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DIFFUSION APPROXIMATION FOR THE ADVECTION OF PARTICLES IN A STRONGLY TURBULENT RANDOM ENVIRONMENT

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In this paper we prove several theorems concerning the motion of a particle in a random environment. The trajectory of a particle is the solution of the differential equation $dx(t)/dt = V(x(t))$, where $V(x) = v + \varepsilon^{1-\alpha}F(x)$, $0 \le \alpha < 1$, v is a constant vector, F is a mean-zero fluctuation field and $\varepsilon^{1-\alpha}$ is a parameter measuring the size of the fluctuations. We show that both in case of a motion of a single particle and of a particle system considered in the macroscopic coordinate system moving along with velocity v [i.e., $x \sim (x - vt)/\varepsilon^{\alpha}$, $t \sim t/\varepsilon^2$] the diffusion approximation holds provided that F is divergence free. Moreover we show how to renormalize trajectories to obtain a similar result for non-divergence-free fields. These results generalize theorems due to Khasminskii and to Kesten and Papanicolaou.

1. Introduction. Let $V(x)$ be a d-dimensional random velocity field, and let $x(t)$ be the particle trajectory in R^d satisfying

$$\frac{dx}{dt}(t) = V(x(t)), \qquad x(0) = x.$$

The following question arises naturally. Under what hypotheses on the field V does the particle have diffusive behavior observed over a long time? More specifically, assume that time $t \sim \varepsilon^{-2}$ and

$$V(x) = v + \varepsilon^{1-\alpha}F(x).$$

Here $v \in R^d$ is a constant nonzero vector, F is a mean-zero, stationary random field and $0 \le \alpha \le 1$ is a fixed parameter with $\varepsilon^{1-\alpha}$ measuring the size of the fluctuations given by F when $\varepsilon \ll 1$. Since the field F is stationary we can assume, with no loss of generality, that initially the particle is at the origin. Define the "scaled" process

$$x_{\varepsilon}(t) = \varepsilon^{\alpha}\left[x\left(\frac{t}{\varepsilon^2}\right) - v\frac{t}{\varepsilon^2}\right]. \tag{1}$$

It satisfies the differential equation

$$\frac{dx_{\varepsilon}}{dt}(t) = \frac{1}{\varepsilon}F\left(\frac{x_{\varepsilon}(t)}{\varepsilon^{\alpha}} + v\frac{t}{\varepsilon^2}\right). \tag{2}$$

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We will show that if $F = (F_1, \dots, F_d)$ is divergence free, that is, $\sum_{p=0}^{d} \partial_p F_p = 0$, and sufficiently smooth, then x_ε behaves, for ε small and $\alpha < 1$, like a diffusion process. In general if

$$\frac{dx_\varepsilon}{dt}(t) = \frac{1}{\varepsilon} V\left(\frac{t}{\varepsilon^2}, \frac{x_\varepsilon(t)}{\varepsilon^\alpha}\right), \qquad x_\varepsilon(0) = x \tag{3}$$

and V is a random velocity field stationary in time, we will show (see Theorem 3) how to scale x_ε properly in order to obtain diffusive behavior in the limit. These results generalize theorems of Khasminskii [8] and Kesten and Papanicolaou [7] which cover the case when $\alpha = 0$.

In addition we study the limiting behavior of stochastic flows corresponding to (3), when V is a divergence-free velocity field. More specifically, let us consider the solutions $T_\varepsilon(t, x)$ of the advection equations

$$\partial_t T_\varepsilon(t, x) = \frac{1}{\varepsilon}\left(V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon^\alpha}\right), \nabla\right) T_\varepsilon(t, x), \qquad T_\varepsilon(0, x) = T_0(x) \tag{4}$$

as stochastic processes in the space of tempered distributions. Using Mitoma's characterization of weak compactness in that space, we will prove weak convergence of T_ε as $\varepsilon \downarrow 0$ to a solution $\overline{T}$ of a diffusion equation

$$\partial_t \overline{T} = \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q} \partial_{p,q}^2 \overline{T}, \qquad \overline{T}(0) = T_0,$$

if $0 < \alpha < 1$, and to a solution of the Itô stochastic differential equation

$$d\overline{T}(t) = \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q} \partial_{p,q}^2 \overline{T}(t)\, dt + K_{\overline{T}(t)}^{1/2}\, d\beta(t), \qquad \overline{T}(0) = T_0,$$

for $\alpha = 0$. Here $\{\beta(t)\}_{t \ge 0}$ is a cylindrical Brownian motion in $L^2(R^d)$; $K_{\overline{T}(t)}^{1/2}$ is the square root of a nonnegative definite trace class operator $K_{\overline{T}(t)}$: $L^2(R^d) \to L^2(R^d)$ given by the formula

$$K_{\overline{T}(t)} f(x) = \sum_{p,q=1}^{d} \int \Gamma_{p,q}(x - y)\, \partial_{x_p} \overline{T}(t, x)\, \partial_{y_q} \overline{T}(t, y) f(y)\, dy, \qquad f \in L^2(R^d). \tag{5}$$

Coefficients $a_{p,q}$ and functions $\Gamma_{p,q}$, $p, q = 1, \dots, d$, are given explicitly in formulas (6) and (25), respectively.

In order to be able to define the operator by means of formula (5), we will also need to prove that the process $\{\overline{T}(t)\}_{t \ge 0}$ takes its values in $H^1(R^d)$ space.

The results mentioned above are related to the problem of a diffusion in a turbulent medium treated by a different method in [4].

Our paper is divided into three parts. In Sections 2–7 we deal with the case of a single particle. In Sections 8–11 we discuss the problem of a flow of particles. Sections 2 and 8 briefly outline the proofs for both situations. The last part of this work consists of appendices including the proofs of facts omitted in the main exposition.

2. Formulation of the main result and sketch of the proof. Below we formulate precisely the problem for the motion of a single particle, briefly described in the Introduction, and outline the strategy for its proof. Denote by $(\Omega, \mathscr{F}, P)$ a probability space. Let $\{x_\varepsilon(t)\}_{t\geq 0}$ be given by (3), where α is assumed to belong to the interval $[0, 1)$. Throughout this article we will let N denote the greatest integer less than or equal to $1/(1-\alpha)$. We formulate the following conditions, which are assumed to be fulfilled by the random field V.

(C1) The random field V is strictly stationary in time and space; that is, for any $t_1, \ldots, t_m \in R$, $x_1, \ldots, x_m \in R^d$ and each $h \in R$ and $k \in R^d$, the joint distribution of

$$V(t_1 + h, x_1 + k), \ldots, V(t_m + h, x_m + k)$$

is the same as that of

$$V(t_1, x_1), \ldots, V(t_m, x_m).$$

(C2) For $C, \varrho > 0$, let $\mathscr{V}_a^b(C, \varrho)$ denote the σ-algebra generated by the sets of the form

$$[\omega\colon\ V(t, x; \omega) \in A],$$

where $a \leq t \leq b$, $\mid x \mid \leq C(1 + t^{\varrho})$ and A is a Borel set in R^d. Let

$$\beta(h; C, \varrho) = \sup_t \sup_{A \in \mathscr{V}_{t+h}^{\infty}(C,\varrho),\ B \in \mathscr{V}_0^t(C,\varrho)} \frac{\mid P(A \cap B) - P(A)P(B) \mid}{P(B)}.$$

We will assume that there exist such $0 \leq \varrho < 1$ and $C > 0$ that for any $m \geq 0$ there is a C_m such that

$$h^m \beta(h; C, \varrho) \leq C_m \quad \text{for all } h \geq 0.$$

The above restriction on ϱ is essential for the application of our main theorem (formulated below) to the situation described in the first paragraph of the introduction. See also Remark 4 after Lemma 1.

(C3) The random field V has $N+1$ spatial derivatives and there is a constant $C > 0$ so that

$$\sum_{0 \leq |\mathbf{k}| \leq N+1} \mid D^{\mathbf{k}} V(t, x) \mid \leq C < +\infty$$

Here $D^{\mathbf{k}} V(t, x) = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} V(t, x)$ and $\mathbf{k} = (k_1, \ldots, k_d)$, $\mid \mathbf{k} \mid = \sum_{p=1}^d k_p$.

(C4) The random field V is divergence free; that is,

$$\operatorname{div} V(t, x) = \sum_{p=1}^{d} \partial_{x_p} V_p(t, x) \equiv 0.$$

The following theorem holds.

THEOREM 1. *Suppose that V satisfies the assumptions made in conditions (C1)–(C4). Then the family of processes $\{x_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, is weakly convergent as $\varepsilon \downarrow 0$ to a Brownian motion with a covariance matrix whose entries are given by the following formulas*:

$$(6)\quad a_{pq} = \int_0^\infty E\{V_p(t,0)V_q(0,0) + V_q(t,0)V_p(0,0)\}\,dt, \qquad p, q = 1, \ldots, d.$$

Theorem 1 implies the following result for the "scaled" trajectories described by (2).

THEOREM 2. *Suppose that F is a strictly stationary random field with mean zero satisfying the following properties*:

(i) *The uniform mixing coefficient*

$$\beta(h; v) = \sup_{t\geq 0} \sup_{A\in\mathscr{F}_{t+h}^{\infty}(v),\ B\in\mathscr{F}_0^t(v)} \frac{|P(A\cap B) - P(A)P(B)|}{P(B)}$$

is decaying faster than any power; *that is, for any $m \geq 0$ there is a $C_m > 0$ such that*

$$\beta(h)h^m \leq C_m.$$

Here $\mathscr{F}_s^t(v)$ denotes the σ-algebra generated by the sets of the form

$$[\omega\colon\ F(y,\omega) \in A], \qquad A \text{ is Borel measurable in } R^d,\ y \text{ satisfies } s \leq \frac{(y,v)}{|v|} \leq t,$$

where $(\cdot,\cdot)$ stands for the usual Euclidean inner product of vectors.

(ii) *The field F has $N+1$ derivatives,*

$$\sum_{0\leq|\mathbf{k}|\leq N+1} |D^{\mathbf{k}}F| \leq C < +\infty$$

and $\operatorname{div} F \equiv 0$. *Here $D^{\mathbf{k}}F(y) = \partial_{y_1}^{k_1}\cdots\partial_{y_d}^{k_d}F$ for $\mathbf{k} = (k_1, \ldots, k_d)$.*

Then the family of processes $\{x_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, given by the solutions of (2) *is weakly convergent as $\varepsilon \downarrow 0$ to a Brownian motion with a covariance matrix whose entries are given by the formulas*

$$a_{pq} = \int_0^\infty E\{F_p(vt)F_q(0) + F_q(vt)F_p(0)\}\,dt, \qquad p, q = 1, \ldots, d.$$

In order to conclude Theorem 1 we will prove a more technical result stated below as our next theorem. It holds for a more general class of random fields than those satisfying condition (C1); therefore we introduce a weaker version of this condition as follows:

(C1′) The random field V is strictly stationary in the variable t for each fixed $x \in R^d$.

THEOREM 3. *Suppose that a random field $V(t, x)$ satisfies assumptions* (C1′), (C2) *and* (C3). *Then there exists a family of d-dimensional processes $\{\beta_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, weakly convergent as $\varepsilon \downarrow 0$ to the standard d-dimensional Brownian motion and such that*

$$\begin{aligned} z_\varepsilon(t) = x_\varepsilon(t) &- \int_0^t \left[\frac{1}{\varepsilon} c_0\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right) + \cdots + \frac{1}{\varepsilon^{N\alpha - N+1}} c_N\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right)\right] ds \\ &- \int_0^t \sigma\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right) d\beta_\varepsilon(s) \end{aligned} \tag{7}$$

converges weakly to 0. *Here*

$$c_n(x) = \int_0^\infty du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} \overline{W}_n(u_1, u_2, \ldots, u_n, 0, x)\, du_n,$$

$$\begin{aligned} W_n(s_1, s_2, \ldots, s_{n+1}, x) &= \sum_{p=1}^d \partial_{x_p} \widetilde{W}_{n-1}(s_1, s_2, \ldots, s_n, x) V_p(s_{n+1}, x) \qquad \textit{for } n = 1, \ldots, N, \\ W_0(s_1, x) &= V(s_1, x), \\ \overline{W}_n &= E W_n, \\ \widetilde{W}_n &= W_n - \overline{W}_n, \end{aligned} \tag{8}$$

$\sigma(x)$ is a symmetric, nonnegative definite $d \times d$-matrix-valued function such that

$$\sigma^2(x) = a(x) \tag{9}$$

and $a = [a_{pq}]_{d\times d}$ is a matrix whose entries are given by the following formulas:

$$\begin{aligned} a_{pq}(x) \\ = \int_0^\infty E\{V_p(t, x)V_q(0, x) + V_q(t, x)V_p(0, x)\}\, dt, \qquad p, q = 1, \ldots, d. \end{aligned} \tag{10}$$

REMARK 1. According to a well-known result of Oleinik [see [13], Theorem 5.2.3, page 132], there exists a unique Lipschitzian $d \times d$-matrix-valued function $\sigma(x)$, $x \in R^d$, satisfying (9) for $a(x)$ given by (10).

REMARK 2. Theorem 3 says that the solution $\{x_\varepsilon(t)\}_{t\geq 0}$ of (3) behaves as $\varepsilon \downarrow 0$ approximately in the weak sense as the solution of the Itô stochastic differential equation

$$dx_\varepsilon(t) = b_\varepsilon\left(\frac{x_\varepsilon(t)}{\varepsilon^\alpha}\right)dt + \sigma\left(\frac{x_\varepsilon(t)}{\varepsilon^\alpha}\right)d\beta(t),$$
$$x_\varepsilon(0) = 0,$$

where

$$b_\varepsilon(x) = \frac{1}{\varepsilon}c_0(x) + \cdots + \frac{1}{\varepsilon^{N\alpha-N+1}}c_N(x) \tag{11}$$

and $c_0, \dots, c_N$, σ are defined as above.

REMARK 3. Notice that if V satisfies the stronger condition (C1), then $c_0, \dots, c_N$ become constants as well as a_{pq}, $p, q = 1, \dots, d$. Theorem 3 asserts then that the scaled processes

$$y_\varepsilon(t) = x_\varepsilon(t) - \left(\frac{1}{\varepsilon}c_0 + \cdots + \frac{1}{\varepsilon^{N\alpha-N+1}}c_N\right)t, \qquad t \ge 0,$$

become eventually as $\varepsilon \downarrow 0$ a Brownian motion with the covariance matrix given by (10). When in addition V is divergence free, then

$$c_0 = \cdots = c_N = 0.$$

Indeed, for $n = 1, \dots, N$,

$$\begin{aligned}\overline{W}_n = EW_n &= E\left\{\sum_{p=1}^d \partial_{y_p}\widetilde{W}_{n-1}(s_1, s_2, \dots, s_n, y)V_p(s_{n+1}, y)\right\}, \\ &= -E\{\overline{W}_{n-1}\operatorname{div} V\} = 0.\end{aligned}$$

The original family of processes $\{x_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, is therefore weakly convergent as $\varepsilon \downarrow 0$ to a Brownian motion with covariance matrix a, as asserted in Theorem 1.

The proof of Theorem 3 will be done under the additional, yet not essential, assumption that a is strongly nondegenerate. As in [13] one can easily overcome this difficulty.

The main line of the argument contained in the proof can be divided into three steps. In step 1 we verify that for $\varepsilon > 0$ the processes

$$y_\varepsilon(t) = x_\varepsilon(t) - \int_0^t\left[\frac{1}{\varepsilon}c_0\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right) + \cdots + \frac{1}{\varepsilon^{N\alpha-N+1}}c_N\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right)\right]ds, \qquad t \ge 0,$$

are tight. We proceed with this step, proving first a Chentsov-type criterion for tightness, formulated in Lemma 2 (Section 3). Its application to this family is possible by means of Lemma 1. In step 2, carried out in Section 6, we construct the family of processes $\{\beta_\varepsilon(t)\}_{t\ge0}$, $\varepsilon > 0$, approximating the standard d-dimensional Brownian motion and verify, using the criterion mentioned above, that the integrals

$$\left\{\int_0^t \sigma\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right)d\beta_\varepsilon(s)\right\}_{t\ge 0}, \qquad \varepsilon > 0,$$

form a tight family as $\varepsilon \downarrow 0$. Identification of the limit is done by means of the martingale characterization of Brownian motion. The final step is the proof that the finite-dimensional distributions of $\{z_\varepsilon(t)\}_{\varepsilon>0}$ defined by (7) tend to 0 as $\varepsilon \downarrow 0$. This property combined with the results of steps 1 and 2 enables us to claim that the conclusion of Theorem 3 holds.

3. Basic lemmas. Here we present the basic lemmas needed for the proof of Theorem 3. The proofs of those facts can be found in Appendix A.

LEMMA 1. *Let us fix $T > 0$ and let $0 \le u \le s \le T$. Assume that $Y(s)$ is a $\mathscr{V}_0^{s/\varepsilon^2}(C, \varrho)$-measurable random vector function. Let $\xi_\varepsilon(u)$ be $\mathscr{V}_0^{u/\varepsilon^2}(C, \varrho)$-measurable, and let $|\xi_\varepsilon(u)|/\varepsilon^\alpha \le C(1 + u^\varrho/\varepsilon^{2\varrho})$, for $0 < \varepsilon < \varepsilon_0(T)$, where $\varepsilon_0(T)$ is sufficiently small constant depending on T. Then for any nonnegative integers k and n there are a constant $C(d, k, n, M_{n+k})$ depending only on $d, k, n, M_{n+k} = \max_{|\mathbf{k}|\le n+k} \sup |D^{\mathbf{k}} V(t, y)|$ and exponents $\alpha_1(k, n), \dots, \alpha_{n+1}(k, n) > 0$ depending only on k and n such that for a multi-index $\mathbf{k} = (k_1, \dots, k_d)$ with $|\,\mathbf{k}\,| = k$ and $0 \le u \le s \le s_{n+1} \le \dots \le s_1 \le T$, $0 < \varepsilon < \varepsilon_0(T)$,*

$$\left| E\left\{ D^{\mathbf{k}} \widetilde{W}_n\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_{n+1}}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) Y\left(\frac{s}{\varepsilon^2}\right)\right\}\right|$$

$$\le C(d, k, n, M_{n+k}) \beta^{\alpha_1(k,n)}(s_1 - s_2) \cdots \beta^{\alpha_{n+1}(k,n)}(s_{n+1} - s) E \left| Y\left(\frac{s}{\varepsilon^2}\right)\right|.$$

Here

$$D^{\mathbf{k}} V(t, y) = \partial_{y_1}^{k_1} \cdots \partial_{y_d}^{k_d} V(t, y),$$

$$D^{\mathbf{k}} \widetilde{W}(s_1, \dots, s_{n+1}, y) = \partial_{y_1}^{k_1} \cdots \partial_{y_d}^{k_d} \widetilde{W}(s_1, \dots, s_{n+1}, y).$$

REMARK 4. The function $\xi_\varepsilon(u)$ is the obvious prototype for $x_\varepsilon(u)$. Notice that $x_\varepsilon(u)$ is $\mathscr{V}_0^{u/\varepsilon^2}(C, \varrho)$-measurable, if $\varrho > (1 + \alpha)/2$ and $C > |\,F\,| + 1$. Indeed,

$$\frac{|\,x_\varepsilon(u)\,|}{\varepsilon^\alpha} \le \frac{1}{\varepsilon^{1+\alpha}} \int_0^u \left| V\left(\frac{u_1}{\varepsilon^2}, \frac{x_\varepsilon(u_1)}{\varepsilon^\alpha}\right)\right| du_1$$

$$\le C \frac{u}{\varepsilon^{1+\alpha}} \le C\left(1 + \frac{u^\varrho}{\varepsilon^{2\varrho}}\right),$$

provided that ε is sufficiently small. The only reason we have ξ_ε in the statement of this lemma is that it will also be needed for $\xi_\varepsilon(u) \equiv x$, where $|\,x\,| \le Cu/\varepsilon$.

The next ingredient of our proof is the following lemma establishing a weak compactness criterion in $C[0, +\infty)$ modelled on classical theorems due to Chentsov and Kolmogorov.

LEMMA 2. *Suppose that $\mathscr{F}$ is a family of probabilistic measures on $C[0,+\infty)$ satisfying the following*:

(i) $\lim_{M\uparrow\infty}\sup_{\mu\in\mathscr{F}}\mu[x: \mid x(0)\mid > M]=0$;

(ii) *for any $T>0$ and $\nu>0$ there is a constant $C(T,\nu)$ such that, for all $\mu\in\mathscr{F}$, $T\ge u\ge t\ge s\ge 0$,*

$$E^{\mu}\{\mid x(u)-x(t)\mid^2\mid x(t)-x(s)\mid^{\nu}\}\le C(T,\nu)(u-t)E^{\mu}\mid x(t)-x(s)\mid^{\nu}.$$

Here E^{μ} denotes the expectation with respect to the measure μ.

Then $\mathscr{F}$ is weakly compact.

4. Auxiliary computations. In this section we suppress the subscript ε on x. Consider a partition $0=t_0<t_1<\cdots<t_M=T$ such that $\Delta t_i = t_{i+1}-t_i=\varepsilon^{\gamma}$, $i=0,\dots,M-2$, where $1<\gamma<2$ is to be defined later. The last interval $[t_{M-1},t_M]$ is of length less than or equal to ε^{γ}. Assuming that $s<t$, we have

$$\begin{aligned}
x(t)-x(s)
&=\frac{1}{\varepsilon}\int_s^t V\left(\frac{u_1}{\varepsilon^2},\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1\\
&=\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1+\frac{1}{\varepsilon}\int_s^t \widetilde{W}_0\left(\frac{u_1}{\varepsilon^2},\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1\\
&=\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1+\sum_i\frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}} \widetilde{W}_0\left(\frac{u_1}{\varepsilon^2},\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1+o(1).
\end{aligned}\tag{12}$$

The summation is taken over those i's for which $s<t_i<t_{i+1}<t$. So the left-hand side of (12) up to a quantity of magnitude $o(1)$ is equal to

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1+\sum_i\frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}} \widetilde{W}_0\left(\frac{u_1}{\varepsilon^2},\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1\\
&\quad=\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1+\sum_i\frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}} \widetilde{W}_0\left(\frac{u_1}{\varepsilon^2},\frac{x(t_{i-1})}{\varepsilon^{\alpha}}\right)du_1\\
&\qquad+\sum_i\frac{1}{\varepsilon^{2+\alpha}}\int_{t_i}^{t_{i+1}}du_1\int_{t_{i-1}}^{u_1}W_1\left(\frac{u_1}{\varepsilon^2},\frac{u_2}{\varepsilon^2},\frac{x(u_2)}{\varepsilon^{\alpha}}\right)du_2\\
&\quad=\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^{\alpha}}\right)du_1+\sum_i\frac{1}{\varepsilon^{2+\alpha}}\int_{t_i}^{t_{i+1}}du_1\int_{t_{i-1}}^{u_1}\overline{W}_1\left(\frac{u_1}{\varepsilon^2},\frac{u_2}{\varepsilon^2},\frac{x(u_2)}{\varepsilon^{\alpha}}\right)du_2\\
&\qquad+\sum_i\frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}} \widetilde{W}_0\left(\frac{u_1}{\varepsilon^2},\frac{x(t_{i-1})}{\varepsilon^{\alpha}}\right)du_1\\
&\qquad+\sum_i\frac{1}{\varepsilon^{2+\alpha}}\int_{t_i}^{t_{i+1}}du_1\int_{t_{i-1}}^{u_1}\widetilde{W}_1\left(\frac{u_1}{\varepsilon^2},\frac{u_2}{\varepsilon^2},\frac{x(u_2)}{\varepsilon^{\alpha}}\right)du_2
\end{aligned}$$

Applying the procedure of "freezing" time at t_{i-1} and expanding $\widetilde{W}_1$ around t_{i-1}, we get a formula involving W_2. Centering W_2 by subtracting its average, we arrive at an analogue of (12) involving, in addition to all the terms on the right-hand side of (12), the mean $\overline{W}_2$ and the fluctuation $\widetilde{W}_2$. Repeating this procedure N times we get

$$\begin{aligned}
&x(t)-x(s)\\
&\quad=\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^\alpha}\right)du_1\\
&\qquad+\sum_i \frac{1}{\varepsilon^{2+\alpha}}\int_{t_i}^{t_{i+1}}du_1\int_{t_{i-1}}^{u_1}\overline{W}_1\left(\frac{u_1}{\varepsilon^2},\frac{u_2}{\varepsilon^2},\frac{x(u_2)}{\varepsilon^\alpha}\right)du_2+\cdots\\
&\qquad+\sum_i \frac{1}{\varepsilon^{N+1+N\alpha}}\int_{t_i}^{t_{i+1}}du_1\\
&\qquad\qquad\times\int_{t_{i-1}}^{u_1}du_2\cdots\int_{t_{i-1}}^{u_N}\overline{W}_N\left(\frac{u_1}{\varepsilon^2},\ldots,\frac{u_{N+1}}{\varepsilon^2},\frac{x(u_{N+1})}{\varepsilon^\alpha}\right)du_{N+1}\\
&\qquad+\sum_i \frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}}\widetilde{W}_0\left(\frac{u_1}{\varepsilon^2},\frac{x(t_{i-1})}{\varepsilon^\alpha}\right)du_1+\cdots\\
&\qquad+\sum_i \frac{1}{\varepsilon^{N+1+N\alpha}}\int_{t_i}^{t_{i+1}}du_1\\
&\qquad\qquad\times\int_{t_{i-1}}^{u_1}du_2\cdots\int_{t_{i-1}}^{u_N}\widetilde{W}_N\left(\frac{u_1}{\varepsilon^2},\ldots,\frac{u_{N+1}}{\varepsilon^2},\frac{x(t_{i-1})}{\varepsilon^\alpha}\right)du_{N+1}\\
&\qquad+\sum_i \frac{1}{\varepsilon^{N+2+(N+1)\alpha}}\int_{t_i}^{t_{i+1}}du_1\\
&\qquad\qquad\times\int_{t_{i-1}}^{u_1}du_2\cdots\int_{t_{i-1}}^{u_{N+1}}W_{N+1}\left(\frac{u_1}{\varepsilon^2},\ldots,\frac{u_{N+2}}{\varepsilon^2},\frac{x(u_{N+2})}{\varepsilon^\alpha}\right)du_{N+2}\\
&\qquad+o(1).
\end{aligned}\tag{13}$$

Denote

$$\begin{aligned}
L&=x(t)-x(s)-\frac{1}{\varepsilon}\int_s^t c_0\left(\frac{x(u_1)}{\varepsilon^\alpha}\right)du_1\\
&\quad-\sum_i \frac{1}{\varepsilon^{2+\alpha}}\int_{t_i}^{t_{i+1}}du_1\int_{t_{i-1}}^{u_1}\overline{W}_1\left(\frac{u_1}{\varepsilon^2},\frac{u_2}{\varepsilon^2},\frac{x(u_2)}{\varepsilon^\alpha}\right)du_2-\cdots\\
&\quad-\sum_i \frac{1}{\varepsilon^{N+1+N\alpha}}\int_{t_i}^{t_{i+1}}du_1\\
&\qquad\times\int_{t_{i-1}}^{u_1}du_2\cdots\int_{t_{i-1}}^{u_N}\overline{W}_N\left(\frac{u_1}{\varepsilon^2},\ldots,\frac{u_{N+1}}{\varepsilon^2},\frac{x(u_{N+1})}{\varepsilon^\alpha}\right)du_{N+1}.
\end{aligned}$$

Our first claim is that L is, up to a quantity of order $o(1)$, equal to

$$x(t)-x(s)-\int_s^t\left[\frac{1}{\varepsilon}c_0\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right)ds+\cdots+\frac{1}{\varepsilon^{N\alpha-N+1}}c_N\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right)\right]ds.$$

Applying a change of variables and condition (C1′) on V, we obtain the following string of equalities for $1 \le n \le N+1$ and any admissible i:

$$
\begin{aligned}
&\frac{1}{\varepsilon^{n+(n-1)\alpha}} \int_{t_i}^{t_{i+1}} du_1 \int_{t_{i-1}}^{u_1} du_2 \cdots \int_{t_{i-1}}^{u_{n-1}} \overline{W}_{n-1}\left(\frac{u_1}{\varepsilon^2}, \ldots, \frac{u_n}{\varepsilon^2}, \frac{x(u_n)}{\varepsilon^\alpha}\right) du_n \\
&\quad = \sum_i \frac{1}{\varepsilon^{n+(n-1)\alpha}} \int_{t_i}^{t_{i+1}} du_1 \\
&\qquad \times \int_{t_{i-1}}^{u_1} du_2 \cdots \int_{t_{i-1}}^{u_{n-1}} \overline{W}_{n-1}\left(\frac{u_1 - u_n}{\varepsilon^2}, \ldots, \frac{u_{n-1} - u_n}{\varepsilon^2}, 0, \frac{x(u_n)}{\varepsilon^\alpha}\right) du_n \\
(14)\quad &\quad = \sum_i \frac{1}{\varepsilon^{n+(n-1)\alpha}} \varepsilon^2 \int_{t_i}^{t_{i+1}} du_1 \int_{t_{i-1}}^{u_1} du_2 \cdots \int_{t_{i-1}}^{u_{n-2}} du_{n-1} \\
&\qquad \times \int_0^{(u_{n-2}-u_{n-1})/\varepsilon^2} \overline{W}_{n-1}\left(\frac{u_1 - u_{n-1}}{\varepsilon^2}, \ldots, \frac{u_{n-2} - u_{n-1}}{\varepsilon^2}, \right. \\
&\qquad\qquad \left. v_{n-1}, 0, \frac{x(u_{n-1})}{\varepsilon^\alpha}\right) dv_{n-1}.
\end{aligned}
$$

Repeating this procedure $n-1$ times and taking into account that the first iterated integral is taken from t_i to t_{i+1} not from t_{i-1} to t_i, we get that the right-hand side of (14) is equal to

$$
\begin{aligned}
&\frac{1}{\varepsilon^{n+(n-1)\alpha}} \varepsilon^{2(n-1)} \int_{t_{i-1}}^{t_i} du_1 \int_{(t_i-u_1)/\varepsilon^2}^{(t_{i+1}-u_1)/\varepsilon^2} dv_1 \\
&\qquad \times \int_0^{v_1} dv_2 \cdots \int_0^{v_{n-2}} \overline{W}_{n-1}\left(v_1, \ldots, v_{n-1}, 0, \frac{x(u_1)}{\varepsilon^\alpha}\right) dv_{n-1} \\
&\quad + \frac{1}{\varepsilon^{n+(n-1)\alpha}} \varepsilon^{2(n-1)} \int_{t_i}^{t_{i+1}} du_1 \int_0^{(t_{i+1}-u_1)/\varepsilon^2} dv_1 \\
&\qquad \times \int_0^{v_1} dv_2 \cdots \int_0^{v_{n-2}} \overline{W}_{n-1}\left(v_1, \ldots, v_{n-1}, 0, \frac{x(u_1)}{\varepsilon^\alpha}\right) dv_{n-1}.
\end{aligned}
$$

Summing the right-hand side of (14) over all admissible i's, we get, up to a term of order $o(1)$,

$$
\begin{aligned}
&\sum_i \frac{1}{\varepsilon^{n+(n-1)\alpha}} \varepsilon^{2(n-1)} \int_{t_{i-1}}^{t_i} du_1 \int_0^{(t_{i+1}-u_1)/\varepsilon^2} dv_1 \\
&\qquad \times \int_0^{v_1} dv_2 \cdots \int_0^{v_{n-2}} \overline{W}_{n-1}\left(v_1, \ldots, v_{n-1}, 0, \frac{x(u_1)}{\varepsilon^\alpha}\right) dv_{n-1}.
\end{aligned}
$$

The absolute value of the difference between this term and

$$
\frac{1}{\varepsilon^{(n-1)\alpha-n+2}} \int_s^t c_{n-1}\left(\frac{x_\varepsilon(u)}{\varepsilon^\alpha}\right) du
$$

is again, up to a term of magnitude $o(1)$, less than or equal to

$$\sum_i \frac{1}{\varepsilon^{(n-1)\alpha-n+2}} \int_{t_{i-1}}^{t_i} du_1 \int_{(t_{i+1}-u_1)/\varepsilon^2}^{+\infty} dv_1 \tag{15}$$
$$\times \int_0^{v_1} dv_2 \dots \int_0^{v_{n-2}} \left| \overline{W}_{n-1}\left(v_1, \dots, v_{n-1}, 0, \frac{x(u_1)}{\varepsilon^\alpha}\right) \right| dv_{n-1}.$$

Observe that, according to Lemma 1, there exist a constant C and positive exponents $\alpha_1, \dots, \alpha_{n-1} > 0$ such that, for all $x \in R^d$,

$$\left| \overline{W}_{n-1}\left(v_1, \dots, v_{n-1}, 0, \frac{x}{\varepsilon^\alpha}\right) \right| \le C\beta^{\alpha_1}(v_1 - v_2) \cdots \beta^{\alpha_{n-2}}(v_{n-2} - v_{n-1})\beta^{\alpha_{n-1}}(v_{n-1}).$$

So the expression given in (15) is less than or equal to

$$\frac{C}{\varepsilon^{(n-1)\alpha-n+2}} \sum_i \int_{t_i}^{t_{i+1}} du_1 \int_{\varepsilon^{\gamma-2}}^{+\infty} dv_1$$
$$\times \int_0^{v_1} dv_2 \cdots \int_0^{v_{n-2}} \beta^{\alpha_1}(v_1 - v_2) \cdots \beta^{\alpha_{n-2}}(v_{n-2} - v_{n-1})\beta^{\alpha_{n-1}}(v_{n-1})\, dv_{n-1}.$$

Thus it is of order of magnitude $o(1)(t-s)$ and the claim we made about L is indeed true.

The second claim we make here is that the last term before $o(1)$ in (13), that is,

$$\sum_i \frac{1}{\varepsilon^{N+2+(N+1)\alpha}} \int_{t_i}^{t_{i+1}} du_1 \tag{16}$$
$$\times \int_{t_{i-1}}^{u_1} du_2 \cdots \int_{t_{i-1}}^{u_{N+1}} W_{N+1}\left(\frac{u_1}{\varepsilon^2}, \dots, \frac{u_{N+2}}{\varepsilon^2}, \frac{x(u_{N+2})}{\varepsilon^\alpha}\right) du_{N+2},$$

is itself of order $o(1)$. This follows from the fact that W_{N+1} is bounded and thus the multiple integral is of order $O(\varepsilon^{(N+2)\gamma})$. Taking into account the fact that the sum has less than $[(t-s)/\varepsilon^\gamma] + 1$ terms, we get that (16) is of order

$$O(\varepsilon^{(N+1)\gamma-(N+2+(N+1)\alpha)}) = o(1),$$

if only $2 > \gamma > (N + 2 + (N+1)\alpha)/(N+1)$, which is possible since $\alpha < 1 - 1/(N+1)$.

5. Tightness of $y_\varepsilon(t)$. Now we proceed with step 1 of the proof of Theorem 2. According to what we have stated so far and Lemma 2, it is enough to prove that there is a $C(T)$ such that

$$E\left\{\left[\sum_i \frac{1}{\varepsilon^{n+1+n\alpha}} \int_{t_i}^{t_{i+1}} du_1 \right.\right. \tag{17}$$
$$\left.\left. \times \int_{t_{i-1}}^{u_1} \cdots \int_{t_{i-1}}^{u_n} \widetilde{W}_n\left(\frac{u_1}{\varepsilon^2}, \dots, \frac{u_{n+1}}{\varepsilon^2}, \frac{x(t_{i-1})}{\varepsilon^\alpha}\right) du_{n+1}\right]^2 Y\right\}$$
$$\le C(T)(t-s)EY,$$

for $0 \le n \le N$ and Y $\mathscr{V}_0^{s/\varepsilon^2}$-measurable and nonnegative (here and in the sequel we will omit writing C and ϱ in the notation of σ-algebras). Notice that the left-hand side of (17) is equal to

$$\begin{aligned}(18)\quad &\frac{2}{\varepsilon^{2n+2+2n\alpha}} \sum_{i\le j} \int_{t_i}^{t_{i+1}} du_1 \int_{t_{i-1}}^{u_1} \cdots \int_{t_{i-1}}^{u_n} du_{n+1} \int_{t_j}^{t_{j+1}} du_1' \\ &\quad\times \int_{t_{j-1}}^{u_1'} \cdots \int_{t_{j-1}}^{u_n'} E\left\{ \widetilde{W}_n\left(\frac{u_1}{\varepsilon^2}, \dots, \frac{u_{n+1}}{\varepsilon^2}, \frac{x(t_{i-1})}{\varepsilon^\alpha}\right) \right. \\ &\quad\quad \left. \times \widetilde{W}_n\left(\frac{u_1'}{\varepsilon^2}, \dots, \frac{u_{n+1}'}{\varepsilon^2}, \frac{x(t_{j-1})}{\varepsilon^\alpha}\right) Y \right\} du_{n+1}'.\end{aligned}$$

We will distinguish three terms I_1, I_2 and I_3 in the expression described by formula (18), corresponding to the summation ranges $i \le j-2$, $i = j-1$ and $i = j$, respectively.

By Lemma 1, term I_1 can be estimated as follows:

$$\begin{aligned} I_1 &\le \frac{2C}{\varepsilon^{2n+2+2n\alpha}} \sum_{i\le j-2} \int_{t_i}^{t_{i+1}} du_1 \int_{t_{i-1}}^{u_1} \cdots \int_{t_{i-1}}^{u_n} du_{n+1} \int_{t_j}^{t_{j+1}} du_1' \\ &\quad\times \int_{t_{j-1}}^{u_1'} \cdots \int_{t_{j-1}}^{u_n'} \beta^{\alpha_1}\left(\frac{u_1'-u_2'}{\varepsilon^2}\right) \cdots \beta^{\alpha_{n+1}}\left(\frac{u_{n+1}'-t_{j-1}}{\varepsilon^2}\right) du_{n+1}' EY \\ &\le \frac{C}{\varepsilon^{2n+2+2n\alpha}} \varepsilon^{2n\gamma} \beta^{\min \alpha_k}\left(\frac{2\varepsilon^{\gamma-2}}{n+1}\right) EY(t-s)^2,\end{aligned}$$

since at least one $u_k' - u_{k+1}' \ge (t_{j+1} - t_{j-1})/(n+1) = 2\varepsilon^\gamma/(n+1)$. Hence it is of order $o(1)$.

Terms I_2 and I_3 can be estimated in the same manner, so we will deal with the latter one only. By Lemma 1,

$$\begin{aligned}(19)\quad I_3 &= \frac{2}{\varepsilon^{2n+2+2n\alpha}} \sum_{i} \int_{t_i}^{t_{i+1}} du_1 \int_{t_{i-1}}^{u_1} \cdots \int_{t_{i-1}}^{u_n} du_{n+1} \int_{t_i}^{t_{i+1}} du_1' \\ &\quad\times \int_{t_{i-1}}^{u_1'} \cdots \int_{t_{i-1}}^{u_n'} E\left\{ \widetilde{W}_n\left(\frac{u_1}{\varepsilon^2}, \dots, \frac{u_{n+1}}{\varepsilon^2}, \frac{x(t_{i-1})}{\varepsilon^\alpha}\right) \right. \\ &\quad\quad \left. \times \widetilde{W}_n\left(\frac{u_1'}{\varepsilon^2}, \dots, \frac{u_{n+1}'}{\varepsilon^2}, \frac{x(t_{i-1})}{\varepsilon^\alpha}\right) Y \right\} du_{n+1}' \\ &\le 2C\varepsilon^{(2n+1)\gamma-2n-2-2n\alpha} EY(t-s).\end{aligned}$$

If $n > 0$, we choose $2 > \gamma > (2n\alpha + 2n + 2)/(2n+1)$ so the right-hand side of (19) is of the form $o(1)EY(t-s)$. If $n = 0$, there can be no such γ. Then we estimate, using Lemma 1, as follows:

$$\sum_i \frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} E\left\{\widetilde{W}_0\left(\frac{u_1}{\varepsilon^2}, \frac{x(t_{i-1})}{\varepsilon^\alpha}\right)\widetilde{W}_0\left(\frac{u_1'}{\varepsilon^2}, \frac{x(t_{i-1})}{\varepsilon^\alpha}\right)Y\right\} du_1\, du_1'$$
$$\le \frac{4C}{\varepsilon^2} EY \sum_i \int_{t_i}^{t_{i+1}}\int_{t_i}^{\varrho_1} \beta\left(\frac{u_1-u_1'}{\varepsilon^2}\right) du_1\, du_1'$$
$$\le 4CEY(t-s)\int_0^{+\infty}\beta(u)\,du.$$

This proves tightness of $\{y_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, and thus step 1 is concluded.

6. Brownian motion approximation. Let us consider $0 = t_0 < t_1 < \cdots < t_m < \cdots$ a partition of $[0, +\infty)$ such that $\varepsilon^\gamma = \Delta t_i$ for all i.

Set, for $t_i \le t \le t_{i+1}$,

$$(\beta_\varepsilon(t))_p = (\beta_\varepsilon(t_i))_p + \frac{1}{\varepsilon}\sum_{q=1}^{d}(\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\int_{t_{i+1}}^{t+\varepsilon^\gamma} \widetilde{W}_0^q\left(\frac{u}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right) du,$$

where σ was defined by formulas (9) and (10).

THEOREM 4. *The family of processes $\{\beta_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, converges weakly, as $\varepsilon \downarrow 0$, to the standard Brownian motion.*

PROOF. First we verify tightness of the above family using Lemma 2. Since the computation is mostly along the lines of Section 4, we will only highlight its most important points. We need to check that, for any bounded, nonnegative and continuous function $\psi(x_1, \ldots, x_M)$ defined on $(R^d)^M$, the following condition holds for some constant $C > 0$:

$$E\{|\beta_\varepsilon(t) - \beta_\varepsilon(s)|^2\, \psi(\beta_\varepsilon(s_1), \ldots, \beta_\varepsilon(s_M))\} \le C(t-s)E\psi(\beta_\varepsilon(s_1), \ldots, \beta_\varepsilon(s_M)), \tag{20}$$

for all $0 \le s_1 \le \cdots \le s_M \le s < t \le T$. Denoting $\psi(\beta_\varepsilon(s_1), \ldots, \beta_\varepsilon(s_M))$ by Ψ, we can write the left-hand side of (20) as, up to a term of magnitude $o(1)$, equal to

$$E\left\{\left[\sum_{p=1}^{d}\frac{1}{\varepsilon}\sum_i\sum_{q=1}^{d}(\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\int_{t_i}^{t_{i+1}} \widetilde{W}_0^q\left(\frac{u}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right) du\right]^2\Psi\right\}, \tag{21}$$

where the summation ranges over those i's where $s < t_i < t_{i+1} < t$. For the sake of notational clarity let us define $U^{qq'}(u, u', x) = \widetilde{W}_0^q(u, x)\widetilde{W}_0^{q'}(u', x) +$

$\widetilde{W}_0^q(u', x)\widetilde{W}_0^{q'}(u, x)$. Then by Lemma 1 we can conclude that (21) equals

$$\frac{1}{\varepsilon^2}\sum_i \sum_{p,q,q'=1}^d E\left\{(\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)(\sigma^{-1})^{pq'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\right.$$
$$\left.\times \int_{t_{i+1}}^{t_{i+2}} du \int_{t_{i+1}}^{u} U^{qq'}\left(\frac{u}{\varepsilon^2}, \frac{u'}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right) du'\,\Psi\right\} + o(1)(t-s)^2 E\Psi$$

$$(22)\qquad \begin{aligned} &= \frac{1}{\varepsilon^2}\sum_i \sum_{p,q,q'=1}^d E\left\{\left((\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)(\sigma^{-1})^{pq'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\right)\right.\\ &\qquad \times \int_{t_{i+1}}^{t_{i+2}} du \int_{t_{i+1}}^{u}\left[U^{qq'}\left(\frac{u}{\varepsilon^2}, \frac{u'}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\right.\\ &\qquad\qquad \left.\left. - EU^{qq'}\left(\frac{u}{\varepsilon^2}, \frac{u'}{\varepsilon^2}, \frac{x}{\varepsilon^\alpha}\right)\Big|_{x=x_\varepsilon(t_i)}\right] du'\,\Psi\right\}\\ &\quad + \frac{1}{\varepsilon^2}\sum_i \sum_{p,q,q'=1}^d E\left\{(\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)(\sigma^{-1})^{pq'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\right.\\ &\qquad \left.\times \int_{t_{i+1}}^{t_{i+2}} du \int_{t_{i+1}}^{u} EU^{qq'}\left(\frac{u}{\varepsilon^2}, \frac{u'}{\varepsilon^2}, \frac{x}{\varepsilon^\alpha}\right)\Big|_{x=x_\varepsilon(t_i)} du'\,\Psi\right\}\\ &\quad + o(1)(t-s)^2 E\Psi\\ &= (o(1)+d)E\Psi(t-s) + o(1)(t-s)^2 E\Psi. \end{aligned}$$

The final equality in (22) holds by application of Lemma 1 and the argument applied in Section 4 that the tails of integrals defining $a_{qq'}$, $q, q' = 1, \dots, d$, go to 0. Thus (22) implies that (20) indeed holds, so $\{\beta_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, is a tight family in $C[0, +\infty)$. As for the second and finishing step of the proof of Theorem 4, we need to verify the following three conditions:

$$\text{(i)}\qquad \lim_{\varepsilon\downarrow 0} E\{[(\beta_\varepsilon)_p(t) - (\beta_\varepsilon)_p(s)]\Psi\} = 0 \quad \text{for all } t > s;$$

$$\text{(ii)}\qquad \begin{aligned} &\lim_{\varepsilon\downarrow 0} E\{\{[(\beta_\varepsilon)_p(t) - (\beta_\varepsilon)_p(s)][(\beta_\varepsilon)_q(t) - (\beta_\varepsilon)_q(s)] - \delta_{pq}(t-s)\}\Psi\}\\ &\quad = 0 \quad \text{for all } t > s; \end{aligned}$$

$$\text{(iii)}\qquad \limsup_{\varepsilon\downarrow 0} E(\beta_\varepsilon)_p^4(t) < +\infty \quad \text{for all } t > 0.$$

The computations for each of the above three conditions are rather standard and once again go along the lines presented in Section 4. We only outline briefly the arguments for each point.

(i) By Lemma 1,

$$E\{[(\beta_\varepsilon)_p(t)-(\beta_\varepsilon)_p(s)]\Psi\}$$
$$=E\left\{\left[\sum_i \frac{1}{\varepsilon}\sum_{q=1}^{d}(\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\int_{t_{i+1}}^{t_{i+2}}\widetilde{W}_0^q\left(\frac{u}{\varepsilon^2},\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)du\right]\Psi\right\}=o(1).$$

(ii) By Lemma 1,

$$E\{[(\beta_\varepsilon)_p(t)-(\beta_\varepsilon)_p(s)][(\beta_\varepsilon)_q(t)-(\beta_\varepsilon)_q(s)]\Psi\}$$
$$\begin{aligned}(23)\quad &=E\left\{\sum_i \frac{1}{\varepsilon}\sum_{r=1}^{d}(\sigma^{-1})^{pr}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\int_{t_{i+1}}^{t_{i+2}}\widetilde{W}_0^r\left(\frac{u}{\varepsilon^2},\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)du\right.\\ &\qquad\left.\times\frac{1}{\varepsilon}\sum_{r'=1}^{d}(\sigma^{-1})^{pr'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\int_{t_{i+1}}^{t_{i+2}}\widetilde{W}_0^{r'}\left(\frac{u'}{\varepsilon^2},\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)du'\,\Psi\right\}+o(1).\end{aligned}$$

Now, subtracting and adding the mean of $U^{rr'}(u/\varepsilon^2, u'/\varepsilon^2, x/\varepsilon^\alpha)$ evaluated at $x=x_\varepsilon(t_i)$ and applying the identical computation as in (22), we get easily that the expression on the right-hand side of (23) is equal to

$$\begin{aligned}&\sum_i\sum_{r,r'=1}^{d}E\left\{(\sigma^{-1})^{pr}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)(\sigma^{-1})^{qr'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)a_{rr'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\Psi\right\}\Delta t_i+o(1)\\ &\qquad=\delta_{pq}(t-s)E\Psi+o(1),\end{aligned}$$

which is precisely what we want.

(iii) Write

$$g_i^q=(\sigma^{-1})^{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\int_{t_{i+1}}^{t_{i+2}}\widetilde{W}_0^q\left(\frac{u}{\varepsilon^2},\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)du.$$

Using this notation, we can write

$$(24)\qquad E\mid\beta_\varepsilon(t)\mid^4=\frac{24}{\varepsilon^4}\sum_{i_1\le i_2\le i_3\le i_4}\sum_{q_1,q_2,q_3,q_4=1}^{d}Eg_{i_1}^{q_1}g_{i_2}^{q_2}g_{i_3}^{q_3}g_{i_4}^{q_4}.$$

We distinguish two terms J_1 and J_2 in the expression described by (24) corresponding to the summation ranges $i_1\le i_2\le i_3<i_4$ and $i_1\le i_2\le i_3=i_4$, respectively. By Lemma 1 the first term is of order $o(1)$. The second term is equal to

$$\begin{aligned}&\frac{24}{\varepsilon^4}\sum_{i_1\le i_2\le i}\sum_{q_1,q_2,q_3,q_4=1}^{d}Eg_{i_1}^{q_1}g_{i_2}^{q_2}Eg_i^{q_3}g_i^{q_4}\\ &\quad+\frac{24}{\varepsilon^4}\sum_{i_1\le i_2\le i}\sum_{q_1,q_2,q_3,q_4=1}Eg_{i_1}^{q_1}g_{i_2}^{q_2}(g_i^{q_3}g_i^{q_4}-Eg_i^{q_3}g_i^{q_4}).\end{aligned}$$

Once again the second term can be estimated by $o(1)$ by means of Lemma 1. The first term is treated by an argument identical to that used for proving (ii) of order $O(1)$, which ends the proof of (iii).

Using the classical characterization of Wiener measure in terms of martingales (see [13]) both parts of our argument prove that $\{\beta_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, is weakly convergent to a standard Brownian motion as ε tends to 0. □

7. The end of the proof of Theorem 3. We want to see that the family of processes $\{z_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, defined by (7) weakly converges to 0 as $\varepsilon \downarrow 0$. It will be done in two steps. First, of course, we check tightness and then we will make a simple observation that

$$\lim_{\varepsilon\downarrow 0} E \mid z_\varepsilon(t) \mid^2 = 0.$$

This of course guarantees that the finite-dimensional distributions of $\{z_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, tend to 0 as $\varepsilon \downarrow 0$, so our proof will be completed.

To prove tightness, notice that we already know that the family of processes

$$\left\{x_\varepsilon(t) - \int_0^t b_\varepsilon(x_\varepsilon(s))\,ds\right\}_{t\geq 0}, \qquad \varepsilon > 0,$$

where b_ε is defined by (11), is tight. Thus we only need to verify tightness of $\{\int_0^t \sigma(x_\varepsilon(s)/\varepsilon^\alpha)\,d\beta_\varepsilon(s)\}_{t\geq 0}$, $\varepsilon > 0$. Since σ is Lipschitzian we get, for $t_i \leq s \leq t_{i+1}$,

$$\begin{aligned}\left|\sigma\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right) - \sigma\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\right| &\leq C\left|\frac{x_\varepsilon(s) - x_\varepsilon(t_i)}{\varepsilon^\alpha}\right| \\ &\leq C\varepsilon^{\gamma-1-\alpha} = o(1).\end{aligned}$$

Writing $\tilde\sigma(s) = \sigma(x_\varepsilon(t_i)/\varepsilon^\alpha)$, for $t_i \leq s \leq t_{i+1}$, we can write

$$\begin{aligned}&E\left|\int_0^t\left[\sigma\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right) - \tilde\sigma(s)\right]d\beta_\varepsilon(s)\right|^2 \\ &\quad = E\sum_{p=1}^d\left\{\frac{1}{\varepsilon}\sum_i\sum_{q=1}^d\int_{t_i}^{t_{i+1}}\left[\sigma_{pq}\left(\frac{x_\varepsilon(s)}{\varepsilon^\alpha}\right) - \tilde\sigma_{pq}(s)\right](\sigma^{-1})^{qq'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)\right. \\ &\qquad\qquad \left.\times\,\widetilde W_0^q\left(\frac{s+\varepsilon^\gamma}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)ds\right\}^2 + o(1) = o(1),\end{aligned}$$

by means of Lemma 1. The summation goes over those i's for which $t_i < t$. Thus only tightness of $\{\int_0^t \tilde\sigma(s)\,d\beta_\varepsilon(s)\}_{t\geq 0}$, $\varepsilon > 0$, needs to be shown. Observe that

$$\left(\int_0^t \tilde\sigma(s)\,d\beta_\varepsilon(s)\right)_p = \sum_i\sum_{q=1}^d \sigma_{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)[(\beta_\varepsilon(t_{i+1}))_q - (\beta_\varepsilon(t_i))_q] + o(1)$$

$$= \sum_i \sum_{q,q'=1}^{d} \frac{1}{\varepsilon} \sigma_{pq}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)(\sigma^{-1})^{qq'}\left(\frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right)$$

$$\times \int_{t_{i+1}}^{t_{i+2}} \widetilde{W}_0^{q'}\left(\frac{u}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right) du + o(1)$$

$$= \frac{1}{\varepsilon} \sum_i \int_{t_{i+1}}^{t_{i+2}} \widetilde{W}_0^{p}\left(\frac{u}{\varepsilon^2}, \frac{x_\varepsilon(t_i)}{\varepsilon^\alpha}\right) du + o(1).$$

However, we have already established tightness of this process in Section 5. Having that in mind we may notice that, according to the auxiliary computations made in Section 4,

$$E \mid z_\varepsilon(t) \mid^2 = o(1),$$

which by our previous remark ends the proof.

8. Formulation of the result for flows, and sketch of the proof. In what follows we will assume that the field V satisfies conditions (C1)–(C4) stated in Section 2. Let

$$R_{p,q}(t, x) = E\{V_p(t, x)V_q(0, 0)\}, \qquad p, q = 1, \dots, d,$$

denote the autocorrelation matrix of the field. Let

$$\Gamma_{p,q}(x) = \int_{-\infty}^{+\infty} R_{p,q}(u, x)\, du, \qquad p, q = 1, \dots, d. \tag{25}$$

Let $\mathscr{S}(R^d)$ denote the space of Schwartz test functions with the Frechét space structure imposed by the norms (see [5])

$$\|f\|_{\mathbf{k},\mathbf{l}}^2 = \sum_{|\mathbf{k}| \le k,\ |\mathbf{l}| \le l} \int \mid x^{\mathbf{l}} D^{\mathbf{k}} f(x) \mid^2 \, dx.$$

Here we adopt customary notation for powers, derivatives and norms with multiindices:

$$\mathbf{k} = (k_1, \dots, k_d), \qquad \mathbf{l} = (l_1, \dots, l_d),$$

$$x^{\mathbf{l}} = x_1^{l_1} \cdot \dots \cdot x_d^{l_d},$$

$$D^{\mathbf{k}} f = \partial_1^{k_1} \cdots \partial_d^{k_d} f,$$

$$\mid \mathbf{l} \mid = \sum l_i, \qquad \mid \mathbf{k} \mid = \sum k_i.$$

For a Hilbert space H let $L^2_{\mathrm{loc},w}([0, +\infty); H)$ denote the space of H-valued, locally square integrable functions equipped with the topology which is the direct limit of weak topologies on $L^2([0, T]; H)$, for $T > 0$ (as in [10]). The symbol $\tau c(H)$ stands for the space of trace class operators on the Hilbert space H (see [1]), and $\tau c^+(H)$ denotes all positive definite trace class operators.

For each $\varepsilon > 0$ let $T_\varepsilon(t, x)$ denote the solution of the advection equation (4). It can be viewed as a stochastic process with values in $\mathscr{S}'$ the space of tempered distributions or, equivalently, as a Borel measure on the topological vector space $C([0, +\infty), \mathscr{S}')$.

The theorem we aim to prove is as follows.

THEOREM 5. *Under the assumptions made about V the family of processes $\{T_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, is weakly convergent in $C([0, +\infty), \mathscr{S}')$-space as $\varepsilon \downarrow 0$ to a process $\overline{T}$ such that the following hold*:

(i) *If $0 < \alpha < 1$ and $\lim_{|x|\to+\infty}\Gamma(x) = 0$, then $\overline{T}$ is deterministic and its trajectories satisfy the following diffusion equation*:

$$\partial_t \overline{T} = \tfrac{1}{2}\sum_{p,q=1}^{d} a_{p,q}\,\partial^2_{p,q}\overline{T},$$

$$\overline{T}(0) = T_0.$$

(ii) *If $\alpha = 0$ and the matrix $[a_{pq}]_{d\times d}$ defined by* (6) *is positive definite, then the law of $\overline{T}$ in $C([0, +\infty), \mathscr{S}') \cap L^2_{\mathrm{loc},w}([0, +\infty); H^1(R^d))$ is identical with that of the solution of the Itô stochastic differential equation*

$$d\overline{T}(t) = \tfrac{1}{2}\sum_{p,q=1}^{d} a_{p,q}\,\partial^2_{p,q}\overline{T}(t)\,dt + K^{1/2}_{\overline{T}(t)}\,d\beta(t), \tag{26}$$

$$\overline{T}(0) = T_0,$$

where β is a cylindrical $L^2(R^d)$-valued Brownian motion in the sense of [14, *page* 60].

REMARK 5. For any $g \in H^1(R^d)$, $K_g^{1/2}$ is a Hilbert–Schmidt operator, being the unique square root of $K_g \in \tau c^+(L^2(R^d))$ given by

$$K_g f(x) = \sum_{p,q=1}^{d} \int \Gamma_{p,q}(x-y)\,\partial_{x_p} g(x)\,\partial_{y_q} g(y) f(y)\,dy.$$

The proof that K_g is indeed nonnegative definite is presented in Appendix B.

The proof of Theorem 5, as is usual in the case of weak convergence results, will be done in two steps. First we verify tightness of the family $\{T_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, in $C([0, +\infty), \mathscr{S}')$ and then we go about the identification of its limit. As for the first step, according to [11], it is enough to show that $\{\int T_\varepsilon(t, x)\phi(x)\,dx\}_{t\geq 0}$, $\varepsilon > 0$, is a tight family of processes in $C([0, +\infty), R)$. This is again obtained by an application of Lemma 2. In order to put ourselves in a position of being able to use that lemma we have to do some preparatory computations in the spirit of Section 4. This step is carried out in Section 9. Now tightness becomes a direct consequence of Lemma 1 stated earlier. Identification of the limit will be done in two stages. First we prove the uniqueness of the limiting measure in $C([0, +\infty), \mathscr{S}')$ by proving that the moments

$$E\{(T(t_1), \phi_1) \cdots (T(t_m), \phi_m)\},$$

for any $t_1 \le \cdots \le t_m$, $\phi_1, \ldots, \phi_m \in \mathscr{S}$ are uniquely determined. Here we use the notation

$$(f, g) = \int f(x)g(x)\,dx,$$

for any $f, g \in L^2(R^d)$.

Having established this fact we will proceed with proving that the limiting measure is supported in $L^2_{\mathrm{loc},w}([0, +\infty); H^1(R^d))$. Thus it is possible to apply Yor's martingale representation theorems in Hilbert spaces (see [14]) and establish Itô's stochastic differential equation corresponding to the distribution law of our limiting measure.

9. Auxiliary computations. Write

$$Y_\varepsilon(t) = \int T_\varepsilon(t, x)\phi(x)\,dx.$$

The following computation will enable us to estimate

$$E \mid Y_\varepsilon(t) - Y_\varepsilon(s) \mid^2 \psi(Y_\varepsilon(s_1), \ldots, Y_\varepsilon(s_m)), \tag{27}$$

for ψ: $R^m \to R$ nonnegative, continuous and $s_1 \le \cdots \le s_m \le s$. The estimate of (27) by $C(t-s)E\psi$ is, according to Lemma 2, everything we need to claim tightness of $\{Y_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, in $C[0, +\infty)$.

Divide $[0, T]$ with $0 = t_0 < \cdots < t_M \le T < t_{M+1}$ and $\Delta t_i = \varepsilon^\gamma$ as in Section 4. Using (4) we can write that $\int[T_\varepsilon(t, x) - T_\varepsilon(s, x)]\phi(x)\,dx$ equals, up to a term of order $o(1)$, to

$$\begin{aligned} &\sum_i \int [T_\varepsilon(t_{i+1}, x) - T_\varepsilon(t_i, x)]\phi(x)\,dx \\ &\quad = -\frac{1}{\varepsilon}\sum_i \sum_{p=1}^d \int_{t_i}^{t_{i+1}} du \int T_\varepsilon(u, x)\Phi_p^\varepsilon(u, x)\,dx, \end{aligned} \tag{28}$$

where

$$\Phi_p^\varepsilon(u, x) = V_p\left(\frac{u}{\varepsilon^2}, \frac{x}{\varepsilon^\alpha}\right)\partial_{x_p}\phi(x).$$

Replacing u in $T_\varepsilon(u, x)$ by t_{i-1} and using (4) again to express the error, we get that the left-hand side of (28) is equal to

$$\begin{aligned} &-\frac{1}{\varepsilon}\sum_i \sum_{p_1=1}^d \int_{t_i}^{t_{i+1}} du \int T_\varepsilon(t_{i-1}, x)\Phi_{p_1}^\varepsilon(u, x)\,dx \\ &\qquad + \frac{1}{\varepsilon^2}\sum_i \sum_{p_1, p_2=1}^d \int_{t_i}^{t_{i+1}} du \int_{t_{i-1}}^u du_1 \int T_\varepsilon(u_1, x)\Phi_{p_1, p_2}^\varepsilon(u, u_1, x)\,dx, \end{aligned}$$

where

$$\Phi_{p_1, p_2}^\varepsilon(u, u_1, x) = V_{p_2}\left(\frac{u_1}{\varepsilon^2}, \frac{x}{\varepsilon^\alpha}\right)\partial_{x_{p_2}}\Phi_{p_1}^\varepsilon(u, x).$$

Repeating the above procedure N times, we get that the right-hand side of (28) is equal to

$$\sum_{n=1}^{N}\frac{(-1)^n}{\varepsilon^n}\sum_i\sum_{p_1,\dots,p_n=1}^{d}\int_{t_i}^{t_{i+1}}du\int_{t_{i-1}}^{u}du_1\cdots\int_{t_{i-1}}^{u_{n-2}}du_{n-1} \tag{29}$$
$$\times\int T_\varepsilon(t_{i-1},x)\Phi^\varepsilon_{p_1,\dots,p_n}(u,u_1,\dots,u_{n-1},x)\,dx+R,$$

where

$$\Phi^\varepsilon_{p_1,\dots,p_n}(u,\dots,u_{n-1},x)=V_{p_n}\left(\frac{u_{n-1}}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right)\partial_{x_{p_n}}\Phi^\varepsilon_{p_1,\dots,p_{n-1}}(u,\dots,u_{n-2},x),$$

and the remainder term R in (29) has a form identical to any term of the sum with $n=N+1$ and t_{i-1} replaced by u_{N+1}. Because T_ε are solutions of an advection equation which can be written in a divergence form, one notices that

$$\sup_x\mid T_\varepsilon(t,x)\mid\le\sup_x\mid T_0(x)\mid.$$

By this fact and consecutive applications of the product formula for derivatives, the integrand appearing in the nth term of the sum on the right-hand side of (29) can be written down as a sum of n terms, each estimated by $O(1/\varepsilon^{k\alpha})$, $k=0,1,\dots,n-1$. Performing multiple integration, taking into account that $\Delta t_i=\varepsilon^\gamma$ and adding along all intervals $[t_i,t_{i+1}]$, one can estimate each term as $O(\varepsilon^{(n-1)\gamma-k\alpha-n})$. Since $\alpha<1$ the magnitude of each term is $o(1)$ as long as $k\le n-2$, $n\ge 2$.

If $k=n-1$, we get the expression

$$\frac{(-1)^{n+1}}{\varepsilon^{n+1+n\alpha}}\sum_i\int_{t_i}^{t_{i+1}}du\cdots\int_{t_{i-1}}^{u_{n-2}}du_{n-1}$$
$$\times\int W_{n-1}\left(\frac{u}{\varepsilon^2},\dots,\frac{u_{n-1}}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right)\cdot\nabla\phi(x)T_\varepsilon(t_{i-1},x)\,dx,$$

where W_n has precisely the same meaning as given in formula (8). As shown in Section 5, all these terms are insignificant in further computations provided that $n\ge 1$, since they all are of order $o(1)$ for those n. Thus in the sequel we will deal with the W_0 term only. An easy observation made using the formula for the remainder term in (29) yields immediately that it is of order $o(1)$ since $\alpha<1-1/(N+1)$. Thus one can write (28) as

$$\sum_i\frac{1}{\varepsilon^2}\int_{t_i}^{t_{i+1}}du\int V\left(\frac{u}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right)\cdot\nabla\phi(x)T_\varepsilon(t_{i-1},x)\,dx$$
$$+\sum_i\sum_{p,q=1}^{d}\frac{1}{\varepsilon^2}\int_{t_i}^{t_{i+1}}du\int_{t_{i-1}}^{u}du_1$$
$$\times\int V_p\left(\frac{u}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right)V_q\left(\frac{u_1}{\varepsilon^2},\frac{x}{\varepsilon^\alpha}\right)\partial^2_{x_p,x_q}\phi(x)T_\varepsilon(t_{i-1},x)\,dx+o(1).$$

10. Tightness and properties of limiting measure. We can estimate

$$E[Y_\varepsilon(t) - Y_\varepsilon(s)]^2 \psi(Y_\varepsilon(s_1), \ldots, Y_\varepsilon(s_m)) \tag{30}$$

in a fashion identical to that in Section 5. There is a constant C such that (30) is less than or equal to $C(t-s)E\Psi$, where $\Psi = \psi(Y_\varepsilon(s_1), \ldots, Y_\varepsilon(s_m))$. Hence $\{Y_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, is tight in $C[0, +\infty)$ space according to Lemma 2, and therefore $\{T_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, is tight in $C([0,+\infty), \mathscr{S}'(R^d))$ by Mitoma's theorem (see [11]). Let $\overline{T}$ denote the possible weak limit. Notice that since $\|T_\varepsilon\|_{L^2} \le \|T_0\|_{L^2}$ the limiting measure is actually supported in

$$C([0,+\infty), \mathscr{S}'(R^d)) \cap L^2_{\mathrm{loc},w}([0,+\infty); L^2(R^d)).$$

Therefore we can assume that the trajectories of $\{\overline{T}(t)\}_{t\ge 0}$ are in $L^2(R^d)$ almost surely. Moreover they are weakly continuous in $L^2(R^d)$; that is, $t \mapsto (\overline{T}(t), \phi)$ is continuous for any $\phi \in L^2(R^d)$. Reasoning along the lines of Section 6, one finds immediately that

$$M_\phi(t) = (\overline{T}(t), \phi) - (T_0, \phi) - \tfrac{1}{2} \sum_{p,q=1}^{d} \int_0^t a_{p,q}(\overline{T}(s), \partial^2_{p,q}\phi)\, ds$$

is a mean-zero martingale for any $\phi \in \mathscr{S}(R^d)$. Consider two cases.

CASE 1 ($0 < \alpha < 1$). In this case we have

$$EM^2_\phi(t) \le t \sum_{p,q=1}^{d} \iint \Gamma_{p,q}\left(\frac{x-y}{\varepsilon^\alpha}\right) |\, \partial_{x_p}\phi(x)\, |\, |\, \partial_{y_q}\phi(y)\, |\; dx\, dy \big(\sup_x T_0(x)\big)^2 + o(1).$$

The right-hand side of this estimate vanishes as we allow $\varepsilon \downarrow 0$, so we get the first part of our theorem.

CASE 2 ($\alpha = 0$). For any $g \in L^2(R^d)$ and $\phi, \psi \in \mathscr{S}(R^d)$, write

$$K_g(\phi, \psi) = \sum_{p,q=1}^{d} \iint \Gamma_{p,q}(x-y)\, \partial_{x_p}\phi(x)\, \partial_{y_q}\phi(y)\, g(x) g(y)\, dx\, dy. \tag{31}$$

For fixed ϕ and ψ,

$$\lim_{\varepsilon\downarrow 0} \int_0^t K_{T_\varepsilon(s)}(\phi, \psi)\, ds = \int_0^t K_{\overline{T}(s)}(\phi, \psi)\, ds \tag{32}$$

(the proof is contained in Appendix C). Thus we see that $\{\overline{T}(t)\}_{t\ge 0}$ satisfies the following conditions:

(D1) $\overline{T}(t) \in \mathrm{L}^2(R^d)$, $t \ge 0$, $\|\overline{T}(t)\| \le \|T_0\|$;
(D2) $t \mapsto (\overline{T}(t), \phi)$, $t \ge 0$ is continuous for any $\phi \in L^2(R^d)$;
(D3) M_ϕ is a zero-mean martingale for any $\phi \in H^2(R^d)$;
(D4) $M_\phi(t) M_\psi(t) - \int_0^t K_{\overline{T}(s)}(\phi, \psi)\, ds$ is a zero-mean martingale for any $\phi, \psi \in \mathscr{S}(R^d)$.

Our next claim is that there is a unique probability measure μ generated by $\overline{T}(t)$ on $C([0,+\infty),\mathscr{S}'(R^d))\cap L^2_{\mathrm{loc},w}([0,+\infty);L^2(R^d))$ satisfying (D1)–(D4) and supported in $C([0,+\infty),\mathscr{S}'(R^d))\cap L^2_{\mathrm{loc},w}([0,+\infty),H^1(R^d))$. Before going any further let us introduce some additional notation. Let $\mathscr{T}_0^b$ denote the σ-algebra generated by $\overline{T}(t)$, $0\le t\le b$. Notice that the above conditions imply the following lemma.

LEMMA 3. *Let $\theta_1(t,x),\dots,\theta_m(t,x)$ be random fields, such that* (a) *$\theta_k(s,x)$ are $\{\mathscr{T}_0^t\}$-measurable for all $t\ge s\ge 0$ and $x\in R^d$, $k=1,\dots,m$, and* (b) *$\theta_k\in C^\infty(R\times R^d)$, $\theta_k(t,\cdot)$, $\partial_t\theta_k(t,\cdot)\in H^2(R^d)$ for each $t\in R$, $k=1,\dots,m$. Then the following hold*:

(i) *$(N_{\theta_1},\dots,N_{\theta_m})$ is a continuous martingale with respect to $\{\mathscr{T}_0^t\}$*;
(ii) *the joint quadratic variation* (*see* [9], *page* 46) *is*

$$\langle N_{\theta_k},N_{\theta_l}\rangle_t=\int_0^t K_{\overline{T}(s)}(\theta_k(s),\theta_l(s))\,ds.$$

Here

$$N_\theta(t)=(\overline{T}(t),\theta(t))-(T_0,\theta(0))\\ -\int_0^t\left\{\frac12\sum_{p,q=1}^d a_{p,q}(\overline{T}(s),\partial^2_{p,q}\theta(s))+(\overline{T}(s),\partial_s\theta(s))\right\}ds.$$

For the proof of this lemma see Appendix D. The claim made above, if proven, implies, as shown in the next section, that $\{T_\varepsilon(t)\}_{t\ge0}$, $\varepsilon>0$ are indeed weakly convergent as $\varepsilon\downarrow 0$. Moreover since $\{\overline{T}(t)\}_{t\ge0}$ is supported in $L^2([0,+\infty);H^1(R^d))$, there is a cylindrical Brownian motion $\{\beta(t)\}_{t\ge0}$ in $L^2(R^d)$ such that

$$N_\phi(t)=\int_0^t\langle K^{1/2}_{\overline{T}(s)}(\phi),d\beta(s)\rangle_{L^2(R^d)}$$

(for the definition of the stochastic integral with respect to a cylindrical Brownian motion see [14]). It is worthwhile to notice here that for $g\in H^1(R^d)$ the bilinear form K_g given by (31) defines an operator in $\tau c^+(L^2(R^d))$

$$K_g\phi(x)=\sum_{p,q=1}^d\int\Gamma_{p,q}(x-y)\,\partial_{x_p}g(x)\,\partial_{y_q}g(y)\phi(y)\,dy.$$

Hence $\{\overline{T}(t)\}_{t\ge0}$ satisfies Itô's stochastic differential equation (26).

11. Uniqueness. To show uniqueness of the distribution law for $\overline{T}$, we will establish uniqueness of the conditional moments

$$E\{\overline{T}(t,x_1)\cdots\overline{T}(t,x_m)\mid\mathscr{T}_0^s\}=u_m(t,x_1,\dots,x_m),\qquad t\ge s.\tag{33}$$

This, in turn, easily implies that all finite-dimensional distributions of $\{(\overline{T}(t),\phi)\}_{t\ge0}$ are uniquely defined so the uniqueness of the distributions of

$\overline{T}(t)$ will follow. To simplify formulas, we will prove (33) only for $s = 0$, that is, when $\mathscr{T}_0^0$ is trivial. We conduct the proof inductively on m. For $m = 1$ it is obvious since $u_1(t, x_1)$ satisfies

$$\partial_t u_1 = \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q} \partial^2_{p,q} u_1,$$
$$u_1(0, x) = T_0(x).$$

Assume now that all $u_k(t, x_1, \dots, x_k)$ are uniquely determined for $k \le m - 1$. For $\phi_1, \dots, \phi_m \in \mathscr{S}(R^d)$ define functions θ_l: $[0, t] \to \mathscr{S}(R^d)$, $1 \le l \le m$, by

$$\theta_l(t) = \phi_l, \qquad \partial_s \theta_l(s) + \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q} \partial^2_{p,q} \theta_l(s) = 0, \qquad s \le t.$$

Letting $G(t, x)$ denote the fundamental solutions of the equation

$$\partial_t v(t, x) = \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q} \partial^2_{x_p, x_q} v(t, x),$$

we can express θ_l using convolution of $G(t - s)$ with ϕ_l, that is,

$$\theta_l(s) = G(t - s) * \phi_l, \qquad l = 1, \dots, m.$$

By Lemma 3 and Itô's formula for bounded quadratic variation martingales (see [9], page 64, Theorem 2.3.11),

$$E\{[(\overline{T}(t), \theta_1(t)) - (T_0, \theta_1(0))] \cdots [(\overline{T}(t), \theta_m(t)) - (T_0, \theta_m(0))]\}$$
$$= \sum_{k<l} \int_0^t E\Big\{(\overline{T}(s), \theta_1(s)) \cdots \overbrace{(\overline{T}(t), \theta_k(s))} \cdots \overbrace{(\overline{T}(t), \theta_l(s))}$$
$$\cdots (\overline{T}(t), \theta_m(s)) K_{\overline{T}(s)}(\theta_k(s), \theta_l(s))\Big\}\, ds \tag{34}$$
$$= \sum_{k<l} \int_0^t ds \int \cdots \int G_m(t - s, x_1 - y_1, \dots, x_m - y_m) u_m(s, x_1, \dots, x_m)$$
$$\times \Gamma(y_k - y_l)\, \nabla_{y_k} \otimes \nabla_{y_l} (\theta_1 \otimes \cdots \otimes \theta_m)(y_1, \dots, y_m)\, dx_1 \cdots dx_m\, dy_1 \cdots dy_m.$$

Here the overbrace stands for skipping the factor in multiplication, $G_m(t, x_1, \dots, x_m) = G(t, x_1) \cdots G(t, x_m)$ and $\Gamma(y_k - y_l)\, \nabla_{y_k} \otimes \nabla_{y_l}$ denotes

$$\sum_{p,q=1}^{d} \Gamma_{p,q}(y_k - y_l)\, \partial_{y_k^p} \partial_{y_l^q},$$

$y_k = (y_k^1, \dots, y_k^d)$, $k = 1, \dots, m$.

Because we are only proving uniqueness of u_m satisfying (34) we can drop all $(T_0, \theta_k(0))$, $k = 1, \dots, m$, in (34), since by the induction assumption all u_k, $k \le m - 1$, are uniquely determined. Equation (34) written in a differential form looks as follows:

$$\partial_t u_m = \sum_{k>l} \Gamma(x_k - x_l) \nabla_{x_k} \otimes \nabla_{x_l} u_m + \tfrac{1}{2} \sum_k \sum_{p,q=1}^{d} a_{p,q} \partial^2_{x_k^p, x_k^q} u_m,$$

$$u_m(0) = 0.$$

As is well known (see, e.g., [6]), there is only one weak solution for this problem in $L^2((R^d)^m)$, namely, $u_m \equiv 0$.

The last step of the proof consists of verifying that the measure generated by $\{\overline{T}(t)\}_{t\geq 0}$ is actually supported in $L^2_{\mathrm{loc},w}([0,+\infty); H^1(R^d))$. This heuristically follows immediately from Lemma 3, where substituting $\overline{T}$ in place of θ we get

$$E\int_0^t \left\{ \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q}(\partial^2_{p,q}\overline{T}(s), \overline{T}(s)) + (\partial_s \overline{T}(s), \overline{T}(s)) \right\} ds = E\|\overline{T}(t)\|^2 - E\|T_0\|^2.$$

Since $(\partial_s \overline{T}(s), \overline{T}(s)) = \frac{1}{2}(d/ds)\|\overline{T}(s)\|^2$, integration by parts yields

$$E\int_0^t \tfrac{1}{2} \sum_{p,q=1}^{d} a_{p,q}(\partial_p \overline{T}, \partial_q \overline{T}) \leq \tfrac{1}{2} E\|T_0\|^2.$$

By the fact that $T_0 \in L^2(R^d)$ and the matrix $[a_{p,q}]_{d\times d}$ is positive definite, we can conclude that $\overline{T} \in L^2([0,t]; H^1(R^d))$, for any $t > 0$. The above argument is made precise in Appendix E.

COROLLARY. 1. *The measure induced by process* $\{\overline{T}(t)\}_{t\geq 0}$ *is supported in*

$$C([0,+\infty); L^2(R^d)) \cap L^2_{\mathrm{loc},w}([0,+\infty); H^1(R^d))$$

and

$$\|\overline{T}(t)\| = \|T_0\| \quad \textit{almost surely}. \tag{35}$$

PROOF. Since (26) holds we may apply Itô's formula in Hilbert space to $\{\|\overline{T}(t)\|^2\}$ (see [14], page 68). As a consequence one can observe that $\{\|\overline{T}(t)\|^2\}$ is a continuous trajectory martingale, so

$$E\|\overline{T}(t)\|^2 = \|T_0\|^2;$$

but at the same time, according to property (D1), $\|\overline{T}(t)\| \leq \|T_0\|$, $t \geq 0$. Hence (35) follows. □

APPENDIX A

Proofs of mixing and tightness lemmas. In this appendix we prove two lemmas formulated in Section 3. To prove Lemma 1, we will need the following result (based on Lemma 1, page 109 of [7]).

LEMMA 4. *Suppose that $U(t,x)$ is a jointly measurable random field in $R \times R^d$ with zero mean, continuous with respect to the variable x and bounded by a constant K. Assume that it is $\mathscr{V}_t^{+\infty}(C,\varrho)$-measurable for $|x| \le C(1+t^\varrho)$, where $\mathscr{V}_t^{+\infty}(C,\varrho)$ is the σ-algebra defined in condition* (C2). *Let $V(s)$ be $\mathscr{V}_0^s$-measurable and let ξ_ε be as in Lemma* 1.

Then for any $T>0$ there is an $\varepsilon_0(T)>0$ sufficiently small that

$$\left| E\left\{ U\left(\frac{t}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) V\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \le 2K\beta\left(\frac{t-s}{\varepsilon^2}\right) E\left| V\left(\frac{s}{\varepsilon^2}\right)\right|,$$

for $0 \le u \le s \le t \le T$ and $0 < \varepsilon < \varepsilon_0(T)$.

PROOF. We will omit writing C and ϱ in the notation of σ-algebras $\mathscr{V}_s^t$. Let M be a fixed positive integer. Define

$$A(\mathbf{k}) = \left[\omega\colon \frac{k_j}{M} \le (\xi_\varepsilon)_j(u) < \frac{k_j+1}{M},\ j=1,\dots,d\right], \qquad \mathbf{k}=(k_1,\dots,k_d);$$

$A(\mathbf{k})$ is $\mathscr{V}_0^{s/\varepsilon^2}$-measurable. This follows from $\mathscr{V}_0^{u/\varepsilon^2}$-measurability of $\xi_\varepsilon(u)$. By the assumptions made about ξ_ε in Lemma 1, one can see that, for large enough M, $A(\mathbf{k})$ is nonempty provided that

$$\left|\frac{\mathbf{k}}{M\varepsilon^\alpha}\right| \le C\left(1+\frac{t^\varrho}{\varepsilon^{2\varrho}}\right).$$

Then

$$\begin{aligned}
&\left| E\left\{ U\left(\frac{t}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) V\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \\
&\quad = \left| \sum_{\mathbf{k}} E\left\{ U\left(\frac{t}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) V\left(\frac{s}{\varepsilon^2}\right)\chi_{A(\mathbf{k})}\right\}\right| \\
&\quad \le \sum_{\mathbf{k}} \left| E\left\{\left[U\left(\frac{t}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) - U\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{k}}{M\varepsilon^\alpha}\right)\right] V\left(\frac{s}{\varepsilon^2}\right)\chi_{A(\mathbf{k})}\right\}\right| \\
&\qquad + \sum_{\mathbf{k}} \left| E\left\{ U\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{k}}{M\varepsilon^\alpha}\right) V\left(\frac{s}{\varepsilon^2}\right)\chi_{A(\mathbf{k})}\right\}\right|.
\end{aligned} \tag{36}$$

Notice that $U(t/\varepsilon^2, \mathbf{k}/M\varepsilon^\alpha)$ is $\mathscr{V}_t^{+\infty}$-measurable. The remark after Lemma 1 from [2, page 171] allows us to estimate the second term on the right-hand side of (36) by

$$2K\beta\left(\frac{t-s}{\varepsilon^2}\right)\sum_{\mathbf{k}} E\left\{\chi_{A(\mathbf{k})}\left| V\left(\frac{s}{\varepsilon^2}\right)\right|\right\} = 2KE\left| V\left(\frac{s}{\varepsilon^2}\right)\right| \beta\left(\frac{t-s}{\varepsilon^2}\right).$$

The first term on the right-hand side of (36) vanishes as we allow M to go to infinity. The proof of Lemma 4 is therefore concluded. □

Now we are ready to prove Lemma 1. We will proceed by induction on n. For $n = 0$ this is just Lemma 4. Suppose then that Lemma 1 holds for $n-1$. We want to prove it for n. If $|\mathbf{l}| = l$, then

$$\begin{aligned}
&\left| E\left\{ D^{\mathbf{l}} \widetilde{W}_n\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_{n+1}}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) Y\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \\
&\quad \le \left| E\left\{ D^{\mathbf{l}} W_n\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_{n+1}}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) Y\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \\
&\qquad + E\left\{ \left| E\left\{ D^{\mathbf{l}} W_n\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_{n+1}}{\varepsilon^2}, \frac{x}{\varepsilon^\alpha}\right)\right\}\right|_{x=\xi_\varepsilon(u)} Y\left(\frac{s}{\varepsilon^2}\right)\right\}.
\end{aligned} \tag{37}$$

The first term on the right-hand side can be estimated in the following way

$$\begin{aligned}
&\left| E\left\{ D^{\mathbf{l}} W_n\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_{n+1}}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) Y\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \\
&\quad \le \sum_{q=1}^{d} \left| E\left\{ D^{\mathbf{l}}\left[\partial_q \widetilde{W}_{n-1}\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_n}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) V_q\left(\frac{s_{n+1}}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right)\right] Y\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \\
&\quad \le d \max_{|\mathbf{l}| \le l+1} C(d, |p|, n-1, M_{n-1+|\mathbf{l}|}) \\
&\qquad \times \max_{|\mathbf{l}| \le l, q \le d} |D^{\mathbf{l}} V_q| \, \beta^{\min_{\mathbf{l}} \alpha_1(|\mathbf{l}|, n-1)}(s_1 - s_2) \cdots \beta^{\min_{\mathbf{l}} \alpha_n(|\mathbf{l}|, n-1)}(s_n - s_{n+1}) \\
&\qquad \times E\left| Y\left(\frac{s}{\varepsilon^2}\right)\right|.
\end{aligned} \tag{38}$$

As for the second term, we apply the same estimate. So finally the left-hand side of (37) can be estimated by twice the right-hand side of (38).

On the other hand, we can apply Lemma 4 to

$$\begin{aligned}
&\left| E\left\{ D^{\mathbf{l}} \widetilde{W}_n\left(\frac{s_1}{\varepsilon^2}, \dots, \frac{s_{n+1}}{\varepsilon^2}, \frac{\xi_\varepsilon(u)}{\varepsilon^\alpha}\right) Y\left(\frac{s}{\varepsilon^2}\right)\right\}\right| \\
&\quad \le \sup |D^{\mathbf{l}} \widetilde{W}_n| \, \beta\left(\frac{s_{n+1} - s}{\varepsilon^2}\right) E\left| Y\left(\frac{s}{\varepsilon^2}\right)\right|.
\end{aligned} \tag{39}$$

This fact together with the previous estimate means that the left-hand side of (36) can be estimated by the geometric mean of the right-hand sides of (38) and (39). Thus we can conclude the assertion of Lemma 1.

Now we turn to the proof of Lemma 2. It is enough to prove the version of this lemma on a compact interval $[0, T]$ instead of $[0, +\infty)$ (see [12] for details). If $\nu = 0$, we get

$$E^\mu |x(u) - x(t)|^2 \le C(u - t) \quad \text{for all } \mu \in \mathscr{F},\ 0 \le t \le u \le T.$$

For $\nu = 1$,

$$\begin{aligned}
E^\mu |x(u) - x(t)|^2 |x(t) - x(s)| &\le C(u - t) E^\mu |x(t) - x(s)| \\
&\le C(t - u)\{E^\mu |x(t) - x(s)|^2\}^{1/2} \\
&\le C(u - t)(t - s)^{1/2} \le C(u - s)^{3/2}.
\end{aligned}$$

By the classical result of Chentsov (see [2], Theorem 15.6, page 127) this implies that for any $\varepsilon, \eta > 0$ there is a $\delta > 0$ such that, for $\mu \in \mathscr{F}$,

$$\mu[x:\ w''_x(\delta) > \varepsilon] < \eta$$

(see [2], page 129), where

$$w''_x(\delta) = \sup_{0 \le u - s \le \delta} \sup_{s \le t \le u} \min[|\, x(u) - x(t)\,|, |\, x(t) - x(s)\,|].$$

However, for $x \in C[0, T]$ one can prove that

$$w''_x(\delta) < w_x(\delta) < 4w''_x(\delta),$$

where $w_x(\delta) = \sup[\ |\, x(u) - x(t)\,|\ |\ 0 \le u - t \le \delta]$. Thus, by Theorem 8.2 of [2, page 55], $\mathscr{F}$ is weakly compact.

APPENDIX B

The proof that K_g is nonnegative definite. First observe that K_g is self-adjoint. Its kernel is clearly symmetric:

$$\begin{aligned}
&\sum_{p,q=1}^{d} \Gamma_{p,q}(x - y)\,\partial_{x_p} g(x)\,\partial_{y_q} g(y)\\
&\quad = \sum_{p,q=1}^{d} \int_{-\infty}^{+\infty} E\{V_p(u, x)V_q(0, y)\}\,\partial_{x_p} g(x)\,\partial_{y_q} g(y)\,du\\
&\quad = \sum_{p,q=1}^{d} \int_{-\infty}^{+\infty} E\{V_p(0, x)V_q(u, y)\}\,\partial_{x_p} g(x)\,\partial_{y_q} g(y)\,du\\
&\quad = \sum_{p,q=1}^{d} \Gamma_{p,q}(y - x)\,\partial_{x_p} g(x)\,\partial_{y_q} g(y).
\end{aligned}$$

Notice also that, for $\phi \in L^2(R^d)$,

$$\begin{aligned}
(K_g\phi, \phi) &= \iint \sum_{p,q=1}^{d} \Gamma_{p,q}(x - y)\,\partial_{x_p} g(x)\,\partial_{y_q} g(y)\,\phi(x)\phi(y)\,dx\,dy\\
&= \sum_{p,q=1}^{d} \int_{-\infty}^{+\infty} du \iint R_{p,q}(u, x - y)\,\partial_{x_p} g(x)\,\partial_{y_q} g(y)\,\phi(x)\phi(y)\,dx\,dy\\
&= \sum_{p,q=1}^{d} \int \hat{R}_{p,q}(0, \xi)\widehat{\partial_p g\phi}(\xi)\overline{\widehat{\partial_q g\phi}(\xi)}\,d\xi \ge 0,
\end{aligned}$$

where the hat symbol stands for the Fourier transform. We used the fact that the Fourier transform of the autocorrelation matrix of a process is a nonnegative definite matrix at any point.

APPENDIX C

Weak continuity of the map $g \mapsto K_g$. Observe that the functional

$$h \mapsto \sum_{p,q=1}^{d} \int_0^T ds \iint \Gamma_{p,q}(x-y)\, \partial_{x_p}\theta(x)\, \partial_{y_q}\theta(y)\, h(s,x,y)\, dx\, dy$$

is continuous in $L^2_w([0,T];L^2(R^{2d}))$. The map

$$L^2_w([0,T];L^2(R^d)) \ni g(s,\cdot) \mapsto g(s,\cdot) \otimes g(s,\cdot) \in L^2_w([0,T];L^2(R^{2d}))$$

is also continuous. Since $\{T_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$, are weakly convergent to $\overline{T}$ as $\varepsilon \downarrow 0$ in $L^2_w([0,T];L^2(R^d))$, therefore (32) is true.

APPENDIX D

The proof of the martingale property. We present here the proof of Lemma 3. First we verify that for any A which is $\mathscr{T}_0^s$-measurable,

$$E\Big\{(\overline{T}(t),\theta(t)) - (\overline{T}(s),\theta(s)) \\ - \int_s^t \Big[\frac{1}{2}\sum_{p,q=1}^{d} a_{p,q}(\overline{T}(u),\partial^2_{p,q}\theta(u)) + (\overline{T}(u),\partial_u\theta(u))\Big]\,du\Big\}\chi_A = 0.$$

The left-hand side is equal to

$$E\Big\{(\overline{T}(t),\theta(s)) - (\overline{T}(s),\theta(s)) - \int_s^t \frac{1}{2}\sum_{p,q=1}^{d} a_{p,q}(\overline{T}(u),\partial^2_{p,q}\theta(s))\,du\Big\}\chi_A \\ + E\Big\{(\overline{T}(t),\theta(t)) - (\overline{T}(t),\theta(s)) - \int_s^t (\overline{T}(u),\partial_u\theta(u))\,du \\ - \int_s^t \frac{1}{2}\sum_{p,q=1}^{d} a_{p,q}[(\overline{T}(u),\partial^2_{p,q}\theta(u)) - (\overline{T}(u),\partial^2_{p,q}\theta(s))]\,du\Big\}\chi_A \\ = E\Big\{\int_s^t [(\overline{T}(t),\partial_u\theta(u)) - (\overline{T}(u),\partial_u\theta(s))]\,du \\ - \frac{1}{2}\int_s^t \Big[\sum_{p,q=1}^{d} a_{p,q}\int_s^u (\overline{T}(u),\partial^2_{p,q}\partial_{u'}\theta(u'))\,du'\Big]\,du\Big\}\chi_A \\ = \int_s^t \Big\{E\Big[(\overline{T}(t),\partial_u\theta(u)) - (\overline{T}(u),\partial_u\theta(u)) \\ - \int_u^t \frac{1}{2}\sum_{p,q=1}^{d} a_{p,q}(\overline{T}(u'),\partial^2_{p,q}\partial_u\theta(u))\,du'\Big]\chi_A\Big\}\,du = 0.$$

This verifies that $(N_{\theta_1},\dots,N_{\theta_m})$ is indeed a martingale. To end the proof of Lemma 3, we need only to prove the fact that

$$N_{\theta_k}(t)N_{\theta_l}(t) - \int_0^t K_{\overline{T}(s)}(\theta_k(s),\theta_l(s))\,ds$$

is a martingale. The statement about the quadratic variation will follow from Theorem 2.2.12 of [9, page 52].

Let us partition $[0, t]$ with $0 = t_0 < \cdots < t_M = t$. By the foregoing computation we get for any A, $\mathscr{T}_0^s$-measurable,

$$
\begin{aligned}
E[N_{\theta_k}(t) - N_{\theta_k}(s)][N_{\theta_l}(t) - N_{\theta_l}(s)]\chi_A \\
&= \sum_i E[N_{\theta_k}(t_{i+1}) - N_{\theta_k}(t_i)][N_{\theta_l}(t_{i+1}) - N_{\theta_l}(t_i)]\chi_A \\
&= \sum_i E\Bigg[\bigg\{M_{\theta_k(t_i)}(t_{i+1}) - M_{\theta_k(t_i)}(t_i) \\
&\qquad - \int_{t_i}^{t_{i+1}} [M_{\partial_u \theta_k(u)}(t_{i+1}) - M_{\partial_u \theta_k(u)}(u)]\, du\bigg\} \\
&\qquad \times \bigg\{M_{\theta_l(t_i)}(t_{i+1}) - M_{\theta_l(t_i)}(t_i) \\
&\qquad - \int_{t_i}^{t_{i+1}} [M_{\partial_u \theta_l(u)}(t_{i+1}) - M_{\partial_u \theta_l(u)}(u)]\, du\bigg\}\chi_A\Bigg].
\end{aligned}
\tag{40}
$$

As $\max \Delta t_i \to 0$ one easily sees that the only significant part of the right-hand side of (40) is

$$
\begin{aligned}
&\sum_i E[M_{\theta_k(t_i)}(t_{i+1}) - M_{\theta_k(t_i)}(t_i)][M_{\theta_l(t_i)}(t_{i+1}) - M_{\theta_l(t_i)}]\chi_A \\
&\quad = \sum_i E\left\{\int_{t_i}^{t_{i+1}} K_{\overline{T}(u)}(\theta_k(t_i), \theta_l(t_i))\, du\right\}\chi_A \\
&\quad \to E\left\{\int_s^t K_{\overline{T}(u)}(\theta_k(u), \theta_l(u))\, du\right\}\chi_A \quad \text{as } \max \Delta t_i \to 0.
\end{aligned}
$$

APPENDIX E

The support of the limiting measure. Consider $\overline{T}_{\varepsilon,\varrho} = (\phi_\varrho \psi_\varepsilon) * \overline{T}$, where

$$
\phi_\varrho(t) = \frac{1}{\sqrt{2\pi\varrho}} \exp\left(\frac{-t^2}{2\varrho}\right),
$$

$$
\psi_\varepsilon(x) = \left(\frac{1}{\sqrt{2\pi\varepsilon}}\right)^d \exp\left(\frac{-\mid x \mid^2}{2\varepsilon}\right).
$$

One can apply now Lemma 3 with $\theta = \overline{T}_{\varepsilon,\varrho}$ and get

$$
\begin{aligned}
&-E\int_0^t \tfrac{1}{2} \sum_{p,q=1}^d a_{p,q}(\partial_{p,q}^2 \overline{T}_{\varepsilon,\varrho}(s), \overline{T}(s))\, ds \\
&\quad = E\left\{(\overline{T}_{\varepsilon,\varrho}(0), T_0) - (\overline{T}_{\varepsilon,\varrho}(t), \overline{T}(t)) - \int_0^t (\partial_s \overline{T}_{\varepsilon,\varrho}(s), \overline{T}(s))\, ds\right\}.
\end{aligned}
\tag{41}
$$

Letting $\hat{T}(t,\xi)$ denote the Fourier transform of $\overline{T}$ with respect to spatial variables, the left-hand side of (41) can be rewritten as

$$\frac{1}{2}E\int_0^t\left\{\sum_{p,q=1}^d a_{p,q}\int \xi_p\xi_q \exp\left(\frac{-\varepsilon\xi^2}{2}\right)(\phi_\varrho * \hat{T})(\xi,s)\overline{\hat{T}(\xi,s)}\,d\xi\right\}ds.$$

Since $\phi_\varrho * \hat{T}(\cdot,s)\to_{L^2}\hat{T}(\cdot,s)$ as $\varrho\downarrow 0$, passing to the limit with respect to ϱ, one gets

$$\frac{1}{2}E\int_0^t\left\{\int\sum_{p,q=1}^d a_{p,q}\xi_p\xi_q\exp\left(\frac{-\varepsilon\mid\xi\mid^2}{2}\right)\mid\hat{T}(\xi,s)\mid^2\,d\xi\right\}ds. \tag{42}$$

Using Fatou's lemma (applicable since the integrand is nonnegative), we pass to the limit with $\varepsilon\downarrow 0$ and get that the limit of (42) is greater than or equal to

$$\tfrac{1}{2}E\int_0^t\left\{\sum_{p,q=1}^d a_{p,q}\int\xi_p\xi_q\mid\hat{T}(\xi,s)\mid^2\,d\xi\right\}ds. \tag{43}$$

Expression (43) therefore will be estimated from above by whatever we obtain on the right-hand side of (41) making passages to the limit in the order mentioned above.

After taking the appropriate limits on the right-hand side of (41), the two first terms can be easily estimated by $E\|T_0\|^2$ since $\|\overline{T}(t)\|\le\|T_0\|$ [see property (D1) from Section 10]. The remaining last term can be written as

$$\int_0^t ds\int_{-\infty}^{+\infty}du\int\frac{s-u}{2\sqrt{2\pi}\varrho^{3/2}}\exp\left(-\frac{(s-u)^2}{2\varrho}\right)\exp(-\varepsilon\mid\xi\mid^2)\hat{T}(u,\xi)\overline{\hat{T}(s,\xi)}\,d\xi.$$

Substituting $v=(s-u)^2/2\varrho$, we get

$$\begin{aligned}&\int\exp(-\varepsilon\mid\xi\mid^2)\,d\xi\int_0^t ds\int_0^{+\infty}\frac{1}{\sqrt{2\pi\varrho}}\exp(-v)[\hat{T}(s-\sqrt{2\varrho v},\xi)\\&\qquad\qquad-\hat{T}(s+\sqrt{2\varrho v},\xi)]\overline{\hat{T}(s,\xi)}\,dv\\&=\operatorname{Re}\int\exp(-\varepsilon\mid\xi\mid^2)\,d\xi\int_0^{+\infty}\frac{v^{1/2}}{\sqrt{\pi}}\exp(-v)\,dv\\&\qquad\times\frac{1}{\sqrt{2\varrho v}}\left\{\int_{t-\sqrt{2\varrho v}}^t\hat{T}(s-\sqrt{2\varrho v},\xi)\overline{\hat{T}(s,\xi)}\,ds\right.\\&\qquad\left.-\int_0^{\sqrt{2\varrho v}}\hat{T}(s+\sqrt{2\varrho v},\xi)\overline{\hat{T}(s,\xi)}\,ds\right\}.\end{aligned} \tag{44}$$

Notice that

$$\left|\int\exp(-\varepsilon\mid\xi\mid^2)\hat{T}(s\pm\sqrt{2\varrho v},\xi)\overline{\hat{T}(s,\xi)}\,d\xi\right|\le\|T_0\|^2,$$

since

$$\int |\hat{T}(s,\xi)|^2 \, d\xi = \|\overline{T}(s)\|^2 \leq \|T_0\|^2.$$

The right-hand side of (44) therefore can be estimated by $2\|T_0\|^2$ independently of ϱ and ε. Hence our claim is proven.

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