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RECURRENCE AND TRANSIENCE OF BRANCHING DIFFUSION PROCESSES ON RIEMANNIAN MANIFOLDS

BY ALEXANDER GRIGOR'YAN AND MARK KELBERT[1,2]

Imperial College and University of Wales, Swansea

We relate the recurrence and transience of a branching diffusion process on a Riemannian manifold M to some properties of a linear elliptic operator on M (including spectral properties). There is a trade-off between the tendency of the transient Brownian motion to escape and the birth process of the new particles. If the latter has a high enough intensity then it may override the transience of the Brownian motion, leading to the recurrence of the branching process, and vice versa. In the case of a spherically symmetric manifold, the critical intensity of the population growth can be found explicitly.

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1. Introduction. Let M be a noncompact Riemannian manifold, and X_t be the Brownian motion on M generated by $\frac{1}{2}\Delta$ where Δ is the Laplace operator of the Riemannian metric of M. Assuming that X_t is stochastically complete, consider a branching process $\overline{X}_t$ based on X_t. The process $\overline{X}_t$ is determined by the branching rate $Q(x)$ at the point $x \in M$ and by the branching mechanism $\{p_k(x)\}_{k=2}^{k_{\max}}$. Namely, $p_k(x)$ is the probability of branching into k offspring at x. We always assume that functions Q and p_k are continuous, nonnegative and

$$\sum_{k=2}^{k_{\max}} p_k(x) \equiv 1.$$

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[2]Also at International Institute for Earthquake Prediction Theory and Mathematical Geophysics, Warshavskoye sh. 79, k2, Moscow 113556, Russia.
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Construction of such a process is similar to that in Euclidean spaces (see [1], Chapter VI).

We say that a (branching) diffusion process on M is *recurrent* if any nonempty open subset is visited by at least one of the offspring with probability 1, and *transient* otherwise. If the Brownian motion X_t is recurrent then $\overline{X}_t$ is obviously recurrent as well. If X_t is transient then the problem arises how to decide whether $\overline{X}_t$ is recurrent or transient. Let us set

$$q(x) := 2Q(x) \sum_{k=2}^{k_{\max}} (k-1) p_k(x). \tag{1.1}$$

Note that $\frac{1}{2}q(x)$ is the intensity of the population growth of $\overline{X}_t$ at the point $x \in M$. We assume throughout the paper that the function $q(x)$ is finite and continuous on M.

The branching diffusion process on hyperbolic spaces (with constant negative curvature) was studied in detail in [17, 14] (see also [15] for branching Markov processes on $\mathbb{Z}^1$). In papers [17, 14], the only allowed branching is into two offspring; that is, $k_{\max} = 2$, $p_2 \equiv 1$ and $q = 2Q$. This is already an interesting case, which contains most difficulties. However, in general we do not assume even the finiteness of $k_{\max}$.

All our results provide sufficient conditions for the transience or the recurrence of $\overline{X}_t$ in terms of q and other related quantities. There is a trade-off between the tendency of the Brownian motion to escape (due to the transience of X_t) and the birth process of the new particles, which is governed by q. If q is large enough then the branching process $\overline{X}_t$ may be recurrent despite the transience of the Brownian motion X_t.

Before we can state the results, we need to introduce some notation. Consider the operator

$$L = \Delta + q(x).$$

Given a nonempty compact set $K \subset M$ with a smooth boundary, denote by $m_K(x)$ the K-*gauge* of $\overline{X}_t$ (or of L) which is by definition the expected number of the branches (not offspring) of $\overline{X}_t$ that ever hit K, starting from a single particle at x. If $m_K < \infty$, then m_K is the smallest positive solution to the following exterior Dirichlet problem in $\Omega := M \setminus K$:

$$\begin{aligned} Lf &= 0, \\ f|_{\partial K} &= 1. \end{aligned} \tag{1.2}$$

For any precompact open set U, denote by $\lambda(U)$ the bottom eigenvalue of the Dirichlet problem in U for the operator L. The transience of the Brownian motion X_t implies that there exists the minimal positive fundamental solution of the Laplace operator Δ on M, which is called the Green function of Δ and is denoted by $G(x, y)$.

Below we list the main results of this paper, assuming that X_t is transient and $q \not\equiv 0$ (as well as some technical hypotheses, all of them being satisfied if $k_{\max} < \infty$; see Sections 2 and 7):

1. If $m_K < \infty$ then $\overline{X}_t$ is also transient (Theorem 3.4). This implies that $\overline{X}_t$ is transient under either of the following two conditions (Theorem 4.1):

   (i) $\lambda(U) > 0$ for all precompact open sets $U \subset M$.
   (ii) There exists a positive function v on M such that $\Delta v + qv \le 0$.

2. $\overline{X}_t$ is transient provided either of the following two (equivalent) conditions holds (Theorem 6.2):

   (i) There exists a *thin* open set $S \subset M$ such that

$$\int_{M\setminus S} G(x, y)q(y)\, d\mu(y) < \infty. \tag{1.3}$$

   [See Section 6 for the definition of a thin set; the condition (1.3) means that q may take arbitrary *large* values on a *small* set S while $\overline{X}_t$ is transient.]
   (ii) There exists a positive bounded function u on M such that $\Delta u - qu \ge 0$.

3. Let $\{K_l\}_{l\ge 0}$ be an exhausting sequence of compact sets in M. Set $q_l = q\mathbf{1}_{K_{l+1}\setminus K_l}$ and denote by m_l the K_l-gauge of the operator $\Delta + q_l$ and by $\overline{m}_l$ the K_l-gauge of the operator $\Delta + 2q_l$. If, for some $\varepsilon > 0$ and all l,

$$\inf_{x\in\partial K_{l+1}} m_l(x) \ge 1+\varepsilon \quad \text{and} \quad \sup_{x\in K_{l+1}} \overline{m}_l(x) \le \varepsilon^{-1}$$

   then the process $\overline{X}_t$ is K_0-recurrent (Corollary 7.4).
4. Let M be a geodesically complete manifold with nonnegative Ricci curvature. Denote by $V(x, r)$ the volume of the geodesic ball of radius r centered at x, and assume that for all $x \in M$ and $R > r > 0$,

$$\frac{V(x, R)}{V(x, r)} \ge c\left(\frac{R}{r}\right)^{\alpha},$$

   where $c > 0$ and $\alpha > 2$. Let $q(x) \ge b|x|^{-2}$ for large enough $|x|$ where $|x|$ is the distance from x to a reference point o. We claim that if the constant b is large enough then $\overline{X}_t$ is recurrent (Theorem 8.1). Note that the condition $\alpha > 2$ guarantees the transience of the Brownian motion X_t.
5. Let M be spherically symmetric manifold with a pole o (see Section 9 for a precise definition) and let $S(r)$ be the surface area of the geodesic sphere of radius r centered at o. Let the function q be also radial so that we can write $q = q(r)$. Assume that

$$\int^{\infty} S(r)q(r)\, dr = \infty \tag{1.4}$$

and that, for some $R > 0$, the function

$$\mathcal{F}(r) := \frac{S^2(r)q(r)}{(\int_R^r S(t)q(t)\,dt)^2}$$

is monotone decreasing in r. Then $\overline{X}_t$ is recurrent if and only if

$$\lim_{r\to\infty} \mathcal{F}(r) < 4.$$

[See Corollary 9.8; note that if the integral (1.4) converges then $\overline{X}_t$ is transient.]

EXAMPLE 1.1. Let $M = \mathbb{R}^n$, $n > 2$, and

$$q(x) = \frac{c}{|x|^2} \qquad \text{for } |x| > 1. \tag{1.5}$$

Then $\overline{X}_t$ is recurrent if and only if $c > \frac{(n-2)^2}{4}$ (see Section 9). For simple random walks on $\mathbb{Z}^n$ similar results were proved in [20].

Let $M = \mathbb{H}^n$ $(n \geq 2)$ be the n-dimensional hyperbolic space. If $q \equiv c = \text{const}$ then the recurrence of $\overline{X}_t$ is equivalent to $c > \frac{(n-1)^2}{4}$ (see Section 9). Similar results were proved for $\mathbb{H}^2$ in [17] and for $\mathbb{H}^n$ in [14].

More examples can be found in the main body of the paper, in particular, in the table of Example 9.2 in Section 9.

2. Preliminaries. Let $\mathbb{P}_x$ be the probability measure associated with the Brownian motion X_t on a Riemannian manifold M started at the point $x \in M$. The Brownian motion X_t (and the manifold M) is called *stochastically complete* if $\mathbb{P}_x(X_t \in M) \equiv 1$. If M is geodesically complete and satisfies in addition a mild condition on the volume growth of geodesic balls then M is stochastically complete (see [6, 8]).

A branching diffusion process $\overline{X}_t$ based on X_t, is determined by the generator $\frac{1}{2}\Delta$, by the *branching rate* $Q(x)$ (i.e., the exponential distribution of the lifetime of a particle "frozen" at x), and by the *branching mechanism* described by the sequence $\{p_k(x)\}_{k=2}^{k_{\max}}$. Namely, $p_k(x)$ is the probability of creation k offspring at the point x. We make the following assumptions which will be used throughout:

(A) M is a noncompact stochastically complete Riemannian manifold. Functions Q and p_k are nonnegative and continuous on M, and

$$\sum_{k=2}^{k_{\max}} p_k(x) \equiv 1. \tag{2.1}$$

(B) We have

$$C_0 := \sup_x \sum_{k=2}^{k_{\max}} (k-1) p_k(x) < \infty, \tag{2.2}$$

and the series in (2.2) converges locally uniformly (of course, this makes sense to assume only if $k_{\max} = \infty$).

In particular, the *branching intensity*

$$q(x) := 2Q(x) \sum_{k=2}^{k_{\max}} (k-1) p_k(x) \tag{2.3}$$

is finite and continuous on M.

Denote by $\overline{\mathbb{P}}_x$ and $\overline{\mathbb{E}}_x$, respectively, the probability measure and expectation associated with the process $\overline{X}_t$ started with one particle at $x \in M$.

DEFINITION 2.1. For any set $K \subset M$, define the function $\psi_K(x)$ on M as the $\overline{\mathbb{P}}_x$-probability of the event that at least one offspring of the process $\overline{X}_t$ will ever visit K.

DEFINITION 2.2. The process $\overline{X}_t$ is called K-*recurrent* if $\psi_K(x) \equiv 1$. The process $\overline{X}_t$ is called *recurrent* if $\overline{X}_t$ is K-recurrent for all sets K with nonempty interior. The process $\overline{X}_t$ is called K-*transient* if it is not K-recurrent; that is, $\psi_K(x) < 1$ for some x. The process $\overline{X}_t$ is called *transient* if it is not recurrent; that is, $\psi_K(x) < 1$ for some set K with nonempty interior and for some x.

Clearly, $\overline{X}_t$ is recurrent if it is K-recurrent for any compact set K with nonempty interior and smooth boundary. Let us fix throughout this section such K, and set $\Omega := M \setminus K$.

DEFINITION 2.3. We say that $\{U_l\}_{l \geq 0}$ is an *exhausting sequence* in M if U_l are precompact sets with nonempty interior and with smooth boundaries, $\overline{U_l} \subset U_{l+1}$, and the union of all U_l is M.

Given an exhausting sequence $\{U_l\}$ of open sets, the Brownian motion X_t can be obtained as the limit of the processes $X_t^{U_l}$ with the killing condition outside U_l. Similarly, the branching process $\overline{X}_t$ is the limit of the processes $\overline{X}_t^{U_l}$ with the killing condition outside U_l. In particular, for any compact K,

$$\psi_K = \lim_{l \to \infty} \psi_K^{U_l}$$

with the obvious meaning of $\psi_K^{U_l}$.

Define a function P on $M \times [0, 1]$ by

$$P(x,u) := 2Q(x) \sum_{k=2}^{k_{\max}} p_k(x)(1 + u + \cdots + u^{k-2}). \tag{2.4}$$

As follows from hypothesis (B), $P(x,u)$ is finite and jointly continuous on $M \times [0, 1]$. Comparing (2.4) with (2.3) and using (2.2), we obtain

$$C_0^{-1} q(x) \le P(x,u) \le P(x,1) = q(x). \tag{2.5}$$

PROPOSITION 2.4. *The function $u = 1 - \psi_K$ solves the following exterior boundary value problem in Ω*:

$$\begin{aligned} &\Delta u - P(x,u)u(1-u) = 0,\\ &u|_{\partial K} = 0,\\ &0 \le u \le 1, \end{aligned} \tag{2.6}$$

and among all solutions u to (2.6), *$u = 1 - \psi_K$ is the maximal one.*

Respectively, the function $v = \psi_K$ solves the following problem:

$$\begin{aligned} &\Delta v + P(x, 1-v)(1-v)v = 0,\\ &v|_{\partial K} = 1,\\ &0 \le v \le 1, \end{aligned} \tag{2.7}$$

and among all solutions v to (2.6), *$v = \psi_K$ is the minimal one.*

Clearly, (2.6) and (2.7) are equivalent by the change $u = 1 - v$. The boundary conditions are obvious. The fact that $1 - \psi_K$ satisfies the equation in (2.6) follows directly from the strong Markov property (cf. [4]). Let us verify that ψ_K is indeed the minimal solution to (2.7). For that, we need the following comparison lemmas.

LEMMA 2.5 (Generalized maximum principle). *Let $U \subset M$ be a precompact region and let $f, g \in C^2(U) \cap C(\overline{U})$, $f > 0$ in U and $g > 0$ in $\overline{U}$. If*

$$\frac{Lf}{f} \ge \frac{Lg}{g} \qquad \text{in } U, \tag{2.8}$$

then

$$\sup_U \frac{f}{g} = \sup_{\partial U} \frac{f}{g}. \tag{2.9}$$

In particular, if $f \le g$ on ∂U then $f \le g$ in U.

PROOF. Observe that $\frac{Lf}{f} = \frac{\Delta f}{f} + q$. Therefore one can replace in (2.8) L by Δ (in particular, the validity of Lemma 2.5 does not depend on q). It is easy to check that

$$\nabla \frac{f}{g} = (g\nabla f - f\nabla g)g^{-2} \tag{2.10}$$

and

$$\Delta \frac{f}{g} = \left(g\Delta f - f\Delta g - 2\nabla g \nabla f + 2|\nabla g|^2 \frac{f}{g} \right) g^{-2}. \tag{2.11}$$

Assume first that a strict inequality takes place in (2.8), that is,

$$\frac{\Delta f}{f} > \frac{\Delta g}{g} \qquad \text{in } U. \tag{2.12}$$

If $x_0 \in U$ is such that f/g takes its maximum value at x_0 then $\nabla \frac{f}{g}(x_0) = 0$, and (2.10) implies $\nabla f(x_0) = (\frac{f}{g}\nabla g)(x_0)$. Therefore, by (2.11) and (2.12), we obtain at x_0,

$$\Delta \frac{f}{g} = \left(g\Delta f - f\Delta g - 2|\nabla g|^2 \frac{f}{g} + 2|\nabla g|^2 \frac{f}{g} \right) g^{-2} > 0,$$

which contradicts the fact that x_0 is the maximum point of f/g. Hence, f/g takes its maximum value on ∂U whence (2.9) follows.

For the general case, let us slightly reduce U so that $g \in C^2(\overline{U})$. Let C be a positive number such that $C > \sup_U \frac{\Delta g}{g}$, and let φ solve the following Dirichlet problem in U:

$$\Delta\varphi - C\varphi = 0,$$
$$\varphi|_{\partial U} = 1.$$

Then $\varphi > 0$ in $\overline{U}$ and $\frac{\Delta\varphi}{\varphi} = C > \frac{\Delta g}{g}$. Set $f_\varepsilon = f + \varepsilon\varphi$. For any $\varepsilon > 0$, we obtain $\frac{\Delta f_\varepsilon}{f_\varepsilon} > \frac{\Delta g}{g}$ whence by the previous case,

$$\sup_{\partial U} \frac{f_\varepsilon}{g} = \sup_{U} \frac{f_\varepsilon}{g}.$$

Letting $\varepsilon \to 0$, we obtain (2.9). $\square$

LEMMA 2.6 (Comparison principle). *Let $U \subset M$ be a precompact open set. Assume that functions $v_1, v_2 \in C^2(U) \cap C(\overline{U})$ such that $0 \le v_1, v_2 \le 1$, satisfy in U the equation*

$$\Delta v + P(x, 1 - v)(1 - v)v = 0,$$

and $v_1 \le v_2$ on ∂U. Then $v_1 \le v_2$ in U.

PROOF. Assume on the contrary that the set

$$W := \{x \in U : v_1(x) > v_2(x)\}$$

is nonempty. For any $v = v_1$ or v_2, we have

$$\frac{\Delta v}{v} = -P(x, 1 - v)(1 - v).$$

It is clear from (2.4) that $-P(x, 1-v)(1-v)$ is an increasing function of v. Therefore, in W we have

$$\frac{\Delta v_1}{v_1} \geq \frac{\Delta v_2}{v_2}.$$

By Lemma 2.5, we conclude that

$$\sup_W \frac{v_1}{v_2} = \sup_{\partial W} \frac{v_1}{v_2} = 1,$$

whence $v_1 \leq v_2$ in W, which contradicts the assumption. □

COMPLETING THE PROOF OF PROPOSITION 2.4. Take any exhausting sequence $\{U_l\}$ of open sets and observe that the sequence $\{\psi_K^{U_l}\}$ increases by Lemma 2.6 and converges to ψ_K. For any solution v to (2.7), we obtain by Lemma 2.6 $\psi_K^{U_l} \leq v$ whence $\psi_K \leq v$, which was to be proved. □

LEMMA 2.7 (Strong maximum principle). *Let $U \subset M$ be a connected open set. Assume that a function, $v \in C^2(U)$ such that $0 \leq v \leq 1$, satisfies in U the equation*

$$\Delta v + P(x, 1-v)(1-v)v = 0.$$

If $v(x_0) = 1$, at some point $x_0 \in U$ then $v \equiv 1$ in U.

PROOF. Set $u := 1 - v$; $V(x) := P(x, 1-v)v$ and observe that u satisfies in U the equation

$$\Delta u - V(x)u = 0,$$

and $u(x_0) = 0$. Let D be a precompact neighborhood of x_0 with smooth boundary and such that $\overline{D} \subset U$. By the Feynman–Kac formula, we have

$$u(x_0) = \mathbb{E}_{x_0}\left[\exp\left(-\int_0^\tau V(X_t)\,dt\right)u(X_\tau)\right], \tag{2.13}$$

where τ is the first time the Brownian motion X_t hits ∂D. Clearly, if u is not identical 0 on ∂D then by (2.13) $u(x_0) > 0$; hence, $u \equiv 0$ on ∂D. Varying D we see that $u \equiv 0$ in some neighborhood of x_0. Using the connectedness of U we obtain by the standard argument that $u \equiv 0$ in U; that is, $v \equiv 1$. □

As a consequence, we obtain that on any connected component of $\Omega = M \setminus K$, either $\psi_K \equiv 1$ or $\psi_K < 1$.

Finally, let us observe that ψ_K is a superharmonic function in M. Indeed, ψ_K is superharmonic in Ω because it follows from (2.7) that $\Delta\psi_K \leq 0$ in Ω. Since on K, ψ_K is identically equal to its maximal value 1, we conclude that ψ_K is superharmonic on M. Consequently, ψ_K satisfies the strong minimum principle.

3. K-gauge. Consider the differential operator on M,

$$L := \Delta + q(x),$$

where q is defined by (2.3). Observe that if v satisfies (2.7) then $P(x, 1-v) \leq q(x)$ implies $Lv \geq 0$. In particular, $L\psi_K \geq 0$ in $\Omega := M \setminus K$, that is, ψ_K is a L-subharmonic function in Ω.

Consider the process $\overline{X}_t^\Omega$ that stops at K, and denote by N_K the (random) number of the offspring of $\overline{X}_t^\Omega$ that reach K. Note that every individual particle of $\overline{X}_t^\Omega$ either reaches K (and hence is counted for N_K) or goes away to ∞. Alternatively, N_K is the number of the *branches* (not offspring) of the process $\overline{X}_t$, that ever hit K.

DEFINITION 3.1. For any $x \in M$, let us set $m_K(x) := \overline{\mathbb{E}}_x N_K$ and refer to the function $m_K(x)$ as *the K-gauge* of the process $\overline{X}_t$ (or of the operator L).

Clearly, we always have $m_K \geq \psi_K$. Two other equivalent definitions of m_K are given in the following statement. Let τ_K be the first time the Brownian motion X_t hits K, that is,

$$\tau_K := \inf\{t > 0 : X_t \in K\}.$$

PROPOSITION 3.2.
(a) *We have*

$$m_K(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\tau_K<\infty\}} \exp\left(\int_0^{\tau_K} q(X_t)\,dt\right)\right]. \tag{3.1}$$

(b) *Consider the following boundary value problem in* Ω:

$$\begin{aligned} Lf &= 0, \\ f|_{\partial K} &= 1. \end{aligned} \tag{3.2}$$

If $m_K < \infty$ then m_K is the minimal positive solution to (3.2). *Otherwise* (3.2) *has no positive solutions.*

The proof is standard and follows from the Markov property and the Feynman–Kac formula. The part (b) shows that although the K-gauge m_K is originally defined via the process $\overline{X}_t$, it is fully determined by the operator L. Various properties of the gauge for bounded domains in $\mathbb{R}^n$ are studied in [3].

Denote by $h_K(x)$ the $\mathbb{P}_x$-probability that X_t ever hits K, that is,

$$h_K(x) := \mathbb{P}_x(\tau_K < \infty). \tag{3.3}$$

Outside K, $h_K(x)$ can be alternatively defined as the minimal positive solution to the following boundary value problem:

$$\begin{aligned} \Delta h &= 0 \qquad \text{in } \Omega, \\ h|_{\partial K} &= 1. \end{aligned}$$

Let G_Ω be the Green function of the operator Δ in Ω with the Dirichlet boundary condition on $\partial\Omega$; that is, G_Ω is the infimum of all positive fundamental solutions to Δ in Ω. It may happen that there is no positive fundamental solution on Ω, in which case $G_\Omega \equiv \infty$. However, if K has a nonempty interior then G_Ω is finite and is the minimal positive fundamental solution of Δ in Ω. Moreover, G_Ω vanishes at all regular points of $\partial\Omega$.

LEMMA 3.3. *If $m_K < \infty$ then m_K satisfies the following identity*: *for any $x \in \Omega$*,

$$m_K(x) = h_K(x) + \int_\Omega G_\Omega(x, y) m_K(y) q(y)\, d\mu(y). \tag{3.4}$$

Furthermore, *in this case either $m_K \geq 1$ in Ω or*

$$\liminf_{x\to\infty} m_K(x) = 0, \tag{3.5}$$

where $x \to \infty$ means any sequence of x's leaving any compact set.

PROOF. Let $\{U_l\}$ be an exhausting sequence of precompact open sets with smooth boundaries, $K \subset U_0$; set $\Omega_l = U_l \setminus K$. Denote by $m_l(x)$ the solution to the following problem in Ω_l:

$$\begin{aligned} &\Delta m_l + q m_l = 0, \\ &m_l|_{\partial K} = 1, \ m_l|_{\partial U_l} = 0. \end{aligned} \tag{3.6}$$

Then $m_l \to m_K$ as $l \to \infty$. On the other hand, m_l can be represented as $w_l + h_l$ where the functions w_l and h_l solve the following problems in Ω_l:

$$\begin{aligned} &\Delta h_l = 0, \\ &h_l|_{\partial K} = 1, \ h_l|_{\partial U_l} = 0 \end{aligned} \quad \text{and} \quad \begin{aligned} &\Delta w_l = -q m_l, \\ &w_l|_{\partial K} = 0, \ w_l|_{\partial U_l} = 0. \end{aligned}$$

In particular, we obtain

$$w_l(x) = \int_{\Omega_l} G_{\Omega_l}(x, y) m_l(y) q(y)\, d\mu(y)$$

whence

$$m_l(x) = h_l(x) + \int_{\Omega_l} G_{\Omega_l}(x, y) m_l(y) q(y)\, d\mu(y). \tag{3.7}$$

As $l \to \infty$, the sequences $\{m_l\}$, $\{h_l\}$ and $\{G_{\Omega_l}\}$ are increasing (by the comparison principle) so that we can pass to the limit in (3.7) and obtain (3.4).

Set $F = m_K q$ and consider the following boundary value problem in Ω:

$$\begin{aligned} &\Delta v = -F, \\ &v|_{\partial K} = 1. \end{aligned} \tag{3.8}$$

Denote by $v_{\min}$ the minimal nonnegative solution of (3.8) if it exists. The proof will be complete if we verify the following two claims:

(i) $v_{\min}$ exists and is equal to m_K.
(ii) If $\inf h_K < 1$ then $\liminf_{x\to\infty} v_{\min} = 0$.

Indeed, if $h_K \geq 1$ then (3.7) implies $m_k \geq 1$; otherwise, (ii) implies (3.5).

Proof of (i). Using an exhausting sequence $\{U_l\}$ as above, consider function v_l solving the following problem in Ω_l:

$$\Delta v_l = -F,$$
$$v_l|_{\partial K} = 1, \; v_l|_{\partial U_l} = 0.$$

Clearly, we have

$$v_l(x) = h_l(x) + \int_{\Omega_l} G_{\Omega_l}(x, y) F(y)\, d\mu(y).$$

By the comparison principle, $v \geq v_l$ for any nonnegative solution v of (3.8). Letting $l \to \infty$, we obtain

$$v(x) \geq h_K(x) + \int_{\Omega} G_{\Omega}(x, y) F(y)\, d\mu(y) = m_K(x).$$

On the other hand, m_K does satisfy (3.8), whence $m_K = v_{\min}$.

Proof of (ii). Since $v_{\min}$ is a superharmonic function, the minimum principle implies

$$\liminf_{x\to\infty} v_{\min} = \inf v_{\min}.$$

If $\inf v_{\min} > \varepsilon > 0$, then consider the function

$$v = v_{\min} - \varepsilon(1 - h_K).$$

Clearly, v is positive and satisfies (3.8). Also, $v < v_{\min}$ as $h_K < 1$, which contradicts the minimality of $v_{\min}$. $\square$

Now we can prove the following dichotomy in the case of a finite K-gauge.

THEOREM 3.4. *Assume $m_K < \infty$. Then the following dichotomy takes place*:

(i) *Either both $\overline{X}_t$ and X_t are* (K-)*transient*, *and* $\liminf_{x\to\infty} m_K(x) = 0$;
(ii) *or both $\overline{X}_t$ and X_t are* (K-)*recurrent*, *and* $m_K \geq 1$.

In particular, *if X_t is transient and $m_K < \infty$ then $\overline{X}_t$ is also transient*.

PROOF. Indeed, if X_t is transient then $h_K < 1$ in Ω. Hence, by Lemma 3.3, $\liminf_{x\to\infty} m_K(x) = 0$. Since $\psi_K \leq m_K$, this implies also that $\overline{X}_t$ is K-transient.

If X_t is recurrent then obviously $\overline{X}_t$ is recurrent. Therefore, $h_K \equiv 1$ and (3.4) implies $m_K \geq 1$. $\square$

EXAMPLE 3.1. Let K be the ball in $\mathbb{R}^n$ of radius $R > 0$ centered at the origin. Assume that the function $q(x)$ depends only on $r := |x|$ so that we write $q(r)$ for $q(x)$. Then the K-gauge m_K of L also depends only on the radius r; set $m_K(x) = m(r)$. More precisely, the function $m(r)$ is the minimal positive solution to the equation

$$m'' + \frac{n-1}{r} m' + q(r)m = 0$$

in $(R, +\infty)$ with the boundary condition $m(R) = 1$.

Consider first the case $n > 2$ and the function

$$q(r) := \frac{c}{r^2}.$$

If $c \le \frac{(n-2)^2}{4}$ then the minimal positive solution to the above problem is

$$m(r) = \left(\frac{R}{r}\right)^{\nu},$$

where $\nu = \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - c}$. Hence, for such c, the K-gauge $m(r)$ is decreasing and $\inf m = 0$. By Theorem 3.4, the process $\overline{X}_t$ is transient. As we will see in Section 9, if $c > \frac{(n-2)^2}{4}$ then $\overline{X}_t$ is recurrent.

Consider now the case $n = 2$ and the function

$$q(r) = \frac{c}{r^2 \log^2 r},$$

assuming $R > 1$ and $0 < c \le 1/4$. Then there exists $0 < \alpha < 1$ such that $\alpha(1-\alpha) = c$, and an easy computation shows that

$$m(r) = \left(\frac{\log r}{\log R}\right)^{\alpha}.$$

Hence, in this case the K-gauge $m(r)$ is increasing and $\inf m = 1$. Of course, in $\mathbb{R}^2$ any branching process is recurrent.

4. Transience and eigenvalues. For any open set $U \subset M$, define $\lambda(U)$ as the bottom of the L^2-spectrum of the operator $-L$ in U with the Dirichlet boundary condition. In other words,

$$\lambda(U) := \inf_{\varphi} \frac{-(\varphi, L\varphi)_{L^2(U,\mu)}}{(\varphi,\varphi)_{L^2(U,\mu)}} = \inf_{\varphi} \frac{\int(|\nabla\varphi|^2 - q\varphi^2)\,d\mu}{\int \varphi^2\,d\mu}, \tag{4.1}$$

where the inf is taken over all nonzero functions $\varphi \in \mathrm{Lip}_0(U)$. Here $\mathrm{Lip}_0(U)$ is the set of all Lipschitz functions compactly supported in U; the gradient ∇ is understood in the weak sense. If U is precompact then $\lambda(U)$ is the bottom eigenvalue of the following problem in U:

$$\begin{aligned} &Lu + \lambda u = 0,\\ &u|_{\partial U} = 0. \end{aligned}$$

Assuming $q(x) > 0$ on M, define, for any open set $U \subset M$ and a precompact set $K \Subset U$,

$$\nu(K, U) := \inf_{\varphi} \frac{\int_{U\setminus K} |\nabla\varphi|^2 \, d\mu}{\int_{U\setminus K} q\varphi^2 \, d\mu}, \tag{4.2}$$

where the inf is taken over all nonzero functions $\varphi \in \mathrm{Lip}_0(U)$. If U is precompact then $\nu(K, U)$ is the bottom eigenvalue of the following problem in $U \setminus \overline{K}$:

$$\begin{aligned} &\Delta u + \nu q u = 0, \\ &u|_{\partial U} = 0, \ \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial K} = 0, \end{aligned} \tag{4.3}$$

where $\mathbf{n}$ is the inward normal vector field on ∂K (assuming that ∂K is smooth enough).

THEOREM 4.1. *Let $q \not\equiv 0$. Then each of the following conditions implies that $\overline{X}_t$ is transient*:

(i) *There exists a positive L-superharmonic function on M.*

(ii) *Operator L has a positive Green function $G^L(x, y)$, such that $G^L(x, y) < \infty$ for all $x \neq y$.*

(iii) $\lambda(U) > 0$ *for any precompact region $U \subset M$.*

(iv) $q > 0$ *on M and, for some nonempty compact $K \subset M$ with smooth boundary*,

$$\nu(K, M) \geq 1. \tag{4.4}$$

PROOF. (i) Let u be a positive L-superharmonic function on M. Since $\Delta u \leq -qu \leq 0$, u is strictly positive by the strong minimum principle. The hypothesis $q \not\equiv 0$ implies that $u \not\equiv \mathrm{const}$. Hence, M admits a nonconstant positive Δ-superharmonic function u, which means that the Brownian motion X_t is transient.

Let K be any compact set with nonempty interior and smooth boundary. Without loss of generality, we may assume $u \geq 1$ on K. Let $\{U_l\}_{l\geq 1}$ be an exhaustive sequence of precompact open sets with smooth boundaries, such that $K \subset U_l$. Since L admits a global positive supersolution, the Dirichlet problem for L can be solved in any precompact open set (e.g., by Perron's method). Let f_l solve the following Dirichlet problem in $U_l \setminus K$:

$$\begin{aligned} &L f_l = 0, \\ &f_l|_{\partial K} = 1, \ f_l|_{\partial U_l} = 0. \end{aligned}$$

By the strong minimum principle, $f_l > 0$ in $U_l \setminus K$. By the comparison principle of Lemma 2.5, $f_l \leq u$. Again by the comparison principle, the sequence $\{f_l\}$ increases and converges to a positive function f on $M \setminus K$, which solves the exterior problem (3.2). Therefore, $m_K \le f \le u < \infty$, and $\overline{X}_t$ is K-transient by Theorem 3.4.

Alternatively, one can conclude the proof without referring to Theorem 3.4. Indeed, replacing u by $u - \inf u$, one can assume from the very beginning that $\inf u = 0$. Hence, $\inf m_K = 0$, which implies $\inf \psi_K = 0$ and the K-transience of the process $\overline{X}_t$.

(ii) Take any nonnegative nonzero function $f \in C_0^\infty(M)$ and set

$$u(x) = \int_M G^L(x, y) f(y)\, d\mu(y). \tag{4.5}$$

The Green function G^L has the same local singularity as the classical fundamental solution of the Laplace operator in the Euclidean space (see [19]); hence, the integral in (4.5) converges. Since $Lu = -f \le 0$, the function u is positive and L-superharmonic, whence (i) is satisfied.

(iii) The condition $\lambda(U) > 0$ implies that L satisfies the maximum principle in any precompact region U and that the Dirichlet problem in U,

$$Lu = 0,$$
$$u|_{\partial U} = \varphi,$$

has a unique (weak) solution for any $\varphi \in C(\partial U)$ (see [2, 16]). Take an exhaustive sequence $\{U_l\}$ and construct a positive function u_l in U_l such that $Lu_l = 0$ (e.g., by solving a Dirichlet problem in U_l with any positive boundary data). Fix a point x_0 which belongs to all U_l and normalize u_l so that $u_l(x_0) = 1$. The sequence $\{u_l\}$ is compact in the local sup-topology (see [16]) and hence has a subsequence converging locally uniformly to a nonnegative function u satisfying $Lu = 0$ on M. Since $u(x_0) = 1$, u is strictly positive by the strong minimum principle. Hence, (i) is satisfied.

(iv) The condition (4.4) implies that, for any precompact open set U containing K,

$$\nu := \nu(K, U) > 1.$$

Then, for any $\varphi \in \mathrm{Lip}_0(U)$,

$$\int_{U\setminus K} |\nabla\varphi|^2\, d\mu \ge \nu \int_{U\setminus K} q\varphi^2\, d\mu,$$

whence

$$\frac{\int_{U\setminus K} (|\nabla\varphi|^2 - q\varphi^2)\, d\mu}{\int_{U\setminus K} \varphi^2\, d\mu} \ge (\nu - 1)\frac{\int_{U\setminus K} q\varphi^2\, d\mu}{\int_{U\setminus K} \varphi^2\, d\mu} \ge (\nu - 1) \inf_{U\setminus K} q. \tag{4.6}$$

Let us use the extended notation $\lambda(U, M)$ for the bottom of the spectrum of the operator L in U defined by (4.1). Then

$$\inf_{\varphi \in \mathrm{Lip}_0(U)} \frac{\int_{U\setminus K} (|\nabla\varphi|^2 - q\varphi^2)\,d\mu}{\int_{U\setminus K} \varphi^2\,d\mu} = \lambda(U', M')$$

where $U' := U \setminus \overset{o}{K}$, and $M' := M \setminus \overset{o}{K}$ is the manifold with boundary ∂K.

By (4.6) we conclude that $\lambda(U', M') > 0$. By part (ii) applied to the manifold M' with boundary, the process $\overline{X}'_t$ on M' with reflection on ∂K is transient. Since $\overline{X}_t$ and $\overline{X}'_t$ coincide outside K, the process $\overline{X}_t$ on M is also transient. $\square$

REMARK 4.1. By [16, 22], the condition (iii) is equivalent to the existence of a positive solution to the equation $\Delta u + qu = 0$ on M.

REMARK 4.2. The finiteness of G^L is not necessary for the transience of $\overline{X}_t$ (see Example 3.1 with $c = \frac{(n-2)^2}{4}$ and Example 9.1 in Section 9).

REMARK 4.3. As follows from Theorem 4.1, if $P(x, u) \not\equiv 0$ and $\lambda(U) > 0$ for any precompact region U then the problems (2.6), (2.7) have nontrivial solutions. Although this result is obvious from the probabilistic approach adopted here, it can be also obtained by purely analytic methods.

The following statement is related to Theorem 4.1(iii).

PROPOSITION 4.2. *If $m_K < \infty$ then $\lambda(U) > 0$, for any precompact open set $U \subset \Omega$.*

REMARK 4.4. It could happen that m_K is finite while $\overline{X}_t$ is recurrent. Theorem 4.1(v) and Proposition 4.2 imply that in this case $\lambda(U) \le 0$ for some precompact open set U which intersects K.

PROOF OF PROPOSITION 4.2. Without loss of generality, we can assume that U has smooth boundary. Let u be the first Dirichlet eigenfunction of L in U; we can assume that $u > 0$ in U. Suppose that $\lambda := \lambda(U) \le 0$. Since $Lm_K = 0$ and $Lu + \lambda u = 0$, we obtain in U

$$\frac{Lm_K}{m_K} = 0 \le -\lambda = \frac{Lu}{u}.$$

By Lemma 2.5, we conclude

$$\sup_{\partial U} \frac{u}{m_K} = \sup_{U} \frac{u}{m_K}.$$

However, the right-hand side here is positive whereas the left-hand side vanishes by $u|_{\partial U} = 0$. This contradiction completes the proof. $\square$

Finally, let us show on a diagram the relations between the hypotheses considered in this and the previous sections (assuming that $q > 0$ and X_t is transient):

$$
\begin{array}{ccccc}
\boxed{\nu(K,M)\ge 1} & \Longrightarrow & \boxed{\lambda(U)>0 \quad \forall U \Subset M} & & \\
 & & \Downarrow & & \\
\boxed{G^L<\infty} & \Longrightarrow & \boxed{\exists u>0: Lu\le 0} & & \\
 & & \Downarrow & & \\
 & & \boxed{m_K<\infty} & \Longrightarrow & \boxed{\overline{X}_t \text{ is } K\text{-transient}}. \\
 & & \Downarrow & & \\
 & & \boxed{\lambda(U)>0 \quad \forall U\Subset\Omega} & &
\end{array} \tag{4.7}
$$

5. Some properties of the hitting probability. Here we establish a relation between the hitting probabilities h_K and ψ_K, and prove some properties of ψ_K. Recall that ψ_K is defined in Definition 2.1 whereas h_K is defined by (3.3).

Let $K \subset M$ be a closed set (so far not necessarily compact) with nonempty interior and with smooth boundary. Let $\{U_l\}_{l=1}^{\infty}$ be an exhausting sequence in M. Set $\Omega = M \setminus K$, $\Omega_l = \Omega \cap U_l$ and consider the following functions:

$$\psi_l = \psi_K^{\Omega_l}, \qquad G_l = G_{\Omega_l}$$

and

$$f_l = P(\cdot, 1 - \psi_l)(1 - \psi_l), \qquad f_K = P(\cdot, 1 - \psi_K)(1 - \psi_K). \tag{5.1}$$

Let h_l be a (weak) solution to the following boundary value problem in Ω_l:

$$\begin{aligned} &\Delta h_l = 0, \\ &h_l|_{\partial K} = 1, \ h_l|_{\partial U_l} = 0. \end{aligned}$$

LEMMA 5.1. *In the above notation we have, for any $x \in \Omega_l$,*

$$\psi_l(x) = h_l(x) + \int_{\Omega_l} G_l(x, \cdot) f_l \psi_l \, d\mu. \tag{5.2}$$

PROOF. Indeed, ψ_l solves the boundary value problem in Ω_l,

$$\begin{aligned} &\Delta \psi_l + f_l \psi_l = 0, \\ &\psi_l|_{\partial K} = 1, \ \psi_l|_{\partial U_l} = 0. \end{aligned}$$

Therefore, the difference $w = \psi_l - h_l$ satisfies

$$\begin{aligned} &\Delta w = -f_l \psi_l, \\ &w|_{\partial K} = 0, \ w|_{\partial U_l} = 0, \end{aligned}$$

whence we obtain $w = G_l(f_l \psi_l)$ that is equivalent to (5.2). $\square$

LEMMA 5.2. *Let $K \subset M$ be a compact set. If x is a point in $\Omega = M \setminus K$ such that $\psi_K(x) < 1$ then*

$$\psi_K(x) = h_K(x) + \int_\Omega G_\Omega(x,\cdot)P(\cdot, 1-\psi_K)(1-\psi_K)\psi_K\,d\mu. \tag{5.3}$$

PROOF. Using the above notation (5.3) can be rewritten in the form

$$\psi_K(x) = h_K(x) + \int_\Omega G_\Omega(x,\cdot) f_K \psi_K\,d\mu. \tag{5.4}$$

Let us show that (5.4) follows from (5.2) by passing to the limit as $l \to \infty$. Since $\psi_l \to \psi_K$, $f_l \to f_K$, $h_l \to h_K$ and $G_l \to G_\Omega$, we obtain, by Fatou's lemma,

$$\psi_K(x) \geq h_K(x) + \int_\Omega G_\Omega(x,\cdot) f_K \psi_K\,d\mu. \tag{5.5}$$

We cannot use the monotone convergence theorem here because the sequences $\{G_l\}$ and $\{\psi_l\}$ are increasing while $\{f_l\}$ is decreasing as one can see from (5.1) and (2.4). Without the hypothesis $\psi_K(x) < 1$, it can actually happen that $h_K(x) < 1$ and $\psi_K \equiv 1$; in this case, $f_K \equiv 0$ and the strict inequality takes place in (5.5).

However, in any case (5.5) implies

$$\int_\Omega G_\Omega(x,\cdot) f_K \psi_K\,d\mu \leq 1, \tag{5.6}$$

which will help us to justify below the use of the dominated convergence theorem, assuming $\psi_K(x) < 1$.

Let Ω' be the connected component of Ω that contains the point x. Since $G_\Omega(x,\cdot) = 0$ away from Ω', the integration in (5.4) [and in (5.2)] can be restricted to Ω'.

It follows from the strong maximum principle (cf. Lemma 2.7) that $\psi_K < 1$ in Ω'. Let K' be a precompact open neighborhood of K. Then there exists $0 < \delta < 1$, such that $\psi_K \leq 1 - \delta$ on $\partial K' \cap \Omega'$ and hence

$$\psi_K \leq 1 - \delta \qquad \text{in } \Omega' \setminus K'.$$

This inequality, (5.1) and (2.5) imply that in $\Omega' \setminus K'$ the following inequalities hold:

$$f_l \leq P(\cdot, 1) \leq C_0 P(\cdot, 1-\psi_K) \leq C_0 \delta^{-1} f_K,$$

where C_0 is the constant from the hypothesis (B). Therefore, we have in $\Omega' \setminus K'$

$$G_l(x,\cdot) f_l \psi_l \leq C_0 \delta^{-1} G_\Omega(x,\cdot) f_K \psi_K.$$

By (5.6), the right-hand side here is integrable in Ω' so that we can apply the dominated convergence theorem that yields

$$\int_{\Omega'\setminus K'} G_l(x,\cdot) f_l \psi_l\,d\mu \to \int_{\Omega'\setminus K'} G_\Omega(x,\cdot) f_K \psi_K\,d\mu. \tag{5.7}$$

Since $f_l \le P(\cdot, 1) = q$ and hence

$$G_l(x, \cdot) f_l \psi_l \le G_\Omega q,$$

the compactness of K' implies that the sequence $\{G_l(x, \cdot) f_l \psi_l\}$ is uniformly bounded in $K' \setminus K$ by the integrable function $G_\Omega q$. Hence, applying again the dominated convergence theorem, we obtain

$$\int_{K' \setminus K} G_l(x, \cdot) f_l \psi_l \, d\mu \to \int_{K' \setminus K} G_\Omega(x, \cdot) f_K \psi_K \, d\mu.$$

Combining with (5.7) we conclude that we can pass to the limit in (5.2), whence (5.4) follows. □

LEMMA 5.3. *Let $K \subset M$ be a closed set such that for some $x \in \Omega = M \setminus K$,*

$$\int_\Omega G_\Omega(x, \cdot) q \, d\mu < \infty. \tag{5.8}$$

Then, for this point x,

$$\psi_K(x) = h_K(x) + \int_\Omega G_\Omega(x, \cdot) P(\cdot, 1 - \psi_K)(1 - \psi_K)\psi_K \, d\mu. \tag{5.9}$$

PROOF. Using the notation from Lemma 5.1, all we need is to justify the passage to the limit in the identity [cf. (5.2)],

$$\psi_l(x) = h_l(x) + \int_{\Omega_l} G_l(x, \cdot) P(\cdot, 1 - \psi_l)(1 - \psi_l)\psi_l \, d\mu. \tag{5.10}$$

Indeed, we have $G_l \le G_\Omega$; $\psi_l \le 1$, and by (2.5) $P(\cdot, 1 - \psi_l) \le q$. Therefore, the integrand in (5.10) is bounded from above by the integrable function $G_\Omega(x, \cdot) q$, and the claim follows by the dominated convergence theorem. □

LEMMA 5.4. *We have*

$$\liminf_{x \to \infty} \psi_K(x) = 0, \tag{5.11}$$

provided one of the following two conditions holds:

(a) *Either K is compact and $\psi_K \not\equiv 1$;*
(b) *or K is closed*, (5.8) *holds for all $x \in \Omega$, and $h_K \not\equiv 1$.*

PROOF. In the case (a), let Ω' be a component of Ω such that $\psi_K < 1$ in Ω'. By the minimum principle, Ω' is noncompact. By Lemma 5.2 we have, for any $x \in \Omega'$,

$$\psi_K(x) = h_K(x) + \int_\Omega G_\Omega(x, \cdot) F \, d\mu, \tag{5.12}$$

where

$$F := P(\cdot, 1 - \psi_K)(1 - \psi_K)\psi_K.$$

In the case (b), let Ω' be a component of Ω such that $h_K < 1$ in Ω'. Clearly, Ω' is noncompact. By Lemma 5.3, we have (5.12) for all $x \in \Omega$, in particular for all $x \in \Omega'$.

In both cases, consider the following boundary value problem in Ω':

$$\begin{aligned} \Delta v &= -F, \\ v|_{\partial\Omega'} &= 1. \end{aligned} \tag{5.13}$$

In the same way as in the proof of Lemma 3.3, one can verify that the minimal nonnegative solution $v_{\min}$ to (5.13) exists and is given by

$$v_{\min}(x) = h_K(x) + \int_{\Omega'} G_{\Omega'}(x, \cdot)F\, d\mu.$$

Comparing with (5.12) we obtain $v_{\min} \equiv \psi_K$ in Ω'.

Let us prove that

$$\inf_{\Omega'} v_{\min} = 0. \tag{5.14}$$

Assuming on the contrary $\inf \psi_K > \varepsilon > 0$, consider the function

$$v := v_{\min} - \varepsilon(1 - h_K),$$

which is also a positive solution to (5.13). Observe that

$$\inf_{\Omega'} h_K < 1,$$

which in the case (a) follows from $h_K \le \psi_K$, whereas in the case (b) this is true by the hypothesis. Therefore, the solution v is smaller than $v_{\min}$, which contradicts the minimality of $v_{\min}$.

Since Ω' is noncompact, ψ_K is superharmonic in Ω', and $\psi_K > 0$ in $\overline{\Omega'}$, (5.11) follows from (5.14) by the strong minimum principle. □

SECOND PROOF OF (a). Unlike the first proof, this proof does not use the hypothesis (2.2). Let $\{K_l\}_{l \ge 0}$ be an exhaustive sequence of compact sets such that $K_0 = K$. Assuming $\inf v_{\min} > \varepsilon > 0$, the minimum principle implies $\psi_K > \varepsilon$ everywhere. Therefore, for any index $l \ge 1$, there exists a time T_l such that

$$\forall x \in K_l, \qquad \overline{\mathbb{P}}_x\big(\exists t < T_l : \overline{X}_t \cap K \ne \varnothing\big) > \varepsilon/2. \tag{5.15}$$

Define $T(x) = T_l$ if $x \in K_l \setminus K_{l-1}$. Define a (random) sequence $\{x_n\}$ of points as follows. Set $x_0 = x \notin K$ and define x_{n+1} as the nearest to K point of $\overline{X}_{T(x_n)}$ (with the starting point at x_n). The sequence $\{x_n\}$ stops as soon as one offspring hits K before $T(x_n)$.

We claim that

$$\overline{\mathbb{P}}_x(x_n \notin K) \le (1-\varepsilon/2)^n. \tag{5.16}$$

If by the lemma of Borel–Cantelli one of x_n is in K with probability 1, this completes the proof.

For $n = 1$ (5.16) is true by (5.15). For the inductive step, we use the Markov property as follows:

$$\begin{aligned}\overline{\mathbb{P}}_x(x_{n+1} \notin K) &= \sum_{l\ge 1} \overline{\mathbb{P}}_x(x_{n+1} \notin K \text{ and } x_n \in K_l \setminus K_{l-1})\\ &\le \sum_{l\ge 1} \sup_{y\in K_l\setminus K_{l-1}} \overline{\mathbb{P}}_y(x_1 \notin K)\overline{\mathbb{P}}_x(x_n \in K_l \setminus K_{l-1})\\ &\le (1-\varepsilon/2)\sum_{l\ge 1}\overline{\mathbb{P}}_x(x_n \in K_l\setminus K_{l-1})\\ &= (1-\varepsilon/2)\overline{\mathbb{P}}_x(x_n \notin K)\\ &\le (1-\varepsilon/2)^{n+1},\end{aligned}$$

which was to be proved. $\square$

6. Transience and the Green function. In this section, $K \subset M$ is a compact set with nonempty interior, $\Omega = M \setminus K$.

DEFINITION 6.1. We say that a set $S \subset M$ is *thin* if $h_S(x) < 1$ for some $x \in M$.

Observe that if S is thin then $\inf h_S = 0$. Some conditions for thinness and examples of thin sets can be found in [9]. For example, if X_t is transient then any compact set is thin.

THEOREM 6.2. *Let $K \subset M$ be a compact set with nonempty interior. The process $\overline{X}_t$ is K-transient provided either of the following two conditions takes place* (*assuming $q \not\equiv 0$*):

(i) *X_t is transient and there exists a thin open set S such that, for all $x \in M$,*

$$\int_{M\setminus S} G(x,y)q(y)\,d\mu(y) < \infty. \tag{6.1}$$

(ii) *There exists a positive bounded function $u \in C^2(M)$ satisfying on M the inequality*

$$\Delta u - qu \ge 0. \tag{6.2}$$

REMARK 6.1. Condition (6.1) is trivially satisfied if

$$\int_M G(x, y)q(y)\, d\mu(y) < \infty \tag{6.3}$$

as one can take $S = \varnothing$. Another trivial example is when $\operatorname{supp} q$ is thin. Indeed, take S to be a small neighborhood of $\operatorname{supp} q$, which is still thin. Clearly, the integral in (6.1) vanishes. The latter example shows that even if the values of q may be arbitrarily large, the process $\overline{X}_t$ remains transient because the support of q is small enough.

Observe that by making q large enough within a compact set $U \subset \Omega$, one can achieve $\lambda(U) < 0$ and hence $m_K = \infty$ (see Proposition 4.2). In this case, Theorem 3.4 is not applicable whereas Theorem 6.2 guarantees the transience.

REMARK 6.2. Let us recall for comparison that, by Theorem 4.1, if $q \not\equiv 0$ and if there exists a positive function $v \in C^2(M)$ satisfying on M the inequality

$$\Delta v + qv \leq 0,$$

then $\overline{X}_t$ is transient.

REMARK 6.3. The conditions (i) and (ii) are in fact equivalent. Indeed, the function u satisfying (6.2) is a bounded nonconstant subharmonic function on M. The existence of such a function implies that the Brownian motion X_t is transient (see [8], Theorem 5.1). By [9], Theorem 4.1, under the transience of the Brownian motion, the existence of a positive bounded solution to (6.2) is equivalent to the existence of an open thin set S satisfying (6.1).

Nevertheless, below we give independent proofs of (i) and (ii) as both are short.

PROOF OF THEOREM 6.2. (i) Without loss of generality, we can assume that the boundary of S is smooth. Fix some $0 < \varepsilon < 1$ and denote $S' = \{x \in M : h_S(x) > \varepsilon\}$. The set S' contains $\overline{S}$ and is also thin because $h_{S'} \leq \varepsilon^{-1} h_S$. Consider the cut-off function $\varphi \in C(M)$ such that

$$\varphi(x) = \begin{cases} 1, & x \notin S', \\ 0, & x \in S. \end{cases}$$

Denote $Q' = \varphi Q$, $q' = \varphi q$, and let $\overline{X}'_t$ be the branching process based on X_t with the branching rate Q' and the branching mechanism $\{p_k\}$. Since q' vanishes on S, it follows from (6.1) that

$$\int_M G(x, y)q'(y)\, d\mu(y) < \infty. \tag{6.4}$$

Let $K \subset M$ be a compact set with nonempty interior. The thinness of S' implies that the set $S'' := K \cup S'$ is thin as well (see [8], Proposition 4.2). By Lemma 5.4(b), (6.4) and $\inf h_{S''} < 1$ imply that $\overline{X}'_t$ is S''-transient. However, outside S'' the processes $\overline{X}_t$ and $\overline{X}'_t$ coincide. Therefore, $\overline{X}_t$ is S''-transient as well.

(ii) We can assume that $0 < u < 1$. Since u is a nonconstant subharmonic function, we obtain by the strong maximum principle,

$$c := \max_K u < \sup_M u.$$

Set $w = u - c$. Then w satisfies

$$\begin{aligned} &\Delta w - qw \geq 0 \qquad \text{on } M, \\ &w|_K \leq 0, \\ &0 < \sup_M w < 1. \end{aligned}$$

Let $\{U_l\}$ be an exhausting sequence and set $\varphi_l := 1 - \psi_K^{U_l}$. Then φ_l solves in U_l the problem

$$\begin{aligned} &\Delta\varphi_l - P(x, \varphi_l)\varphi_l(1 - \varphi_l) = 0, \\ &\varphi_l|_{\partial K} = 0, \\ &\varphi_l|_{\partial U_l} = 1 \end{aligned}$$

[cf. (2.6)]. Since $P(x, \cdot) \leq q(x)$, this implies $\Delta\varphi_l - q(x)\varphi_l \leq 0$ in U_l. By the comparison principle for the operator $\Delta - q$ in $U_l \setminus K$, we conclude $w \leq \varphi_l$. As $l \to \infty$ we obtain $w \leq 1 - \psi_K$, whence $\psi_K \not\equiv 1$ and $\overline{X}_t$ is K-transient. $\square$

Set

$$G_{\Omega,q}(x) := \int_\Omega G_\Omega(x, y)q(y)\,d\mu(y).$$

The next statement shows some relations between $G_{\Omega,q}$, m_K, and the transience of $\overline{X}_t$.

PROPOSITION 6.3.

(i) *Assume that for all* $x \in \Omega$,

$$G_{\Omega,q}(x) < \infty. \tag{6.5}$$

Then X_t is transient if and only if $\overline{X}_t$ is transient.

(ii) *Assume that*

$$c := \sup_{x\in\Omega} G_{\Omega,q}(x) < 1. \tag{6.6}$$

Then for all $x \in M$,

$$m_K(x) \leq (1 - c)^{-1} h_K(x). \tag{6.7}$$

Furthermore, if $\inf m_K < 1$ *then* $\overline{X}_t$ *is K-transient, and if* $\inf m_K \geq 1$ *then* $\overline{X}_t$ *is K-recurrent.*

(iii) *Let there exist a nonempty precompact subset D of Ω such that*

$$\inf_{x \in D} \int_D G_\Omega(x, y) q(y) \, d\mu(y) > 1. \tag{6.8}$$

Then $m_K \equiv \infty$.

REMARK 6.4. If one assumes instead of (6.5) a stronger hypothesis (6.3) then (i) follows from Theorem 6.2.

PROOF OF PROPOSITION 6.3. (i) By Lemma 5.3, we have

$$\psi_K(x) = h_K(x) + \int_\Omega G_\Omega(x, y) F(y) \, d\mu(y), \tag{6.9}$$

where

$$F = P(\cdot, 1 - \psi_K)(1 - \psi_K)\psi_K \le q(1 - \psi_K)\psi_K.$$

If X_t is recurrent then $\overline{X}_t$ is trivially K-recurrent. If $\overline{X}_t$ is K-recurrent then $\psi_K \equiv 1$. Substituting this into (6.9) and observing that $F \equiv 0$, we obtain $h_K \equiv 1$, so that X_t is also K-recurrent.

(ii) Denote by J the integral operator on functions on Ω given by

$$Jf(x) = \int_\Omega G_\Omega(x, y) f(y) q(y) \, d\mu(y).$$

Then

$$\sup |Jf| \le c \sup |f|,$$

where c is defined by (6.6). Hence, $\|J\| \le c < 1$ in the sense of the operator norm in $C(\Omega)$. Therefore, the series $\mathrm{Id} + J + J^2 + \cdots$ absolutely converges to $(\mathrm{Id} - J)^{-1}$. The function $(\mathrm{Id} - J)^{-1} h_K$ is the minimal positive solution to the integral equation (3.4), whence

$$m_K = (\mathrm{Id} - J)^{-1} h_K \le (1 - c)^{-1} h_K(x).$$

The second claim follows from Theorem 3.4.

(iii) Assume $m_K < \infty$ and set

$$C = \inf_{x \in D} \int_D G_\Omega(x, y) q(y) \, d\mu(y) > 1.$$

Then by (3.4),

$$m_K(x) \ge h_K(x) + \inf_D m_k \int_D G_\Omega(x, y) q(y) \, d\mu(y),$$

whence

$$\inf_D m_K \ge \inf_D h_K + C \inf_D m_K.$$

However, this is impossible as $C > 1$ and $\inf_D h_K > 0$. $\square$

7. Recurrence and K-gauge. In this section, we introduce an additional assumption about the branching mechanism:

(C) For all $x \in M$,

$$\sum_{k=2}^{k_{\max}} k(k-1)p_k(x) < \infty, \tag{7.1}$$

and the series in (7.1) converges locally uniformly.

Of course, if $k_{\max}$ is finite then (C) trivially holds. Define function $\overline{q}$ by

$$\overline{q}(x) := 2Q(x)\sum_{k=2}^{k_{\max}} k(k-1)p_k(x) \tag{7.2}$$

and observe that $\overline{q}(x)$ is finite and continuous on M. On the other hand, comparing with (2.3), we obtain $\overline{q}(x) \geq 2q(x)$.

Fix a nonempty compact set $K \subset M$ with a smooth boundary, and consider the moment generating function

$$w(x,s) = \sum_{k=0}^{\infty} g_k(x)s^k,$$

where $g_k(x)$ is the probability that the process $\overline{X}_t$ started at the point x will eventually produce k branches that hit K. Recall that if N_K is the (random) number of all branches of $\overline{X}_t$ that ever hit K then the K-gauge of $\overline{X}_t$ is given by

$$m_K(x) := \overline{\mathbb{E}}_x N_K = \sum_{k=1}^{\infty} k g_k(x) = \left.\frac{\partial w}{\partial s}\right|_{s=1}.$$

Similarly, introduce the *quadratic K-gauge* v_K of $\overline{X}_t$ defined by

$$v_K(x) := \overline{\mathbb{E}}_x N_K^2 = \sum_{k=1}^{\infty} k^2 g_k(x) = \left.\frac{\partial^2 w}{\partial s^2}\right|_{s=1} + \left.\frac{\partial w}{\partial s}\right|_{s=1}.$$

It is clear that $m_K(x) \leq v_K(x)$.

LEMMA 7.1. *If $v_K < \infty$ then $v = v_K$ is the minimal positive solution to the boundary value problem in $\Omega := M \setminus K$:*

$$\begin{aligned} Lv &= -\overline{q}m_K^2, \\ v|_{\partial K} &= 1, \end{aligned} \tag{7.3}$$

where $\overline{q}$ is defined by (7.2). *Otherwise*, (7.3) *has no positive solution.*

REMARK 7.1. Hence, the quadratic K-gauge is determined by the pair $(L, \overline{q})$, in contrast to the K-gauge which is fully determined by the operator L.

PROOF OF LEMMA 7.1. The function $w(x,s)$ satisfies the following equation (cf. [4]):

$$\Delta w(x,s) + \eta\big(x, w(x,s)\big) = 0, \tag{7.4}$$

where

$$\eta(x,z) := 2Q(x)\Bigg[\sum_{k=2}^{k_{\max}} p_k(x) z^k - z\Bigg].$$

Differentiating (7.4) with respect to s twice at $s = 1$, we obtain again that m_K satisfies the equation $Lm_K = 0$ and that v_K satisfies the equation in (7.3). The boundary value is obvious. □

Now we will prove the main theorem of this section.

THEOREM 7.2. *Assume that the hypotheses* (A), (B) *and* (C) *hold and X_t is transient. Let $\{K_l\}_{l=0}^{\infty}$ be an exhausting sequence of compact sets in M, each with nonempty interior and smooth boundary. Set*

$$q_l := q\mathbf{1}_{K_{l+1}\setminus K_l} \quad \textit{and} \quad \overline{q}_l := \overline{q}\mathbf{1}_{K_{l+1}\setminus K_l},$$

and denote by m_l the K_l-gauge of the operator $L_l := \Delta + q_l$ and by v_l the quadratic K_l-gauge of the pair $(L_l, \overline{q}_l)$. If for some $\varepsilon > 0$ and all $l = 0, 1, 2, \dots$,

$$\inf_{x\in\partial K_{l+1}} m_l(x) \geq 1 + \varepsilon \quad \textit{and} \quad \sup_{x\in\partial K_{l+1}} v_l(x) \leq \varepsilon^{-1}, \tag{7.5}$$

then the process $\overline{X}_t$ is K_0-recurrent.

PROOF. For any index l, denote by $\overline{X}_t^l$ the process with the branching rate $Q\mathbf{1}_{K_{l+1}\setminus K_l}$ and with the same branching mechanism $\{p_k\}$ as $\overline{X}_t$. Then the branching intensity of $\overline{X}_t^l$ is equal to q_l.

Fix an integer n and construct a random branching tree $\Gamma_n \subset M$ with the root at a point $x^{(0)} \in \partial K_n$ and so that the descendants of jth generation will lie on ∂K_{n-j}. If $x_1^{(j)}, x_2^{(j)}, \dots$ are those descendants then the next generation is constructed as follows. Set $l = n - j - 1$ and consider the independent copies of the process $\overline{X}_t^l$ started at $x_1^{(j)}, x_2^{(j)}, \dots$, respectively. Note that $x_i^{(j)} \in \partial K_{l+1}$. For each process, identify all points on ∂K_l where the branches of the process hit K_l for the first time, and let $x_1^{(j+1)}, x_2^{(j+1)}, \dots$ be all such points, across all starting points (see Figure 1). If there is no hitting of K_l then the tree Γ_n terminates at this step.

Let $x \in \partial K_{l+1}$ be a point of the tree Γ_n, and let N_l be the number the descendants of x in Γ_n. Clearly, we have

$$\overline{\mathbb{E}}_x N_l = m_l(x) \quad \text{and} \quad \overline{\mathbb{E}}_x N_l^2 = v_l(x).$$

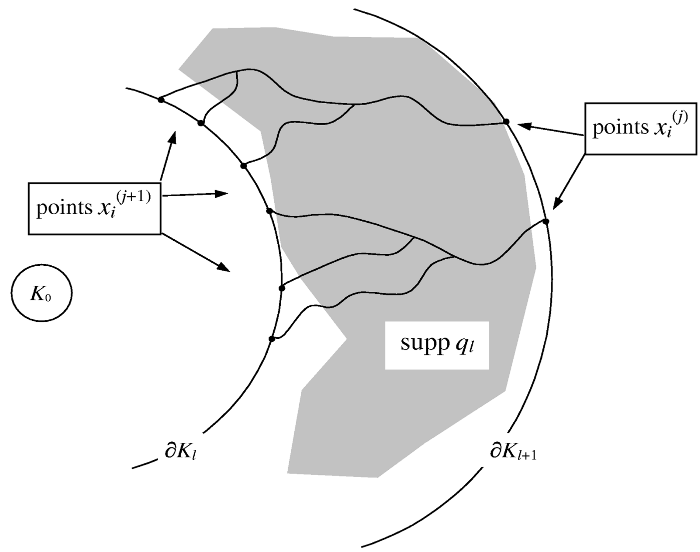

FIG. 1. *Construction of tree* Γ_n.

By (7.5), we have for all $x \in \partial K_{l+1}$,

$$\overline{\mathbb{E}}_x N_l \geq 1 + \varepsilon \quad \text{and} \quad \overline{\mathbb{E}}_x N_l^2 \leq \varepsilon^{-1}.$$

By a version of the theorem of Galton and Watson, one observes, using the uniform upper bound on the second moment, that the tree Γ_n eventually survives with the probability at least $\pi(\varepsilon) > 0$, uniformly in n.

Since the branching rate of $\overline{X}_t$ dominates that of $\overline{X}_t^l$, the process $\overline{X}_t$ started at $x \in \partial K_n$ hits K_0 with the probability at least $\pi(\varepsilon)$. Therefore, $\psi_{K_0}(x) \geq \pi(\varepsilon)$ for all $x \in \partial K_n$ and thus for all $x \in M$. By Lemma 5.4(a), $\psi_{K_0} \equiv 1$, that is, the process $\overline{X}_t$ is K_0-recurrent. $\square$

Under some additional hypotheses, the quadratic K-gauge of the pair $(L, \overline{q})$ can be estimated from above by the K-gauge of the operator $\Delta + 2q$ as is stated in the following lemma.

LEMMA 7.3. *Assume in addition to the hypothesis* (C) *that*

$$C_1 := \sup_x \sum_{k=2}^{k_{\max}} k(k-1) p_k(x) < \infty. \tag{7.6}$$

Let m_K be the K-gauge of the operator $L = \Delta + q$, v_K be the quadratic K-gauge of the operator L and $\overline{m}_K$ be the K-gauge of the operator $\Delta + 2q$. Then the following estimate holds:

$$v_K(x) \leq (C_1 \sup m_K) \overline{m}_K(x), \tag{7.7}$$

for all $x \in M$.

PROOF. If the right-hand side of (7.7) is infinite then there is nothing to prove. Otherwise, by Proposition 3.2, $\overline{m}_K$ is the minimal positive solution to the following boundary value problem in $\Omega := M \setminus K$:

$$\Delta f + 2qf = 0,$$
$$f|_{\partial K} = 1,$$

which is equivalent to

$$Lf = -qf, \quad f|_{\partial K} = 1. \tag{7.8}$$

Denote by G_Ω^L the Green function of the operator L on Ω, which is defined similarly to G_Ω (see Section 3). We consider G_Ω^L as an integral operator acting on nonnegative functions on Ω as follows:

$$G_\Omega^L[f](x) = \int_\Omega G_\Omega^L(x, y) f(y)\, d\mu(y).$$

Similarly to the proof of Lemma 3.3, we obtain, from (7.8),

$$\overline{m}_K = m_K + G_\Omega^L[q\overline{m}_K] \tag{7.9}$$

(which in particular implies the finiteness of G_Ω^L) and from (7.3),

$$v_K = m_K + G_\Omega^L[\overline{q} m_K^2]. \tag{7.10}$$

From (2.1), (2.3), (7.2) and (7.6) we obtain $\overline{q} \le C_1 q$. From (7.9) and (7.10) we obtain

$$\begin{aligned} v_K &\le m_K + C_1(\sup m_K) G_\Omega^L[q m_K] \\ &\le m_K + C_1(\sup m_K) G_\Omega^L[q\overline{m}_K] \\ &\le C_1(\sup m_K)\overline{m}_K, \end{aligned}$$

which was to be proved. □

COROLLARY 7.4. *Under the hypotheses of Theorem* 7.2, *assume in addition that* (7.6) *holds, and denote by* $\overline{m}_l$ *the* K_l*-gauge of the operator* $\Delta + 2q_l$. *If for some* $\varepsilon > 0$ *and all* $l = 0, 1, 2, \ldots,$

$$\inf_{x \in \partial K_{l+1}} m_l(x) \ge 1 + \varepsilon \quad \text{and} \quad \sup_{x \in K_{l+1}} \overline{m}_l(x) \le \varepsilon^{-1} \tag{7.11}$$

then the process $\overline{X}_t$ *is* K_0*-recurrent.*

PROOF. Indeed, all we need is to verify that $\sup_{\partial K_{l+1}} v_l$ is finite uniformly in l. Outside K_{l+1} the function $\overline{m}_l$ is harmonic and minimal, which implies by the maximum principle that $\overline{m}_l(x) \le \varepsilon^{-1}$ for all $x \in M$. By Lemma 7.3, we obtain, for all $x \in M$,

$$v_l(x) \le C_1(\sup \overline{m}_l)^2 \le C_1 \varepsilon^{-2},$$

whence the claim follows. $\square$

Theorem 7.2 and Corollary 7.4 provide implicitly the conditions on how big the intensity of branching q should be for $\overline{X}_t$ to be recurrent while X_t is transient. Unfortunately, applications of these results require quite subtle estimates of m_l from below and of v_l (or $\overline{m}_l$) from above, which may be difficult to obtain. In the next section, we consider a different approach although in a more restrictive setting.

8. Recurrence and the Green function. In this section, M is a geodesically complete noncompact Riemannian manifold. On such manifolds, all geodesic balls are precompact sets. Denote by $B(x,r)$ the (open) geodesic ball of radius r centered at $x \in M$, and by $V(x,r)$ its Riemannian volume. Let us introduce the following hypotheses:

(a) The doubling volume property: for all $x \in M$ and $r > 0$,

$$V(x,2r) \le C_V V(x,r). \tag{8.1}$$

(b) The volume growth assumption: for some $\alpha > 2$ and for all $x \in M$ and $R > r > 0$,

$$\frac{V(x,R)}{V(x,r)} \ge C_V^{-1}\left(\frac{R}{r}\right)^{\alpha}. \tag{8.2}$$

(c) The Green function estimate: for all $x \ne y$,

$$C_G^{-1}\frac{d^2}{V(x,d)} \le G(x,y) \le C_G \frac{d^2}{V(x,d)}, \tag{8.3}$$

where $d = d(x,y)$ is the geodesic distance between the points x, y.

EXAMPLE 8.1. Let manifold M have nonnegative Ricci curvature at all points and in all directions. Then the doubling volume property (8.1) holds, and M is stochastically complete. Moreover, the Green function $G(x,y)$ admits the following estimate (cf. [18, 7]):

$$C^{-1}\int_d^\infty \frac{r\,dr}{V(x,r)} \le G(x,y) \le C\int_d^\infty \frac{r\,dr}{V(x,r)}. \tag{8.4}$$

In particular, the Brownian motion X_t is transient if and only if

$$\int^\infty \frac{r\,dr}{V(x,r)} < \infty. \tag{8.5}$$

If in addition (8.2) holds with $\alpha > 2$ then (8.4) implies (8.3). Therefore, in this setting all conditions (a)–(c) are satisfied.

THEOREM 8.1. *Let M be a geodesically complete noncompact Riemannian manifold, and assume that the hypotheses* (A), (B) and (a)–(c) *are satisfied. Let $o \in M$ be a reference point, and set $|x| := d(x,o)$. Let the branching intensity satisfy*

$$q(x) \geq \frac{b}{|x|^2}, \tag{8.6}$$

for all $|x|$ large enough. There exists a positive constant $b_0 = b_0(\alpha, C_V, C_G)$ such that if $b > b_0$ then $\overline{X}_t$ is recurrent.

In particular, if M has a nonnegative Ricci curvature and (8.2) holds with $\alpha > 2$ then (8.6) implies the recurrence of $\overline{X}_t$ provided b is large enough. Before the proof of Theorem 8.1, let us describe some consequences of the hypotheses (a), (b) and (c).

Eigenvalue of the Laplace operator. For any precompact open set $U \subset M$, denote by $\lambda_0(U)$ the first eigenvalue of the Laplace operator in U with the Dirichlet boundary condition [not to be confused with $\lambda(U)$, which relates to the operator L rather than Δ]. The condition (a) implies that for any ball $B(x,R)$,

$$\lambda_0\big(B(x,R)\big) \leq \frac{C_\lambda}{R^2}. \tag{8.7}$$

Indeed, let $\varphi \in \mathrm{Lip}_0(B(x,R))$ be a test function which is equal to 1 in $B(x,R/2)$ and decays from 1 to 0 between $\partial B(x,R/2)$ and $\partial B(x,R)$ linearly in the radius. Then $|\nabla\varphi| \leq 2/R$, whence

$$\lambda_0\big(B(x,R)\big) \leq \frac{\int |\nabla\varphi|^2\,d\mu}{\int \varphi^2\,d\mu} \leq \frac{4}{R^2}\frac{V(x,R)}{V(x,R/2)} \leq \frac{4C_V}{R^2}.$$

Integration of the Green function. The conditions (a) and (c) imply that for any ball $B(x,R)$,

$$\int_{B(x,R)} G(x,y)\,d\mu(y) \leq C_I R^2. \tag{8.8}$$

Indeed, denoting $r = d(x,y)$, we have

$$\begin{aligned}\int_{B(x,R)} G(x,y)\,d\mu(y) &\leq C_G \sum_{k=0}^{\infty} \int_{\{R2^{-k-1}<r\leq R2^{-k}\}} \frac{r^2 d\mu(y)}{V(x,r)} \\ &\leq C_G \sum_{k=0}^{\infty} \frac{R^2 4^{-k}}{V(x,R2^{-k-1})} V(x,R2^{-k}) \\ &\leq 2C_G C_V R^2.\end{aligned}$$

Harnack inequality. It follows from [11], Proposition 10.1, that the conditions (a) and (c) imply the *Harnack inequality* for harmonic functions in M: If u is a nonnegative function in an arbitrary ball $B(x,R)$ satisfying in this ball $\Delta u = 0$ then

$$\sup_{B(x,R/2)} u \le H_0 \inf_{B(x,R/2)} u, \tag{8.9}$$

where $H_0 = H_0(C_V, C_G)$ does not depend on x, R.

A result of [10] (see also [12]) says that (a)–(c) and (8.9) imply the following: If u is a nonnegative solution in $B(x,R)$ to the equation

$$\Delta u - qu = 0,$$

where q is a nonnegative continuous function, then

$$\sup_{B(x,\delta R)} u \le H \inf_{B(x,\delta R)} u, \tag{8.10}$$

where $\delta = \delta(\alpha, C_V, C_G) \in (0, 1/2)$,

$$H = H_0 \exp\left(C \sup_{x' \in B(x,R)} \int_{B(x,R)} G(x', y) q(y)\, d\mu(y) \right),$$

and $C = C(\alpha, C_V, C_G) > 0$. Note that H may depend on the ball in question, but the dependence is explicit.

PROOF OF THEOREM 8.1. Without loss of generality we can assume that $q(x) = \frac{b}{|x|^2}$ provided $|x| \ge r_0 > 0$. Let $K \subset M$ be any precompact open set with smooth boundary, and let us prove that $\overline{X}_t$ is K-recurrent. Assuming the contrary, we have, by Lemma 5.4, $\liminf_{x\to\infty} v(x) = 0$, where $v := \psi_K$. Hence, for $u = 1 - \psi_K$ we have $\limsup_{x\to\infty} u(x) = 1$, and there exists a sequence $\{x_n\}_{n\ge1} \to \infty$ such that $u(x_n) > 1/2$.

Set $r_n = |x_n|$ and $R_n = r_n/4$. We can assume that r_n are large enough so that the balls $B(x_n, 2R_n)$ do not intersect $B(o, r_0) \cup K$. For any $y \in B(x_n, R_n)$ we have

$$q(y) = \frac{b}{|y|^2} \le \frac{2b}{r_n^2},$$

whence using (8.8), we obtain for any $x' \in B(x_n, R_n)$,

$$\begin{aligned} \int_{B(x_n,R_n)} G(x', y) q(y)\, d\mu(y) &\le \frac{2b}{r_n^2} \int_{B(x',2R_n)} G(x', y)\, d\mu(y) \\ &\le \frac{2b}{r_n^2} C_I (2R_n)^2 \le C_I b. \end{aligned} \tag{8.11}$$

The function u satisfies outside K equation (2.6), that is,

$$\Delta u - P(x,u) v u = 0. \tag{8.12}$$

Denoting $q'(x) := P(x,u)v$ we rewrite it as

$$\Delta u - q'u = 0.$$

Since $q' \le q$, we conclude from (8.11) that u satisfies in $B(x_n, R_n)$ the Harnack inequality (8.10) with the constant $H = H_0 \exp(CC_I b)$ that does not depend on n. Since $u(x_n) > 1/2$, (8.10) yields the following estimate in $B_n := B(x_n, \delta R_n)$:

$$\inf_{B_n} u \ge \frac{1}{2H}. \tag{8.13}$$

Let us rewrite (8.12) as follows:

$$\Delta v + P(x,u)uv = 0.$$

In the ball B_n we have the lower bound (8.13) for u. By hypotheses (B) and (2.5), we have $P(x,u) \ge C_0^{-1} q(x)$, which implies for all $x \in B_n$,

$$P\big(x, u(x)\big)u(x) \ge (2HC_0)^{-1} q(x) =: cq(x)$$

and

$$\Delta v + cqv \le 0.$$

Let w be the first eigenfunction of Δ in B_n, that is, w satisfies in B_n,

$$\begin{aligned} &\Delta w + \lambda w = 0, \\ &w|_{\partial B_n} = 0, \end{aligned}$$

where $\lambda = \lambda_0(B_n) \le C_\lambda \delta^{-2} R_n^{-2}$. Also, we may assume $w > 0$ in B_n. Since in B_n,

$$q(x) = \frac{b}{|x|^2} \ge \frac{b}{2r_n^2} = \frac{b}{32R_n^2},$$

we obtain

$$-\frac{\Delta w}{w} = \lambda \le \frac{C_\lambda}{\delta^2 R_n^2} \le \left(\frac{32C_\lambda}{\delta^2 cb}\right) cq \le cq \le -\frac{\Delta v}{v},$$

assuming that b is large enough, namely,

$$b \ge b_0 := \frac{32C_\lambda}{\delta^2 c}.$$

It follows from the comparison principle of Lemma 2.5 that

$$\sup_{\partial B_n} \frac{w}{v} = \sup_{B_n} \frac{w}{v}. \tag{8.14}$$

However, the right-hand side of (8.14) is positive whereas the left-hand side vanishes by $w|_{\partial B_n} = 0$. This contradiction completes the proof. $\square$

9. Spherically symmetric manifolds. Let $o \in M$ be a fixed reference point and let $n = \dim M$. Outside the cut locus of o, one can introduce on M the polar coordinates (r, θ), where r is the distance from a given point x to o, and $\theta \in \mathbb{S}^{n-1} \subset T_o M$ is the unit tangent vector at o in the direction of the geodesic ray from o to x. The manifold M is called *spherically symmetric* if the cut locus of o is empty and if the metric of M in the polar coordinates has the form

$$ds^2 = dr^2 + \sigma^2(r)\, d\theta^2, \tag{9.1}$$

with some function $\sigma(r)$ depending only on r. Here $d\theta$ refers to the standard metric on $\mathbb{S}^{n-1}$. Note that any smooth positive function σ is suitable for the metric (9.1) provided it satisfies certain conditions at 0 (see [5]).

Denote by B_r the (open) geodesic ball of radius r centered at o, and let $S(r)$ be the surface area of the sphere ∂B_r. In terms of the function $\sigma(r)$ from (9.1), we have $S(r) = \omega_n \sigma(r)^{n-1}$. For a function $u = u(r)$, the Laplace operator has the form

$$\Delta u = u'' + \kappa(r) u',$$

where $\kappa(r) = (n-1)\frac{\sigma'}{\sigma} = \frac{S'}{S}$ (see [8], Section 3). The Green function $G(o, x)$ of Δ depends only on r [where $x = (r, \theta)$], and is given by

$$G(o, x) = G(r) := \int_r^\infty \frac{dt}{S(t)} \tag{9.2}$$

(see [8], (4.8)). In particular, the Brownian motion X_t is transient if and only if $G(r) < \infty$ for all positive r, and X_t is stochastically complete if and only if

$$\int^\infty G(r) S(r)\, dr = \infty.$$

In this section, we investigate the transience and the recurrence of the branching process $\overline{X}_t$, under the following standing assumption:

(D) M is a geodesically and stochastically complete, noncompact, spherically symmetric manifold. Also, $k_{\max} = 2$, and the function $q(x)$ is positive and depends only on r.

In particular, we have also $p_2(x) \equiv 1$ and $P(x, u) \equiv 2Q = q(x)$. Hence, the hypotheses (A)–(C) are satisfied. We will use the notation $q(r)$ for the function $q(x)$.

If $K = B_{r_0}$ then the hitting probability ψ_K is radial and hence satisfies the following ODE in $[r_0, +\infty)$:

$$\begin{aligned} &v'' + \kappa(r) v' + q v(1 - v) = 0, \\ &v(r_0) = 1,\ 0 \le v(r) \le 1. \end{aligned} \tag{9.3}$$

Therefore, B_{r_0}-recurrence of the process $\overline{X}_t$ is equivalent to the fact that (9.3) has a unique solution $v \equiv 1$. Observe that the process $\overline{X}_t$ is recurrent if and only if it is B_r-recurrent for all $r > 0$, which follows from Lemma 5.4(a).

A simple condition for the transience of $\overline{X}_t$ follows immediately from Theorem 6.2.

COROLLARY 9.1. *If*

$$\int^{\infty} G(r)S(r)q(r)\,dr < \infty, \tag{9.4}$$

then $\overline{X}_t$ is transient.

For example, if $S(r) = Cr^{\alpha}$, $\alpha > 1$, then $G(r) = \frac{1}{C(\alpha-1)} r^{1-\alpha}$, and (9.4) becomes

$$\int^{\infty} rq(r)\,dr < \infty.$$

In particular, this condition is satisfied for $q(r) = \frac{\mathrm{const}}{r^{2+\varepsilon}}$, $\varepsilon > 0$.

COROLLARY 9.2. *Assume that $S\sqrt{q}$ is monotone increasing and*

$$\int^{\infty} \sqrt{q(r)}\,dr < \infty. \tag{9.5}$$

Then $\overline{X}_t$ is transient.

PROOF. Substituting (9.2) in (9.4) and interchanging the integrals, we obtain that (9.4) is equivalent to

$$\int^{\infty} \frac{1}{S(r)} \left(\int_R^r S(t)q(t)\,dt \right) dr < \infty, \tag{9.6}$$

for some $R > 0$. Let R be so large that

$$\int_R^{\infty} \sqrt{q(t)}\,dt \le 1.$$

Then we have

$$\int_R^r S(t)q(t)\,dt = \int_R^r \left(S\sqrt{q}\right)\sqrt{q}\,dt \le S(r)\sqrt{q(r)} \int_R^{\infty} \sqrt{q}\,dt \le S(r)\sqrt{q(r)}.$$

Therefore, (9.6) follows from (9.5). □

A more sophisticated (but less transparent) condition for transience follows from Theorem 4.1. In the next statement we use the notation $\nu(K, U)$ defined in (4.2).

COROLLARY 9.3. *If $\nu(B_R, M) \ge 1$ for some $R > 0$, then $\overline{X}_t$ is transient.*

The main result of this section is the following matching condition for the recurrence of $\overline{X}_t$.

THEOREM 9.4. *Let* (D) *hold*, *and assume that*

$$\nu(B_R, M) \le 1 - \varepsilon, \tag{9.7}$$

for some $\varepsilon > 0$ *and for all large enough* R. *Then* $\overline{X}_t$ *is recurrent*.

For the proof of Theorem 9.4, we will need the following lemmas.

LEMMA 9.5. *The function* $R \mapsto \nu(B_R, M)$ *is monotone increasing*.

PROOF. Since

$$\nu(K, M) = \inf_U \nu(K, U),$$

where the inf is taken over all precompact U containing $\overline{K}$; it suffices to show that $\nu(B_R, U)$ increases in R assuming that U is a large ball centered at o. Let φ be the first eigenfunction for the eigenvalue problem (4.3) with $K = B_R$. Then φ is radial so that $\varphi|_{\partial K} = \text{const}$. Therefore, φ can be continuously extended by a constant to the interior of B_R, and for any $r < R$ we have

$$\nu(B_r, U) \le \frac{\int_{U \setminus B_r} |\nabla\varphi|^2 \, d\mu}{\int_{U \setminus B_r} q\varphi^2 \, d\mu} \le \frac{\int_{U \setminus B_R} |\nabla\varphi|^2 \, d\mu}{\int_{U \setminus B_R} q\varphi^2 \, d\mu} = \nu(B_R, U),$$

whence the claim follows. □

LEMMA 9.6. *The function* $r \mapsto \psi_{B_R}(r)$ *is monotone decreasing*.

PROOF. Indeed, $\psi = \psi_{B_R}$ satisfies the equation

$$\psi'' + \kappa\psi' + q\psi(1 - \psi) = 0.$$

If ψ is not monotone decreasing then there is a point $r > R$ of a local minimum of ψ. At this point, $\psi' = 0$, whence we find

$$\psi'' = -q\psi(1 - \psi) < 0.$$

However, at the minimum we have $\psi'' \ge 0$. □

PROOF OF THEOREM 9.4. As follows from Lemma 9.5, the condition (9.7) holds for all $R > 0$. Fix some $r > 0$ and set $\psi = \psi_{B_r}$. We will prove that for any $R > r$ there exists a point $x \in M \setminus B_R$, such that $\psi(x) \ge \varepsilon/2$ where ε is the same as in (9.7). Since ψ is radial and monotone decreasing, this will imply $\liminf_{x\to\infty} \psi > 0$ and hence the B_r-recurrence of $\overline{X}_t$ by Lemma 5.4.

As follows from (9.7), for any ball B_R there exists a larger ball B_ρ such that

$$\nu := \nu(B_R, B_\rho) \le 1 - \varepsilon/2.$$

Let u be the first eigenfunction of the problem (4.3) for the couple (B_R, B_ρ); that is,

$$\begin{aligned} &\Delta u + \nu q u = 0, \\ &u|_{\partial B_\rho} = 0, \ \frac{\partial u}{\partial \mathbf{n}}\bigg|_{\partial B_R} = 0. \end{aligned} \tag{9.8}$$

We may assume that $u > 0$ in B_ρ. Recall that ψ satisfies outside B_r the equation

$$\Delta\psi + q\psi(1-\psi) = 0.$$

Multiplying this equation by u and integrating over $B_\rho \setminus B_R$, we obtain

$$\begin{aligned} 0 &= \int_{B_\rho\setminus B_R} \left[u\Delta\psi + qu\psi(1-\psi)\right] d\mu \\ &= \int_{\partial B_\rho} \left[\frac{\partial u}{\partial \mathbf{n}}\psi - \frac{\partial \psi}{\partial \mathbf{n}} u\right] d\mu' - \int_{\partial B_R} \left[\frac{\partial u}{\partial \mathbf{n}}\psi - \frac{\partial \psi}{\partial \mathbf{n}} u\right] d\mu' \\ &\quad + \int_{B_\rho\setminus B_R} \psi\Delta u \, d\mu + \int_{B_\rho\setminus B_R} qu\psi(1-\psi)\, d\mu \\ &\geq -\int_{B_\rho\setminus B_R} (\nu - 1 + \psi) qu\psi \, d\mu. \end{aligned} \tag{9.9}$$

Here $\mathbf{n}$ is the inward normal vector field on ∂B_ρ and ∂B_R, and μ' is the surface area on a hypersurface. We have used (9.8) and the facts that

$$\frac{\partial u}{\partial \mathbf{n}}\bigg|_{\partial B_\rho} \geq 0 \quad \text{and} \quad \frac{\partial \psi}{\partial \mathbf{n}}\bigg|_{\partial B_R} \geq 0,$$

the latter being true by Lemma 9.6. As follows from (9.9), there exists $x \in B_\rho \setminus B_R$ such that

$$\psi(x) \geq 1 - \nu \geq \varepsilon/2,$$

which was to be proved. $\square$

Let $\mathrm{Lip}_0[R, +\infty)$ denote the set of all Lipschitz functions on $[R, +\infty)$ which vanish for all large enough values of the argument.

COROLLARY 9.7.

(i) *If for some* $R > 0$ *and for all functions* $\varphi \in \mathrm{Lip}_0[R, +\infty)$,

$$\int_R^\infty (\varphi')^2 S(r)\, dr \geq \int_R^\infty \varphi^2 S(r) q(r)\, dr, \tag{9.10}$$

then $\overline{X}_t$ *is transient.*

(ii) *If, for some* $\varepsilon > 0$ *and for all large enough* R *there exists a nonzero function* $\varphi \in \mathrm{Lip}_0[R, +\infty)$ *such that*

$$\int_R^\infty (\varphi')^2 S(r)\,dr \le (1-\varepsilon)\int_R^\infty \varphi^2 S(r) q(r)\,dr, \tag{9.11}$$

then $\overline{X}_t$ *is recurrent.*

PROOF. Since the first eigenfunction of (4.3) is radial, (9.10) implies that, for all $\varphi \in \mathrm{Lip}_0(M)$,

$$\int_{M\setminus B_R} |\nabla\varphi|^2\,d\mu \ge \int_{M\setminus B_R} q\varphi^2\,d\mu.$$

Hence, $\nu(B_R, M) \ge 1$, and $\overline{X}_t$ is transient by Theorem 4.1 (or Corollary 9.3).

Similarly, (9.11) implies $\nu(B_R, M) \le 1-\varepsilon$, and $\overline{X}_t$ is recurrent by Theorem 9.4. □

EXAMPLE 9.1. Let $M = \mathbb{R}^3$. We claim that the following are true:

(i) If $q(r) \le \frac{1}{4r^2}$ for all r large enough then $\overline{X}_t$ is transient.
(ii) If $q(x) \ge \frac{1+\varepsilon}{4r^2}$ for all r large enough then $\overline{X}_t$ is recurrent (where $\varepsilon > 0$).

(i) Indeed, (9.10) will follow for the given q if we prove that

$$\int_R^\infty (\varphi')^2 r^2\,dr \ge \int_R^\infty \frac{1}{4r^2}\varphi^2 r^2\,dr,$$

which indeed follows from the Hardy inequality (see, e.g., [13]) by the change $r \mapsto r - R$.

Note that if $q(r) \le \frac{1-\varepsilon}{4r^2}$ then the operator $L = \Delta + q$ possesses a finite Green function G^L (see [21]). Therefore, the transience of $\overline{X}_t$ can also be obtained by Theorem 4.1. However, for $q = \frac{1}{4r^2}$, we have $G^L \equiv \infty$ whereas $\overline{X}_t$ is still transient.

(ii) Condition (9.11) (with ε replaced by $\varepsilon/2$) will follow from

$$\int_R^\infty (\varphi')^2 r^2\,dr \le \left(1 - \frac{\varepsilon}{2}\right)\int_R^\infty \frac{1+\varepsilon}{4r^2}\varphi^2 r^2\,dr = \alpha\int_R^\infty \varphi^2\,dr, \tag{9.12}$$

where $\alpha := \frac{1}{4}(1-\varepsilon/2)(1+\varepsilon) > \frac{1}{4}$. For parameters $\rho \gg R$ and $\beta > \frac{1}{2}$, consider the function

$$\varphi(r) := (r^{-\beta} - \rho^{-\beta})_+.$$

As $\rho \to \infty$, we have

$$\int_R^\infty (\varphi')^2 r^2\,dr = \beta^2 \int_R^\rho r^{-2\beta}\,dr = \frac{\beta^2}{2\beta - 1} R^{1-2\beta} + o(1)$$

and

$$\int_R^\infty \varphi^2\,dr = \int_R^\rho (r^{-\beta} - \rho^{-\beta})^2\,dr = \frac{R^{1-2\beta}}{2\beta - 1} + o(1).$$

Therefore, taking $\beta \in (\frac{1}{2}, \sqrt{\alpha})$ and ρ large enough, we obtain (9.12).

Let us introduce the following notation:

$$\mathcal{F}_R(r) := \frac{S^2(r)q(r)}{(\int_R^r S(t)q(t)\,dt)^2} \tag{9.13}$$

and

$$\underline{a}(R) := \inf_{r>R} \mathcal{F}_R(r) \quad \text{and} \quad \overline{a} := \limsup_{r\to\infty} \mathcal{F}_R(r). \tag{9.14}$$

In the next statements, we will assume that

$$\int^\infty S(r)q(r)\,dr = \infty. \tag{9.15}$$

Under this assumption, the number $\overline{a}$ in (9.14) does not depend on R. Note that if (9.15) is false then $\overline{X}_t$ is transient if and only if so is X_t. Indeed, if X_t is transient then $G(r)$ is finite and decreasing. Therefore, the convergence of the integral in (9.15) implies (9.4), and $\overline{X}_t$ is transient by Corollary 9.1.

COROLLARY 9.8. *Let* (9.15) *hold.*

(i) *If* $\underline{a}(R) \geq 4$ *for some* $R > 0$ *then* $\overline{X}_t$ *is transient.*
(ii) *If* $\overline{a} < 4$ *then* $\overline{X}_t$ *is recurrent.*
(iii) *Assume that, for some* $R > 0$, *the function* $\mathcal{F}_R(r)$ *is monotone decreasing in* $r \in [R, +\infty)$, *and set*

$$a := \lim_{r\to\infty} \mathcal{F}_R(r).$$

Then $\overline{X}_t$ *is transient if and only if* $a \geq 4$.

PROOF. (i) Let us prove that, if α, β are positive continuous functions on $(0, \infty)$ such that $\int^\infty \beta(r)\,dr = \infty$, the following inequality holds for all $\varphi \in \mathrm{Lip}_0[R, +\infty)$:

$$\int_R^\infty \left(\frac{d\varphi}{dr}\right)^2 \alpha(r)\,dr \geq \frac{a}{4} \int_R^\infty \varphi^2(r)\beta(r)\,dr, \tag{9.16}$$

where

$$a := \inf_{r>R} \frac{\alpha(r)\beta(r)}{(\int_R^r \beta(s)\,ds)^2}. \tag{9.17}$$

By the change of variables,

$$t = t(r) = \int_R^r \beta(s)\, ds,$$

we reduce (9.16) to

$$\int_0^\infty \left(\frac{d\varphi}{dt}\right)^2 \alpha(r)\beta(r)\, dt \geq \frac{a}{4}\int_0^\infty \varphi(r)^2\, dt. \tag{9.18}$$

By definitions of a and t, we have

$$\alpha(r)\beta(r) \geq at^2,$$

so that (9.18) is true by the classical Hardy inequality,

$$\int_0^\infty \left(\frac{d\varphi}{dt}\right)^2 t^2\, dt \geq \frac{1}{4}\int_0^\infty \varphi^2\, dt. \tag{9.19}$$

Substituting in (9.16) $\alpha = S$ and $\beta = Sq$, we obtain the first claim of Corollary 9.8. Indeed, if $\underline{a}(R) \geq 4$ then (9.16) implies (9.10), whence $\overline{X}_t$ is transient by Corollary 9.7.

(ii) For any (large) $T > 0$ and $\gamma > \frac{1}{2}$ consider the function

$$\varphi(t) = \begin{cases} t^{-\gamma}, & t > T, \\ T^{-\gamma}, & t \leq T, \end{cases}$$

for which one easily computes

$$\int_0^\infty \left(\frac{d\varphi}{dt}\right)^2 t^2\, dt = \frac{\gamma^2}{2\gamma - 1} T^{1-2\gamma} = \frac{\gamma}{2}\int_0^\infty \varphi^2\, dt.$$

Setting $\gamma = \frac{1+\varepsilon}{2}$ and changing the variables as above we obtain

$$\int_R^\infty \left(\frac{d\varphi}{dr}\right)^2 \frac{1}{\alpha(r)\beta(r)}\left(\int_R^r \beta(s)\, ds\right)^2 \alpha(r)\, dr \leq \frac{(1+\varepsilon)}{4}\int_R^\infty \varphi^2(r)\beta(r)\, dr.$$

Note that $d\varphi/dr$ vanishes for $r < r(T) =: r_1$. Introducing notation

$$\widetilde{a} := \sup_{r > r_1} \frac{\alpha(r)\beta(r)}{(\int_R^r \beta(t)\, dt)^2},$$

we obtain

$$\int_R^\infty \left(\frac{d\varphi}{dr}\right)^2 \alpha(r)\, dr \leq \frac{\widetilde{a}(1+\varepsilon)}{4}\int_R^\infty \varphi^2(r)\beta(r)\, dr.$$

If $\overline{a} < 4$ then also $\widetilde{a} < 4$ provided T (and hence r_1) is large enough. Therefore, for small enough ε, this implies (9.11), and hence the recurrence of $\overline{X}_t$ follows by Corollary 9.7.

(iii) If $a \geq 4$ then by the monotonicity of $\mathcal{F}_R$, we have $\mathcal{F}_R(r) \geq 4$ whence $\underline{a}(R) = \inf \mathcal{F}_R \geq 4$. If $a < 4$ then also $\overline{a} = \limsup \mathcal{F}_R(r) = a < 4$. □

The following statement shows, for a given $S(r)$, the critical branching intensity q for the recurrence/transience of $\overline{X}_t$.

COROLLARY 9.9. *Assume $G(r) < \infty$, and let for large r,*

$$q(r) = \frac{b}{S^2(r)G^2(r)}, \tag{9.20}$$

where b is a positive constant. Then $\overline{X}_t$ is transient if and only if $b \leq \frac{1}{4}$.

REMARK 9.1. For the function (9.20) we have

$$\int_R^r GSq\,dr = b \log \frac{G(R)}{G(r)} \quad \text{and} \quad \int^\infty GSq\,dr = \infty,$$

so that the condition (9.4) does not hold.

PROOF OF COROLLARY 9.9. Let R be so large that (9.20) holds for $r \geq R$. For such r, we have

$$\int_R^r S(t)q(t)\,dt = b\int_R^r \frac{dt}{S(t)G^2(t)} = -b\int_R^r \frac{dG(t)}{G^2(t)} = b\left(\frac{1}{G(r)} - \frac{1}{G(R)}\right).$$

Since $\lim_{r\to\infty} G(r) \to 0$, condition (9.15) holds. Also, we obtain

$$\mathcal{F}_R(r) = \frac{S^2(r)q(r)}{(\int_R^r S(t)q(t)\,dt)^2} = \frac{1}{b(1 - G(r)/G(R))^2}.$$

Clearly, $\mathcal{F}_R$ is a decreasing function and

$$\lim_{r\to\infty} \mathcal{F}_R(r) = \frac{1}{b}.$$

Hence, the claim follows from Corollary 9.8. □

EXAMPLE 9.2. Here are some concrete examples of computations by (9.2), (9.20) and applications of Corollary 9.9:

Parameter	$\boldsymbol{\alpha > 1}$	$\boldsymbol{\alpha > 0}$	$\boldsymbol{\alpha > 1}$	$\boldsymbol{\alpha > 0,\ 0 < \beta < 1}$
$S(r)$	Cr^α	$C\exp(\alpha r)$	$Cr\log^\alpha r$	$Cr^{1-\beta}\exp(\alpha r^\beta)$
$G(r)$	$\frac{1}{C(\alpha-1)}r^{1-\alpha}$	$\frac{1}{C\alpha}\exp(-\alpha r)$	$\frac{1}{C(\alpha-1)}\log^{1-\alpha} r$	$\frac{1}{C\alpha\beta}\exp(-\alpha r^\beta)$
$q(r)$	$\frac{c}{r^2}$	c	$\frac{c}{r^2\log^2 r}$	$\frac{c}{r^{2(1-\beta)}}$
c	$b(\alpha-1)^2$	$b\alpha^2$	$b(\alpha-1)^2$	$b\alpha^2\beta^2$
$\overline{X}_t$ transient ⇔	$c \leq \frac{(\alpha-1)^2}{4}$	$c \leq \frac{\alpha^2}{4}$	$c \leq \frac{(\alpha-1)^2}{4}$	$c \leq \frac{\alpha^2\beta^2}{4}$

For instance, for $M = \mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$, which matches the first data column in the table with $\alpha = n - 1$. Hence, the branching process in $\mathbb{R}^n$ with the branching intensity $q(r) = \frac{c}{r^2}$ is transient if and only if $c \le \frac{(n-2)^2}{4}$. Similarly one can read the other columns in the table.

EXAMPLE 9.3. Let $M = \mathbb{H}^n$ (= the hyperbolic space) and $q \equiv c = \text{const}$. Then $S(r) = \omega_n \sinh^{n-1} r$ and

$$\mathcal{F}_R(r) = \frac{1}{c}\left(\frac{\sinh^{n-1} r}{\int_R^r \sinh^{n-1} t\, dt}\right)^2 \searrow \frac{(n-1)^2}{c} =: a.$$

Hence, by Corollary 9.8, the transience of $\overline{X}_t$ is equivalent to $c \le \frac{(n-1)^2}{4}$.

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IMPERIAL COLLEGE
180 QUEEN'S GATE
LONDON SW7 2BZ
UNITED KINGDOM
E-MAIL: a.grigoryan@ic.ac.uk

UNIVERSITY OF WALES, SWANSEA
SINGLETON PARK
SWANSEA SA2 8PP
UNITED KINGDOM
E-MAIL: m.kelbert@swansea.ac.uk