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PERSISTENT SURVIVAL OF ONE-DIMENSIONAL CONTACT PROCESSES IN RANDOM ENVIRONMENTS

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Consider an inhomogeneous contact process on $\mathbf{Z}^1$ in which the recovery rates $\delta(x)$ at site x are i.i.d. random variables (bounded above) while the infection rate is a constant ε. The condition $u\mathbf{P}(-\log \delta(x) > u) \to +\infty$ as $u \to +\infty$ implies the survival of the process for *every* $\varepsilon > 0$.

1. Introduction and main results. In recent years, there have been a number of papers on contact processes in random environments (see [3, 4, 6–8]). These processes are of intrinsic interest as examples of interacting particle systems with random parameters. They are also closely related to various disordered systems arising in statistical mechanics (see, e.g., [1]).

In this paper, we will be primarily concerned with the one-dimensional contact process in a random environment. This is a Markov process with state space $\{0, 1\}^{\mathbf{Z}}$ with transitions

$$1 \to 0 \quad \text{at site } x \text{ with rate } \delta(x)$$

and

$$0 \to 1 \quad \text{at site } x \text{ with rate } \rho(x)\eta(x+1) + \lambda(x)\eta(x-1),$$

where η is the state of the process. We will assume that the local environments $(\delta(x), \rho(x), \lambda(x))$ are i.i.d. random vectors on some $(\Omega, \mathscr{F}, \mathbf{P})$ with (for simplicity) the $\rho(x)$'s and the $\lambda(x)$'s taking the nonrandom values ε_1 and ε_2. We shall also assume that the $\delta(x)$'s are strictly positive. We denote by $\eta_t^j = \{\eta_t^j(x)\colon x \in \mathbf{Z}\}$ the contact process at time t with initial state $\eta_0^j(x) = 1$ for $x = j$ and 0 for $x \neq j$ and by S^j the event that η_t^j survives for all $t > 0$ (i.e., for every $t > 0$, η_t^j is not identically zero).

We say that the contact process survives with infection rates ε_1 and ε_2 if either of the following equivalent statements is valid:

1. For $\mathbf{P}$-a.e. $\{\delta(x)\colon x \in \mathbf{Z}\}$, S^j has positive probability for every j.
2. For a set of $\{\delta(x)\colon x \in \mathbf{Z}\}$ of positive $\mathbf{P}$-probability, S^0 has positive probability.

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We leave the demonstration of the equivalence as an exercise. Under mild conditions on the tails (near ∞ and 0) of the common distribution of the $\delta(x)$'s, one has extinction (nonsurvival) for sufficiently small ε_1 and ε_2 and survival for sufficiently large ε_1 and ε_2. In particular, [7] implies the former result on extinction if $\mathbf{E}(-\log \delta(x)) < \infty$ and [8] implies the latter result on survival if $\mathbf{E}(\delta(x)^2) < \infty$.

However, sufficiently fat tails can change these conclusions. From [7], one has extinction for *all* $\varepsilon_1, \varepsilon_2$ (no matter how large) if one has a sufficiently fat tail at ∞ (and not too big a tail at 0) so that $\mathbf{E}(-\log \delta(x)) = -\infty$. The main result of this paper is the next theorem, which gives conditions that imply the opposite extreme, that is, survival for all positive $\varepsilon_1, \varepsilon_2$ (no matter how small). We call this *persistent survival*. Equivalently (by a scaling of time) one may regard $\varepsilon_1, \varepsilon_2$ as fixed, take $\delta(x) = \theta\delta_0(x)$ and ask for conditions on $\{\delta_0(x)\}$ which imply survival for all θ (no matter how big). The next theorem provides such a condition, valid even when one of $\varepsilon_1, \varepsilon_2$ vanishes.

THEOREM 1. *Suppose that*

$$u\mathbf{P}(-\log \delta(x) > u) \to \infty \quad \text{as } u \to \infty \tag{1.1}$$

and also (*for simplicity*) *that* $\mathbf{P}(\delta(x) \le \bar{\delta}) = 1$ *for some* $\bar{\delta} < \infty$. *Then the contact process survives as long as* $\max(\varepsilon_1, \varepsilon_2) > 0$.

Hypothesis (1.1) requires a bit more on the tail at 0 than what is needed to have $\mathbf{E}([-\log \delta(x)]^+) = \infty$, which is equivalent to

$$\int_1^{+\infty} \mathbf{P}(-\log \delta(x) > u)\, du = \infty. \tag{1.2}$$

On the other hand we do not need to require *both* ε_1 and ε_2 to be positive; survival occurs even in a process with infection spreading only in one direction.

We remark that the effects of fat tails on d-dimensional contact processes in random environments are quite different for $d \ge 2$ than for $d = 1$. In particular, for $d \ge 2$, one always has survival for sufficiently large transmission rates, regardless of how fat the tail at ∞ of the recovery rate distribution. This is because there is always an embedded one-dimensional system with recovery rates bounded above. On the other hand, such an embedding argument can be used for $d \ge 2$ to give a fairly simple proof that a fat tail at 0 for the recovery rate does imply persistent survival (see [1, 3]). A sufficient condition on the tail for $d \ge 2$ (obtainable by essentially the arguments of Theorem 1.8 of [1]) is that

$$u^d\mathbf{P}(-\log \delta(x) > u) \to \infty \quad \text{as } u \to \infty. \tag{1.3}$$

Although a proof of persistent survival for the $d = 1$ contact process in a random environment, as given in this paper, is to our knowledge new, there have been a number of closely related results for $d = 1$. For example, it follows from Liggett's results [8] that $\mathbf{E}(-\log \delta(x)) = +\infty$ implies what might be called persistent *weak* survival. Namely, that for all positive ε_1, ε_2, in the process η_t^* (with initial state $\dots 111000 \dots$), the location r_t of the rightmost 1 satisfies $\limsup_{t\to\infty} r_t = +\infty$. From our point of view, these results give the "wrong" conclusion for the right process (i.e., the contact process). As pointed out by the referee, it appears to be an open problem to determine whether $\mathbf{E}(-\log \delta(x)) = +\infty$ suffices [as a replacement for (1.1)] to yield the right conclusion, that is, persistent survival as in Theorem 1.

There are also $d = 1$ results of [1] which give the right conclusion but for the "wrong" processes. A contact process on $\mathbf{Z}$ may be regarded [5] (see Section 3 below) as a directed percolation model on $\mathbf{Z} \times [0, \infty) = \{(x, t)\}$ (directed in the positive t direction) in which survival corresponds to percolation. The results of [1] apply to (among others) undirected percolation models on $\mathbf{Z} \times (-\infty, \infty)$ (in which, roughly speaking, infection can go backward as well as forward in time) and imply persistent percolation (see especially Theorems 1.7 and 3.2 of [1]) under the stronger requirements than (1.1) that

$$\frac{u}{\log u}\mathbf{P}(-\log \delta(x) > u) \to \infty \quad \text{as } u \to \infty, \tag{1.4}$$

providing that also $\mathbf{E}(\delta(x)) < \infty$. Thus in addition to extending the [1] result on persistent percolation to the directed model, we also succeed in eliminating (for the contact process and for the models of [1]) the $\log u$ factor from (1.4), which one previously suspected was not needed because of the $d \geq 2$ condition (1.3).

The remainder of the paper is organized as follows. In Section 2, we state and prove the analogue of Theorem 1 for discrete time contact processes (i.e., for directed percolation). The proof is based on a construction which relates percolation in these models to coverings of a half-line by random intervals. In Section 3, we prove the continuous time result, Theorem 1, by reducing its proof to the discrete time result of Section 2.

2. Survival for discrete time. In this section we consider (doubly) directed percolation on $\mathbf{Z}^+ \times \mathbf{Z}^+ = \{(x, t)\colon\ x, t \in \{0, 1, 2, \dots\}\}$. There are only two type of edges, both directed: right edges from (x, t) to $(x + 1, t)$ which are open with probability ε (not depending on x or t) and otherwise closed, and up edges from (x, t) to $(x, t + 1)$ which are open with probability p_x (not depending on t) and otherwise closed. The edges are open or closed independently of each other. For given ε and $\mathbf{p} = \{p_x\colon\ x \in \mathbf{Z}^+\}$, we denote by $\mathbf{Q} = \mathbf{Q}_{\varepsilon,\,p}$ the probability distribution for the independent percolation system. We take $\varepsilon \in (0, 1)$ and the p_x's as independent random variables with a common distribution supported on $[0, 1)$. We denote by $\mathbf{P}$ the probability distribution for the random environment $\mathbf{p}$.

Let $\hat{S}^x$ denote the event (for the percolation model with given $\varepsilon, \mathbf{p}$) that there is an infinite directed path of open edges starting from $(x, 0)$. Note that since $\varepsilon < 1$, any such infinite path must ($\mathbf{P}$-a.s.) have t-coordinates tending to infinity. Let $\hat{S}$ denote the union of $\hat{S}^x$ over all x in $\mathbf{Z}^+$; $\hat{S}$ is the event that percolation occurs (for given ε and $\mathbf{p}$). The percolation model may be regarded as a discrete time contact process in which open right edges spread infection (to the right) while *closed* up edges correspond to recovery. From that point of view, survival means that $\mathbf{Q}_{\varepsilon,\mathbf{p}}(\hat{S}) > 0$ for $\mathbf{P}$-a.e. $\mathbf{p}$. The following theorem is our analogue of Theorem 1. We note that it has no hypothesis analogous to the upper bound $\bar{\delta}$ on the $\delta(x)$'s in Theorem 1. Indeed, in this discrete time context p_x is allowed to vanish with positive probability. In continuous time of course, analogously letting $\delta(x)$ take the value $+\infty$ with positive probability would preclude survival.

THEOREM 2. *Suppose that*

$$u\mathbf{P}(-\log(1-p_x) > u) \to \infty \quad \text{as } u \to \infty. \tag{2.1}$$

Then percolation occurs (*a.s.*) *for any* $\varepsilon > 0$. *That is*,

$$\mathbf{Q}_{\varepsilon,\mathbf{p}}(\hat{S}) = 1 \quad \text{for } \mathbf{P}\text{-a.e. } \mathbf{p}.$$

REMARK. We leave it as an exercise to show that the conclusion of the theorem also implies that $\mathbf{Q}_{\varepsilon,\mathbf{p}}(\hat{S}^0) > 0$ for $\mathbf{P}$-a.e. $\mathbf{p}$.

PROOF OF THEOREM 2. Our strategy is to relate directed percolation to covering of the half-line $\mathbf{R}^+$ by random intervals (see Proposition 3 below). This follows the approach of [2] to a different percolation model, except there the object was to prove *absence* of percolation.

Let us denote by $\tilde{\mathbf{P}}$ the joint distribution of $\mathbf{p}$ and the $(\varepsilon,\mathbf{p})$-percolation model, and let us denote by $\tilde{S}$ the event in the joint probability space $\tilde{\Omega}$ that $\hat{S}$ occurs for the percolation model. Thus

$$\tilde{\mathbf{P}}(\tilde{S}) = \int \left[\mathbf{Q}_{\varepsilon,\mathbf{p}}(\hat{S})\right]\mathbf{P}(d\mathbf{p}). \tag{2.2}$$

To obtain the conclusion of the theorem, it clearly suffices to show that $\tilde{\mathbf{P}}(\tilde{S}) = 1$.

To do so we make a recursive construction in $\tilde{\Omega}$. We start with step 0 (see Figure 1). Let Y_0 be the largest y such that *every* up edge between $(0,0)$ and $(0,y)$ is open. Let D_0^i $(i = 1,\dots,Y_0)$ be the largest d such that *every* right edge between $(0,i)$ and (d,i) is open and let

$$R_0 = \max\left(D_0^1,\dots,D_0^{Y_0}\right), \tag{2.3}$$

providing $Y_0 \geq 1$. If $Y_0 = 0$, set $R_0 = 0$. For $1 \leq x \leq R_0$, let H_0^x be the largest i such that $D_0^i \geq x$; for $x > R_0$, let $H_0^x = 0$. Note that H_0^x is monotonic (nonincreasing) in x. We set $K_0 = R_0$ and note that if $K_0 \geq 1$, then each site (x, H_0^x) with $1 \leq x \leq K_0$ is reached from $(0,0)$ by an open directed path and there is no (yet known) obstruction to continuing the open directed path past (x, H_0^x) anywhere into the northeast quadrant, other than a single closed right edge on the horizontal line at height H_0^x. We say that step 0 fails if $K_0 = 0$ and succeeds if $K_0 \geq 1$. This completes the zero-th step of the construction.

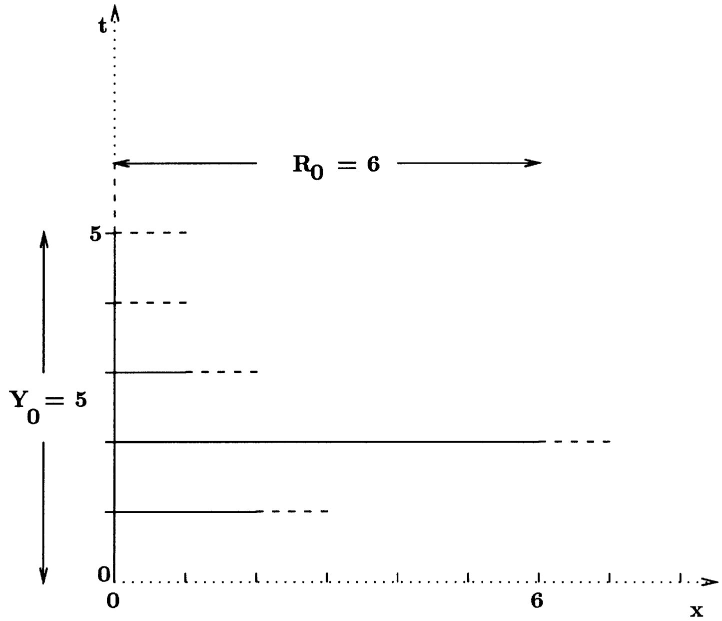

FIG. 1. *Step 0 of the recursive construction. Edges with complete lines are open and edges with dashed lines are closed. In this example,* $(D_0^1, \dots, D_0^{Y_0}) = (2, 6, 1, 0, 0)$ *and* $(H_0^1, H_0^2, \dots) = (3, 2, 2, 2, 2, 2, 0, 0, \dots)$.

The jth step is quite similar (see Figure 2). Given H_{j-1}^x for $x \geq j$ and $K_{j-1} \geq j - 1$, we proceed as follows. Let Y_j be the largest y such that every up edge between (j, H_{j-1}^j) and $(j, H_{j-1}^j + y)$ is open. Let D_j^i $(i = H_{j-1}^j + 1, \dots, H_{j-1}^j + Y_j)$ be the largest d such that every right edge between (j, i) and $(j + d, i)$ is open and let

$$R_j = \max\Bigl(D_j^i \colon H_{j-1}^j + 1 \leq i \leq H_{j-1}^j + Y_j\Bigr), \tag{2.4}$$

providing $Y_j \geq 1$. If $Y_j = 0$, set $R_j = 0$. For $j + 1 \leq x \leq j + R_j$, let H_j^x be the largest i such that $j + D_j^i \geq x$; for $x > j + R_j$, let $H_j^x = H_{j-1}^x$. Again H_j^x is nonincreasing in x. We now set $K_j = \max(K_{j-1}, j + R_j)$. If $K_j \geq j + 1$ (i.e., if either $R_j > 0$ or else K_{j-1} already exceeded j before the jth step), then we say that step j succeeds; otherwise step j fails. This completes the jth step of the construction.

If all the steps $0, 1, \dots, j$ succeed, then there is an open directed path from $(0, 0)$ to each (x, H_j^x) with $j + 1 \leq x \leq K_j$ which can be continued (to the northeast, as after step 0). If step l (with $l < j$) fails but steps $l + 1, \dots, j$ all succeed, then there is an open directed path from $(l + 1, 0)$ to each (x, H_j^x) with $j + 1 \leq x \leq K_j$, which can be continued. If step j fails, then $H_j^x \equiv 0$ for $x \geq j + 1$ and step $j + 1$, as already defined, searches for directed paths from $(j + 1, 0)$. We conclude that if all but finitely many steps succeed, then

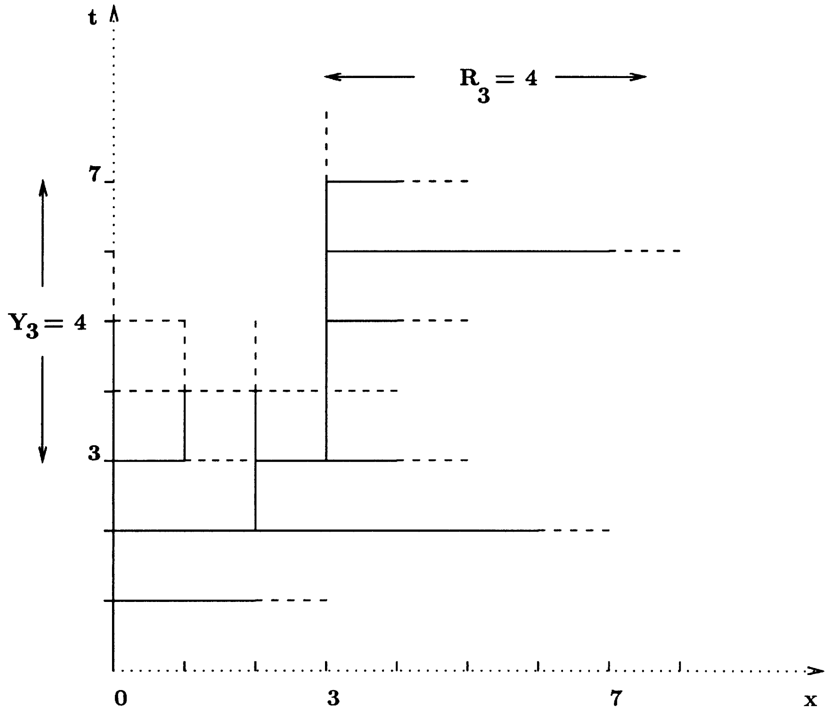

FIG. 2. *The recursive construction after step* 3. *In this example*, Y_3 *and* R_3 *are as indicated*, $H_2^3 = 3$ *and* $(D_3^i\colon\ H_2^3 + 1 \le i \le H_2^3 + Y_3) = (0, 1, 4, 1)$ *while* $(H_3^x\colon\ x \ge 4) = (7, 6, 6, 6, 0, 0, \dots)$. *Also* $Y_0 = 5$, $R_0 = 6$ *as in Figure* 1, *while* $Y_1 = 1$, $R_1 = 0$ *and* $Y_2 = 2$, $R_2 = 2$.

directed percolation (i.e., the event $\tilde{S}$) occurs in $\tilde{\Omega}$. This is so because for some l, there will be an open directed path from $(l + 1, 0)$ to the vertical line $y \times \mathbf{Z}^+$ for every $y \ge l + 1$ and thus there must be an infinite open directed path from $(l + 1, 0)$.

From our definitions of R_j, K_j and step failure, we may restate this conclusion as follows. Consider the collection of closed intervals on the *continuous* half line $\mathbf{R}^+ = [0, \infty)$ of the form $I_j = [j, j + R_j]$. Then directed percolation occurs if the I_j's cover all but a bounded portion of $\mathbf{R}^+$; that is, if

$$[0, \infty) \setminus \bigcup_{j=0}^{\infty} I_j \quad \text{is bounded.} \tag{2.5}$$

A crucial feature of our recursive construction in $\tilde{\Omega}$ is that the R_j's are independent and identically distributed random variables. This is because after $j - 1$ steps, we have no information about the value of p_j or about the status of the edges to the northeast of (j, H_{j-1}^j), except for right edges on the horizontal line at height H^j_{j-1}. The common distribution of the R_j's is that of a random variable [on some probability space $(\Omega^*, \mathscr{F}^*, \mathbf{P}^*)$]

$$R = \begin{cases} \max(D^1, \dots, D^Y), & \text{if } Y \geq 1, \\ 0, & \text{if } Y = 0, \end{cases} \tag{2.6}$$

where $D^1, D^2, \dots$ are i.i.d. with the geometric distribution

$$\mathbf{P}^*(D^i \geq d) = \varepsilon^d \quad \text{for } d = 0, 1, 2, \dots, \tag{2.7}$$

and Y (independent of the D^i's) has the distribution (geometric, conditional on a random parameter p_x)

$$\mathbf{P}^*(Y \geq y) = \mathbf{E}\big((p_x)^y\big) \quad \text{for } y = 0, 1, 2, \dots. \tag{2.8}$$

Here $\mathbf{E}$ denotes expectation with respect to the $\mathbf{P}$ of the random environment as in (2.1), and (2.6) is a consequence of (2.3) and (2.4).

According to Proposition 3 (stated and proved later in this section), to show that (2.5) occurs $\tilde{\mathbf{P}}$-almost surely, we need only verify that for any $\varepsilon > 0$ in (2.7),

$$\liminf_{r \to \infty} r\mathbf{P}^*(R > r) > 1. \tag{2.9}$$

We will show that this is a consequence of (2.1).

From (2.6)–(2.8), we have for $r = 1, 2, \dots,$

$$\begin{aligned} \mathbf{P}^*(R \geq r) &= 1 - \sum_{y=0}^{\infty} \mathbf{P}^*(Y = y)\big[\mathbf{P}^*(D^1 < r)\big]^y \\ &= 1 - \sum_{y=0}^{\infty} \mathbf{E}\big((1 - p_x)(p_x)^y\big)(1 - \varepsilon^r)^y \\ &= 1 - \mathbf{E}\left[\frac{1 - p_x}{1 - p_x(1 - \varepsilon^r)}\right] \\ &= \mathbf{E}\left[\frac{p_x \varepsilon^r}{1 - p_x + p_x \varepsilon^r}\right]. \end{aligned} \tag{2.10}$$

Setting $V = -\log(1 - p_x)$ and $c = -\log \varepsilon$, the last expression in (2.10) may be rewritten, using integration by parts, to yield

$$\begin{aligned} \mathbf{P}^*(R \geq r) &= \mathbf{E}\left[\frac{1 - e^{-V}}{1 - e^{-V} + e^{-V+cr}}\right] \\ &= \int_0^{\infty} e^{v-cr}\big[e^{v-cr} - e^{-cr} + 1\big]^{-2} \mathbf{P}(V > v)\, dv. \end{aligned} \tag{2.11}$$

However, from (2.1), we have that for any $\gamma < \infty$, there is some $v_0 < \infty$ so that

$$\mathbf{P}(V > v) \geq \gamma/v \quad \text{for } v \geq v_0. \tag{2.12}$$

Inserting this into (2.11), letting $v' = v - cr$ and taking a limit yields

$$\begin{aligned}\liminf_{r\to\infty} r\mathbf{P}^*(R > r) &= \liminf_{r\to\infty} r\mathbf{P}^*(R \geq r)\\ &\geq \liminf_{r\to\infty}\int_{v_0 - cr}^{\infty} e^{v'}[e^{v'} - e^{-cr} + 1]^{-2}\frac{\gamma r}{v' + cr}\,dv'\\ &\geq \frac{\gamma}{c}\int_{-\infty}^{\infty} e^{v'}[e^{v'} + 1]^{-2}\,dv' = \frac{\gamma}{c}.\end{aligned}$$

Since (2.12) is valid for any $\gamma < \infty$, we conclude that for any $c < \infty$ (i.e., for any $\varepsilon > 0$), $\liminf r\mathbf{P}^*(R > r)$ is $+\infty$ and hence (2.9) has been verified. It only remains to prove the following proposition about random coverings.

PROPOSITION 3. *Let $R_0, R_1, R_2, \dots$ be i.i.d. nonnegative integer-valued random variables on $(\Omega^*, \mathscr{F}^*, \mathbf{P}^*)$ such that*

$$\liminf_{r\to\infty} r\mathbf{P}^*(R_0 > r) > 1. \tag{2.13}$$

Then the intervals $[j, j + R_j]$ almost surely cover all but a bounded portion of $\mathbf{R}^+$.

PROOF. Let A_k be the event that $(k, k+1)$ is not covered by any of the random intervals $[j, j + R_j]$. We must show that, under (2.13), only finitely many A_k's occur. However,

$$\begin{aligned}\mathbf{P}^*(A_k) &= \mathbf{P}^*(R_0 \leq k, R_1 \leq k - 1, \dots, R_k \leq 0)\\ &= \prod_{i=0}^{k}\mathbf{P}^*(R_0 \leq i) = \prod_{i=0}^{k}(1 - \mathbf{P}^*(R_0 > i))\\ &\leq \exp\left(-\sum_{i=0}^{k}\mathbf{P}^*(R_0 > i)\right)\\ &\leq \exp(-c\log k) \quad \text{for large } k,\end{aligned} \tag{2.14}$$

where the last inequality follows from (2.13), with $c > 1$. Summing over k and applying the Borel–Cantelli lemma completes the proof. □

3. Survival for continuous time. In this section we prove the continuous time result, Theorem 1, by showing that it is a consequence of the discrete time Theorem 2, proved in the last section. Our strategy is to construct a coupling between the continuous and discrete time processes. The coupling is based on the well-known graphical representation of the contact process as a continuous time directed percolation model [5].

We first note that to prove Theorem 1, we may, without loss of generality, assume that $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ so that (as in our discrete time model) infection is transmitted only to the right, and then take as state space $\{0, 1\}^{\mathbf{Z}^+}$. The graphical representation then utilizes independent Poisson processes on the time line $\mathbf{R}^+$: one for each $x \in \mathbf{Z}^+$ with rate $\delta(x)$ which generates "cuts" at the locations $\{(x, U_n^x): n = 1, 2, \dots\}$ in $\mathbf{Z}^+ \times \mathbf{R}^+$ and one for each $\langle x, x+1 \rangle$ with rate ε_2 which generates "arrows" from (x, T_n^x) to $(x+1, T_n^x)$ for $n = 1, 2, \dots$. Cuts correspond to recovery from infection and arrows to rightward spread of infection. The contact process survival event S^0 corresponds to the graphical event that there is an infinite directed path starting from $(0, 0)$ moving upward in time without crossing cuts and moving rightward in space only over arrows. We need to show that S^0 has positive probability.

In order to construct our coupling, we will generate the Poisson process of cuts as follows. We begin by dividing $\mathbf{Z}^+ \times \mathbf{R}^+$ into portions above and below the 45° line:

$$\mathbf{L}^+ = \{(x, s): s > x\}, \qquad \mathbf{L}^- = \{(x, s): s \le x\}. \tag{3.1}$$

We first generate the cuts in $\mathbf{L}^-$; these will play no role in our coupling. Then, independently of the cuts in $\mathbf{L}^-$, we will generate the cuts in $\mathbf{L}^+$ in a rather special way. Roughly speaking, this will be based on the "thinning" property of Poisson processes. To generate a process of rate $\delta(x)$, one may begin with a Poisson process of higher rate $2\bar{\delta}$ and then thin it out by independently rejecting the original higher rate Poisson occurrences with probability $1 - \delta(x)/(2\bar{\delta})$. We say "roughly speaking" because we will focus on certain intervals of the form $\{x\} \times (j, j+2]$ in $\mathbf{L}^+$. We only generate and reject or accept the *earliest* higher rate occurrence in those intervals, after which we generate occurrences at the correct rate $\delta(x)$.

The intervals in $\mathbf{L}^+$ we focus on will be indexed by (x, t) in $\mathbf{Z}^+ \times \mathbf{Z}^+$ and defined as

$$\mathscr{I}(x, t) = \{x\} \times (x + 2t, x + 2t + 2]. \tag{3.2}$$

For each such interval, we take (independently for different intervals) independent random variables $H_0(x, t), \eta(x, t), H_1(x, t), H_2(x, t), \dots,$ where $H_0(x, t)$ is exponential with mean $(2\bar{\delta})^{-1}$, $\eta(x, t)$ takes the values 0 and 1 with probabilities $1 - \delta(x)/(2\bar{\delta})$ and $\delta(x)/(2\bar{\delta})$ and each $H_j(x, t)$ for $j \ge 1$ is exponential with mean $\delta(x)^{-1}$. The cuts within $\mathscr{I}(x, t)$ are then located as follows. Define

$$U_m(x, t) = H_0(x, t) + \cdots + H_m(x, t). \tag{3.3}$$

If $H_0(x, t) > 2$, then there are no cuts within $\mathscr{I}(x, t)$. If $H_0(x, t) \le 2$ and $\eta(x, t) = 0$, then the set of cut locations within $\mathscr{I}(x, t)$ is

$$\{(x, x + 2t + U_m(x, t)): m \ge 1 \text{ and } U_m(x, t) \le 2\}. \tag{3.4}$$

If $H_0(x, t) \le 2$ and $\eta(x, t) = 1$, then in addition to the cuts given by (3.4), there is one more cut located at $(x, x + 2t + H_0(x, t))$. Note that there are no cuts within $\mathscr{I}(x, t)$ if $\eta(x, t) = 0$ *and* $H_1(x, t) > 2$, regardless of the value of $H_0(x, t)$.

With the above process for generating all cuts and with the Poisson processes of arrows independent of all cuts, we can now determine the open and closed edges of our coupled directed percolation model on $\mathbf{Z}^+ \times \mathbf{Z}^+$. We define the up edge from (x, t) to $(x, t+1)$ to be open if $\eta(x, t) = 0$ *and*

$H_1(x,t) > 2$; otherwise, it is closed. This up edge from (x,t) is thus open with probability

$$p_x = \left(1 - \frac{\delta(x)}{2\bar{\delta}}\right)\exp[-2\delta(x)]. \tag{3.5}$$

We define the right edge from (x,t) to $(x+1,t)$ to be open if $H_0(x,t) > 2$ *and* there is at least one arrow from the upper half of $\mathscr{I}(x,t)$ to the bottom half of $\mathscr{I}(x+1,t)$ (i.e., for some n, the arrow time coordinate T_n^x is in $(x+2t+1, x+2t+2]$); otherwise the right edge from (x,t) to $(x+1,t)$ is closed. This right edge from (x,t) is thus open with probability

$$\varepsilon = \exp\left[-2(2\bar{\delta})\right]\left[1 - \exp(-\varepsilon_2)\right]. \tag{3.6}$$

The point of these definitions is twofold. First, this is an *independent* directed percolation model—that is, all the edges are open or closed independently of each other. Second, each open edge in the discrete time model implies the existence of a corresponding path for the spread of infection in the original model. Specifically, an open up edge from (x,t) implies there are no cuts within $\mathscr{I}(x,t)$, so any infection present anywhere in the bottom half of $\mathscr{I}(x,t)$ will still be present everywhere in the top half of $\mathscr{I}(x,t)$. Similarly, an open right edge from (x,t) implies there are no cuts within $\mathscr{I}(x,t)$ and an arrow from the upper half of $\mathscr{I}(x,t)$ to the lower half of $\mathscr{I}(x+1,t)$; thus any infection present anywhere in the bottom half of $\mathscr{I}(x,t)$ will be spread to some part of the bottom half of $\mathscr{I}(x+1,t)$. We conclude that occurrence of the event $\hat{S}^0$ in the discrete time model [i.e., existence of an infinite directed open path starting at $(0,0)$] implies the occurrence of the survival event S^0 in the continuous time model [i.e., survival for all time of an infection starting at $(0,0)$].

To complete the proof of Theorem 1, we need to show that $\hat{S}^0$ has positive probability for any $\varepsilon_2 > 0$. This we will do by applying Theorem 2 (and the remark following it). Our discrete time model is of the type treated in Theorem 2, with ε given by (3.6) strictly positive, and i.i.d. p_x's given by (3.5). It remains only to show that the condition (2.1) on the distribution of p_x, needed for our discrete time result, follows from the hypothesis (1.1) on the distribution of $\delta(x)$. However, this is an immediate consequence of inequalities following easily from (3.5):

$$p_x \geq \left(1 - \frac{\delta(x)}{2\bar{\delta}}\right)(1 - 2\delta(x)) \geq 1 - C\delta(x), \tag{3.7}$$

where $C = (2\bar{\delta})^{-1} + 2$, so that

$$-\log(1 - p_x) \geq -\log\delta(x) - C', \tag{3.8}$$

where $C' = \log C$. The proof is now complete. □

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