ON STATES OF EXIT MEASURES
FOR SUPERDIFFUSIONS[1]

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We consider the exit measures of (L, α)-superdiffusions, $1 < \alpha \leq 2$, from a bounded smooth domain D in $\mathbf{R}^d$. By using analytic results about solutions of the corresponding boundary value problem, we study the states of the exit measures. (Abraham and Le Gall investigated earlier this problem for a special case $L = \Delta$ and $\alpha = 2$.) Also as an application of these analytic results, we give a different proof for the critical Hausdorff dimension of boundary polarity (established earlier by Le Gall under more restrictive assumptions and by Dynkin and Kuznetsov for general situations).

1. Introduction. A super-Brownian motion $X = (X_t, P_\mu)$ on $\mathbf{R}^d$ is a branching measure-valued Markov process describing the evolution of a random cloud. It is related (via Laplace transition functionals) to the equation

$$(1.1) \qquad \frac{\partial u}{\partial t} = \Delta u - u^\alpha \quad \text{on } \mathbf{R}^+ \times \mathbf{R}^d,$$

where Δ is the Laplace operator and $1 < \alpha \leq 2$. The process X can be obtained as a limit of branching Brownian particle systems by speeding up the branching rate, decreasing the mass of particles and increasing the number of particles. [We refer to Dynkin (1994) for more detail.]

It is well known that if $d < 2/(\alpha - 1)$, the states X_t of X are absolutely continuous (with respect to the Lebesgue measure on $\mathbf{R}^d$), whereas in the case $d \geq 2/(\alpha - 1)$ they are singular measures. [See, e.g., Dawson and Hochberg (1979), Dawson, Fleischmann and Roelly (1991) and Fleischmann (1988).]

An enhanced model of superdiffusions (of which super-Brownian motion is a special case) was introduced by Dynkin (1993). For every open set D in $\mathbf{R}^d$, as a special case of Dynkin's construction, there corresponds a random exit measure X_D describing, before taking a limit, the mass distribution of the particle systems at the first exit time from D [see, e.g., Dynkin (1991)]. The exit measures X_D play a role similar to that of random exit points from D in

the diffusion theory. The exit measure X_D is related to the boundary value problem

$$(1.2) \qquad \begin{aligned} Lu &= u^\alpha \quad \text{in } D, \\ u &= \nu \quad \text{on } \partial D, \end{aligned}$$

where L is a differential operator of the form (2.1) and ν is a finite measure on the boundary ∂D of D. Problem (1.2) was investigated probabilistically by, for example, Dynkin (1991), Dynkin and Kuznetsov (1995) and Le Gall (1993, 1994b). For analytic treatment of (1.2), we refer to Gmira and Véron (1991) and Sheu (1994).

In this paper we will study the states of the random exit measures X_D for (L, α) superdiffusions. We observe that if $d < (\alpha + 1)/(\alpha - 1)$, the states of X_D are absolutely continuous with respect to the surface area on ∂D (see Theorem 3.3), whereas in the case $d > (\alpha + 1)/(\alpha - 1)$, they are singular (see Theorem 4.3). [For the special case $L = \Delta$ and $\alpha = 2$, the same results were obtained earlier by Abraham and Le Gall (1993).] Our approach depends on some analytic results about solutions of the problem (1.2). Also as an application of these analytic results, we establish in Section 5 that the critical Hausdorff dimension of the boundary polarity is $d - (\alpha + 1)/(\alpha - 1)$, which confirms a conjecture stated in Dynkin (1994). [By using the relation between Hausdorff dimension and the Bessel capacity, Dynkin and Kuznetsov (1994) obtained the same results. The case $L = \Delta$ and $\alpha = 2$ was also treated by Le Gall (1994a).]

We write $d(E, F)$ for the Euclidean distance between two subsets, E and F, of $\mathbf{R}^d$. Moreover, if E is a Borel set, the notation $M(F)$ stands for the set of all finite measures on E. If Y is a random variable on a probability space $(\Omega, \mathscr{F}, P)$, PY is the expected value of Y with respect to the probability measure P. The notation c always denotes a constant which may change values from line to line.

2. Diffusion and superdiffusion. Throughout this paper we consider a differential operator in $\mathbf{R}^d$ of the form

$$(2.1) \qquad L = \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

such that:

1. The functions $a_{ij} = a_{ji}$ and b_i are bounded smooth functions in $\mathbf{R}^d$.
2. There exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x) u_i u_j \geq c \sum u_i^2 \quad \text{for all } x \in \mathbf{R}^d \text{ and all } u_1, u_2, \ldots, u_d.$$

Then there exists a continuous Markov process $\xi = (\xi_t, \Pi_x)$ in $\mathbf{R}^d$ with the property that for every continuous function f with compact support, the function

$$u_t(x) = \Pi_x f(\xi_t)$$

is the solution of the initial-value problem:

$$\frac{\partial u}{\partial t} = Lu \quad \text{in } \mathbf{R}^+ \times \mathbf{R}^d,$$

$$u \to f \quad \text{as } t \to 0.$$

[See, e.g., Stroock and Varadhan (1979).] We call ξ the L-diffusion.

To every L of the form (2.1) and every α, $1 < \alpha \leq 2$, there corresponds a Markov process $X = (X_t, P_\mu)$ in $M(\mathbf{R}^d)$ such that for every positive Borel function f on $\mathbf{R}^d$ and every $\mu \in M(\mathbf{R}^d)$, we have

$$(2.2) \qquad P_\mu \exp\{-\langle f, X_t \rangle\} = \exp\{-\langle v_t, \mu \rangle\},$$

where v_t satisfies the equation

$$(2.3) \qquad v_t(x) + \Pi_x \int_0^t v_{t-s}^\alpha(\xi_s)\, ds = \Pi_x f(\xi_t), \qquad x \in \mathbf{R}^d.$$

[For every Borel function f on $\mathbf{R}^d$ and $\nu \in M(\mathbf{R}^d)$, $\langle f, \nu \rangle$ denotes the integral of f with respect to ν.] Moreover, for every open set D in $\mathbf{R}^d$, there exists a random exit measure X_D such that for every positive Borel function f and every $\mu \in M(\mathbf{R}^d)$,

$$(2.4) \qquad P_\mu \exp\{-\langle f, X_D \rangle\} = \exp\{-\langle v, \mu \rangle\},$$

where v satisfies the equation

$$(2.5) \qquad v(x) + \Pi_x \int_0^{\tau_D} v^\alpha(\xi_s)\, ds = \Pi_x f(\xi_{\tau_D})$$

and τ_D is the first exit time of ξ from D [see, e.g., Dynkin (1991)]. Following Dynkin, we call $X = (X_t, X_D, P_\mu)$ the (L, α) superdiffusion. Note that (2.4) and (2.5) imply

$$(2.6) \qquad P_\mu \langle f, X_D \rangle = \Pi_\mu f(\xi_{\tau_D}).$$

3. Absolutely continuous states of X_D. From this point on we consider an (L, α) superdiffusion $X = (X_t, X_D, P_\mu)$ and always assume that D is a bounded smooth domain in $\mathbf{R}^d$. Let $S(dz)$ be the surface area on the boundary ∂D of D. For every $z \in \mathbf{R}^d$ and every $\varepsilon > 0$, let $Q_\varepsilon(z)$ be the cube in $\mathbf{R}^d$ with center z and edge length ε. Denote by $\mathscr{B}(M(\partial D))$ the σ-algebra on $M(\partial D)$ generated by $f \to \langle f, \mu \rangle$, $f \in C(\partial D)$. To prove absolute continuity of X_D, we need the following two lemmas. The first one is a modification of Lemma 3.4.2.2 in Dawson (1993).

LEMMA 3.1. *Let Y be a random measure defined on a probability space* $(\Omega, \mathscr{F}, P)$ *with values in* $(M(\partial D), \mathscr{B}(M(\partial D)))$. *Assume that*:

(a) *There exists a Borel subset $N \subset \partial D$ of surface area zero such that for each $z \in \partial D \setminus N$, there is a sequence $\varepsilon_n(z) \to 0$ and as $n \to \infty$,*

$$\frac{Y(Q_{\varepsilon_n}(z))}{S(Q_{\varepsilon_n}(z))} \quad \text{converges to } \mathscr{X}(z) \text{ weakly},$$

where $\mathscr{X}(z)$ is a random variable with $P\mathscr{X}(z) < \infty$.
   (b) *$P\langle f, Y \rangle = \int_{\partial D} f(z) P\mathscr{X}(z) S(dz)$, for all $f \in C(\partial D)$.*

Then Y is almost surely an absolutely continuous measure [with respect to $S(dz)$] on ∂D.

PROOF. With some suitable changes, the proof is the same as that of Lemma 3.4.2.2 in Dawson (1993). We use Wheeden and Zygmund [(1977), Corollary (10.50)] to replace the Lebesgue density theorem, quoted in Dawson. We omit the proof and refer the reader to Dawson (1993) for more detail. $\square$

We replace (1.2) by an equivalent integral equation

$$(3.1) \quad u(x) + \int_D g(x, y) u^\alpha(y)\, dy = \int_{\partial D} k(x, z) \mu(dz), \qquad x \in D,$$

where $g(x, y)$ is the Green function of L in D and $k(x, z)$ is the Poisson kernel. Note that quotients of Green functions are uniformly bounded [see Hueber and Sieveking (1982)] and that there is a constant c depending only on L and D such that

$$(3.2) \qquad k(x, z) \le c\rho(x)\|x - z\|^{-d}, \qquad x \in D \text{ and } z \in \partial D,$$

where $\rho(x) = d(x, \partial D)$ and $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^d$ [see, e.g., Maz'ya (1972), Lemma 6, or Dynkin and Kuznetsov (1994)]. Therefore, the same arguments as that of Sheu [(1994), Lemmas 2.2 and 2.3] imply that if $d < (\alpha + 1)/(\alpha - 1)$, for every $\mu \in M(\partial D)$, there exists a solution (which means a positive solution) of (3.1).

LEMMA 3.2. *Assume $d < (\alpha + 1)/(\alpha - 1)$. Let μ_n be a sequence of finite measures on ∂D and, for each n, let u_n be a solution of (3.1) with μ replaced by μ_n. If μ_n converges weakly to μ_∞ in $M(\partial D)$, then there exists a subsequence $n_k \to \infty$ such that u_{n_k} converges pointwise to a function u_∞ in D and u_∞ satisfies (3.1) with μ replaced by μ_∞.*

PROOF. We show that the family u_n^α is relatively weakly compact in $L^1(D, \rho(x)\, dx)$. By the Dunford–Pettis theorem [see, e.g., Dunford and Schwartz (1958), IV.8, Corollary 11] we need to prove that for any $\varepsilon > 0$, it is possible to find $\delta > 0$ such that for any n and any measurable set $E \subset D$,

$$(3.3) \qquad \int_E \rho(x)\, dx < \delta \quad \text{implies} \quad \int_E u_n^\alpha(x) \rho(x)\, dx < \varepsilon.$$

Note that for any $E \subset D$ and $a > 0$, we have

(3.4) $\qquad \int_E u_n^\alpha(x)\rho(x)\,dx \le a^\alpha \int_E \rho(x)\,dx + \int_{\{u_n > \alpha\}} u_n^\alpha(x)\rho(x)\,dx$

and

(3.5) $\qquad\qquad \int_{\{u_n > a\}} u_n^\alpha(x)\rho(x)\,dx = -\int_a^\infty \lambda^\alpha\,d\beta_n(\lambda),$

where $\beta_n(\lambda) = \int_{\{u_n > \lambda\}} \rho(x)\,dx$, $\lambda > 0$.

We estimate $\beta_n(\lambda)$. Set $h_n(x) = \int_{\partial D} k(x, z)\mu_n(dz)$. By (3.1), we have $u_n \le h_n$ and so $\beta_n(\lambda) \le \gamma_n(\lambda)$, where $\gamma_n(\lambda) = \int_{\{h_n > \lambda\}} \rho(x)\,dx$. Note that

$$\lambda\gamma_n(\lambda) \le \int_{\{h_n > \lambda\}} h_n(x)\rho(x)\,dx = \int_{\partial D} \mu_n(dz)\int_{\{h_n > \lambda\}} k(x, z)\rho(x)\,dx$$

(3.6) $\qquad\qquad \le \mu_n(\partial D) \sup_{z \in \partial D} \int_{\{h_n > \lambda\}} k(x, z)\rho(x)\,dx$

$$\le c \sup_{z \in \partial D} \int_{\{h_n > \lambda\}} k(x, z)\rho(x)\,dx,$$

where $c = \sup_n \mu_n(\partial D) < \infty$, by assumption. Choose $\tilde{\alpha} \in (\alpha, (d + 1)/(d - 1))$ and let $z \in \partial D$. By Hölder's inequality, we have

(3.7) $\qquad\qquad \int_{\{h_n > \lambda\}} k(x, z)\rho(x)\,dx \le (A)^{1/\tilde{\alpha}}(\gamma_n(\lambda))^{1/\tilde{\alpha}'},$

where $A = \int_D k(x, z)^{\tilde{\alpha}}\rho(x)\,dx$ and $1/\tilde{\alpha} + 1/\tilde{\alpha}' = 1$. By (3.2), we have

(3.8) $\qquad A \le c\int_D \rho(x)^{\tilde{\alpha}+1}\|x - z\|^{-d\tilde{\alpha}}\,dx \le c\int_D \|x - z\|^{-\tilde{\alpha}(d-1)+1}\,dx.$

Since D is bounded and $\tilde{\alpha} < (d + 1)/(d - 1)$, (3.8) implies that A is bounded for $z \in \partial D$. Combining (3.6)–(3.8) we get $\lambda\gamma_n(\lambda) \le c\gamma_n(\lambda)^{1/\tilde{\alpha}'}$, and so

(3.9) $\qquad\qquad\qquad \beta_n(\lambda) \le \gamma_n(\lambda) \le c\lambda^{-\tilde{\alpha}}$

for all $\lambda > 0$.

Since $\alpha < \tilde{\alpha}$, we have, by integration by parts and (3.9),

$$-\int_a^\infty \lambda^a\,d\beta_n(\lambda) = a^\alpha\beta_n(a) + \alpha\int_a^\infty \beta_n(\lambda)\lambda^{\alpha-1}\,d\lambda$$

(3.10) $$\le ca^{\alpha-\tilde{\alpha}} + \alpha\int_a^\infty \lambda^{\alpha-\tilde{\alpha}-1}\,d\lambda$$

$$\le ca^{\alpha-\tilde{\alpha}}.$$

Therefore the condition in (3.3) follows easily from (3.4), (3.5) and (3.10), and this implies that $\{u_n^\alpha\}$ is relatively weakly compact.

Assume we choose a subsequence $n_k \to \infty$ such that u_{n_k} converges to w weakly in $L^1(D, \rho(x)\,dx)$. Fix $x \in D$. Since $(g(x, y))/(\rho(y))$ is bounded in y,

$$\int_D g(x, y) u_{n_k}^\alpha(y)\,dy \to \int_D g(x, y) w(y)\,dy.$$

Since μ_n converges to μ_∞, $h_n(x)$ converges to $h_\infty(x)$, where

$$h_\infty(x) = \int_{\partial D} k(x, y) \mu_\infty(dy).$$

Passing to a limit in (3.1) with $u = u_{n_k}$ and $\mu = \mu_{n_k}$, we obtain $u_{n_k}(x) \to u_\infty(x)$ and

$$u_\infty(x) + \int_D g(x, y) w(y)\,dy = h_\infty(x).$$

It remains to prove that $u_\infty^\alpha = w$. To do this, it suffices to show that $u_{n_k}^\alpha$ converges weakly in $L^1(D, \rho(x)\,dx)$ to u_∞^α. Let $f \in L^\infty(D, \rho(x)\,dx)$ and let K be an arbitrary compact set in D. Then

$$(3.11) \qquad \left| \int_D u_{n_k}^\alpha(x) f(x) \rho(x)\,dx - \int_D u_\infty^\alpha(x) f(x) \rho(x)\,dx \right| \le I + J,$$

where

$$I = \left| \int_K u_{n_k}^\alpha(x) f(x) \rho(x)\,dx - \int_K u_\infty^\alpha(x) f(x) \rho(x)\,dx \right|$$

and

$$J = \left| \int_{D \setminus K} u_{n_k}^\alpha(x) f(x) \rho(x)\,dx \right| + \left| \int_{D \setminus K} u_\infty^\alpha(x) f(x) \rho(x)\,dx \right|.$$

Note that $u_n \le h_n$ and, by (3.2), $h_n(x) \le c \int_{\partial D} \rho(x) \| x - z \|^{-d} \mu_n(dz) \le c$ for all n and $x \in K$. For fixed K, the bounded convergence theorem implies that $I \to 0$ as $n_k \to \infty$. By Fatou's lemma, $J \le c \sup_{n_k} \int_{D / K} u_{n_k}^\alpha(x) \rho(x)\,dx$. Since u_{n_k} satisfies condition in (3.3), $J \to 0$ as $K \uparrow D$. Letting $k \to \infty$ and then $K \uparrow D$ in (3.11), we get

$$\int_D u_{n_k}^\alpha(x) f(x) \rho(x)\,dx \to \int_D u_\infty^\alpha(x) f(x) \rho(x)\,dx,$$

which completes the proof of Lemma 3.2. $\square$

We write $\mu \in M_c(D)$ if $\mu \in M(D)$ and μ has a compact support in D.

THEOREM 3.3. *Assume $d < (\alpha + 1)/(\alpha - 1)$. If $\nu \in M_c(D)$, then X_D is, P_ν-a.s., an absolutely continuous measure on ∂D.*

PROOF. Fix $\nu \in M_c(D)$ and let K be the support of ν. It suffices to show that the random measure X_D satisfies conditions Lemma 3.1(a) and (b). To

verify Lemma 3.1(a), we choose $z \in \partial D$ and let $\lambda > 0$. For every $\varepsilon > 0$, define a function f_ε on ∂D by

$$f_\varepsilon(y) = \begin{cases} \dfrac{1}{S(Q_\varepsilon(z))}, & \text{for } y \in Q_\varepsilon(z) \cap \partial D, \\ 0, & \text{for } y \notin Q_\varepsilon(z) \cap \partial D. \end{cases}$$

We have, by (2.4) and (2.5),

$$P_\mu \exp\{-\lambda\langle f_\varepsilon, X_D \rangle\} = \exp\{-\langle v_{\lambda, \varepsilon}, \mu \rangle\},$$

where $v_{\lambda, \varepsilon}$ satisfies (3.1) with $\mu(dy)$ replaced by $\lambda f_\varepsilon(y)S(dy)$. Clearly as $\varepsilon \to 0$, $\lambda f_\varepsilon(y)S(dy)$ converges weakly to $\lambda\delta_z(dy)$. By Lemma 3.2, there exists a sequence $\varepsilon_n \to 0$ such that $v_{\lambda, \varepsilon_n}(x) \to v_\lambda(x)$ for all $x \in D$ and v_λ satisfies the equation

$$(3.12) \qquad v_\lambda(x) + \int_D g(x, y)v_v^\alpha(y)\, dy = \lambda k(x, z), \qquad x \in D.$$

Note that $v_{\lambda, \varepsilon}$ are uniformly bounded on K [see, e.g., Dynkin (1991), Lemma 3.1 and Theorem 0.5]. We have, by the bounded convergence theorem, $\langle v_{\lambda, \varepsilon_n}, \nu \rangle \to \langle v_\lambda, \nu \rangle$ and so

$$P_\nu \exp\{-\lambda\langle f_{\varepsilon_n}, X_D \rangle\} \to \exp\{-\langle v_\lambda, \nu \rangle\}.$$

Thus, $\langle f_{\varepsilon_n}, X_D \rangle$ converges weakly to some $\mathscr{X}(z)$ and

$$(3.13) \qquad P_\nu \exp\{-\lambda\mathscr{X}(z)\} = \exp\{-\langle v_\lambda, \nu \rangle\}.$$

Note that, by (3.12), we have

$$(3.14) \qquad \frac{v_\lambda(x)}{\lambda} = k(x, z) - \frac{1}{\lambda}\int_D g(x, y)v_\lambda^\alpha(y)\, dy$$

and

$$(3.15) \qquad 0 \le \frac{1}{\lambda}\int_D g(x, y)v_\lambda^\alpha(y)\, dy \le \lambda^{\alpha-1}\int_D g(x, y)k^\alpha(y, z)\, dy.$$

We show $\int_D g(x, y)k^\alpha(y, z)\, dy$ is bounded in $x \in K$. Let $2\delta = d(K, \partial D)$ and set $K_\delta = \{x \in D, d(x, K) \le \delta\}$. Then it suffices to check that both A and B are bounded on K, where

$$A = \int_{K_\delta} g(x, y)k^\alpha(y, z)\, dy$$

and

$$B = \int_{D\setminus K_\delta} g(x, y)k^\alpha(y, z)\, dy.$$

By (3.2), on K_δ, $k(y, z)$ is bounded and so

$$A \le c\int_D g(x, y)\, dy \le c, \qquad x \in K.$$

To estimate B, we fix $x_0 \in K$. Then there exists a constant c such that $g(x_0, y) \le c\rho(y)$ for all $y \in D \setminus K_\delta$ [see, e.g., Dautray and Lions (1990), II.4, Property 4] and $(g(x, y))/(g(x_0, y)) \le c$ for all $x \in K$ and $y \in D \setminus K_\delta$ [see Doob (1984), 1 XII, Section 2]. Therefore $(g(x, y))/\rho(y))$ is bounded for $x \in K$ and $y \in D \setminus K_\delta$. Thus if $x \in K$,

$$B \le c\int_D k^\alpha(y, z)\rho(y)\,dy,$$

which, by taking $\tilde{\alpha} = \alpha$ in (3.8), is bounded. Combining with (3.13)–(3.15), we obtain that

$$(3.16) \qquad P_\nu \mathscr{H}(z) = \lim_{\lambda \to 0} \frac{\langle v_\lambda, \mu \rangle}{\lambda} = \int_D k(x, z)\nu(dx).$$

Moreover, for every $f \in C(\partial D)$, by (2.6) and (3.16),

$$P_\nu \langle f, X_D \rangle = \int \nu(dx)k(x, z)f(z)S(dz) = \int_{\partial D} P_\nu \mathscr{H}(z)f(z)S(dz),$$

which completes the proof of the theorem. □

REMARK. Abraham and Le Gall (1993) obtained the same result for $L = \Delta$, $\alpha = 2$ and $\nu = \delta_x$, $x \in D$.

4. Singular state of X_D. Let F be a subset of the boundary ∂D of D. Consider the following boundary value problem:

$$(4.1) \qquad \begin{aligned} Lu &= u^\alpha \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D \setminus F. \end{aligned}$$

[We write $u = f$ on $K \subset \partial D$ if for every $z \in K$, $\lim_{x \to z} u(x) = f(z)$.]

LEMMA 4.1. *If u is a solution of the boundary value problem* (4.1), *then*

$$(4.2) \qquad u(x) \le cd(x, F)^{-2/(\alpha - 1)}, \qquad x \in D,$$

where c is a constant depending only on L, α and D.

PROOF. Our proof is a modification of that of step 1 for Sheu [(1994), Theorem 3]. We sketch only the main steps. Let u be a solution of (4.1). Put $w(x) = u(x) - 1$, $x \in D$, and $h(x) = g(w(x))1_{\overline{D} \setminus F}(x)$ for all $x \in \mathbf{R}^d$, where

$$g(r) = \begin{cases} 0, & \text{if } r < 0, \\ r^2/2, & \text{if } 0 \le r < 1, \\ r - \frac{1}{2}, & \text{if } r \ge 1. \end{cases}$$

On D, we have either $w = u - 1 \le 1$ or $h(x) = u(x) - 3/2$. Since D is bounded, it suffices to show that

$$(4.3) \qquad h(x) \le cd(x, F)^{-2/(\alpha - 1)}, \qquad x \in D.$$

To prove (4.3), by Dynkin [(1991), Lemma 3.1 and Theorem 0.5], we need to check that

(4.4) $-Lh + h^\alpha \le 0$ on $\mathbf{R}^d \setminus F$.

Note that in D,

$$-Lh + h^\alpha = -g'(w)Lw - g''(w)\sum_{i,j}a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + h^\alpha$$

$$\le -g'(w)Lw + h^\alpha,$$

where the last inequality follows from the assumptions on L. Then (4.4) follows from arguments similar to those of Sheu (1994). $\square$

Notation $B_\varepsilon(z)$ stands for the ball with radius ε centered at z.

LEMMA 4.2.   *Let K be a compact set in D. Then there exist two constants c and ε_0 (depending only on L, α, D and K) such that if u_ε is a solution of (4.1) with $F = \overline{B}_\varepsilon(z) \cap \partial D$ for some $z \in \partial D$ and $\varepsilon < \varepsilon_0$, then we have*

(4.5) $u_\varepsilon(x) \le c\varepsilon^{d-(\alpha+1)/(\alpha-1)}$, $x \in K$.

PROOF.   By choosing ε small enough we can assume $d(K, \partial D) \ge 2\varepsilon$. Let u_ε be a solution of (4.1) with $F = \overline{B}_\varepsilon(z) \cap \partial D$ for some $z \in \partial D$. Let $\tau = \inf\{t, \xi_t \notin D\}$ and $\tau_\varepsilon = \min\{\inf\{t, \|\xi_t - z\| \ge 2\varepsilon\}, \tau\}$. Since $Lu - u^\alpha = Lu - u^{\alpha-1}u$, we have

(4.6) $u_\varepsilon(x) = \Pi_x u_\varepsilon(\xi_{\tau_\varepsilon})\exp\left\{-\int_0^{\tau_\varepsilon} u^{\alpha-1}(\xi_s)\,ds\right\}$, $x \in K$.

[See, e.g., Wentzell (1981), Microtheorem 13.5.] Since $u_\varepsilon = 0$ on $\partial D \setminus F$, (4.6) implies that

(4.7) $u_\varepsilon(x) \le \Pi_x\left[u_\varepsilon(\xi_{\tau_\varepsilon}), \tau_\varepsilon < \tau\right]$.

On $\tau_\varepsilon < \tau$, we have, by Lemma 4.1,

$$u_\varepsilon(\xi_{\tau_\varepsilon}) \le cd(\xi_{\tau_\varepsilon}, F)^{-2/(\alpha-1)} \le c\varepsilon^{-2/(\alpha-1)},$$

and so, by (4.7),

(4.8) $u_\varepsilon(x) \le c\varepsilon^{-2/(\alpha-1)}\Pi_x[\tau_\varepsilon < \tau]$.

The same arguments as in Abraham and Le Gall [(1991), Theorem 3.1] imply that for ε sufficiently small,

(4.9) $\Pi_x[\tau_\varepsilon < \tau] \le c\Pi_x[\xi_\tau \in B_{5\varepsilon}(z) \cap \partial D]$, $x \in K$.

By (4.9) and (3.2), we have $\Pi_x[\tau_\varepsilon < \tau] \le c\varepsilon^{d-1}$, $x \in K$. Our conclusion follows from the above inequality and (4.8). $\square$

REMARK.   It follows from Dynkin (1991) that solutions of $Lu = u^\alpha$ in D are locally uniformly bounded in D. Therefore if $d \le (\alpha+1)/(\alpha-1)$, the estimate (4.5) does not give the best possible lower bound.

THEOREM 4.3. *Assume $d > (\alpha + 1)/(\alpha - 1)$. If $\mu \in M_c(D)$, then X_D is, P_μ-a.s., singular with respect to the surface area S on ∂D.*

PROOF. Fix $\mu \in M_c(D)$ and put $K = \text{supp}(X_D)$. For all $n \geq 1$, let $\{B_{n,i}\}_{i \in I_n}$ be an open covering of ∂D and $\text{diam}(B_{n,i}) = 2^{-n}$. By the regularity of ∂D, we can assume the cardinality of I_n is less than $c2^{n(d-1)}$, where c is a constant independent of n. Set

$$H_n = \sum_{i \in I_n} 1_{\{B_{n,i} \cap K \neq \varnothing\}}$$

and

$$v_{n,i}(x) = -\log P_{\delta_x}\left[X_D(B_{n,i}) = 0\right].$$

Then

$$
\begin{aligned}
P_\mu H_n &= \sum_{i \in I_n} P_\mu\left[X_D(B_{n,i}) > 0\right] = \sum_{i \in I_n}\left(1 - P_\mu\left[X_D(B_{N,i}) = 0\right]\right) \\
&= \sum_{i \in I_n}\left(1 - \exp\{-\langle v_{n,i}, \mu\rangle\}\right) \leq \sum_{i \in I_n}\langle v_{n,i}, \mu\rangle.
\end{aligned}
$$

(4.10)

Note that $v_{n,i}$ is, as $\lambda \to \infty$, the limit of the functions

$$v_{n,i,\lambda}(x) = -\log P_{\delta_x}\exp\{-\lambda X_D(B_{n,i})\}$$

and $v_{n,i,\lambda}$ is a solution of $Lu = u^\alpha$ in D with $u = 0$ on $\partial D \setminus \overline{B}_{n,i}$. Note that similar results as in Dynkin [(1992), Theorem 1.2] hold for elliptic case. Therefore, $v_{n,i}$ is a solution of (4.1) with $F = \overline{B}_{n,i} \cap \partial D$. By Lemma 4.2, we have, for n sufficiently large,

$$\langle v_{n,i}, \mu\rangle \leq c(2^{-n})^{d-(\alpha+1)/(\alpha-1)}, \qquad i \in I_n.$$

Therefore, by (4.10),

$$P_\mu H_n \leq \sum_{i \in I_n} c(2^{-n})^{(d-(\alpha+1)/(\alpha-1))} \leq c(2^{-n})^{-2/(\alpha-1)},$$

which implies, for n sufficiently large,

$$P_\mu\left[(2^{-n})^{2/(\alpha-1)} H_n\right] \leq c < \infty.$$

By Fatou's lemma, $\liminf(2^{-n})^{2/(\alpha-1)}H_n < \infty$, P_μ-a.s., and so the Hausdorff dimension of K is less or equal to $2/(\alpha - 1)$. Since $\dim(\partial D) = d - 1$ and $d - 1 > 2/(\alpha - 1)$, X_D is, P_μ-a.s., singular. $\square$

REMARK. (a) The same result was obtained by Abraham and Le Gall [(1993), Theorem 6.1] for the special case $L = \Delta$ and $\alpha = 2$. Our proof is the same in spirit.

(b) Assume $d = (\alpha + 1)/(\alpha - 1)$. By using Brownian path-valued processes, Abraham and Le Gall (1993) obtained a lower upper bound for $P_\mu[X_D(B_{n,i}) \neq 0]$ and proved that X_D is singular for $L = \Delta$ and $\alpha = 2$.

5. Critical Hausdorff dimension for boundary polarity. In this section we consider $X = (X_t, P_\mu)$, a (L, α) superdiffusion in D [for more detail, see Dynkin (1993)]. The range of X is the smallest closed set $\mathscr{R}_D$ in $\mathbf{R}^d$ such that for every $t \geq 0$, $\mathrm{supp}(X_t) \subset \mathscr{R}_D$. A set $F \subset \partial D$ is said to be ∂-polar if

$$P_{\delta_x}[\mathscr{R}_D \cap F = \varnothing] = 1 \quad \text{for all } x \in D.$$

Dynkin and Kuznetsov (1994) obtained that a closed subset F of ∂D is ∂-polar if and only if there is no nonzero solution of the problem (4.1). Combining with Sheu [(1994), Theorem 1(A)] (which is still true for general L), we obtain that if $d < (\alpha + 1)/(\alpha - 1)$, there are no ∂-polar sets.

Let $h_\beta(s) = s^\beta$, $s > 0$, $\beta > 0$ and write $h_\beta - m(F)$ for the h_β-Hausdorff measure of F [for a definition of h_β-Hausdorff measure, see Dynkin (1991)].

THEOREM 5.1. *Assume $d > (\alpha + 1)/(\alpha - 1)$ and let F be a closed subset of ∂D. Put $\beta_0 = d - (\alpha + 1)/(\alpha - 1)$.*

(a) *If $h_{\beta_0}\text{-}m(F) = 0$, then F is ∂-polar.*
(b) *If $h_s\text{-}m(F) > 0$ for some $\beta_0 < s \leq d - 1$, then F is not ∂-polar.*

PROOF. (a) Fix $x \in D$ and let $\delta > 0$. By assumption, there is a covering $\{B_{\varepsilon_i}(z_i)\}$, with $z_i \in \partial D$, of F such that

$$\sum_i h_{\beta_0}(\varepsilon_i) < \delta.$$

(We can assume ε_i are small enough such that if $K = \{x\}$, the conclusions of Lemma 4.2 hold.) Set $B_i = B_{\varepsilon_i}(z_i)$. Then

$$P_{\delta_x}[\mathscr{R}_D \cap F \neq \varnothing] \leq \sum_i P_{\delta_x}[\mathscr{R}_D \cap \overline{B}_i \neq \varnothing] = \sum_i \left(1 - P_{\delta_x}[\mathscr{R}_D \cap \overline{B}_i = \varnothing]\right)$$

$$= \sum_i \left(1 - \exp\{-v_i(x)\}\right) \leq \sum_i v_i(x),$$

where v_i is the maximum solution of (4.1) with $F = \partial D \cap \overline{B}_{\varepsilon_i}(z_i)$ [see Dynkin and Kuznetsov (1994)]. By Lemma 4.2, there exists a constant c, depending only on L, α and D, such that

$$v_i(x) \leq ch_{\beta_0}(\varepsilon_i) \quad \text{for all } i,$$

and so $P_{\delta_x}[\mathscr{R}_D \cap F \neq \varnothing] \leq c\sum_i h_{\beta_0}(\varepsilon_i) < c\delta$. Since δ is arbitrary, we obtain $P_{\delta_x}[\mathscr{R}_D \cap F \neq \varnothing] = 0$, which completes the proof of (a).

(b) This follows directly from Sheu [(1994), Theorem 1(B)] and Dynkin and Kuznetsov's criterion for ∂-polarity. □

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