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ON THE STRONG LAW OF LARGE NUMBERS

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A version of the SLLN for a large class of means is proved.

The result presented in this paper is closely related to two classical theorems. Namely, it links in some sense the SLLN of Kolmogorov and that of Marcinkiewicz. The celebrated results just mentioned say that the averages, for $0 < \alpha < 2$,

$$\frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} (X_k - c), \qquad n = 1, 2, \ldots,$$

converge a.s. if and only if $\mathbb{E}|X|^\alpha < \infty$, c being 0 or $\mathbb{E}X$, according as $\alpha < 1$ or $\alpha \geq 1$. Let us stress here that the Kolmogorov SLLN ($\alpha = 1$) concerns the arithmetic (Cesàro) means being a regular method of summability, which is not the case when $0 < \alpha < 2$, $\alpha \neq 1$.

We consider a large class of summability methods, which are defined as follows.

Let g be a positive, increasing function and h a positive function such that $\phi(y) \equiv g(y)h(y)$ satisfies the following conditions:

(i) For some $d \geq 0$, ϕ is strictly increasing on $[d, \infty)$ with range $[0, \infty)$.

(ii) There exist C and a positive integer k_0 such that $\phi(y+1)/\phi(y) \leq C$, $y \geq k_0$.

(iii) There exist constants a and b such that

$$\phi(s)^2 \int_s^\infty \frac{1}{\phi(x)^2}\, dx \leq as + b, \qquad s > d.$$

For h and g as above, the (h, g)-transform of a sequence $x = \{\xi_n\}$ is given by

$$\sigma_n(x) = \frac{1}{g(n)} \sum_{k=1}^{n} \frac{1}{h(k)} \xi_k, \qquad n = 1, 2, \ldots. \tag{1}$$

If $\sigma_n(x) \to \xi$, then we say that (ξ_n) is summable (limitable) to ξ by the method (h, g) and write $(h, g) - \lim \xi_k = \xi$. It should be stressed here that the above class of sequence transformations includes several regular summability methods such as Cesàro means $[h(y) = 1,\ g(y) = y]$ or logarithmic means $[h(y) = y,\ g(y) = \log y]$ but also embraces nonregular transformations. For example, the

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choice $h(y)=1$ and $g(y)=y^{1/\alpha}$, $0<\alpha<2$, $\alpha\neq 1$, gives the means related to the Marcinkiewicz theorem.

Let us formulate the strong law of large numbers for the means just defined.

THEOREM. *Let h and g be as above and let $(X, X_1, X_2, \dots)$ be a sequence of i.i.d. random variables. Let us put*

$$m_k = \mathbb{E}\big[X_k \mathbb{1}_{\{|X_k|\le\phi(k)\}}\big]. \tag{2}$$

Then the following two conditions are equivalent:

$$(h, g) - \lim(X_k - m_k) = 0 \qquad a.s., \tag{3}$$

$$\mathbb{E}\big[\phi^{-1}(|X|)\big] < \infty, \tag{4}$$

where ϕ^{-1} is the inverse of ϕ.

PROOF. (3) → (4). Since $\lim_{k\to\infty}(m_k/\phi(k)) = 0$, condition (3) implies $\lim_{k\to\infty}(X_k/\phi(k)) = 0$ a.s. Consequently,

$$\sum_{k=1}^{\infty} P\big(|X_k| \ge \phi(k)\big) = \sum_{k=1}^{\infty} P\big(\phi^{-1}(|X|) \ge k\big) < \infty,$$

which implies (4).

(4) → (3). Let $\mathbb{E}[\phi^{-1}(|X|)] < \infty$. Then

$$\sum_{k=1}^{\infty} P\big(|X_k| \ge \phi(k)\big) = \sum_{k=1}^{\infty} P\big(\phi^{-1}(|X|) \ge k\big) \le \mathbb{E}\big[\phi^{-1}(|X|)\big] < \infty.$$

Let us set

$$\overline{X}_n = X_n \mathbb{1}_{\{|X_n|\le\phi(n)\}},$$

so

$$\sum_k P(X_k \neq \overline{X}_k) < \infty.$$

By the Borel–Cantelli lemma, it is enough to show that

$$\frac{1}{g(n)} \sum_{k=1}^{\infty} \frac{(\overline{X}_k - m_k)}{h(k)} \to 0 \qquad \text{a.s.} \tag{5}$$

To this end we estimate the series

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}\overline{X}_k^2}{\phi(k)^2} = \mathbb{E}\sum_{k=1}^{\infty} \frac{X^2}{\phi(k)^2} \mathbb{1}_{\{|X|\le\phi(k)\}}.$$

Note that, since $|X| = \phi(\phi^{-1}(|X|))$,

$$\sum_{k=1}^{\infty} \frac{X^2}{\phi(k)^2} \mathbb{1}_{\{|X| \le \phi(k)\}} \le k_0 + C^2 \sum_{k=k_0+1}^{\infty} \frac{X^2}{\phi(k+1)^2} \mathbb{1}_{\{|X| \le \phi(k)\}}$$
$$\le k_0 + C^2 X^2 \int_{\phi^{-1}(|X|)}^{\infty} \frac{1}{\phi(x)^2}\, dx$$
$$\le k_0 + C^2 a \phi^{-1}(|X|) + C^2 b.$$

[The above short argument with the use of the identity $\phi(\phi^{-1}(|X|)) = |X|$ is due to the referee.]

By (4) we get

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\overline{X}_n^2}{\phi(n)^2} < \infty,$$

which, in turn, implies

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\frac{\overline{X}_n - m_n}{\phi(n)}\right]^2 < \infty.$$

Consequently, the series

$$\sum_{n=1}^{\infty} \frac{\overline{X}_n - m_n}{\phi(n)}$$

converges almost surely and it is enough to apply the Kronecker lemma. □

REMARKS. (a) It is clear that the theorem essentially gives the classical result of Marcinkiewicz [for $\phi(x) = x^{1/\alpha}$].

(b) Our theorem seems to be new also for a particular case of logarithmic means

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(X_k - m_k) \tag{1'}$$

(cf. [1], pages 106 and 314), that is, for $\phi(x) = x \log x$.

It is worth noting here that in this case

$$\mathbb{E}(|X|^{\alpha}) \le \mathbb{E}\phi^{-1}(|X|) \le \mathbb{E}(|X|) \quad \text{for } 0 < \alpha < 1.$$

Thus, the moment condition (4) is situated between that of Kolmogorov and Marcinkiewicz.

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