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SMALL GAPS IN THE RANGE OF STABLE PROCESSES

BY T. S. MOUNTFORD[1] AND S. C. PORT

University of California, Los Angeles

We examine the structure of the range of a stable process with drift near its initial position and derive an integral test to determine the rate of growth of the size of intervals in the complement of the range. This integral test depends on the random number of excursions that the process makes away from the initial point.

0. Introduction. We consider processes

$$X_t = bt + \tilde{X}_t,$$

where $\tilde{X}$ is a strictly stable process with index $\alpha \in (0,1)$, $X_0 = \tilde{X}_0 = 0$ and b is a strictly positive constant; that is to say, X is a stable process with drift. We examine the structure of the range of X close to the origin. The range of X has positive Lebesgue measure [see, e.g., Kesten (1969) or Bretagnolle (1971)]. However, if the Lévy measure puts infinite mass on $(0,\infty)$, the range is a.s. nowhere dense [see, e.g., Mountford and Port (1991)]. Therefore there must be open intervals in the complement of the range arbitrarily close to the origin both to the right and left of the origin. This paper answers the question of how large these intervals can be as we approach the origin.

THEOREM 0.1. *Let X be a stable process with positive drift and Lévy measure which puts mass on $(0,\infty)$. Let h be a continuous, increasing function satisfying $h(0) = 0$. Define the event*

$$A_n = \{\textit{there exists } x \in (0, 2^{-n}) \textit{ so that } \overline{(\textit{Range}(X))} \cap [x, x + h(x)] = \varnothing\}.$$

Given that X makes r excursions from the origin, the event $\bigcap_n A_n$ has probability 0 or 1 according to whether

$$\int_{0+} h(y)^{-\alpha^r}\, dy$$

is finite or infinite.

Here the number of excursions is equal to the number of times the process leaves the origin, which is equal to the number of returns to the origin plus 1. It should be noted that as h is continuous the event A_n is equal to the union

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over rational numbers q less than 2^{-n} of the events $A(q) = \{[q, q + h(q)] \cap \overline{(\text{Range}(X))} = \varnothing\}$. Consequently no measurability problems arise.

In the last section we will show that an excursion of X, conditioned to return to the origin, is the time reversal of an excursion of the dual of X, also conditioned to return to the origin. This and the proof of Theorem 0.1 makes the following immediate.

THEOREM 0.2. *Let X and h be as in Theorem* 0.1. *Define the event*

$$B_n = \left\{\text{there exists } x \in (-2^{-n}, 0) \text{ so that } \overline{\text{Range}(X)} \cap [x - h(-x), x] = \varnothing\right\}.$$

Given that X makes r excursions from the origin, the probability of $\bigcap_n B_n$ is equal to 0 *or* 1 *according to whether*

$$\int_{0+} h(y)^{-\alpha^{r-1}}\, dy$$

is finite or infinite. If $r = 1$, then there must be an interval $(-y, 0)$ which is disjoint from the range of X.

REMARK. This result is uninteresting if X has no jumps to the left.

This work follows previous work by Erickson (1983) and Mountford, O'Hara and Port (1995) examining large-scale structures of gaps.

The paper is organized as follows: Section 1 gathers some bounds on probabilities; these bounds show that, for small x, outside of a set of small probability, X enters the interval $[x, 2x]$ at x and leaves through $2x$ and does not return to the interval before hitting the origin; Section 2 establishes inequalities for the probability of a certain sized gap in the range of r excursions from 0, a given distance from the origin; Section 3 completes the proof of Theorem 0.1. The last section is devoted to showing that the time reversal of an excursion of X from 0 which returns to the origin is equal in law to the law of such an excursion from the origin by the dual of X. Theorem 0.2 is immediate given this result and the proof of Theorem 0.1, and no further comment is required. The rest of this section is spent introducing terms and giving definitions.

Definitions. For a process Y and a time interval I, $R(Y, I)$ or $Y(I)$ will denote the closure of the set $\{x\colon x = Y_s \text{ for some } s \in I\}$.

If the interval $I = [0, \infty)$, then we drop I from the notation.

An interval (p, q) is a *gap* for a process Y if $(p, q) \cap R(Y) = \varnothing$ and $q < \sup_s Y_s$. The interval is called a gap for a family of processes $\{Y^i\}$ if it is a gap for each Y^i. Given a function h on an interval containing 0 and a family of processes, an interval (p, q) is an *h-gap* if it is a gap and $q - p > h(p)$. Given a process or family of processes $\{Y^i\}$ and an increasing, continuous function h, we say that h is a.s. a *gap function* for $\{Y^i\}$ if, for each $x > 0$, there exists an h-gap $\subset [0, x]$, with probability 1. We will use the phrase *h is a.s. not a gap function* in a similar manner.

An interval (p, q), $p \neq q$, is called a *jump at time* t for Y if $Y_{t-} = p$, $Y_t = q$. The interval is a jump of Y if it is a jump of Y for some t. Obviously, given processes Y^i an interval (p, q) can be a gap only if there are jumps (p_i, q_i) for each Y^i, so that $(p, q) \subset (p_i, q_i)$ for each i. A jump or gap (p, q) is said to be *in* interval I if $p \in I$.

Given a process Y, the process Y^* will represent the process Y killed on hitting $(-\infty, 0]$.

Given a process Y (X by default), T_A will denote the first hitting time of the set A, and T_A^* will denote the first hitting time of A by the process Y^*; Q_A (resp., Q_A^*) will denote the quitting time of A by the process Y (resp., Y^*); S_A (resp., S_A^*) will denote the first time after T_A (resp., T_A^*) that the process leaves A.

For simplicity, throughout the paper we will assume that b, the drift of the process X, is equal to 1.

1. We use potential theory to establish some probability bounds.

LEMMA 1.1. *There exists $K < \infty$ such that, for all x small and positive,*

$$P^0[T_0 < T_x \leq \infty] < Kx^{1-\alpha}.$$

PROOF. Define

$$p_x = P^0[T_x < T_0 \leq \infty], \qquad q_x = P^0[T_0 < T_x \leq \infty].$$

Let L_y be the (random) number of times the point y is hit for strictly positive times. Then

$$\begin{aligned} g(x) = E[L_x] &= p_x E[L_x | T_x < T_0] + q_x E[L_x | T_0 < T_x] \\ &= p_x(1 + g(0-)) + q_x g(x) \end{aligned}$$

[see Fitzsimmons and Port (1989)]. Similarly,

$$\begin{aligned} g(0-) = E(L_0) &= p_x E[L_0 | T_x < T_0] + q_x E[L_0 | T_0 < T_x] \\ &= p_x(g(-x)) + q_x(1 + g(0-)). \end{aligned}$$

Solving, we obtain

$$q_x = \frac{g(0-)(1 + g(0-)) - g(x)g(-x)}{(1 + g(0-))^2 - g(x)g(-x)}.$$

Port (1989) gives the bounds

$$|g(0-) - g(-x)| + |g(0-) + 1 - g(x)| \leq Kx^{1-\alpha},$$

for some K. The result follows. $\square$

Similar arguments give the following lemma.

LEMMA 1.2. *There exists $K < \infty$ such that, for all x sufficiently small but positive,*

$$P^0[T_0 < T_{-x}] \leq Kx^{1-\alpha}.$$

COROLLARY 1.3. *There exists $K < \infty$ such that, for all x sufficiently small but positive,*

$$P_0\left[T_{[-x,0]} < T_{-x}\right] \le Kx^{1-\alpha}.$$

PROOF. The point y is irregular for $(-\infty, y)$; also, for each $\varepsilon > 0$, $P^y[T_x < \varepsilon] > 1 - \varepsilon$ for $x - y$ sufficiently small but positive; therefore there exists δ such that $P^y[T_x < T_{(-\infty, y)}] > \frac{1}{2}$ for $x - y \in (0, \delta)$. Thus, applying the strong Markov property at $T_{[-x,0]}$, we obtain $P_0[T_0 < T_{-x}] \ge \frac{1}{2}P_0[T_{[-x,0]} < T_{-x}]$. The corollary now follows from Lemma 1.2. □

COROLLARY 1.4. *There exists $K < \infty$ such that, for all x sufficiently small but positive,*

$$P^0\left[T_{[x,2x]} < T_x\right] < Kx^{1-\alpha}.$$

PROOF. By the strong Markov property,

$$\begin{aligned} P^0[T_0 \le T_x] &\ge \int_{(x,2x]} P^0\left[X_{T_{[x,2x]}} \in dy\right] P^y\left[T_0 \le T_{[0,y]}\right] \\ &= \int_{(x,2x]} P^0\left[X_{T_{[x,2x]}} \in dy\right] P^0\left[T_{-y} \le T_{[-y,0]}\right] \\ &\ge \tfrac{1}{2} P^0\left[X_{T_{[x,2x]}} \in (x,2x]\right]. \end{aligned}$$

The corollary follows from Corollary 1.3 (if x is small enough) and Lemma 1.1. □

LEMMA 1.5. *There exists $K < \infty$ such that, for all x sufficiently small but positive,*

$$P^0\left[\tfrac{3}{2}x \wedge T_{[2x,\infty)} < T_{[x,2x]}\right] \le Kx^{1-\alpha}.$$

PROOF. We have

$$P^0\left[T_{[2x,\infty)} < T_{[x,2x]}\right] \le P\left[X_{3x/2} \notin [x,2x]\right] + P\left[T_{[2x,\infty)} \le \frac{3x}{2}\right].$$

Using the decomposition $X_t = t + \tilde{X}_t$, these last two probabilities can be bounded by

$$P^0\left[|\tilde{X}_{3x/2}| > \frac{x}{2}\right] + P^0\left[\sup_{0 \le s \le x/2} \tilde{X}_s > \frac{x}{2}\right] \le K' P^0\left[|\tilde{X}_{3x/2}| > \frac{x}{2}\right] \le Kx^{1-\alpha}. \quad \square$$

LEMMA 1.6. *We have $P[T_{(-\infty,0]} < x] < Kx^{1-\alpha}$ for some $K < \infty$.*

PROOF. Let $A(n)$ be the event $\{\inf\{X_s\colon s \in [2^{-n}x, 2^{-n+1}x]\} \le 0\}$. Clearly, as the event $\{T_{(\infty,0]} < x\}$ is equal to $\bigcup_n A(n)$, we have $P|T_{(-\infty,0]} < x] \le \sum_{n=1}^{\infty} P[A(n)]$. Now

$$\begin{aligned} P[A(n)] &\le P\Big[\inf_{0 \le s \le 2^{-n+1}x} \tilde{X}_s \le -2^{-n}x\Big] \\ &\le kP\big[\tilde{X}_{x2^{-n+1}} < -x2^{-n}\big] \le Kx^{1-\alpha}(2^{-n})^{1-\alpha} \end{aligned}$$

and so the lemma follows. □

Lemmas 1.6 and 1.5 enable us to extend the previous results to X^*, the process obtained by killing X when it hits $(-\infty, 0]$.

PROPOSITION 1.7. *Let X^* be the process X killed upon hitting the negative half-line. Let $T^*_{(\,)}$ be stopping times analogous to $T_{(\,)}$ for process X. There exists finite K such that, for all x positive and sufficiently small, the following hold*:

(a) $P(T^*_x = T^*_{(x,\infty)} < \infty) \ge 1 - Kx^{1-\alpha}$;
(b) $P(T^*_{2x} = T^*_{[2x,\infty)} = Q^*_{[x,2x]} = S^*_{[x,2x]} < \infty) \ge 1 - Kx^{1-\alpha}$.

PROOF. We prove (a):

$$\{T_x = T_{[x,\infty)} < \infty\} \subseteq \{T^*_x = T^*_{[x,\infty)} < \infty\} \cup \left\{T_x > \frac{3x}{2}\right\} \cup \left\{T_{(-\infty,0)} \le \frac{3x}{2}\right\}.$$

Thus

$$\begin{aligned} P\big(T^*_x = T^*_{[x,\infty)} < \infty\big) &\ge P\big[T_x = T_{(x,\infty)} < \infty\big] - P\left[T_x > \frac{3}{2}\right] - P\left[T_{(-\infty,0]} < \frac{3x}{2}\right] \\ &\ge P\big[T_{[x,2x]} < T_{[2x,\infty)}\big] - P\big[T_{[x,2x]} < T_x\big] \\ &\quad - P\left[T_x > \frac{3x}{2}\right] - P\left[T_{(-\infty,0]} < \frac{3x}{2}\right]. \end{aligned}$$

The result now follows from Lemma 1.5, Corollary 1.4, Lemma 1.5 again and Lemma 1.6. Inequality (b) is proved in essentially the same way but also uses Corollary 1.3. □

2. The aim of this section is to prove two propositions.

PROPOSITION 2.1. *Let X^i, for $i = 1, 2, 3, \ldots, r$, be independent copies of X, killed on hitting the negative half-line. There is a constant k_r so that, for x and h sufficiently small*,

$$P\big[\exists \textit{ gap } (p, q) \subset [0, x] \textit{ of length greater than } h \textit{ for the } X^i\big] \le k_r x h^{-\alpha^r}.$$

PROPOSITION 2.2. *Let X^i for $i = 1, 2, 3, \dots, r$, be independent copies of X, killed on hitting the negative half-line (or killed on leaving $[0, x]$). There is a constant k_r so that, for x and h sufficiently small and $h^{\alpha^{r-1}} \le x/2$.*

$$P\big[\exists \text{ gap } (p, q) \subset [0, x] \text{ of length greater than } h \text{ for the } X^i\big] \\ \ge k_r(xh^{-\alpha^r} \wedge 1).$$

Note that the event in question implies the existence of an h-gap in $[0, x]$.

Before proving these propositions, we require some preparatory lemmas and definitions.

Given r independent excursions from the origin killed upon entering the negative axis, $X^1, X^2, \dots, X^r$, and an interval, or collection of intervals, I, we define $J^I(k, i)$ to be the collection of (random) jumps (p, q) such that the following hold:

1. $q - p \in (2^{-i-1}, 2^{-i}]$;
2. $p \in I$;
3. there exists t so that $X^k_{t-} = p$ and $X^k_t = q$.

If the interval I is of the form $[0, t]$, we write $J^t(k, i)$ instead.

LEMMA 2.1. *Let I be an interval or union of intervals. Then $E[|J^I(k, i)|] < K\lambda(I)2^{i\alpha}$. Here $|\ |$ denotes cardinality and $\lambda(\ )$ denotes Lebesgue measure.*

PROOF. Let $N^k(x, A) = \#\{t;\ X^k_t - X^k_{t-} > x,\ X^k_{t-} \in A\}$. Then it follows from elementary properties of Lévy processes that

$$E\big[N^k(x, A)\big] = v(x, \infty)G^*(A),$$

where v is the Lévy measure of X and $G^*(A)$ is the expected occupation time of A for X^k.

From the definition of N and J we have

$$J^I(k, i) \le N^k(2^{-i-1}, I),$$

so

$$\begin{aligned} E\big[J^I(k, i)\big] &\le v(2^{-i-1}, \infty)\int_I g^*(y)\, dy \\ &\le v(2^{-i-1}, \infty)\int_I g(y)\, dy \\ &\le v(2^{-i-1}, \infty)g(0+)\lambda(I) \\ &= K'(2^{-i-1})^{-\alpha}\lambda(I) \\ &= K2^{+i\alpha}\lambda(I). \end{aligned}$$

□

PROOF OF PROPOSITION 2.1. Let $A(r, h, t)$, $0 \le h, t \le 1$, be the event that there exists an interval (p, q) so that the following hold:

(a) $p \in [0, t]$;
(b) $p - p > h$;
(c) $(p, q) \cap \bigcup_{i=1}^{r} R(X^i) = \varnothing$.

The event A is decreasing in h, so it will be sufficient to consider the case $h = 2^{-i}$ for some i.

If A occurs and (p, q) is a corresponding interval, then for each $k \in \{1, 2, \dots, r\}$ there exists an interval $I_k \in J^t(k, i_k)$ with $i_k \le i$ so that $(p, q) \subset I_k$. Therefore,

$$A(r, 2^{-i}, t) \subset \bigcup_{\pi \in S(r)} \bigcup_{i_1 \le i_2 \le \cdots \le i_r \le i} B^t(\pi, i_1, i_2, \dots, i_r),$$

where $S(r)$ is the set of permutations on $\{1, 2, \dots, r\}$ and

$$B^t(\pi, i_1, i_2, \dots, i_r) = \{\text{for } k = 1, 2, \dots, r, \text{ there exist jumps } V_k \in J^t(\pi(k), i_k) \text{ such that } V_1 \cap V_2 \cdots V_r \ne \varnothing\}.$$

By symmetry, the events

$$\bigcup_{i_1 \le i_2 \le \cdots \le i_r \le i} B^t(\pi, i_1, i_2, \dots, i_r)$$

have equal probability as π varies, and so,

$$P\big[A(r, 2^{-i}, t)\big] \le r! P\left[\bigcup_{i_1 \le i_2 \le \cdots \le i_r \le i} B^t(\mathrm{id}, i_1, i_2, \dots, i_r)\right],$$

where id is the identity permutation. Henceforth we suppress id, writing $B^t(i_1, i_2, \dots, i_r)$ for $B^t(\mathrm{id}, i_1, i_2, \dots, i_r)$.

Given an interval $I = (r, t)$ with $t - r \in [2^{-(n+1)}, 2^{-n}]$, we say the enlargement of the interval, $e(I)$, is the interval $(r - 2^{-n}, t + 2^{-n})$. Define a relation R between two intervals as follows:

$$\text{for } I = (p, q),\ J = (r, t), \qquad IRJ \quad \text{if } r \in e(I).$$

Then $B^t(i_1, i_2, \dots, i_r) \subset C^t(i_1, i_2, \dots, i_r)$, where $C^t(i_1, i_2, \dots, i_r)$ is the event

$$\big\{\text{there exist } I_k \in J^t(k, i_k) \text{ so that, for all } k,\ I_k\, R I_{k+1}\big\}.$$

Our proposition will follow from Lemma 2.2.

LEMMA 2.2. *We have*

$$P\left[\bigcup_{i_1 \le i_2 \le \cdots \le i_r \le i} C^t(i_1, i_2, \dots, i_r)\right] \le k_r t 2^{i\alpha^r} \quad \textit{for some finite } k_r.$$

PROOF. We prove this lemma by induction on r and Lemma 2.1.

First, if $r = 1$, then the lemma follows directly from Lemma 2.1. We now assume that Lemma 2.2 has been established for less than or equal to $r - 1$ independent excursions. First note that, by the inductive hypothesis,

$$P\left[\bigcup_{\substack{i_j \le i\alpha^{r-j} \\ i_1 \le i_2 \le \cdots \le i_r \le i}} C^t(i_1, i_2, \dots, i_r)\right] \le k_j t\left(2^{i\alpha^r} - j\right)^{\alpha^j} \le k_j t 2^{i\alpha^r} \quad \text{for } j < r.$$

Therefore it remains to show

$$P\left[\bigcup_{\substack{i_j > i\alpha^{r-j}\ \forall j \\ i_1 \le i_2 \le \cdots \le i_r \le i}} C^t(i_1, i_2, \dots, i_r)\right] \le V_r t 2^{i\alpha^r} \quad \text{for some finite } V_r.$$

However, in this case we can use the obvious bound

$$P\left[\bigcup_{\substack{i_j > i\alpha^{r-j}\ \forall j \\ i_1 \le i_2 \le \cdots \le i_r \le i}} C^t(i_1, i_2, \dots, i_r)\right] \le \sum_{\substack{i_j > i\alpha^{r-j}\ \forall j \\ i_1 \le i_2 \le \cdots \le i_r \le i}} P\left[C^t(i_1, i_2, \dots, i_r)\right].$$

Obviously,

$$P\left[C^t(i_1, i_2, \dots, i_r)\right] \le E\left[|N^t(i_1, i_2, \dots, i_r)|\right],$$

where $N^t(i_1, i_2, \dots, i_r)$ is the collection of all r-tuples $(I_1, \dots, I_r)$ such that the following hold:

(a) $I_k \in J^t(k, i_k)$ for each k;
(b) $I_k \, R \, I_{k+1}$ for each $k \le r - 1$.

Furthermore, iterated use of Lemma 2.1 yields

$$\begin{aligned} E\left[|N^t(i_1, i_2, \dots, i_r)|\right] &\le (Kt2^{\alpha i_1})(K2^{-i_1}2^{\alpha i_2}) \cdots (K2^{-i_{r-1}}2^{\alpha i_r}) \\ &= K^r t 2^{-(1-\alpha)i_1} 2^{-(1-\alpha)i_2} \cdots 2^{-(1-\alpha)i_{r-1}} 2^{\alpha i_r}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\substack{i_j > i\alpha^{r-j}\ \forall j \\ i_1 \le i_2 \le \cdots \le i_r}} P\left[C^t(i_1, i_2, \dots, i_r)\right] \\ &\quad \le \sum_{\substack{i_j > i\alpha^{r-j}\ \forall j \\ i_1 \le i_2 \le \cdots \le i_r}} K^r t 2^{-(1-\alpha)i_1} 2^{-(1-\alpha)i_2} \cdots 2^{-(1-\alpha)i_{r-1}} 2^{\alpha i_r} \\ &\quad \le K(r)' t 2^{i\alpha^r} \quad \text{for some } K(r)'. \qquad \square \end{aligned}$$

If we know that our process X makes r excursions from 0, then these excursions, while independent, are not iid. The first $r - 1$ will be distributed as excursions conditioned to return to 0, while the last will be distributed as an excursion conditioned never to return. Accordingly we require Corollary 2.4.

COROLLARY 2.1. *Let $X^1, X^2, \ldots, X^r$ be independent excursions of X from the origin. Suppose each X^k is (separately) conditioned either to return to the origin or not to return to the origin. Then*

$$P\big[\exists \text{ interval } (p,q) \subset [0,x] \text{ of length greater than } h \text{ so that } (p,q) \cap \big(\cup R(X^i)\big) = \varnothing\big] \le u^{-r} k x h^{-\alpha r},$$

where k is the constant of Proposition 2.1 and

$$u = \min\big(P^0[T_0 < \infty], P^0[T_0 = \infty]\big) = \min\left(\frac{g(0)}{g(0+)}, \frac{1}{g(0+)}\right).$$

We also require some preparatory lemmas before proving Proposition 2.2. The two lemmas below are elementary and so are stated without proof.

LEMMA 2.3. *Let X be an excursion from 0. There exists $c_1 > 0$ so that, for each $x \in [0,1]$, $P[T_x = T_{[x,\infty)}] \ge c_1$.*

LEMMA 2.4. *Let X be an excursion from 0. There exists $c_2 > 0$ such that, for all $x \in [0,1]$, $P^x[T_0 \le T_{[0,x]}] \ge c_2$.*

The following lemma is crucial to the proof of Proposition 2.2.

LEMMA 2.5. *Let I be a subinterval of $[0, \frac{1}{2}]$. Then, for $X^{k,e}$ an excursion from zero killed on hitting the negative half-line,*

$$P\big[\text{there exists an } h\text{-gap for } X^{k,e} \text{ in } I\big] \le c_3(\lambda(I)h^{-\alpha} \wedge 1) - c_4(\lambda(I))^{1-\alpha},$$

for $h < \frac{1}{2}$.

PROOF. Let $I = (x,y)$. Lemma 2.3 states that, with probability at least c_1, $T_x = T_{[x,\infty)} < \infty$; furthermore, by the strong Markov property,

$$Y_s = X_{T_x+s} - X_{T_x}$$

is a stable process independent of $\mathscr{F}_{T_x}$. Let $V = \inf\{s > 0; Y_s - Y_{s-} \in (h, 2h)\}$. Mountford and Port (1991) show that

$$W_s = Y_{V-} - Y_{V-s}, \qquad 0 \le s \le V,$$

is a Lévy process killed at rate $v^X(h,2h)$, where v^X is the Lévy measure of X with unit drift and Lévy measure given by $v^W(dy) = v^X(dy)I_{\{y \in (h,2h)^c\}}$.

Thus the probability of the event

$$\{R(Y,V) \cap (Y_{V-}, Y_V) = \varnothing\} \cap \{V \le \lambda(I)/2\}$$

is greater than the probability of the event $\{W$ does not hit the negative axis$\} \cap \{V \le \lambda(I)/2\}$. This last term is at least $c(1 - (\exp^X[-\lambda(I)/(2v^X(h,2h))]))$ for

$$c = P[W \text{ does not hit } (-\infty, 0) | V = \lambda(I)/2].$$

Let the event $A(I, h)$ be the intersection of the following events:

(a) $\{T_{[x,\infty)} = T_x < \infty\}$;
(c) $\{R(Y, V) \cap (Y_{V-}, Y_V) = \varnothing\}$;

and

(b) $\{V < \lambda(I)/2\}$, where V is as defined above.

The above paragraph demonstrates that

$$P[A(I,h)] \geq c\left(1 - \left(\exp\left[-\frac{\lambda(I)}{2v^X(h,2h)}\right]\right)\right) \leq c'(\lambda(I)h^{-\alpha} \wedge 1)$$

for some c'. Let $B(I)$ be the intersection of the following events:

(d) $\{T_{[x,\infty)} = T_x < \infty\}$

and

(e) $\{X_{T_x+s} - X_{T_x} \in [0, \lambda(I)]$ for $s \in [0, \lambda(I)/2]\}$.

Lemmas 1.5 and 1.6, the strong Markov property applied at T_x and the fact that X is a Lévy process imply that $P[B(I)^c] \leq K\lambda(I)^{1-\alpha}$ for some K.

On the event $C(I, h) = A(I, h) \setminus B(I)^c$ we have the following:

(a) $T_0 > T_x + \lambda(I)/2$;
(b) $(X_{(T_x+V)-}, X_{T_x+V}) \cap R(X, T_x + V) = \varnothing$;
(c) $(X_{(T_x+V)-}, X_{T_x+V}) \subset I$.

The bounds given above for the probabilities of the respective events imply that $P[C(I, h)] \geq c'(\lambda(I)h^{-\alpha} \wedge 1) - K\lambda(I)^{1-\alpha}$.

Let $C^*(I, h)$ be the event corresponding to $C(I, h)$ for the excursion X^* killed on hitting the negative axis. Since, on $C(I, h)$, $T_{(-\infty,0]} > T_x + \lambda(I)/2$, we have $P[C^*(I, h)] = P[C(I, h)]$. The random time $T_x^* + V^*$ (for X^*) is a stopping time; and, on the event $C^*(I, h)$, $X_{T_x+V+s} - X_{T_x+V}$ is a Lévy process killed on hitting $(\infty, -X_{T_x+V}]$ but otherwise independent of $\mathscr{F}_{T_x+V}$. So (if I is small),

$$P\left[X_s^* \in \left(X^*_{(T_x+V)-}, X^*_{T_x+V}\right) \text{ for } s > T_x + V \,|\, C(I,h)\right] > c',$$

for some strictly positive c'. The result follows. □

Proof of Proposition 2.2. We use induction on r. The result follows directly from Lemma 2.5 above if $r = 1$. Suppose now that the result holds when the number of excursions is less than or equal to $r - 1$. Let $v = h^\alpha \times v^{\alpha^{r-2}} = h^{\alpha^{r-1}} \leq x/2$; so, by the inductive hypothesis, with probability at least $k_{r-1}(tv^{-\alpha^{r-1}} \wedge 1) = k_{r-1}(th^{-\alpha^r} \wedge 1)$, there is a gap I for the processes $X^2, X^3, \dots, X^r$ of length at least v completely contained in the interval $[0, x]$. Applying Lemma 2.5 to the process X^i and the interval I, we obtain the result. □

3. In this section we establish Theorem 0.1.

We will decompose the process X into a collection of independent excursions X^i. Define a sequence of stopping times by

$$T_0 = 0; \qquad \text{for } i > 0, \qquad T_i = \inf\{t > T_{i-1}: X_t = 0\}.$$

Let R be the random number of excursions from 0: $R = \inf\{i: T_i = \infty\}$. Conditionally if $r = 1$, then X simply has the law of an excursion from the origin conditioned never to return. If $R > 1$, then, for $i \leq R - 1$, the processes Y^i defined by

$$Y_s^i = \begin{cases} X_{T_{i-1}+s}, & \text{for } s < T_i - T_{i-1}, \\ \delta, & \text{for } s \geq T_i - T_{i-1}, \end{cases}$$

where δ is some graveyard state, are independent processes each with law equal to that of an excursion by X from 0, conditioned to return to 0. The process Y^R defined by

$$Y_s^r = X_{T_{R-1}+s}, \qquad s \geq 0,$$

is independent of the $Y^1, \dots, Y^{R-1}$ but has law equal to that of an excursion from 0, conditioned never to return.

It is well known that, for every $s > 0$ and all $i \leq R(\omega)$, there exists a random $\varepsilon(\omega) > 0$ so that

$$R\big(Y^i[s, \infty)\big) \cap \big[0, \varepsilon(\omega)\big) = \varnothing \quad \text{for each } i \leq r(\omega).$$

Therefore, if X^i are the processes obtained by killing the Y^i upon hitting the negative axis, then h is a gap function for $\{Y^i\}$ if and only if h is a gap function for $(X^i\}$. Given $R > 1$, the X^i processes, while independent, are not identically distributed.

It follows that Theorem 0.1 will be proved by the following pair of propositions.

PROPOSITION 3.1. *Let h be a continuous increasing function, zero at the origin, and let X^i, $i = 1, 2, \dots, r$, be independent stable processes with positive drift, killed on hitting the negative axis. Then*

$$\int_{0+} \frac{dx}{h(x)^{\alpha^r}} < \infty$$

implies that a.s. there do not exist arbitrarily small x with

$$(x, x + h(x)) \cap \left(\bigcup_{i=1}^{r} R(X^i) \right) = \varnothing.$$

PROPOSITION 3.2. *Let h be a continuous increasing function, zero at the origin, and let X^i, $i = 1, 2, \dots, r$, be independent stable processes with positive drift, killed on hitting the negative axis. Then*

$$\int_{0+} \frac{dx}{h(x)^{\alpha^r}} = \infty$$

implies that a.s. h is a gap function for $\{X^i\}$.

PROOF OF PROPOSITION 3.1. As h is increasing,

$$\int_0^1 \frac{dx}{h(x)^{\alpha^r}} \geq \frac{1}{2} \sum_{n=1}^{\infty} 2^{-n} h(2^{-n})^{\alpha^r}.$$

So finiteness of the integral implies finiteness of the sum. Corollary 2.1 implies that if X^i, $i = 1, 2, \dots, r-1$, are excursions from the origin, conditioned to return to the origin, while X^r is an excursion conditioned not to return and all r excursions are independent, then $P(A_n) \leq K_r h(2^{-n})^{-\alpha^r} 2^{-n}$, where

$$A_n = \left\{ \text{there exists } (p, q),\ p \in [2^{-(n+1)}, 2^{-n}],\ q - p \geq h(2^{-(n+1)}) \right.$$
$$\left. \text{such that } (p, q) \cap \bigcup_{i=1}^{r} R(X^i) = \varnothing \right\}.$$

The event $\{h$ is a gap function$\}$ is clearly contained in the event $\limsup A_n$, but, by the Borel–Cantelli lemma and the finiteness of the relevant integral, the latter event has probability zero. $\square$

PROOF OF PROPOSITION 3.2. Augmenting the probability space if necessary, we assume the existence of independent stable processes $Z^{i,n}$ which in addition are independent of our r excursions from 0, X^i. Say an index (i, n) is good if the following holds:

$$\text{for the process } X^i, \quad T_{2^{-n}} = T_{[2^{-n}, \infty)}.$$

Define the processes $W^{i,n}$ by

$$W_s^{i,n} = \begin{cases} X_{T_{2^{-n}}+s} - 2^{-n}, & \text{if } (i, n) \text{ is good,} \\ Z_s^{i,n}, & \text{if } (i, n) \text{ is not good,} \end{cases}$$

and the processes $W^{i,n}$ are killed on exiting $[2^{-n}, 2^{-n+1}]$. The W processes are all independent. By Proposition 1.7 and the Borel–Cantelli lemma for all i, (i, n) is good for n large enough and for all i and, for n large enough,

$$2^{-n} + R(W^{i,n}) = R(X^i) \cap [2^{-n}, 2^{-n+1}].$$

Therefore h is a.s. a gap function for X^i, $i = 1, 2, \dots, r$, if A_n occurs infinitely often, where

$$A_n = \{\text{there exists a gap of size } h(2^{n-1}) \text{ for } \{W^{i,n}\},\ i = 1, 2, \dots, r\}.$$

By design, the events $\{A_n\}$ are independent.

The infiniteness of $\int_{0+} h(x)^{-\alpha^r}\, dx$ implies that $\sum_n 2^{-n} h(2^{-n+1})^{-\alpha^r} = \infty$. Let

$$G = \left\{ n \geq 1 \colon h(2^{-n+1})^{\alpha^{r-1}} \geq \frac{2^{-n}}{2} \right\}$$

and

$$B = \left\{ n \geq 1 \colon h(2^{-n+1})^{\alpha^{r-1}} < \frac{2^{-n}}{2} \right\};$$

G (for good) denotes the n's for which Proposition 2.2 yields information. The sum

$$\sum_n 2^{-n} h(2^{-n+1})^{-\alpha^r} = \sum_B 2^{-n} h(2^{-n+1})^{-\alpha^r} + \sum_G 2^{-n} h(2^{-n+1})^{-\alpha^r}.$$

The first sum on the right-hand side is majorized by $\sum_n 2^{-n}(2^{-n}/2)^{-\alpha}$, which is finite. Hence the infiniteness of $\int_{0+} h(x)^{-\alpha^r}\,dx$ implies that $\sum_G 2^{-n} h(2^{-n+1})^{-\alpha^r}$ is infinite. This (and Proposition 2.2) implies that $\sum_n P[A_n] = \infty$. The result now follows from the second Borel–Cantelli lemma. □

4. To prove Theorem 0.2, we require the result that time reversals of excursions of X (that remain) are equal in distribution to excursions of the dual. This result follows (with elementary modifications) from Getoor and Sharpe (1981). We include a short proof for completeness.

Let $\hat{X}_t = -X_t$ be the dual process of X_t; quantities relative to this process will be marked with a caret. Let T be the stopping time $\inf\{t > 0\colon X_t = 0\}$, $0 < h_1 < h_2 < \cdots < h_n$; let $q_0(t, x, y)$ be the transition function for the process X killed at time T; and let $f_0, f_1, \dots, f_n$ be bounded continuous functions.

PROPOSITION 4.1. *We have*

$$E^0\left[\prod_{i=1}^n f_i\big(X_{T-h_i}\big);\, h_n < T < \infty\right] = E^0\left[\prod_{i=1}^n f_i\big(\hat{X}_{h_i}\big);\, h_n < \hat{T} < \infty\right]. \tag{4.1}$$

To establish this proposition, we will need to recall some potential-theoretic facts. These can be found in Port (1989). Let $g^\lambda(x) = \int_0^\infty \exp(-\lambda t) q(t, 0, x)\,dt$, where q is the unkilled transition probability function. Let $g_0^\lambda(x, y) = \int_0^\infty \exp(-\lambda t) q_0(t, x, y)\,dt$. Let $h^\lambda(x) = E^x[\exp(-\lambda t)]$, and $h(x) = P^x[T < \infty]$. Then $h^\lambda(x) = C^\lambda g^\lambda(-x)$, $h(x) = Cg(-x)$, $\hat{C} = C$, $\hat{C}^\lambda = C^\lambda$, $g^\lambda(-x) = \hat{g}^\lambda(x)$, $\hat{q}(t, x, y) = q(t, y, x)$, $\hat{g}_0^\lambda(x, y) = g_0^\lambda(y, x)$, $g_0^\lambda(0, x) = 1 - h^\lambda(0))g^\lambda(x)$ and

$$\frac{1 - h^\lambda(0)}{C^\lambda} = \frac{1 - \hat{h}^\lambda(0)}{\hat{C}^\lambda} = \frac{1}{|b|},$$

where b is the drift, which in our case is 1. Using these facts, we find that

$$[1 - h(0)]\, g(x) C^\lambda g^\lambda(-y) = \big[1 - \hat{h}^\lambda(0)\big]\, \hat{g}^\lambda(y) C \hat{g}(-x). \tag{4.2}$$

To establish (4.1), note that

$$\int_0^\infty E^0\left[\prod_{i=1}^n f_i(X_{T-t-h_i});\, t+h_n<T<\infty\right]\exp(-\lambda t)\,dt$$

$$= \int\int\cdots\int g_0(0,x_n)\prod_{i=1}^n q_0(h_i-h_{i-1},x_i,x_{i-1})$$

$$\times\prod_{i=1}^n f_i(x_i)E^{x_0}[\exp(-\lambda T)]\,dx_0\,dx_1\,\cdots\,dx_n$$

$$= \int\int\cdots\int(1-h(0))cg(x_n)\prod_{i=1}^n q_0(h_i-h_{i-1},x_i,x_{i-1})$$

$$\times\prod_{i=1}^n f_i(x_i)c^\lambda g^\lambda(x_0)\,dx_0\,dx_1\,\cdots\,dx_n.$$

By (4.2) and dual identities, this last expression is equal to

$$\int\int\cdots\int(1-h^\lambda(0))cg(-x_n)c^\lambda g^\lambda(x_0)\prod_{i=1}^n \hat{q}_0(h_i-h_{i-1},x_{i-1},x_i)$$

$$\times\prod_{i=1}^n f_i(x_i)\,dx_0\,dx_1\,\cdots\,dx_n.$$

However, this is equal to

$$\int_0^\infty E^0\left[\prod_{i=1}^n f\big(\hat{X}_{t+h_i}\big);\, t+h_n<\hat{T}<\infty\right]\exp(-\lambda t)\,dt.$$

Therefore, t-a.e.,

$$E^0\left[\prod_{i=1}^n f\big(\hat{X}_{t+h_i}\big);\, t+h_n<\hat{T}<\infty\right] = E^0\left[\prod_{i=1}^n f_i(X_{T-t-h_i});\, t+h_n<T<\infty\right]. \tag{4.3}$$

Equation (4.1) now follows from (4.3) and the fact that X_t and $X_{\hat{t}}$ are standard processes.

Proof of Theorem 0.2. If the process X makes r excursions from 0, then $r-1$ of these are excursions which return to the origin and the last is an excursion that never returns. This latter excursion does not hit an interval $(-\varepsilon(\omega),0]$ for strictly positive, random ε, and so is irrelevant to left gaps. The $r-1$ excursions that return to the origin are, by Proposition 4.1, time reversals of independent excursions of $\hat{X}$, again conditioned to return to the origin; $\hat{X}$ is a process that drifts to the left. The arguments for Proposition 3.1 and 3.2 now imply Theorem 0.2. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024
E-mail: port@math.ucla.edu