-1$ and thus guarantee $\mathcal{F}(\Phi_\beta, \Gamma) = \overline{\mathcal{F}}(\Phi_\beta, \Gamma)$ is $C^{1,\alpha}$. Our next theorem gives the precise criteria.

THEOREM 4.9. *Let $k \ge 2$ be an integer and let $k+1/2 < \beta \le k+3/2$.*

(i) *If α is such that $0 < \beta - (k+1/2) < \alpha \le 1$ and $0 \le m < k$ is an integer, then, for any $C^{k-m,\alpha}$-curve Γ in $\mathbb{R}^2$, there exists a function $u \in \mathcal{L}^{\beta,2}$ such that $u|_\Gamma = 0, \nabla u|_\Gamma = 0, \dots, \nabla^{k-m} u|_\Gamma = 0$, but $\nabla^{k-m+1} u|_\Gamma \ne 0$. In particular, $\mathcal{F}(\Phi_\beta, \Gamma) \ne \overline{\mathcal{F}}(\Phi_\beta, \Gamma)$.*

(ii) *If $0 < \alpha < \beta - (k+1/2) \le 1$, there exists a curve Γ of Hölder class $C^{k,\alpha}$ such that, for all $u \in \mathcal{L}^{\beta,2}$, $u|_\Gamma = 0$ implies $\nabla u|_\Gamma = 0$.*

(iii) *If α is such that $0 < \alpha < \beta - (k+1/2) \le 1$, then there exists a curve Γ of Hölder class $C^{1,\alpha}$ such that $u|_\Gamma = 0$ implies $\nabla^m u|_\Gamma = 0$ for all $m = 1, 2, \dots, k$, and thus $\mathcal{F}(\Phi_\beta, \Gamma) = \overline{\mathcal{F}}(\Phi_\beta, \Gamma)$ holds.*

PROOF. (i) The proof is a modification of that of Theorems 3.6 and 4.8. Using the change of variables $k(s,t)$ as in Theorem 4.8(ii), we choose $u(x,y)$ of compact support to be $u(x,y) = t^{k-m+1}$ near $t = 0$, where $t = t(x,y)$ is the t-coordinate of k^{-1}. With this choice of u, it is clear that $\nabla^{k-m+1} u|_\Gamma \ne 0$. For $\beta = k+1$, thus, all that remains to show is that $\nabla^{k+1} u \in L^2$. A direct calculation gives

$$\begin{aligned}\|\nabla^{k+1} u\|_\infty &\le c \sum_{l=0}^{k-m} t^{k-m-l} \|\nabla^{k+1-l} w\|_\infty \\ &\le c t^{k-m-l} t^{m+\alpha-(k+1-l)} = c t^{\alpha-1}\end{aligned}$$

and thus, since $\alpha > 1/2$, the claim follows in this case.

To prove $u \in \mathcal{L}^{\beta,2}$ in the case when $k+1/2 < \beta \le k+3/2$, $\beta \ne k+1$ and $k > 2$, we will show that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\nabla^k u(z+h) + \nabla^k u(z-h) - 2\nabla^k u(z)|^2 / |h|^{2+2(\beta-k)} \, dz \, dh < \infty.$$

As in the proof of Theorem 4.8, only the integral over the region $\{|z| < 1, |h| < 1\}$ is of interest. Writing $z = (s, t)$ and spliting the integral into two parts, as in Theorem 4.8, we have

$$
\begin{aligned}
I_1 &\le c \int_{|h|<1} \int_{|s|<1} \int_{|t|<|h|} \frac{|\nabla^k u(z+h) - \nabla^k u(z)|^2}{|h|^{2+2(\beta-k)}} \, ds\, dt\, dh \\
&\le c \int_{|h|<1} \int_{|s|<1} \int_{|t|<|h|} \frac{|h|^2 |\nabla^{k+1} u|^2}{|h|^{2+2(\beta-k)}} \, ds\, dt\, dh \\
&\le \int_{|h|<1} |h|^{2(k-\beta)} \int_{|t|<|h|} t^{2(\alpha-1)} \, dt\, dh < \infty
\end{aligned}
$$

for $\alpha > \beta - (k + 1/2)$. For I_2, we have

$$
\begin{aligned}
I_2 &\le c \int_{|h|<1} \int_{|s|<1} \int_{1>t\ge|h|} \frac{|h|^4 |\nabla^{k+2} u|^2}{|h|^{2+2(\beta-k)}} \, ds\, dt\, dh \\
&\le \int_{|h|<1} |h|^{2-2(k-\beta)} \int_{1>t\ge|h|} t^{2(\alpha-2)} \, dt\, dh < \infty
\end{aligned}
$$

for $\alpha > \beta - 3/2$. This completes the proof of part (i).

(ii) We study the case $k = 1$ first and outline the proof where it differs from that of Theorem 3.7. The curve Γ is again the graph of the function

$$
g(x) = \sum_{j=0}^{\infty} a_j \bigl(1 - \cos(2\pi n_j x)\bigr), \qquad x \in [0, 1],
$$

with the choice of $n_j = 2^{j!}$ and $a_j = n_j^{-1-\alpha}$. With this choice of a_j and n_j, Γ has the desired smoothness by the Weierstrass–Hardy theorem [see, e.g., Zygmund (1959), page 48]. From Theorems 2.2 and 1.5 and Lemma 4.2, we observe that we must show that $\nabla u|_\Gamma = 0$ for each $u \in \mathcal{L}^{\beta,2}(\mathbb{R}^2)$ with $u|_\Gamma = 0$.

Lemma 3.8 remains the same. A version of Lemma 3.9 holds with the following changes:

1. $\|\tau_h\|_{-\beta,2} = O(h^{(2\beta-3)/2})$.
2. The effect of the change of variables on the norms (1), as in Theorem 3.7, requires a bound on the $\mathcal{L}^{\beta,2}$-norm of $u(x, y + g_N(x))$ when compared with $\|u\|_{\beta,2}$. We do this by using interpolation between the integer order spaces, applying Theorem 1.1. It is clear from the proof of Theorem 3.7 that, when β is an integer,

$$
\bigl\|u\bigl(x, y + g_N(x)\bigr)\bigr\|_{\beta,2} \le c \|g^{(\beta)}\|_\infty \|u\|_{\beta,2},
$$

and interpolating between $\mathcal{L}^{1,2}$ and $\mathcal{L}^{2,2}$ first and then between $\mathcal{L}^{2,2}$ and $\mathcal{L}^{3,2}$ (Theorem 1.1) gives

$$\|u(x, y + g_N(x))\|_{\beta,2} \le \begin{cases} \|g''\|_\infty^{\beta-1}\|u\|_{\beta,2}, & \text{when } 1 < \beta < 2,\\ \|g''\|_\infty^{3-\beta}\|g'''\|_\infty^{\beta-2}\|u\|_{\beta,2}, & \text{when } 2 < \beta < 3. \end{cases}$$

3. Bounds for the norms of the linear functionals $T_N \in \mathcal{L}^{-\beta,2}$ are given by

$$\|T_N\|_{-\beta,2} \le \begin{cases} \|g''\|_\infty^{\beta-1}(a_{N+1})^{(2\beta-3)/2}, & \text{when } 3/2 < \beta < 2,\\ \|g''\|_\infty^{3-\beta}\|g'''\|_\infty^{\beta-2}(a_{N+1})^{(2\beta-3)/2}, & \text{when } 2 < \beta < 5/2, \end{cases}$$

and $\|T_N\|_{-\beta,2} \to 0$ since the uniform norms of the derivatives of g are bounded by a power of n_N and are small when multiplied by a power of a_{N+1}.

4. The numerical approximations of $\int_{\Gamma_N} f(x)u(x, y)\,dx$ and $\int_{\Gamma_N} f(x)u(x, y + h)\,dx$ are treated using the more general form of Lemma 3.10 [Theorem 8.1 in Pitt, Robeva and Wang (1995)] to the effect that, for any $f \in C^1[0, 1]$ and $1 < \beta < 5/2$,

$$\sup_{\|u\|_{\beta,2}\le 1}\left|\int_0^1 f(x)u(x,0)\,dx - \frac{1}{M}\sum_{j=0}^{M} f\left(\frac{j+\frac{1}{2}}{M}\right)u\left(\frac{j+\frac{1}{2}}{M}, 0\right)\right| = O\left(\frac{1}{M^{(2\beta-1)/2}}\right). \tag{4.4}$$

The rest of the proof of Theorem 3.7 carries over with the obvious change that ∇u is $(\beta - 1, 2)$-quasicontinuous.

For general β in the range $k + 1/2 < \beta \le k + 3/2$, where $k \ge 1$ is an integer, the construction from Theorem 3.7 carries over with the following changes:

1. $\|\tau_h\|_{-\beta,2} = O(h^{\min\{2,(2\beta-3)/2\}})$ and one can show as before that $\|T_N\|_{-\beta,2} \to 0$.
2. To prove that $\int_{\Gamma_N} f(x)\,\partial_2 u(x, y)\,dx$ is small for large N, a more efficient estimate of the integrals $\int_{\Gamma_N} f(x)u(x, y)\,dx$ and $\int_{\Gamma_N} f(x)u(x, y + h)\,dx$ is required than that given by (4.4). Such estimates, based on an Euler–MacLaurin formula, appear in Benhenni (1998).

(iii) Build the curve Γ as in (ii). Then, for any $u \in \mathcal{L}^{\gamma,2}$, $3/2 < \gamma \le 5/2$, $u|_\Gamma = 0$ implies $\nabla u|_\Gamma = 0$. Now let $u \in \mathcal{L}^{\beta,2}$, with $k + 1/2 < \beta \le k + 3/2$, $k \ge 2$, vanish on Γ. Write $\beta = \gamma + (k - 1)$. Since $\gamma < \beta$ implies $\mathcal{L}^{\beta,2} \subset \mathcal{L}^{\gamma,2}$, applying (ii) repeatedly for the function u and its derivatives of order up to $k - 1$ shows that $\nabla^m u|_\Gamma = 0$ for $m = 1, 2, \ldots, k$. Thus, by Theorem 1.5, $u \in \mathcal{L}_{00}^{\beta,2}(\mathbb{R}^2 \setminus \Gamma)$, which is equivalent to $\mathcal{F}(\Phi_\beta, \Gamma) = \overline{\mathcal{F}}(\Phi_\beta, \Gamma)$. □

REMARKS. It is possible to extend these results to other Gaussian fields beyond the Bessel fields. Since all the results depend only on the space $\mathcal{H}(\Phi)$ and its norm, they do not distinguish between $\{\Phi_\beta\}$ and any Gaussian field Φ for which the space $\mathcal{H}(\Phi)$ contains exactly the same functions as $\mathcal{L}^{\beta,2}$ and for which the norm is equivalent to the norm on $\mathcal{L}^{\beta,2}$. This remark covers any stationary field whose spectral density $\Delta(\lambda)$ is bounded above and below by constant multiples of $\Delta_\beta(\lambda)$ and certain nonstationary fields.

Similarly, when two fields Φ and Φ_β have equivalent localized spaces (and norms) $\mathcal{H}(\Phi, D)$ and $\mathcal{H}(\Phi_\beta, D)$, the spectral synthesis in the two spaces $\mathcal{H}(\Phi)$ and $\mathcal{H}(\Phi_\beta)$ will coincide for closed sets $\Gamma \in D$. This method can be used, for example, to prove local results for periodic versions of the Bessel fields or for nonstationary fractional Brownian motions. Sufficient spectral conditions under which this is possible are given in Pitt (1975).

The obvious barrier to extending our results by such methods comes from the fact that the Bessel fields are isotropic as are the spaces $\mathcal{L}^{\beta,2}$. To treat fields that are intrinsically anisotropic, such as solutions of stochastic wave and heat equations, by these methods will require development of a satisfactory potential theory for the corresponding anisotropic function spaces, and, to date, this work has not been done.

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