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# ON CONSERVATION OF PROBABILITY AND THE FELLER PROPERTY[1]

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It is known that any smooth, nondegenerate, second-order elliptic operator on a manifold (dimension $\neq 2$) has the form $\Delta + B$, where $B$ is a vector field and $\Delta$ is the Laplace–Beltrami operator under some Riemannian metric on the manifold. In this paper we give several conditions on the "Ricci curvature" $\mathrm{Ric} - \nabla_B^s$ associated with the operator $\Delta + B$ to ensure that the diffusion semigroup generated by $\Delta + B$ conserves probability and possesses the Feller property.

**1. Introduction and main results.** Let $(M, g)$ be a noncompact, connected, complete Riemannian manifold and let $\Delta$ be the Laplace–Beltrami operator on $M$. Let $B$ be a $C^1$-vector field and let $L = \Delta + B$. In the case where $B$ is a gradient vector field, that is, $B = \nabla h$ for some $C^2$ function $h$ on $M$, we will use $\Delta^h$ to denote the operator $L = \Delta + \nabla h$. Let $(X_t, P^x)$ be a diffusion process with infinitesimal generator $\frac{1}{2}L$ and state space $M \cup \{\partial\}$, the one-point compactification of $M$, and let $(P_t)_{t \geq 0}$ be the transition semigroup of the diffusion process $(X_t, P^x)$ (cf. Ikeda and Watanabe [12]). That is, $P_t f(x) = P^x f(X_t)$ for any positive or bounded measurable function $f$.

(1.1) DEFINITION. (i) It is said that the semigroup $(P_t)_{t \geq 0}$ conserves probability (in this case it is also said that the operator $L$ is stochastically complete or conservative) if $P_t 1 = 1$, for any $t > 0$. That is,

$$P_t(x, M) = P^x(X_t \in M) = 1,$$

for any $x \in M$ and $t \geq 0$.

(ii) It is said that the semigroup $(P_t)_{t \geq 0}$ possesses the Feller property (or has the $C_0$-diffusion property) if $P_t C_0(M) \subset C_0(M)$, for any $t > 0$, where $C_0(M)$ denotes the set of all continuous functions which vanish at $\partial$.

There are many papers written by various authors on the conservativeness and the Feller property. When $L = \Delta$, the Laplace–Beltrami operator, fundamental results on the conservation property of the heat semigroup on a complete Riemannian manifold have been obtained by Gaffney [7], Yau [18], Karp and Li [13], Grigor'yan [9] and others. Azencott [1], Hsu [10] and Takeda [17] presented probabilistic approaches to the stochastic complete-

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ness and the Feller property. For general stochastic flows on a complete manifold, Elworthy [6] and Li [15] gave several criteria for the conservation property and the $C_0$-diffusion property. When the elliptic operator $L = \Delta^h$, a conservation criterion has been established by Bakry [2] recently using the deformed Ricci curvature Ric − Hess $h$, and a different criterion for conservation (resp., the Feller property) has been obtained by Davies [5] using a weighted volume growth condition (resp., conditions involving a kind of modified injectivity radius). More precisely we have the following theorem.

(1.2) THEOREM. (i) (Bakry's criterion [2]). *If $B = \nabla h$ is a gradient vector field and* $\mathrm{Ric} - \mathrm{Hess}\, h \geq -k$, *for some constant $k$, then the semigroup $(P_t)_{t \geq 0}$ conserves probability.*

(ii) (Davies' criterion [5]). *Let $B = \nabla h$ be a gradient vector field and let*

$$\mathrm{Vol}_h(B(p,r)) = \int_{B(p,r)} e^{h(x)}\, dx$$

*be the weighted volume of the geodesic ball $B(p,r)$ centered at $p \in M$ with radius $r > 0$. Suppose that there is a point $p \in M$ and positive constants $a$ and $b$ such that* $\mathrm{Vol}_h(B(p,r)) \leq a \exp(br^2)$, *for all $r > 0$. Then $(P_t)_{t \geq 0}$ conserves probability.*

(1.3) REMARK. (i) Bakry's criterion is an extension of the famous result obtained by Yau [18]. I would like to point out that there is no control on the Ricci curvature of the manifold $M$ itself.

(ii) Davies' criterion is a further extension of Gaffney's criterion [7] (cf. Karp and Li [13] and Grigor'yan [9]). In differential geometry it is well known that a lower bound on the Ricci curvature yields an upper estimate on the volume of a geodesic ball due to Bishop's volume comparison theorem. It follows that Gaffney's condition on the volume growth is satisfied if the Ricci curvature is bounded below; that is, Yau's conservation criterion can also be derived from Gaffney's result in the case of the Laplace–Beltrami operator (cf. Karp and Li [13]). However, it is not obvious that Bakry's criterion can be proved using Davies' criterion for a weighted Laplacian.

This paper relates several results about the probability conservation and the Feller property for general elliptic operators $L$, using a modified Ricci curvature. Our main contributions are the comparison results established in Section 2. As applications of these comparison theorems, we establish several criteria for conservation and the Feller property for general elliptic operators on a smooth manifold by adopting the probabilistic approach considered by Hsu [10]. In particular we give an extension of Bakry's conservation criterion. Before stating our main results, we recall some basic facts about the Bakry–Emery curvature associated with a diffusion operator. Motivated by the classical Lichnerowicz–Bochner–Weitzenböck formula, Bakry and Emery [3] introduced a bilinear map $\Gamma_2$, the "curvature" operator of the diffusion operator $L$, taking the place of the Ricci curvature (which corresponds to the case $L = \Delta$). More precisely, the "curvature" operator $\Gamma_2$ is defined by

$$\Gamma(f,g) = \tfrac{1}{2}\{L(fg) - fLg - gLf\},$$
$$\Gamma_2(f,g) = \tfrac{1}{2}\{L\Gamma(f,g) - \Gamma(Lf,g) - \Gamma(f,Lg)\}. \tag{1.1}$$

It is easily seen that

$$\Gamma(f,g) = \langle \nabla f, \nabla g\rangle,$$
$$\Gamma_2(f,f) = |\text{Hess } f|^2 + (\text{Ric} - \nabla_B^s)(\nabla f, \nabla f), \tag{1.2}$$

where $\nabla_B^s$ is a symmetric, sectional bilinear form on the vector bundle $TM \times TM$ defined by

$$\nabla_B^s(\xi,\eta) = \tfrac{1}{2}\left\{\langle \nabla_\xi B, \eta\rangle + \langle \nabla_\eta B, \xi\rangle\right\} \qquad \forall\ \xi, \eta \in TM. \tag{1.3}$$

In particular if $B = \nabla h$, then $\nabla_B^s = \text{Hess } h$. We are now in a position to state our criteria on probability conservation and the Feller property.

(1.4) THEOREM. *If $B = \nabla h$ is a gradient vector field and* Ric − Hess $h \geq -k$ *for some nonnegative constant $k$, then for any $p \in M$ there are two constants $A_1$ and $A_2 > 0$ such that*

$$\text{Vol}_h(B(p,r)) \leq A_1 \exp\left(A_2 r^2\right) \qquad \forall\ r > 0.$$

Theorem 1.4 shows that the condition on the volume growth in the Davies criterion is satisfied if the deformed Ricci curvature Ric − Hess $h$ is bounded below, from which it follows that Bakry's criterion can also be derived from the above estimate and the Davies criterion.

(1.5) THEOREM. *Assume that there is a point $p \in M$ such that*

$$(\text{Ric} - \nabla_B^s)(x) \geq -k_1^2(d(x,p)),$$
$$|B|(x) \leq k_2(d(x,p)), \tag{1.4}$$

*for any $x \in M$, where $d(x,p)$ denotes the geodesic distance between $x$ and $p$, and $k_i$: $R_+ \to R_+$ $(i = 1,2)$ are two positive, continuous, nondecreasing functions which satisfy the conditions that $\lim_{t\to+\infty} k_i(t) = +\infty$ and*

$$\int_c^\infty \frac{1}{\sqrt{k_1^2(t) + k_2^2(t)}}\, dt = +\infty, \tag{1.5}$$

*for some $c > 0$. Then $(P_t)_{t\geq 0}$ conserves probability.*

Theorem 1.5 is an extension of Varopoulos and Hsu's result; see [10, 11]. The following Theorem 1.6 is an extension of Bakry's theorem [2].

(1.6) THEOREM. *Assume that* Ric $- \nabla_B^s \geq -k$, *for some nonnegative constant $k$. Then $(P_t)_{t\geq 0}$ conserves probability.*

For the Feller property, we have the following theorem.

(1.7) THEOREM. *Assume that* $\mathrm{Ric} - \nabla_B^s \geq -k$, *for some nonnegative constant* $k$, *and there is a point* $o \in M$ *such that*

$$|B|(x) \leq C(d(o, x) + 1), \tag{1.6}$$

*for any* $x \in M$ *and some nonnegative constant* $C$. *Then* $(P_t)_{t \geq 0}$ *possesses the Feller property.*

(1.8) REMARK. We note that there is no control on the Ricci curvature of the manifold $M$ itself, which implies that we may not have any control on the function $\Delta \rho$, where $\rho$ is a distance function with respect to a fixed point. If the Ricci curvature is bounded below or there exists some kind of growth conditions on it, then it is easy to give several criteria for conservation and the Feller property with an additional condition on the growth of $\nabla_B^s$ or $|B|$.

The paper is organized as follows. In Section 2, we shall establish several comparison results which form the key part of this paper. As a consequence, we derive Theorem 1.4. In Section 3, we shall give proofs of Theorems 1.5 and 1.6 following the method in Hsu [10]. The final Section 4 is devoted to the proof of Theorem 1.7.

**2. Comparison theorem.** Throughout this paper we work with an $n$-dimensional, connected, complete Riemannian manifold $M$. Let $p \in M$. We denote by $C(p)$ the cut locus of the manifold $M$ with respect to the point $p$ and let $\rho(x) = d(x, p)$ be the distance function with respect to the fixed point $p$. It is well known that $\rho$ is Lipschitz continuous and smooth within the cut locus $C(p)$. If $x \in M - C(p)$, let $r = d(p, x)$ and let $\gamma_\xi(t) = \exp_p(t\xi)$ be the minimal and normal geodesic connecting $p$ and $x$ such that $\gamma_\xi(0) = p$, $\gamma_\xi(r) = x$, so that $\rho(\gamma_\xi(t)) = t$, for any $t \leq r$. By the use of Gauss's lemma, we know that $\nabla\rho(\gamma_\xi(t)) = \dot{\gamma}_\xi(t)$, for any $t \leq r$, and

$$\begin{aligned} &\langle \dot{\gamma}_\xi(t), B \rangle \big(\gamma_\xi(t)\big) = B\rho\big(\gamma_\xi(t)\big), \qquad \forall\ t \leq r, \\ &\frac{d}{dt} B\rho\big(\gamma_\xi(t)\big) = \nabla_B^s\big(\dot{\gamma}_\xi(t), \dot{\gamma}_\xi(t)\big). \end{aligned} \tag{2.1}$$

Hence

$$(B\rho)(x) = \int_0^r \nabla_B^s\big(\dot{\gamma}_\xi(t), \dot{\gamma}_\xi(t)\big)\, dt + (B\rho)(p), \tag{2.2}$$

for any $x \in M - C(p)$ and $p \in M$. Using a standard method in Riemannian geometry (for details, cf. [16]), we have

$$(\Delta\rho)(x) \leq \int_0^r (n-1)\varphi'(t)^2\, dt - \int_0^r \varphi(t)^2 \operatorname{Ric}\big(\dot{\gamma}_\xi(t), \dot{\gamma}_\xi(t)\big)\, dt, \tag{2.3}$$

for any continuous, piecewise-smooth function $\varphi$ satisfying the conditions that

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(r) = 1. \tag{2.4}$$

It is easily seen that (2.2) and (2.3) imply that

$$
\begin{aligned}
(L\rho)(x) \le & \int_0^r (n-1)\varphi'(t)^2\,dt \\
& - \int_0^r \left[\varphi^2(t)\mathrm{Ric} - \nabla_B^s\right]\left(\dot\gamma_\xi(t), \dot\gamma_\xi(t)\right) dt + (B\rho)(p),
\end{aligned}
\tag{2.5}
$$

for any $p \in M$, $x \in M - C(p)$, where $\varphi$ satisfies (2.4) and $r = d(x, p)$. One can also write (2.5) as

$$
\begin{aligned}
(L\rho)(x) \le & \int_0^r (n-1)\varphi'(t)^2\,dt \\
& - \int_0^r \varphi(t)^2[\mathrm{Ric} - \nabla_B^s]\left(\dot\gamma_\xi(t), \dot\gamma_\xi(t)\right) dt \\
& + \int_0^r \left(1 - \varphi(t)^2\right)\nabla_B^s\left(\dot\gamma_\xi(t), \dot\gamma_\xi(t)\right) dt + (B\rho)(p).
\end{aligned}
\tag{2.6}
$$

Using the fact that

$$
\begin{aligned}
& \int_0^r (1 - \varphi^2(t))\,\nabla_B^s\left(\dot\gamma_\xi(t), \dot\gamma_\xi(t)\right) dt \\
& \quad = -B\rho(p) + 2\int_0^r \langle B, \nabla\rho\rangle\left(\gamma_\xi(t)\right)\varphi(t)\varphi'(t)\,dt,
\end{aligned}
\tag{2.7}
$$

we derive the inequality

$$
\begin{aligned}
(L\rho)(x) \le & \int_0^r (n-1)\varphi'(t)^2\,dt \\
& - \int_0^r \varphi(t)^2[\mathrm{Ric} - \nabla_B^s]\left(\dot\gamma_\xi(t), \dot\gamma_\xi(t)\right) dt \\
& + 2\int_0^r \langle B, \nabla\rho\rangle\left(\gamma_\xi(t)\right)\varphi(t)\varphi'(t)\,dt,
\end{aligned}
\tag{2.8}
$$

for any continuous, piecewise-smooth function $\varphi$ satisfying (2.4).

(2.1) THEOREM (Comparison theorems for distance function). *Let $p \in M$, $\rho(x) = d(p, x)$ be the distance function with respect to the point $p \in M$ and $B$ be a $C^1$-vector vector field on $M$.*

(i) *Assume that* $\mathrm{Ric} - \nabla_B^s \ge -k$, *for some nonnegative constant $k$. Then there is a nonnegative constant $C$ depending only on the manifold $M$, the vector field $B$ and the point $p \in M$ such that*

$$
L\rho \le C + \frac{n-1}{\rho} + k\rho \quad \text{on } M - C(p).
\tag{2.9}
$$

(ii) *Assume* (1.4) *holds,*

$$
\begin{aligned}
(\mathrm{Ric} - \nabla_B^s)(x) &\ge -k_1^2(d(p,x)), \\
|B|(x) &\le k_2(d(p,x)),
\end{aligned}
$$

*where* $k_i$: $R_+ \to R_+$ $(i = 1, 2)$ *are two continuous, increasing functions satisfying the condition that* $\lim_{t\to\infty} k_i(t) = \infty$. *Then*

$$L\rho \le n \frac{G'(\rho)}{G(\rho)} \quad \text{on } M - C(p), \tag{2.10}$$

*where* $G$: $R_+ \to R_+$ *is the solution of the equation*

$$\begin{aligned} &G''(t) - \frac{1}{n}\big(k_1^2(t) + k_2^2(t)\big)G(t) = 0, \\ &G(0) = 0, \qquad G'(0) = 1. \end{aligned} \tag{2.11}$$

(iii) *Assume that* $\mathrm{Ric} - \nabla_B^s \ge -k$, *for some nonnegative constant* $k$, *and there is a point* $o \in M$ *and a nonnegative constant* $C$ *such that* (1.6) *holds*:

$$|B|(x) \le C(d(o, x) + 1) \qquad \forall\ x \in M.$$

*Then*

$$L\rho \le \frac{n-1}{\rho} + \frac{1}{3}(k + 2C)\rho + C(1 + d(o, p)) \quad \text{on } M - C(p), \tag{2.12}$$

*where* $\rho(x) = d(x, p)$.

PROOF. (i) Let

$$C_1 = \max_{x \in B(p, 2)} \{|\mathrm{Ric}|(x) + |\nabla_B^s|(x)\}.$$

First consider the case where $x \in M - C(p)$ and $\rho(x) \le 2$. By using (2.5) with $r = \rho(x)$ we have

$$L\rho(x) \le \int_0^r (n-1)\varphi'(t)^2\, dt + \int_0^r \varphi(t)^2 C_1\, dt + 2C_1 + B\rho(p). \tag{2.13}$$

Let $\varphi$ be the solution of the equation

$$\varphi''(t) - \frac{C_1}{n-1}\varphi(t) = 0, \qquad \varphi(0) = 0, \varphi(r) = 1,$$

to get that

$$\begin{aligned} &\int_0^r (n-1)\varphi'(t)^2\, dt + \int_0^r \varphi(t)^2 C_1\, dt \\ &\qquad = (n-1)\frac{g'(r)}{g(r)} \le \frac{n-1}{r} + C_2, \end{aligned} \tag{2.14}$$

for some positive constant $C_2$ depending only on $n$ and $C_1$, where $g$ is the solution of the equation

$$g''(t) - \frac{C_1}{n-1} g(t) = 0, \qquad g(0) = 0, g'(0) = 1.$$

We next consider the case where $x \in M - C(p)$ but $\rho(x) > 2$. In this case we can choose a continuous, piecewise-smooth function $\varphi$ satisfying $0 \le \varphi(t) \le 1$,

$|\varphi'(t)| \le 1$, for any $t$, and $\varphi(t) = 1$, when $t \ge 2$. By using (2.6) we get

$$\begin{aligned} L\rho(x) &\le \int_0^2 (n-1)\varphi'(t)^2\,dt + \int_0^r k\,dt \\ &\quad + \int_0^2 \left(1-\varphi(t)^2\right)\nabla_B^s\left(\dot\gamma_\xi(t),\dot\gamma_\xi(t)\right)dt + B\rho(p) \\ &\le 2(n-1) + kr + 4C_1 + B\rho(p). \end{aligned} \tag{2.15}$$

It is clear that (2.9) follows from (2.14) and (2.15) immediately.

(ii) By (2.8) we have

$$\begin{aligned} L\rho(x) &\le \int_0^r (n-1)\varphi'(t)^2\,dt + \int_0^r \varphi(t)^2 k_1(t)^2\,dt \\ &\quad + 2\int_0^r |B|\left(\gamma_\xi(t)\right)\varphi(t)\varphi'(t)\,dt \\ &\le \int_0^r n\varphi'(t)^2\,dt + \int_0^r \varphi(t)^2\left(k_1^2(t)+k_2^2(t)\right)dt. \end{aligned}$$

Letting $\varphi$ in the above inequality be the solution of the equation

$$\varphi''(t) - \frac{1}{n}\left(k_1^2(t)+k_2^2(t)\right)\varphi(t) = 0, \qquad \varphi(0)=0,\ \varphi(r)=1,$$

we get

$$L\rho(x) \le n\frac{G'(\rho(x))}{G(\rho(x))}$$

(cf. [8]). Finally we prove (iii). Using (2.8) we get

$$L\rho(x) \le \int_0^r (n-1)\varphi'(t)^2\,dt + \int_0^r \varphi(t)^2 k\,dt + 2\int_0^r |B|\left(\gamma_\xi(t)\right)\varphi(t)\varphi'(t)\,dt.$$

Letting $\varphi(t) = t/r$ and using the fact that

$$|B|\left(\gamma_\xi(t)\right) \le C(d(o,p)+t+1),$$

we obtain

$$L\rho(x) \le \frac{n-1}{r} + \frac{1}{3}kr + 2C\int_0^r [d(o,p)+t+1]\frac{t}{r^2}\,dt,$$

which implies (2.12). Thus we have completed the proof of Theorem 2.1. □

We are now in a position to prove Theorem 1.4.

(2.2) Proof of Theorem 1.4. Denote by $\sqrt{g}_p(t;\xi) = \det A(t,\xi)$, where $A(t,\xi)$ is the solution of the equation

$$A''(t,\xi) + K(t,\xi)A(t,\xi) = 0, \qquad A(0,\xi)=0,\ A'(0,\xi)=I,$$

and let $K(t,\xi)\colon \xi^{\perp}\to\xi^{\perp}$, $\xi^{\perp}=\{\eta\in T_pM\colon \eta\perp\xi\}$ be the curvature operator defined by

$$K(t,\xi)\eta=\tau_t^{-1}R\big(\dot\gamma_\xi(t),\tau_t\eta\big)\dot\gamma_\xi(t),$$

where $\tau_t$ is the parallel translation along $\gamma_\xi$ (cf. [4]). It is known that

$$\Delta\rho\big(\gamma_\xi(t)\big)=\frac{d}{dt}\ln\sqrt{g_p}(t;\xi),\qquad t\le r,$$

$$\Delta^h\rho\big(\gamma_\xi(t)\big)=\frac{d}{dt}\ln\sqrt{g_p}(t;\xi)\exp\big(h\big(\gamma_\xi(t)\big)\big).$$

By (2.9) and the fact that

$$\sqrt{g_p}(t;\xi)=t^{n-1}\bigg(1-t^2\frac{\operatorname{Ric}(\xi,\xi)}{6}+O(t^3)\bigg)\quad\text{as } t\to 0,$$

we get

$$\ln t^{1-n}\sqrt{g_p}(t;\xi)\exp\big(h\big(\gamma_\xi(t)\big)\big)\le h(p)+\int_0^t(C+ks)\,ds\le C_1+C_2t^2,$$

for some constants $C_1$ and $C_2$. Hence we have

$$\operatorname{Vol}_h(B(p,r))\le\int_0^r\int_{S^{n-1}}t^{n-1}C_1\exp\big(C_2t^2\big)\,d\xi\,dt\le A_1\exp\big(A_2r^2\big),$$

for some nonnegative constants $A_1$ and $A_2$. □

**3. Proofs of Theorems 1.5 and 1.6.** The goal of this section is to prove Theorems 1.5 and 1.6. For a given $C^1$-vector field $B$ and $L=\Delta+B$, let $(X_t,P^x)$ be a $\frac12 L$-diffusion process and let $(P_t)_{t\ge0}$ be its transition semigroup. Recall that $(P_t)_{t\ge0}$ conserves probability if and only if $P^x\{\zeta=\infty\}=1$, for some $x\in M$ (hence for all $x\in M$), where $\zeta=\inf\{t\ge0\colon X_t=\partial\}$, that is,

$$P^x\{d(X_t,x)<\infty;t>0\}=1,\tag{3.1}$$

for some $x\in M$ (hence for all $x\in M$).

Let $p\in M$. Then the distance function $\rho(x)=d(x,p)$ is smooth on $M-C(p)$ and Lipschitz continuous on $M$. By using Kendall's decomposition for the Riemann Brownian motion (cf. [14]) and Girsanov's formula, we have

$$\rho(X_t)=\beta_t+\int_0^t\tfrac12L\rho(X_s)1_{M-C(p)}(X_s)\,ds-L_t,\qquad P^p\text{-a.e.},\tag{3.2}$$

where $(\beta_t)$ is a standard Brownian motion and $L_t$ is a continuous increasing process with initial value zero.

(3.1) Proof of Theorem 1.6. By Theorem 2.1 one knows that the assumption $\operatorname{Ric}-\nabla_B^s\ge -k$ implies (2.9),

$$L\rho\le C+\frac{n-1}{\rho}+k\rho\quad\text{on } M-C(p).$$

Let $r_t$ be the solution of the stochastic differential equation

$$dr_t = d\beta_t + \frac{1}{2}\left(C + \frac{n-1}{r_t} + kr_t\right) dt, \qquad r_0 = 0.$$

It is easily seen that

$$P^p\{r_t < \infty, \forall\ t > 0\} = 1. \tag{3.3}$$

By using a comparison theorem from stochastic differential equation theory (cf. [12]), (2.9) and (3.2) we get that

$$P^p\{\rho(X_t) \le r_t, \forall\ t > 0\} = 1.$$

which yields that

$$P^p\{\rho(X_t) < \infty, \forall\ t > 0\} = 1,$$

by (3.3). Hence we have proved Theorem 1.6. □

(3.2) Proof of Theorem 1.5. Let $r_t$ be the solution of the stochastic differential equation

$$dr_t = d\beta_t + \frac{n}{2}\frac{G'(r_t)}{G(r_t)} dt, \qquad r_0 = 0,$$

where $G$ is the solution of (2.11). Then the condition (1.5) implies that

$$P^p\{r_t < \infty, \forall\ t > 0\} = 1$$

(cf. Hsu [10]). By using (3.2), (2.10) and a comparison theorem for stochastic differential equations, we have

$$P^p\{\rho(X_t) \le r_t, \forall\ t > 0\} = 1.$$

By the same reasoning as in the proof of Theorem 1.6, we conclude the proof. □

**4. Proof of Theorem 1.7.** In this section, we prove Theorem 1.7. We shall follow the method used by Hsu [10]. The main difficulty in our case is the fact that we do not have any universal comparison theorem for general differential operators, that is, we lack the fact (b) from the proof of Lemma 3.2 in [10], which plays an essential role in the study of Hsu [10]. Instead, we use the comparison Theorem 2.1(iii) in Section 2. Recall that the semigroup $(P_t)_{t\ge 0}$ possesses the Feller property if and only if

$$\lim_{d(x,o)\to\infty} P^x\{T_K \le t\} = 0, \tag{4.1}$$

for any $t > 0$ and compact subset $K \subset M$, where $T_K$ denotes the hitting time of the subset $K \subset M$, that is,

$$T_K = \inf\{t > 0\colon X_t \in K\},$$

and $o$ is a fixed point in the manifold $M$ (cf. Azencott [1] and Hsu [10]). In fact we only have to prove (4.1) for any $K = B(o, R)$—the geodesic ball with center $o$ and radius $R$—since $M$ is complete. By (2.12), we have

$$L\rho \le \frac{n-1}{\rho} + \frac{1}{3}(k+2C)\rho + C(1+d(o,p)) \quad \text{on } M - C(p),$$

for any $p \in M$ with $\rho(x) = d(x,p)$. By Theorem 1.6, we know that the $\frac{1}{2}L$-diffusion process $(X_t, P^x)$ is a conservation process. Define a sequence of stopping times as follows:

$$\begin{aligned}
\tau &= \inf\{t > 0 : d(X_0, X_t) = 1\},\\
S_0 &= 0,\\
\tau_1 &= \tau,\\
S_1 &= \inf\{t \ge \tau_1 : d(o, X_t) = d(o,x) - 1\},\\
\tau_2 &= \tau_1 \circ \theta_{S_1},\\
S_2 &= \inf\{t \ge \tau_2 + S_1 : d(o, X_t) = d(o,x) - 2\},\\
&\vdots\\
\tau_n &= \tau \circ \theta_{S_{n-1}} = \inf\{t > S_{n-1} : d(X_{S_{n-1}}, X_t) = 1\} - S_{n-1},\\
S_n &= \inf\{t \ge \tau_n + S_{n-1} : d(o, X_t) = d(o,x) - n\}.
\end{aligned}$$

Then we have

$$\{T_{B(o,R)} \le t\} \subset \{\tau_1 + \cdots + \tau_{[d(o,x)-R]} \le t\}, \tag{4.2}$$

where $[a]$ denotes the integral part of $a$. Thus the key point in the proof of Theorem 1.7 is to give a good estimate for $\tau_i$. By (3.2), (2.12) and the strong Markov property, we have

$$\begin{aligned}
d(X_t, X_{S_{i-1}}) = W_t &- (L_t - L_{S_{i-1}})\\
&+ \int_{S_{i-1}}^t \frac{1}{2}\left\{\frac{n-1}{d(X_s, X_{S_{i-1}})} + k_1 d(X_s, X_{S_{i-1}})\right.\\
&\qquad \left. + C\big(1 + d(o, X_{S_{i-1}})\big)\right\} ds,
\end{aligned} \tag{4.3}$$

$P^x$-a.e. on $\{t \ge S_{i-1}\}$, with $k_1 = \frac{1}{3}(k + 2C)$ and $W_t = \beta_t - \beta_{S_{i-1}}$, $t \ge S_{i-1}$. Let $r_t$ be the solution of the stochastic differential equation

$$r_t = W_t + \int_{S_{i-1}}^t \frac{1}{2}\left\{\frac{n-1}{r_s} + k_1 r_s + C(d(o,x) - i + 2)\right\} ds, \qquad t \ge S_{i-1},$$

$$r_{S_{i-1}} = 0.$$

Using (4.3), the fact that $d(X_{S_{i-1}}, o) = d(o,x) - i + 1$, $P^x$-a.e. and a comparison theorem for diffusion processes, we deduce that

$$P^x\{d(X_t, X_{S_{i-1}}) \le r_t : t \ge S_{i-1}\} = 1. \tag{4.4}$$

To estimate $\tau_i$, we define a sequence of stopping times as follows:

$$\begin{aligned} T_0 &= \inf\{t > S_{i-1}: r_t > \tfrac{1}{2}\}, \\ T_1 &= \inf\{t > S_{i-1}: r_t < \tfrac{1}{4}\}, \\ T_{2m} &= T_{2m-1} + T_0 \circ \theta_{T_{2m-1}}, \\ T_{2m+1} &= T_{2m} + T_1 \circ \theta_{T_{2m}}, \\ T &= \inf\{t \geq S_{i-1}: r_t = 1\}. \end{aligned}$$

Then it is easily seen that $T \in [T_{2m}, T_{2m+1}]$, for some $m$, and

$$\begin{aligned} r_T - r_{T_{2m}} = W_T - W_{T_{2m}} &+ \int_{T_{2m}}^{T} C(d(o,x) - i + 2)\, ds \\ &+ \int_{T_{2m}}^{T} \left[ \frac{n-1}{r_s} + k_1 r_s \right] ds, \qquad P^x\text{-a.e.} \end{aligned}$$

It is easy to check that $\frac{1}{4} \leq r_s \leq 1$ on $[T_{2m}, T]$; hence we have

$$r_T - r_{T_{2m}} \leq 2 \sup_{[0,t]} |\hat{\beta}_s| + t k_3(x,i), \qquad P^x\text{-a.e. on } \{T - S_{i-1} \leq t\},$$

where $k_3(x,i) = C(d(o,x) - i + 2) + k_2$, $k_2 = k_1 + 4(n-1)$ and $\hat{\beta}_s = \beta_{S_{i-1}+s} - \beta_{S_{i-1}}$, which implies

$$\tfrac{1}{2} \leq 2 \sup_{[0,t]} |\hat{\beta}_s| + t k_3(x,i), \qquad P^x\text{-a.e. on } \{T - S_{i-1} \leq t\}. \tag{4.5}$$

Noting that $\{\tau_i \leq t\} \subset \{T - S_{i-1} \leq t\}$, which follows from (4.4), and using (4.5), we deduce that

$$\begin{aligned} P^x\left\{\tau_i \leq k_4(x,i)^{-1}\right\} &\leq P^x\left\{T - S_{i-1} \leq k_4(x,i)^{-1}\right\} \\ &\leq P^x\left\{ \sup_{[0, k_4(x,i)^{-1}]} |\hat{\beta}_s| \geq \frac{1}{8} \right\} \\ &\leq \alpha \exp\left( -\frac{1}{\alpha} k_4(x,i) \right), \end{aligned} \tag{4.6}$$

where $k_4(x,i) = 4C(d(o,x) - i + 2) + 4k_2$ and $\alpha$ is a positive constant depending only on $n$. Using (4.6) and the same arguments as in Hsu [10], one can show that the semigroup $(P_t)_{t \geq 0}$ possesses the Feller property. For completeness, we give the details. Let

$$\begin{aligned} n(x,t) &= \inf\left\{ n \in N: \sum_{i=1}^{n} \left(4C(d(o,x) - i + 2) + 4k_2\right)^{-1} > t \right\}, \\ m(x,t) &= d(o,x) - n(x,t) + 1. \end{aligned}$$

It is easily seen by definition that

$$\sum_{i=1}^{n(x,t)-1} \{4C(d(o,x) - i + 2) + 4k_2\}^{-1} \le t,$$

which implies that

$$\int_{m(x,t)+2}^{d(o,x)+2} (4Cs + 4k_2)^{-1}\, ds \le t. \tag{4.7}$$

Using (4.7) and the fact that

$$\sum_{i=1}^{[d(o,x)-R]} \{4C(d(o,x) - i + 2) + 4k_2\}^{-1} \to \infty \quad \text{as } d(o,x) \to \infty,$$

we deduce that $m(x,t) \to \infty$ as $d(o,x) \to \infty$. On the other hand, it is easily seen that (4.2) and (4.6) imply that

$$\begin{aligned} P^x\{T_{B(o,R)} \le t\} &\le P^x\{\tau_1 + \cdots + \tau_{[d(o,x)-R]} \le t\} \\ &\le \int_{m(x,t)}^{d(o,x)+1} \alpha \exp\left(-\frac{1}{\alpha}(4Cs + 4k_2)\right) ds. \end{aligned}$$

Hence we have

$$\lim_{d(o,x)\to\infty} P^x\{T_{B(o,R)} \le t\} = 0$$

by the fact that $m(x,t) \to \infty$ as $d(o,x) \to \infty$. Thus we have proved Theorem 1.7. □

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