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TRANSIENCE, RECURRENCE AND LOCAL EXTINCTION PROPERTIES OF THE SUPPORT FOR SUPERCRITICAL FINITE MEASURE-VALUED DIFFUSIONS[1]

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We consider the supercritical finite measure-valued diffusion, $X(t)$, whose log-Laplace equation is associated with the semilinear equation $u_t = Lu + \beta u - \alpha u^2$, where $\alpha, \beta > 0$, and $L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(\partial^2/(\partial x_i\,\partial x_j)) + \sum_{i=1}^d b_i(\partial/\partial x_i)$. A path $X(\cdot)$ is said to *survive* if $X(t) \not\equiv 0$, for all $t \ge 0$. Since $\beta > 0$, $P_\mu(X(\cdot)$ survives$) > 0$, for all $0 \not\equiv \mu \in \mathscr{M}(R^d)$, where $\mathscr{M}(R^d)$ denotes the space of finite measures on R^d. We define transience, recurrence and local extinction for the support of the supercritical measure-valued diffusion starting from a finite measure as follows. The support is *recurrent* if $P_\mu(X(t, B) > 0$, for some $t \ge 0 \mid X(\cdot)$ survives$) = 1$, for every $0 \not\equiv \mu \in \mathscr{M}(R^d)$ and every open set $B \subset R^d$. For $d \ge 2$, the support is *transient* if $P_\mu(X(t, B) > 0$, for some $t \ge 0 \mid X(\cdot)$ survives$) < 1$, for every $\mu \in \mathscr{M}(R^d)$ and bounded $B \subset R^d$ which satisfy $\text{supp}(\mu) \cap \overline{B} = \varnothing$. A similar definition taking into account the topology of R^1 is given for $d = 1$. The support exhibits *local extinction* if for each $\mu \in \mathscr{M}(R^d)$ and each bounded $B \subset R^d$, there exists a P_μ-almost surely finite random time ζ_B such that $X(t, B) = 0$, for all $t \ge \zeta_B$. Criteria for transience, recurrence and local extinction are developed in this paper. Also studied is the asymptotic behavior as $t \to \infty$ of $E_\mu \int_0^t \langle \psi, X(s)\rangle\, ds$, and of $E_\mu\langle g, X(t)\rangle$, for $0 \le g, \psi \in C_c(R^d)$, where $\langle f, X(t)\rangle \equiv \int_{R^d} f(x) X(t, dx)$. A number of examples are given to illustrate the general theory.

1. Statement of results. In this article, we investigate the transience, recurrence and local extinction properties for the support of supercritical $\mathscr{M}(R^d)$-valued diffusions, where $\mathscr{M}(R^d)$ denotes the space of finite measures on R^d. The $\mathscr{M}(R^d)$-valued diffusion is constructed as follows. Let

$$L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} \quad \text{on } R^d,$$

where $a(x) = \{a_{ij}(x)\}$ is positive definite for each $x \in R^d$ and $a_{ij}, b_i \in C^\alpha(R^d)$, $\alpha \in (0, 1]$. We will always assume that the solution to the martingale problem for L is well posed; that is, we will assume that there exists a conservative diffusion on R^d corresponding to the operator L. We will also always assume that the semigroup corresponding to the diffusion is C_0-preserving; that is, if $f \in C_0(R^d)$, then $T_t f(x) \equiv E_x f(Y(t)) \in C_0(R^d)$, for all

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$t > 0$, where $C_0(R^d) = \{f \in C(R^d): \lim_{|x|\to\infty} f(x) = 0\}$ and where E_x denotes the expectation for the diffusion process $Y(t)$ corresponding to L and starting from $x \in R^d$. [The C_0-preserving property can be expressed probabilistically by the following condition: $\lim_{|x|\to\infty} P_x(\sigma_D \le t) = 0$, for all $t > 0$, where $\sigma_D = \inf\{t > 0\colon Y(t) \in D\}$ and $D \subset R^d$ is a bounded domain. When this condition fails, the diffusion is said to *explode inward from infinity* (see [9], Section 8.4).] For each positive integer n, consider N_n particles, each of mass $1/n$, starting at points $x_i^{(n)} \in R^d$, $i = 1, \dots, N_n$, and performing independent branching diffusions according to the operator L, with branching rate cn, $c > 0$, and branching distribution $\{p_k^{(n)}\}_{k=1}^{\infty}$, where

$$\sum_{k=0}^{\infty} kp_k^{(n)} = 1 + \gamma/n, \qquad \gamma > 0,$$

and

$$\sum_{k=0}^{\infty} (k-1)^2 p_k^{(n)} = m + o(1) \quad \text{as } n \to \infty, m > 0.$$

Let $N_n(t)$ denote the number of particles alive at time t and denote their positions by $\{x_i^n(t)\}_{i=1}^{N_n(t)}$. Define an $\mathscr{M}(R^d)$-valued process $X_n(t)$ by $X_n(t) = (1/n)\sum_{i=1}^{N_n(t)}\delta_{x_i^{(n)}(t)}$. If $X_n(0) = (1/n)\sum_{i=1}^{N_n}\delta_{x_i^{(n)}}$ converges weakly to a measure $\mu \in \mathscr{M}(R^d)$, then $X_n(\cdot)$ converges weakly to an $\mathscr{M}(R^d)$-valued process which can be uniquely characterized as the solution to the following martingale problem [1, 10, 12]. (In what follows, $\langle f, X(t)\rangle = \int_{R^d} f(x)X(t, dx)$, $\alpha = cm$, $\beta = c\gamma$.)

MG. The process $X(t) \in \mathscr{M}(R^d)$ satisfies:

(i) $X(0) = \mu$ a.s.

(ii) For all $f \in C_c^2(R^d)$ and for $f \equiv 1$,

$$M_f(t) \equiv \langle f, X(t)\rangle - \int_0^t \langle Lf, X(s)\rangle\, ds - \beta\int_0^t \langle f, X(s)\rangle\, ds$$

(1.1) is a martingale with increasing process

$$\langle M_f\rangle_t = 2\alpha\int_0^t \langle f^2, X(s)\rangle\, ds.$$

The probability measure corresponding to the solution of the above martingale problem will be denoted by P_μ. (The dependence of P_μ on α and β has been suppressed; we point this out because in the sequel the parameter β will be allowed to vary.) An alternative method of characterizing the measure-valued diffusion is via the following log-Laplace equation:

$$\begin{aligned} &E_\mu \exp\left(-\langle g, X(t)\rangle - \int_0^t \langle \psi, X(s)\rangle ds\right) \\ &\qquad = \exp\left(-\int_{R^d} u(x,t)\mu(dx)\right) \quad \text{for all } 0 \le g, \psi \in C_c^2(R^d), \end{aligned} \tag{1.2}$$

where $u \in C^{2,1}(R^d \times [0,\infty))$ is the unique positive solution of the evolution equation

$$\begin{aligned} &u_t = Lu + \beta u - \alpha u^2 + \psi, \qquad (x,t) \in R^d \times [0,\infty),\\ &u(\cdot,0) = g(\cdot),\\ &u(\cdot,t) \in C_0(R^d). \end{aligned} \tag{1.3}$$

The existence of a classical solution to (1.3) follows from [7], Chapter 6, Theorems 1.4 and 1.5. (Actually, in [7], the results are proved for the case $\psi = 0$, but the same proof holds with $\psi \in C_c(R^d)$.) For the nonnegativity, one can use the type of argument appearing in [5], page 115. The uniqueness follows from the parabolic maximum principle (Proposition 4).

Let $Z(t) = \langle 1, X(t)\rangle$ denote the total mass process. Substituting $f \equiv 1$ in (1.1), it follows that under P_μ, $Z(t)$ is a one-dimensional diffusion on $[0,\infty)$ corresponding to the operator $(\alpha x(\partial^2/\partial x^2) + \beta x(\partial/\partial x))$ and satisfying $Z(0) = \mu(R^d)$. Standard techniques from the theory of one-dimensional diffusions show that

$$\begin{aligned} &P_\mu\Big(Z(t) > 0, \text{for all } t \geq 0, \text{and } \lim_{t\to\infty} Z(t) = \infty\Big)\\ &\qquad = 1 - \exp\left(-\frac{\beta}{\alpha}\mu(R^d)\right),\\ &P_\mu(Z(t) = 0, \text{for all large } t) = \exp\left(-\frac{\beta}{\alpha}\mu(R^d)\right). \end{aligned} \tag{1.4}$$

If $Z(t) > 0$, for all $t \geq 0$, we will say that the path $X(\cdot)$ *survives*, while, if $Z(t) = 0$, for all large t, we will say that it becomes *extinct*.

Remark. The critical measure-valued diffusion is obtained by choosing $\gamma = 0$ in the above construction. In that case, $\beta \equiv c\gamma = 0$ and by (1.4), $X(\cdot)$ dies out with probability 1.

We can now define transience, recurrence and local extinction for the support of supercritical measure-valued diffusions.

Definition. (i) The support of the process is *recurrent* if

$$P_\mu(X(t,B) > 0, \text{for some } t \geq 0 \mid X(\cdot) \text{ survives}) = 1,$$

for every $0 \not\equiv \mu \in \mathscr{M}(R^d)$ and every open set $B \subset R^d$.

(ii) (a) Let $d \geq 2$. The support of the process is *transient* if

$$P_\mu(X(t,B) > 0, \text{for some } t \geq 0 \mid X(\cdot) \text{ survives}) < 1,$$

for every $\mu \in \mathscr{M}(R^d)$ and bounded $B \subset R^d$ which satisfy $\operatorname{supp}(\mu) \cap \overline{B} = \varnothing$.

(b) Let $d = 1$. The support of the process is *transient* if for each bounded $B \subset R$,

$$P_\mu(X(t,B) > 0, \text{for some } t \geq 0 \mid X(\cdot) \text{ survives}) < 1,$$

either for every $\mu \in \mathscr{M}(R^d)$ satisfying $\sup B < \inf \operatorname{supp}(\mu)$ or for every $\mu \in \mathscr{M}(R^d)$ satisfying $\sup(\operatorname{supp}(\mu)) < \inf B$.

REMARK. By the Markov property, it follows that recurrence is equivalent to

$$P_\mu(X(t,B) > 0, \text{ for arbitrarily large } t \mid X(\cdot) \text{ survives}) = 1,$$

for every $0 \not\equiv \mu \in \mathscr{M}(R^d)$ and every open set $B \subset R^d$.

DEFINITION. The support of the process exhibits *local extinction* if for each $\mu \in \mathscr{M}(R^d)$ and each bounded $B \subset R^d$, there exists a P_μ-almost surely finite random time ζ_B such that $X(t, B) = 0$, for all $t \geq \zeta_B$.

We will prove that the support is necessarily either recurrent or transient and we will give the following criterion to distinguish between these two cases: Fix $x_0 \in R^d$ and $R > 0$ and let ϕ denote the minimal positive solution to the equation $Lu + \beta u - \alpha u^2 = 0$ in $R^d - \overline{B}_R(x_0)$ and $\lim_{|x| \to R} \phi(x) = \infty$. (The existence of ϕ is proved in Theorem 1.) Theorem 2 states that either $\inf_{x \in R^d - \overline{B}_R(x_0)} \phi(x) \geq \beta/\alpha$ or $\liminf_{|x| \to \infty} \phi(x) = 0$, and that the support of the measure-valued diffusion is recurrent in the former case and transient in the latter case.

This result is then used to obtain criteria which depend more explicitly on the operator L. In Theorem 3, it is proved that if the underlying diffusion process corresponding to the operator L is recurrent, then the support of the measure-valued process is also recurrent. In order to handle the case when the diffusion process corresponding to L is transient, we define the generalized principal eigenvalue, $\lambda_c \leq 0$, and the generalized principal eigenvalue at infinity, $\lambda_{c,\infty} \leq \lambda_c$, for the operator L. Theorem 4 treats the one-dimensional case and shows that if $\beta < -\lambda_{c,\infty}$ (resp. $\beta > -\lambda_{c,\infty}$), then the support of the measure-valued diffusion is transient (resp. recurrent). Also, if $\beta = -\lambda_{c,\infty} = -\lambda_c$, it is shown that the support of the measure-valued diffusion is transient. Theorem 5 treats the multidimensional case and shows that if $\beta < -\lambda_{c,\infty}$ or if $\beta = -\lambda_{c,\infty} = -\lambda_c$, then the support of the measure-valued diffusion is transient. An example is given to show that in the multidimensional case it is possible to obtain transience even if $\beta > -\lambda_{c,\infty}$.

It may come as a surprise that local extinction is not equivalent to transience; it is, in fact, a stronger condition. In Theorem 6, we show that local extinction occurs if and only if $\beta \leq -\lambda_c$. Thus, if $\lambda_c \neq \lambda_{c,\infty}$ and $\beta \in (-\lambda_c, -\lambda_{c,\infty})$, then the support is transient but does not exhibit local extinction. We also investigate in this paper the behavior of $E_\mu \int_0^t \langle \psi, X(s) \rangle \, ds$ and $E_\mu \langle g, X(t) \rangle$, as $t \to \infty$, where $0 \lneqq \psi, g \in C_c(R^d)$ and $0 \neq \mu \in \mathscr{M}(R^d)$ has compact support. A number of examples are presented to illustrate the general theory.

We now state the results in full.

THEOREM 1. *Fix $R > 0$ and $x_0 \in R^d$ and let $B_R(x_0) = \{x \in R^d\colon |x - x_0| < R\}$. Let $\{\psi_n\}_{n=1}^\infty$ be a nondecreasing sequence of functions satisfying $\psi_n \in$*

$C_c(R^d)$, $\psi_n(x) = n$, *for* $|x - x_0| \le R - 1/n$, $\psi_n(x) = 0$, *for* $|x - x_0| \ge R$, *and* $0 \le \psi_n \le n$. *Let* $u_n(x,t)$ *denote the solution to* (1.3) *with* $g = 0$ *and* $\psi = \psi_n$. *Then*

$$\phi_n(x) \equiv \lim_{t \to \infty} u_n(x,t) \quad \textit{exists, for } x \in R^d,$$

$\phi_n \in C^{2,\alpha}(R^d)$ *and* ϕ_n *is the minimal positive solution of*

$$Lu + \beta u - \alpha u^2 + \psi_n = 0 \quad \textit{in } R^d. \tag{1.5}$$

Furthermore,

$$\phi(x) \equiv \lim_{n \to \infty} \phi_n(x) \textit{ exists on the extended real line, for } x \in R^d,$$

and satisfies

$$\phi(x) = \infty \quad \textit{for } |x - x_0| \le R, \tag{1.6}$$

and

$$\phi(x) \le C + \lambda(|x - x_0| - R)^{-2} \quad \textit{for } |x - x_0| > R, \tag{1.7}$$

where C, λ *are positive constants. Moreover*, $\phi \in C^{2,\alpha}(R^d - \overline{B}_R(x_0))$ *and is the minimal positive solution of*

$$\begin{aligned} &Lu + \beta u - \alpha u^2 = 0 \quad \textit{in } R^d - \overline{B}_R(x_0), \\ &\lim_{|x - x_0| \to R} u(x) = \infty. \end{aligned} \tag{1.8}$$

Substituting $g \equiv 0, \psi = \psi_n$ and $u = u_n$ in (1.2), letting $t \to \infty$ and then $n \to \infty$, and using Theorem 1, we obtain the following corollary.

Corollary 1. *For each* $\mu \in \mathscr{M}(R^d)$,

$$P_\mu\big(X(t, B_R(x_0)) = 0, \textit{ for all } t \ge 0\big) = \exp\left(-\int_{R^d} \phi(x)\mu(dx)\right), \tag{1.9}$$

where ϕ *satisfies* (1.6) *and is the minimal positive solution to* (1.8).

Remark. The fact that $\phi_n(x) = \lim_{t \to \infty} u_n(x,t)$ exists and satisfies (1.5) has been proved in [5] via semigroup techniques in the case that $L = \frac{1}{2}\Delta$ and $\beta = 0$. The proof does not extend to more general operators or to the supercritical case. The proof given in this article for the general case is rather probabilistic.

The next theorem gives necessary and sufficient conditions for transience or recurrence in terms of the behavior of the solution $\phi(x)$ for large $|x|$.

Theorem 2. *Let* $R > 0$. *The minimal positive solution* ϕ *to* (1.8) *satisfies one of the following conditions*:

(i) $\inf_{x \in R^d - \overline{B}_R(x_0)} \phi(x) \ge \beta/\alpha$, *and thus*, $\liminf_{|x| \to \infty} \phi(x) \ge \beta/\alpha$.
(ii) $\inf_{x \in R^d - \overline{B}_R(x_0)} \phi(x) = \liminf_{|x| \to \infty} \phi(x) = 0$.

If (i) *holds, then the support of the supercritical measure-valued diffusion is recurrent; if* (ii) *holds, then the support of the supercritical measure-valued diffusion is transient.*

The proof that the support of the measure-valued diffusion is transient if (ii) holds above in Theorem 2 utilizes the following useful strong transitivity result which we will prove.

PROPOSITION 1. (i) *Assume that* $d \geq 2$ *and let* $B_R(x)$ *denote the ball of radius* R *centered at* $x \in R^d$. *Let* $0 \not\equiv \mu \in \mathscr{M}(R^d)$, *let* $R_0, R_1 > 0$ *and let* $x_0, x_1 \in R^d$. *If* $\bar{B}_{R_0}(x_0) \cap \bar{B}_{R_1}(x_1) = \emptyset$ *and* $\operatorname{supp}(\mu) \cap \bar{B}_{R_0}(x_0) = \emptyset$, *then for all* $t > 0$,

$$P_\mu\Big(X\big(t, B_{R_1}(x_1)\big) > 0, X\big(t, R^d - B_{R_1}(x_1)\big) = 0,$$
$$X\big(s, B_{R_0}(x_0)\big) = 0, \forall\, s \in [0, t]\Big) > 0.$$

(ii) *Assume that* $d = 1$ *and let* $I_i = (c_i, d_i), i = 0, 1$, *where* $-\infty < c_0 < d_0 < c_1 < d_1 < \infty$. *Let* $0 \not\equiv \mu \in \mathscr{M}(R)$ *satisfy* $\operatorname{supp}(\mu) \cap \bar{I}_0 = \emptyset$ *and* $\operatorname{supp}(\mu) \cap (d_0, \infty) \neq \emptyset$. *Then, for all* $t > 0$,

$$P_\mu(X(t, I_1) > 0, X(t, R - I_1) = 0, X(s, I_0) = 0, \forall\, s \in [0, t]) > 0.$$

We now use Theorem 2 to obtain more concrete criteria for transience, recurrence and local extinction. The next theorem shows that if the underlying diffusion process corresponding to L is recurrent, then the support of the supercritical measure-valued diffusion is also recurrent.

THEOREM 3. *Let* ϕ *be as in Theorem* 2. *If* L *corresponds to a recurrent diffusion process, then* $\inf \phi(x) \geq \beta/\alpha$. *Thus, by Theorem* 2, *the support of the supercritical measure-valued diffusion is recurrent.*

In order to state the rest of the results in this paper, we need several definitions and results concerning criticality theory for second order elliptic operators (see [9], Chapter 4). Let $D \subseteq R^d$ be a domain and for $\lambda \in R$ define

$$C_{L-\lambda}(D) = \big\{u \in C^2(D)\colon (L - \lambda)u = 0 \text{ and } u > 0 \text{ in } D\big\}.$$

The operator $L - \lambda$ on D is called *subcritical* if it possesses a positive Green's function; in this case $C_{L-\lambda}(D) \neq \emptyset$. If $L - \lambda$ on D does not possess a positive Green's function, but $C_{L-\lambda}(D) \neq \emptyset$, then $L - \lambda$ on D is called *critical*. If $C_{L-\lambda}(R^d) = \emptyset$, then $L - \lambda$ on R^d is called *supercritical*. There exists a number $\lambda_c(D) \in (-\infty, 0]$ such that $L - \lambda$ on D is subcritical for $\lambda > \lambda_c(D)$, supercritical for $\lambda < \lambda_c(D)$ and either subcritical or critical for $\lambda = \lambda_c(D)$. The number $\lambda_c(D)$ is called the *generalized principal eigenvalue* for L on D; it is monotone nondecreasing as a function of D. When $D = R^d$, we will write $\lambda_c = \lambda_c(R^d)$. Note that $\lambda_c(D) = \inf\{\lambda \in R\colon C_{L-\lambda}(D) \neq \emptyset\}$. We mention that if D is bounded with a smooth boundary and the coefficients of

L are smooth up to ∂D, then $\lambda_c(D)$ is the classical principal eigenvalue. Alternatively, if L is symmetric with respect to a reference density ρ, then $\lambda_c(D)$ equals the supremum of the spectrum of the self-adjoint operator on $L^2(D, \rho\, dx)$ obtained from L via the Friedrichs extension theorem.

If $d \geq 2$, let $\{D_n\}_{n=1}^{\infty}$ be an increasing sequence of bounded domains satisfying $R^d = \bigcup_{n=1}^{\infty} D_n$ and define the *generalized principal eigenvalue at* ∞ by

$$\lambda_{c,\infty} = \lim_{n \to \infty} \lambda_c\left(R^d - \overline{D}_n\right).$$

Since $\lambda_c(D)$ is monotone nondecreasing in D, it follows that $\lambda_{c,\infty}$ is independent of $\{D_n\}_{n=1}^{\infty}$. If $d = 1$, define the *generalized principal eigenvalue at* $\pm\infty$ by

$$\lambda_{c,+\infty} = \lim_{n \to \infty} \lambda_c((n, \infty)) \quad \text{and} \quad \lambda_{c,-\infty} = \lim_{n \to \infty} \lambda_c((-\infty, -n)).$$

If L is symmetric with respect to a reference density ρ, then $\lambda_{c,\infty}$ is equal to the supremum of the essential spectrum of the self-adjoint operator on $L^2(R^d, \rho\, dx)$ obtained from L via the Friedrichs extension theorem [8]. If $d = 1$, then L is always symmetric with respect to an appropriate reference density ρ and $\lambda_{c,+\infty}$ ($\lambda_{c,-\infty}$) is the supremum of the essential spectrum of the self-adjoint operator on $L^2((0,\infty), \rho\, dx)$ [$L^2((-\infty, 0), \rho\, dx)$] obtained from L via the Friedrichs extension theorem.

For use in the proofs of the theorems, we note that the above theory holds just as well if the operator L is replaced by $L + V$, where $V \in C^{\alpha}(D)$ and is bounded from above. The only difference is that now $\lambda_c(D)$ may take positive values.

With the above definitions in place, we now turn to the case in which the underlying diffusion process is transient. The next theorem treats the one-dimensional case.

THEOREM 4. *Let* $d = 1$ *and assume that* $L = \frac{1}{2}a(d^2/dx^2) + b(d/dx)$ *corresponds to a transient diffusion*; *that is*, *assume either that*

$$I^{+} \equiv \int_0^{\infty} \exp\left(-\int_0^x \frac{2b}{a}(y)\, dy\right) dx < \infty$$

or that

$$I^{-} \equiv \int_{-\infty}^{0} \exp\left(-\int_0^x \frac{2b}{a}(y)\, dy\right) dx < \infty.$$

Let ϕ *be as in Theorem* 2, *let* $\lambda_{c,\pm\infty}$ *denote the generalized principal eigenvalue at* $\pm\infty$ *for L and let* λ_c *denote the generalized principal eigenvalue for L on R. Then* $\lim_{x \to \infty} \phi(x)$ *and* $\lim_{x \to -\infty} \phi(x)$ *exists. Furthermore*, *the following results hold*:

(i) *If* $I^{+} = \infty$ *or if* $I^{+} < \infty$ *and* $\beta > -\lambda_{c,+\infty}$, *then* $\lim_{x \to +\infty} \phi(x) \geq \beta/\alpha$. *Similarly*, *if* $I^{-} = \infty$, *or if* $I^{-} < \infty$ *and* $\beta > -\lambda_{c,-\infty}$, *then* $\lim_{x \to -\infty} \phi(x) \geq \beta/\alpha$. *Thus*, *by Theorem* 2, *the support of the supercritical measure-valued diffusion* *is recurrent if both of the following conditions hold*:

(a) $I^+ = \infty$ *or* $\beta > -\lambda_{c,+\infty}$;

(b) $I^- = \infty$ *or* $\beta > -\lambda_{c,-\infty}$.

(ii) *If* $I^+ < \infty$ *and either* $\beta < -\lambda_{c,+\infty}$ *or* $\beta = -\lambda_{c,+\infty} = -\lambda_c$, *then* $\lim_{x\to+\infty}\phi(x) = 0$. *Similarly, if* $I^- < \infty$ *and either* $\beta < -\lambda_{c,-\infty}$ *or* $\beta = -\lambda_{c,-\infty} = -\lambda_c$, *then* $\lim_{x\to-\infty}\phi(x) = 0$. *Thus, by Theorem* 2, *the support of the supercritical measure-valued diffusion is transient if at least one of the following two conditions holds*:

(a) $I^+ < \infty$ *and either* $\beta < -\lambda_{c,+\infty}$ *or* $\beta = -\lambda_{c,+\infty} = -\lambda_c$;

(b) $I^- < \infty$ *and either* $\beta < -\lambda_{c,-\infty}$ *or* $\beta = -\lambda_{c,-\infty} = -\lambda_c$.

REMARK. Theorem 4 completely characterizes transience or recurrence except in the case that $I^+ < \infty$ and $\beta = -\lambda_{c,+\infty} < -\lambda_c$, or that $I^- < \infty$ and $\beta = -\lambda_{c,-\infty} < -\lambda_c$. The condition $\lambda_{c,\pm\infty} = \lambda_c$ will hold, in particular, if L has constant coefficients.

The method used in the proof of Theorem 4 carries over immediately to the radially symmetric multidimensional case and gives the following corollary.

COROLLARY 2. *Let* L *be radially symmetric on* R^d, $d \ge 2$, *and correspond to a transient diffusion. Let* ϕ *be as in Theorem* 2, *let* $\lambda_{c,\infty}$ *denote the generalized principal eigenvalue at* ∞ *for* L *and let* λ_c *denote the generalized principal eigenvalue for* L *on* R^d.

(i) *If* $\beta > -\lambda_{c,\infty}$, *then* $\lim_{|x|\to\infty}\phi(x) \ge \beta/\alpha$. *Thus, by Theorem* 2, *the support of the supercritical measure-valued diffusion is recurrent*.

(ii) *If* $\beta < -\lambda_{c,\infty}$ *or if* $\beta = -\lambda_{c,\infty} = -\lambda_c$, *then* $\lim_{|x|\to\infty}\phi(x) = 0$. *Thus, by Theorem* 2, *the support of the supercritical measure-valued diffusion is transient*.

The theorem and proposition which follow show that only one direction of Theorem 4 carries over to the multidimensional case.

THEOREM 5. *Let* L *correspond to a transient diffusion on* R^d, $d \ge 2$. *Let* ϕ *be as in Theorem* 2, *let* $\lambda_{c,\infty}$ *denote the generalized principal eigenvalue at* ∞ *for* L *and let* λ_c *denote the generalized principal eigenvalue for* L *on* R^d.

If $\beta < -\lambda_{c,\infty}$ *or if* $\beta = -\lambda_{c,\infty} = -\lambda_c$, *then* $\liminf_{|x|\to\infty}\phi(x) = 0$. *Thus, by Theorem* 2, *the support of the supercritical measure-valued diffusion is transient*.

PROPOSITION 2. *Let* $L = \frac{1}{2}\Delta + b(x)\cdot\nabla$, *where* $b_j(x) \equiv 0$, *for* $j \ne 1$, $b_1(x) = 0$, *for* $x < 0$, *and* $b_1(x) = \gamma > 0$, *for* $x > 1$. *Then* $\lambda_{c,\infty} = 0$, *but the support of the supercritical measure-valued diffusion is transient if* $\beta \in (0, \gamma^2/2)$.

The next theorem gives necessary and sufficient conditions for local extinction.

THEOREM 6. *Let λ_c denote the generalized principal eigenvalue for L on R^d. The support of the supercritical measure-valued diffusion exhibits local extinction if and only if $\beta \le -\lambda_c$.*

REMARK 1. The measure-valued diffusion may also be defined in the case that α and β are positive functions instead of constants. In this case, especially if α and β are not bounded away from zero, most of the results and proofs in this paper must be modified considerably. However, Theorem 6 and its proof go through directly after an appropriate reformulation. Let $\lambda_c^{(\beta)}$ denote the generalized principal eigenvalue for $L + \beta$ on R^d. (If β is constant, then $\lambda_c^{(\beta)} = \lambda_c + \beta$.) Then the proof of Theorem 6 shows that local extinction occurs if and only if $\lambda_c^{(\beta)} \le 0$.

REMARK 2. From Theorems 4–6, it follows that if the support exhibits local extinction, then it is transient. Here is an alternative way to see this without appealing to any of these theorems. For any bounded set B, it follows from the Markov property and the definition of local extinction that there exists a measure $0 \ne \mu \in \mathscr{M}(R^d)$ such that $P_\mu(X(t, B) = 0, \text{ for all } t \ge 0) > 0$. Thus the support cannot be recurrent. Since by Theorem 2 the support must be either recurrent or transient, we conclude that it is transient.

Theorems 4–6 show that the support can be transient and yet not exhibit local extinction. Indeed, for $d \ge 2$, this will occur if the diffusion corresponding to L is transient and $-\lambda_c < \beta < -\lambda_{c,\infty}$ and, for $d = 1$, this will occur if $I^+ < \infty$ and $-\lambda_c < \beta < -\lambda_{c,+\infty}$ or if $I^- < \infty$ and $-\lambda_c < \beta < -\lambda_{c,-\infty}$. See Example 2 at the end of this section.

In order to state the next theorem, we need the following additional facts concerning criticality theory (see [9], Chapter 4). Let $0 \lneqq \psi \in C_c(R^d)$. If $L - \lambda$ is subcritical, then there exists a minimal positive solution f_ψ to

$$(L - \lambda) f = -\psi \quad \text{in } R^d. \tag{1.10}$$

The solution is given by $f_\psi(x) = \int_{R^d} G_\lambda(x, y)\psi(y)\, dy$, where $G_\lambda(x, y)$ is the Green's function for $L - \lambda$ on R^d. If $L - \lambda$ is not subcritical, then (1.10) has no positive solution. In fact then, a fortiori, there is no positive function f satisfying $Lf \le 0$ and $Lf \not\equiv 0$. If $L - \lambda_c$ is critical, then $C_{L-\lambda_c}(R^d)$ is one dimensional and its unique element up to constant multiples will be denoted by ϕ_c and is called the *ground state*. Let $\tilde{L}$ denote the formal adjoint of L. Then λ_c is also the generalized principal eigenvalue of $\tilde{L}$ on R^d, and $\tilde{L} - \lambda_c$ is critical if and only if $L - \lambda_c$ is critical. In the critical case, the ground state of $\tilde{L} - \lambda_{c_1}$ will be denoted by $\tilde{\phi}_c$. If $\int_{R^d} \phi_c \tilde{\phi}_c\, dx < \infty$, then $L - \lambda_c$ is called *product L^1-critical*. (The definitions and results above hold even if $L1 \not\equiv 0$; however, in this article, L corresponds to a diffusion process and satisfies $L1 = 0$. If such an L corresponds to a recurrent diffusion, then $\lambda_c = 0$. If such an L satisfies $\lambda_c = 0$, then subcriticality, criticality and product L^1-criticality for $L - \lambda_c = L$ are equivalent to transience, recurrence and positive recurrence.)

THEOREM 7. *Let* $0 \lneq \psi, g \in C_c(R^d)$ *and let* $0 \neq \mu \in \mathscr{M}(R^d)$ *have compact support.*

(a) (i) *Assume that* $\lambda_c = 0$. *Then for all* $\beta > 0$,

$$E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty.$$

(ii) *Assume that* $\lambda_c < 0$. *If* $\beta \in (0, -\lambda_c)$ *or if* $\beta = -\lambda_c$ *and* $L - \lambda_c$ *is subcritical, then*

$$E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \int_{R^d} f_\psi \, d\mu,$$

where f_ψ *is the minimal positive solution to* $(L + \beta) f = -\psi$ *in* R^d.
If $\beta \in (-\lambda_c, \infty)$ *or if* $\beta = -\lambda_c$ *and* $L - \lambda_c$ *is critical, then*

$$E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty.$$

(b) *Let* $\rho \in R$.
(i) $\lim_{t \to \infty} e^{\rho t} E_\mu \langle g, X(t) \rangle = 0$, *if* $\rho < (-\lambda_c - \beta)$, *and* $\lim_{t \to \infty} e^{\rho t} E_\mu \langle g, X(t) \rangle = \infty$, *if* $\rho > (-\lambda_c - \beta)$.
(ii) *If* $L - \lambda_c$ *is subcritical or if* $L - \lambda_c$ *is critical, but not product* L^1-*critical, then*

$$\lim_{t \to \infty} \exp((-\lambda_c - \beta) t) E_\mu \langle g, X(t) \rangle = 0.$$

If $L - \lambda_c$ *is product* L^1-*critical, then*

$$\lim_{t \to \infty} \exp((-\lambda_c - \beta) t) E_\mu \langle g, X(t) \rangle = \left(\int_{R^d} \phi_c \, d\mu \right) \left(\int_{R^d} \tilde{\phi}_c g \, dx \right),$$

where ϕ_c *and* $\tilde{\phi}_c$, *normalized by* $\int_{R^d} \phi_c \tilde{\phi}_c \, dx = 1$, *are the ground states of* $L - \lambda_c$ *and* $\tilde{L} - \lambda_c$.

REMARK. Note that if $\beta = -\lambda_c$ and $L - \lambda_c$ is critical, then by Theorem 7, $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$, yet by Theorem 6, $\int_0^\infty \langle \psi, X(t) \rangle \, dt < \infty$ a.s.-P_μ.

We now give several examples to illustrate the theorems. The claims made in the examples concerning criticality and generalized principal eigenvalues are elaborated upon in the Appendix at the end of the paper. In the examples, it is assumed that $0 \lneq \psi, g \in C_c(R^d)$ and that $0 \neq \mu \in \mathscr{M}(R^d)$ has compact support.

EXAMPLE 1. Let $L = \frac{1}{2}(d^2/dx^2) + b_0(d/dx)$ on R, where $b_0 \neq 0$ is a constant. Then L corresponds to a transient diffusion, $\lambda_c = \lambda_{c,+\infty} = \lambda_{c,-\infty} = -b_0^2/2$ and $L - \lambda_c$ is critical, but not product L^1-critical. If $\beta < b_0^2/2$, then the Green's function for $L + \beta = L - (-\beta)$ is given by

$$G_{-\beta}(x, y) = \frac{2\pi}{\sqrt{b_0^2 - 2\beta}} \exp\Big(-\big(b_0^2 - 2\beta\big)^{1/2}|y - x| - b_0(x - y)\Big).$$

Therefore, by (1.10) and Theorems 4, 6 and 7, it follows that:

(i) If $\beta \in (0, b_0^2/2)$, then the support of $X(\cdot)$ is transient and exhibits local extinction. Also,

$$E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \int_{R^d} \mu(dx) \int_{R^d} \psi(y) \frac{2\pi}{\sqrt{b_0^2 - 2\beta}} \times \exp\Big(-\big(b_0^2 - 2\beta\big)^{1/2}|y - x| - b_0(x - y)\Big) \, dy.$$

(ii) If $\beta = b_0^2/2$, then the support of $X(\cdot)$ is transient and exhibits local extinction. Also, $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$ (however, by the local extinction property, $\int_0^\infty \langle \psi, X(t) \rangle \, dt < \infty$ a.s.-P_μ).

(iii) If $\beta > b_0^2/2$, then the support of $X(\cdot)$ is recurrent and we have $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$.

Furthermore,

$$\lim_{t \to \infty} e^{\rho t} E_\mu \langle g, X(t) \rangle = \begin{cases} 0, & \text{if } \rho \le b_0^2/2 - \beta, \\ \infty, & \text{if } \rho > b_0^2/2 - \beta. \end{cases}$$

EXAMPLE 2. Let $L = \frac{1}{2}\Delta + kx \cdot \nabla$ on $R^d, d \ge 1$, where $k > 0$. Then L corresponds to a transient diffusion, $\lambda_{c,\infty} = -\infty$, if $d \ge 2$, and $\lambda_{c,\pm\infty} = -\infty$, if $d = 1$, $\lambda_c = -kd$ and $L - \lambda_c$ is product L^1-critical. The ground states ϕ_c for $L - \lambda_c$ and $\tilde{\phi}_c$ for $\tilde{L} - \lambda_c$, normalized by $\int_{R^d} \phi_c \tilde{\phi}_c \, dx = 1$, are given by $\phi_c(x) = (k/\pi)^{d/2} \exp(-k|x|^2)$ and $\tilde{\phi}_c(x) = 1$. Thus, it follows from Theorems 4–7 that:

(i) If $\beta \in (0, kd)$, then the support of $X(\cdot)$ is transient and exhibits local extinction. Also,

$$E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \int_R \mu(dx) \int_R G_{-\beta}(x, y) \psi(y) \, dy,$$

where $G_{-\beta}(x, y)$ is the Green's function for $\frac{1}{2}\Delta + kx \cdot \nabla + \beta$ on R^d.

(ii) If $\beta = kd$, then the support of $X(\cdot)$ is transient and exhibits local extinction. Also $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$ (however, by the local extinction property, $\int_0^\infty \langle \psi, X(t) \rangle \, dt < \infty$ a.s.-P_μ).

(iii) If $\beta > kd$, then the support of $X(\cdot)$ is transient but does not exhibit local extinction. Also $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$.

Furthermore,

$$\lim_{t \to \infty} \exp((kd - \beta)t) E_\mu \langle g, X(t) \rangle = \left(\int_{R^d} \left(\frac{k}{\pi} \right)^{d/2} \exp(-k|x|^2) \mu(dx) \right) \left(\int_{R^d} g(x) \, dx \right).$$

EXAMPLE 3. Let $L = \frac{1}{2}\Delta - kx \cdot \nabla$ on $R^d, d \ge 1$, where $k > 0$. Then L corresponds to a recurrent diffusion $\lambda_c = 0$ and $L - \lambda_c$ is product L^1-critical. The ground states ϕ_c for $L - \lambda_c$ and $\tilde{\phi}_c$ for $\tilde{L} - \lambda_c$, normalized by $\int_{R^d} \phi_c \tilde{\phi}_c \, dx = 1$, are given by $\phi_c \equiv 1$ and $\tilde{\phi}_c(x) = (k/\pi)^{d/2} \exp(-k|x|^2)$. Thus, from Theorems 3 and 7, it follows that for all $\beta > 0$, the support of $X(\cdot)$ is recurrent and $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$. Furthermore,

$$\lim_{t \to \infty} \exp(-\beta t) E_\mu \langle g, X(t) \rangle = \mu(R^d) \int_{R^d} g(y) \left(\frac{k}{\pi} \right)^{d/2} \exp\left(-k|y|^2\right) dy.$$

EXAMPLE 4 (Supercritical super-Brownian motion). Let $L = \frac{1}{2}\Delta$ on R^d. If $d = 1$ or 2, then L corresponds to a recurrent diffusion, while if $d \ge 3$, then L corresponds to a transient diffusion and $\lambda_{c,\infty} = 0$. Also, $\lambda_c = 0$ and $L - \lambda_c$ is critical, but not product L^1-critical, if $d \le 2$, and subcritical if $d \ge 3$. Thus, by Theorem 3, Corollary 2 and Theorem 7, it follows that for all $\beta > 0$, the support of $X(\cdot)$ is recurrent and $E_\mu \int_0^\infty \langle \psi, X(t) \rangle \, dt = \infty$. Furthermore,

$$\lim_{t \to \infty} e^{\rho t} E_\mu \langle g, X(t) \rangle = \begin{cases} 0, & \text{if } \rho \le -\beta, \\ \infty, & \text{if } \rho > -\beta. \end{cases}$$

NOTE ON THE NOTATION. In the sequel, the notation P_μ and $X(t)$ will always refer to the supercritical measure-valued diffusion, while the notation P_x and $Y(t)$ will refer to the diffusion process corresponding to the operator L.

2. Proof of Theorem 1. For the proof of Theorem 1, we will need the following elliptic maximum principle for the semilinear equation.

PROPOSITION 3. *Let $D \subset R^d$ be a bounded $C^{2,\alpha}$-domain and let $v_1, v_2 \in C^{2,\alpha}(D) \cap C(\overline{D})$ satisfy $v_1, v_2 > 0$ in D, $Lv_1 + \beta v_1 - \alpha v_1^2 \le \min(0, Lv_2 + \beta v_2 - \alpha v_2^2)$ in D and $v_1 \ge v_2$ on ∂D. Then $v_1 \ge v_2$ in $\overline{D}$.*

PROOF. Let $w = v_1 - v_2$. Then w satisfies

$$(2.1) \qquad (L + \beta - \alpha v_1 - \alpha v_2) w \le 0 \quad \text{in } D, \qquad w \ge 0 \quad \text{on } \partial D.$$

Also, the function v_1 satisfies

$$(2.2) \qquad (L + \beta - \alpha v_1 - \alpha v_2) v_1 \le -\alpha v_1 v_2 \le 0.$$

Now, if v_1 were strictly positive on $\overline{D}$, then (2.2) would allow one to invoke the generalized maximum principle ([9], Theorem 3.2.2) and conclude from (2.1) that $w \ge 0$ in D. However, we only know that $v_1 \ge 0$ on $\overline{D}$. Thus, we proceed as follows. Since $v_1 v_2 \ge 0$ and $v_1 v_2 \not\equiv 0$, it follows from (2.2) that the operator $L + \beta - \alpha v_1 - \alpha v_2$ on D is subcritical ([9], Theorem 4.3.9; recall the two sentences following (1.10)). It then follows from [9], Theorem 4.3.2, that the principal eigenvalue of the operator $L + \beta - \alpha v_1 - \alpha v_2$ on D with the Dirichlet boundary condition is negative. However, then, by [9], Theorem

3.6.6, it follows that

$$E_x \exp\left(\int_0^{\tau_D} (\beta - \alpha v_1 - \alpha v_2)(Y(t))\,dt\right) < \infty, \tag{2.3}$$

where $\tau_D = \inf\{t \geq 0 : Y(t) \notin D\}$. By the Feynman–Kac formula and (2.1), we have

$$w(x) \geq E_x \exp\left(\int_0^{t \wedge \tau_D} (\beta - \alpha v_1 - \alpha v_2)(Y(s))\,ds\right) w(Y(t \wedge \tau_D)), \qquad x \in D. \tag{2.4}$$

Letting $t \to \infty$ in (2.4) and using (2.3) to invoke the dominated convergence theorem, we obtain

$$w(x) \geq E_x \exp\left(\int_0^{\tau_D} (\beta - \alpha v_1 - \alpha v_2)(Y(s))\,ds\right) w(Y(\tau_D)).$$

Thus, from (2.1) it follows that $w \geq 0$ in $\overline{D}$. □

We will also need the following parabolic maximum principle for the semilinear equation, which is easier to prove than its elliptic counterpart, since the zeroth order term need only be bounded from above in order to invoke the classical parabolic maximum principle.

PROPOSITION 4. (i) *Let $D \subset R^d$ be a bounded region and let $0 \leq v_1, v_2 \in C^{2,1}(D \times (0,\infty)) \cap C(\overline{D} \times [0,\infty))$ satisfy $Lv_1 + \beta v_1 - \alpha v_1^2 - (v_1)_t \leq Lv_2 + \beta v_2 - \alpha v_2^2 - (v_2)_t$ in $D \times (0,\infty)$, $v_1(x,0) \geq v_2(x,0)$, for $x \in D$, and $v_1(x,t) \geq v_2(x,t)$, for $x \in \partial D$ and $t > 0$. Then $v_1 \geq v_2$ in $\overline{D} \times [0,\infty)$.*

(ii) *Let $0 \leq v_1,\ v_2 \in C^{2,1}(R^d \times (0,\infty)) \cap C(R^d \times [0,\infty))$ satisfy $Lv_1 + \beta v_1 - \alpha v_1^2 - (v_1)_t \leq Lv_2 + \beta v_2 - \alpha v_2^2 - (v_2)_t$ in $R^d \times (0,\infty)$, $v_1(x,0) \geq v_2(x,0)$, for $x \in R^d$, and $\lim_{|x|\to\infty} v_2(x,t) = 0$, for $t > 0$. Then $v_1 \geq v_2$ in $R^d \times [0,\infty)$.*

PROOF. We will prove (i); the proof of (ii) is similar. Let $w = v_1 - v_2$. Then $(L + \beta - \alpha v_1 - \alpha v_2)w - w_t \leq 0$ in $D \times (0,\infty)$, $w(x,0) \geq 0$, for $x \in D$, and $w(x,t) \geq 0$, for $x \in \partial D$ and $t > 0$. Since $\beta - \alpha(v_1 + v_2)$ is bounded from above, we can invoke the classical parabolic maximum principle [11] and conclude that $w \geq 0$ in $\overline{D} \times [0,\infty)$. □

We will now prove the various claims of Theorem 1 in the following order. We will first prove the existence of a minimal positive solution ϕ to (1.8) which satisfies the bounds in (1.7). Then we will prove the existence of a minimal positive solution ϕ_n to (1.5). After that, we will prove that $\phi(x) = \lim_{n\to\infty} \phi_n(x)$, for all $x \in R^d$, where ϕ has been extended to all of R^d by (1.6). Finally, we will prove that $\phi_n(x) = \lim_{t\to\infty} u_n(x,t)$, for $x \in R^d$. For convenience, we will assume that $x_0 = 0$ and we will write $B_r \equiv B_r(0)$.

Fix $m > R$ and consider the semilinear elliptic problem

$$
\begin{aligned}
Lv + \beta v - \alpha v^2 &= 0 \quad \text{in } B_m - \overline{B}_R,\\
v &= m \quad \text{on } \partial B_R,\\
v &= 0 \quad \text{on } \partial B_m.
\end{aligned}
\tag{2.5}
$$

Recall that a function v^+ satisfying $Lv^+ + \beta v^+ - \alpha(v^+)^2 \le 0$ in $B_m - \overline{B}_R$, $v^+ \ge m$ on ∂B_R and $v^+ \ge 0$ on ∂B_m is called an *upper solution* to (2.5). Similarly, a function v^- satisfying the reverse inequalities is called a *lower solution* to (2.5). By the theory of upper and lower solutions [13], if v^+ and v^- are upper and lower solutions satisfying $v^- \le v^+$, then there exists a solution v_m to (2.5) satisfying $v^- \le v_m \le v^+$. Clearly, the function $v^- \equiv 0$ is a lower solution to (2.5). We now construct an upper solution to (2.5) which is independent of m. Let g be a smooth function on $R^d - \overline{B}_R$ which satisfies $g(x) = (R - |x|)^{-2}$, for $R < |x| < R + 1$, $0 \le g(x) \le (R - |x|)^{-2}$, for $R + 1 \le |x| < R + 2$, and $g(x) = 0$, for $|x| \ge R + 2$. One can check that if $C, \lambda > 0$ are chosen sufficiently large, then the function $v^+(x) \equiv C + \lambda g(x)$, for $x \in R^d - \overline{B}_R$, is indeed an upper solution to (2.5) for all $m > R$. Thus, there exists a solution v_m to (2.5) which satisfies

$$
0 = v^-(x) \le v_m(x) \le v^+(x) \le C + \lambda(|x| - R)^{-2}. \tag{2.6}
$$

By Proposition 3, v_m is monotone nondecreasing in m. Define

$$
\phi(x) = \lim_{m \to \infty} v_m(x) \quad \text{for } x \in R^d - B_R. \tag{2.7}
$$

By standard arguments (Sobelev embedding and Schauder estimates [3]) it follows that ϕ satisfies $L\phi + \beta\phi - \alpha\phi^2 = 0$ in $R^d - \overline{B}_R$. From the monotonicity of v_m and the fact that $v_m = m$ on ∂B_R, we conclude that $\lim_{|x| \to R} \phi(x) = \infty$. This proves that ϕ is a nonnegative solution to (1.8). The strict positivity of ϕ follows from the strong maximum principle and the minimality of ϕ follows from the construction and Proposition 3. The estimate (1.7) follows from (2.6) and (2.7).

We now turn to the proof of the existence of a minimal positive solution ϕ_n to (1.5). Fix $m > 0$ and consider the elliptic problem

$$
\begin{aligned}
Lv + \beta v - \alpha v^2 + \psi_n &= 0 \quad \text{in } B_m,\\
v &= 0 \quad \text{on } \partial B_m.
\end{aligned}
\tag{2.8}
$$

Then $v^- \equiv 0$ is a lower solution for (2.8) and, for C sufficiently large and independent of m, $v^+ \equiv C$ is an upper solution to (2.8). Thus, by the method of upper and lower solutions, there exists a solution v_m to (2.8) which satisfies $0 \le v_m \le C$. By Proposition 3, v_m is monotone nondecreasing in m. Define

$$
\phi_n(x) = \lim_{m \to \infty} v_m(x), \qquad x \in R^d.
$$

By standard arguments (Sobelev embedding and Schauder estimates), ϕ_n satisfies $L\phi_n + \beta\phi_n - \alpha\phi_n^2 + \psi_n = 0$ in R^d. This proves that ϕ_n is a nonnegative solution to (1.5). The strict positivity of ϕ_n follows from the strong maximum principle, and the minimality of ϕ_n follows from the construction and Proposition 3.

We now prove that $\phi(x) = \lim_{n\to\infty} \phi_n(x)$, for all $x \in R^d$, where ϕ has been extended by (1.6). By Proposition 3, ϕ_n is monotone nondecreasing in n. Define

$$w(x) = \lim_{n\to\infty} \phi_n(x), \qquad x \in R^d.$$

By Proposition 3, it also follows that $\phi_n \le \phi$, for all n. Thus, by standard arguments again, w satisfies $Lw + \beta w - \alpha w^2 = 0$ in $R^d - \overline{B}_R$. Clearly, $w \ge 0$. We now show that $\lim_{|x|\to R} w(x) = \infty$. Let $\varepsilon, \delta, \lambda > 0$ and define

$$z(x) = \lambda(|x| - R + 2\varepsilon)^{-2} \quad \text{for } R - \varepsilon < |x| < R + \delta.$$

One can check that there exists a $\rho > 0$ such that if $\varepsilon, \delta, \lambda \in (0, \rho)$, then

$$Lz + \beta z - \alpha z^2 \ge 0 \quad \text{for } R - \varepsilon < |x| < R + \delta. \tag{2.9}$$

Choose λ even smaller, if necessary, so that

$$z(x) \le \phi_1(x) \quad \text{for } |x| = R + \delta, \tag{2.10}$$

where ϕ_1 is the minimal positive solution to (1.5) with $n = 1$. Extend z in a smooth way to $|x| \le R - \varepsilon$. Since $\psi_n(x) = n$, for $|x| \le R - 1/n$, it is clear that for sufficiently large n,

$$Lz + \beta z - \alpha z^2 + \psi_n \ge 0 \quad \text{for } |x| < R - \varepsilon. \tag{2.11}$$

From (2.9) and (2.11), it follows that for sufficiently large n, $Lz + \beta z - \alpha z^2 + \psi_n \ge 0$, for $|x| < R + \delta$, and from (2.10), we have $z(x) \le \phi_n(x)$, for $|x| = R + \delta$ and $n \ge 1$. Therefore, by Proposition 3, $z(x) \le \phi_n(x)$, for $|x| \le R + \delta$ and n sufficiently large. Thus $z(x) \le w(x)$, for $|x| \le R + \delta$. Consequently, $\liminf_{|x|\to R} w(x) \ge \lambda/4\varepsilon^2$. Since ε may be chosen arbitrarily small, we conclude that $\lim_{|x|\to R} w(x) = \infty$. We have now proved that w is a positive solution to (1.8). By the construction of ϕ_n and w, and by Proposition 3, it follows that w is the minimal positive solution of (1.8). Thus, $w = \phi$ on $R^d - \overline{B}_R$. A proof similar to but simpler than the proof above that $\lim_{|x|\to R} w(x) = \infty$ can be made to show that $w(x) = \infty$ for $|x| < R$; this is left to the reader.

We now prove that $\phi_n(x) = \lim_{t\to\infty} u_n(x,t)$. From (1.2) and (1.3), it follows that u_n is monotone nondecreasing in t. Define

$$w_n(x) = \lim_{t\to\infty} u_n(x,t) \quad \text{for } x \in R^d. \tag{2.12}$$

Applying Proposition 4(ii) with $v_1(x,t) = \phi_n(x)$ and $v_2(x,t) = u_n(x,t)$, it follows that $u_n(x,t) \le \phi_n(x)$. Thus,

$$w_n(x) \le \phi_n(x) \quad \text{for } x \in R^d. \tag{2.13}$$

We will prove that $w_n = \phi_n$. Since $Lu_n - (u_n)_t = \alpha u_n^2 - \beta u_n - \psi_n$, it follows that for each $t > 0$, $u_n(Y(s), t-s) - \int_0^s (\alpha u_n^2 - \beta u_n - \psi_n)(Y(r), t-r)\, dr$

is a local martingale up to time t under P_x. Let $m > 0$ and define $\tau_m = \inf(t \geq 0: |Y(t)| \geq m)$. Then

$$u_n(x,t) = E_x u_n(Y(t \wedge \tau_m), t - t \wedge \tau_m) - E_x \int_0^{t \wedge \tau_m} (\alpha u_n^2 - \beta u_n - \psi_n)(Y(r), t - r)\, dr.$$

Let $t \to \infty$ above. Since $E_x \tau_m < \infty$, it follows from (2.12), (2.13) and the dominated convergence theorem that

$$w_n(x) = E_x w_n(Y(\tau_m)) + E_x \int_0^{\tau_m} (\beta w_n - \alpha w_n^2 + \psi_n)(Y(r))\, dr. \tag{2.14}$$

Let $f_n(x) = E_x w_n(Y(\tau_m))$ and let $g_n(x) = E_x \int_0^{\tau_m} (\beta w_n - \alpha w_n^2 + \psi_n)(Y(r))\, dr$, for $|x| \leq m$. Then, as is well known, f_n is the unique solution to $Lf_n = 0$ in $|x| < m$ and $f_n = w_n$ on $|x| = m$, while g_n is the unique solution to $Lg_n = -(\beta w_n - \alpha w_n^2 + \psi_n)$ in $|x| < m$ and $g_n = 0$ on $|x| = m$. Since m is arbitrary, it follows from this and (2.14) that w_n satisfies $Lw_n + \beta w_n - \alpha w_n^2 + \psi_n = 0$ in R^d and, by construction, $w_n \geq 0$ on R^d. It then follows by the strong maximum principle that $w_n > 0$ on R^d. Thus, w_n is a positive solution to (1.5). Since ϕ_n is the minimal positive solution to (1.5), it follows from (2.13) that $w_n = \phi_n$. □

3. Proofs of Proposition 1, Theorem 2 and Theorem 3.

PROOF OF PROPOSITION 1. We will prove (i); the proof of (ii) is similar. Let μ, R_0, R_1, x_0, and x_1 be as in the statement of the proposition. Choose $\delta > 0$ such that $\overline{B}_{R_0+\delta}(x_0) \cap \overline{B}_{R_1}(x_1) = \varnothing$ and $\overline{B}_{R_0+\delta}(x_0) \cap \operatorname{supp}(\mu) = \varnothing$. The proposition hinges on the following three claims:

$$P_\mu\Big(X\big(t, B_{R_1/2}(x_1)\big) > 0, X\big(s, B_{R_0+\delta}(x_0)\big) = 0, \forall s \in [0,t)\Big) > 0 \quad \text{for all } t > 0; \tag{3.1}$$

$$H_1(t,\nu) \equiv P_\nu\Big(X(t, R^d) = 0, X\big(s, B_{R_0}(x_0)\big) = 0, \forall s \in [0,t)\Big) > 0, \tag{3.2}$$

for all $t > 0$ and all $\nu \in \mathscr{M}(R^d)$ satisfying $\operatorname{supp}(\nu) \subset R^d - B_{R_0+\delta}(x_0)$;

$$H_2(t,\nu) \equiv P_\nu\Big(X\big(t, B_{R_1}(x_1)\big) > 0, X\big(s, R^d - B_{R_1}(x_1)\big) = 0, \forall s \in [0,t)\Big) > 0, \tag{3.3}$$

for all $t > 0$ and all $0 \not\equiv \nu \in \mathscr{M}(R^d)$ satisfying $\operatorname{supp}(\nu) \subset \overline{B}_{R_1/2}(x_1)$.

Recall that $X(t)$ is a multiplicative process; that is, for $\mu_1, \mu_2 \in \mathscr{M}(R^d)$, $X(\cdot)$ under $P_{\mu_1+\mu_2}$ is equal in distribution to the independent sum of $X(\cdot)$ under P_{μ_1} and $X(\cdot)$ under P_{μ_2}. This follows from (1.2). Using this fact and the

Markov property, we have for $t > 0$,

$$
\begin{aligned}
&P_\mu\Big(X\big(t, B_{R_1}(x_1)\big) > 0,\, X\big(t, R^d - B_{R_1}(x_1)\big) = 0,\\
&\qquad\qquad X\big(s, B_{R_0}(x_0)\big) = 0, \forall\, s \in [0,t)\Big)\\
&\quad \ge E_\mu\Bigg(H_1\bigg(\frac{t}{2}, X\bigg(\frac{t}{2}, \cdot \cap \big(R^d - B_{R_1/2}(x_1)\big)\bigg)\bigg)\\
&\qquad \times H_2\bigg(\frac{t}{2}, X\bigg(\frac{t}{2}, \cdot \cap B_{R_1/2}(x_1)\bigg)\bigg);\\
&\qquad X\bigg(\frac{t}{2}, B_{R_1/2}(x_1)\bigg) > 0,\, X\big(s, B_{R_0+\delta}(x_0)\big) = 0, \forall\, s \in \bigg[0, \frac{t}{2}\bigg]\Bigg).
\end{aligned} \tag{3.4}
$$

The proposition follows from (3.1)–(3.4).

We will prove (3.1) and (3.2); the proof of (3.3) is similar to that of (3.1) and is left to the reader. [The proof of (3.1) involves the function ϕ defined in Theorem 1. To prove (3.3) one must use instead the minimal positive solution $\hat{\phi}$ to $L\hat{\phi} + \beta\hat{\phi} - \alpha\hat{\phi}^2 = 0$ in $B_{R_1}(x_1), \lim_{|x-x_1| \to R_1} \hat{\phi}(x) = \infty$.]

Proof of (3.1). Let ψ_n, ϕ_n and ϕ be as in Theorem 1 with $R = R_0 + \delta$. Let $0 \le g_j \in C_c(R^d)$, $j = 1, 2$, satisfy $0 \le g_j \le \beta/\alpha$, $g_1 = g_2$ on $R^d - B_{R_1/2}(x_1)$ and $g_1 < g_2$ on $B_{R_1/2}(x_1)$. Let $u_{n,j}$ denote the solution to (1.3) with $\psi = \psi_n$ and $g = g_j$. Let $v(t) = 2(2 - e^{-\beta t})^{-1}$ and note that

$$\beta v - \beta v^2 = v_t. \tag{3.5}$$

Let $f_n(x) = \phi_n(x) + \beta/\alpha$ and note that, from (1.5), it follows that

$$L f_n + \beta f_n - \alpha f_n^2 + \psi_n \le 0. \tag{3.6}$$

Define $w_n(x,t) = v(t) f_n(x)$. Using (3.5) and (3.6) for the first inequality below and using the fact that $f_n \ge \beta/\alpha$ and $v \ge 1$ for the second one, we have

$$L w_n + \beta w_n - \alpha w_n^2 - (w_n)_t \le (v - v^2)\big(\alpha f_n^2 - \beta f_n\big) - v\psi_n \le -\psi_n. \tag{3.7}$$

Since $w_n(x,0) \ge 2(\beta/\alpha)$ and $u_{n,j}(x,0) \le \beta/\alpha$, it follows from (3.7) and Proposition 4 that $u_{n,j}(x,t) \le w_n(x,t) = v(t)(\phi_n(x) + \beta/\alpha)$. Since ϕ_n converges monotonically to ϕ, we conclude that

$$u_{n,j}(x,t) \le 2(2 - e^{-\beta t})^{-1}\bigg(\phi(x) + \frac{\beta}{\alpha}\bigg). \tag{3.8}$$

Applying (1.2) and (1.3) with $u = u_{n,j}, \psi = \psi_n$ and $g = g_j$, and letting $n \to \infty$, we obtain for $t > 0$,

$$
\begin{aligned}
&E_\mu\Big(\exp\big(-\langle g_j, X(t)\rangle\big); X\big(s, B_{R_0+\delta}(x_0)\big) = 0, \forall\, s \in [0,t]\Big)\\
&\qquad = \lim_{n\to\infty} \exp\bigg(-\int_{R^d} u_{n,j}(x,t)\, d\mu\bigg).
\end{aligned} \tag{3.9}
$$

We will now prove that

$$\liminf_{n\to\infty}\big(u_{n,2}(x,t)-u_{n,1}(x,t)\big)>0 \quad \text{for all } t>0 \text{ and } x\in R^d-\overline{B}_{R_0+\delta}(x_0). \tag{3.10}$$

Then (3.1) follows from (3.9) and (3.10), since $g_1=g_2$ on $R^d-B_{r_1/2}(x_1)$ and $g_2>g_1$ on $B_{R_1/2}(x_1)$.

To prove (3.10), let $z_n=u_{n,2}-u_{n,1}$. By Proposition 4,

$$z_n\ge 0. \tag{3.11}$$

We have

$$Lz_n-(z_n)_t=(\alpha u_{n,1}+\alpha u_{n,2}-\beta)z_n. \tag{3.12}$$

Let $\delta_1>\delta$ satisfy $\overline{B}_{R_0+\delta_1}(x_0)\cap\overline{B}_{R_1}(x_1)=\varnothing$ and $\overline{B}_{R_0+\delta_1}(x_0)\cap\operatorname{supp}(\mu)=\varnothing$. Define $\tau=\inf\{t\ge 0: Y(t)\in\overline{B}_{R_0+\delta_1}(x_0)\}$. By (3.12) and the Feynman–Kac formula ([2], Theorem 2.2) we have

$$\begin{aligned} z_n(x,t)=E_x\bigg[&\exp\bigg(\int_0^{s\wedge\tau}(\beta-\alpha u_{n,1}-\alpha u_{n,2})(Y(r),t-r)\,dr\bigg)\\ &\times z_n(Y(s\wedge\tau),t-s\wedge\tau)\bigg] \quad \text{for } 0\le s\le t. \end{aligned} \tag{3.13}$$

Substituting $s=t$ in (3.13), using (3.11) and using the fact that $z_n(x,0)=(g_2-g_1)(x)$, we obtain

$$\begin{aligned} z_n(x,t)\ge E_x\bigg(&\exp\bigg(\int_0^{t\wedge\tau}(\beta-\alpha u_{n,1}-\alpha u_{n,2})(Y(r),t-r)\,dr\bigg)\\ &\times(g_2-g_1)(Y(t));\tau>t\bigg) \quad \text{for } x\in R^d-\overline{B}_{R_0+\delta_1}(x_0) \text{ and } t>0. \end{aligned} \tag{3.14}$$

Recall that $g_2\ge g_1$ on R^d and $g_2>g_1$ on $B_{R_1/2}(x_1)$. Also, $P_x(Y(t)\in B_{R_1/2}(x_1),\ \tau>t)>0$, for all $x\in R^d-\overline{B}_{R_0+\delta_1}(x_0)$ and $t>0$. By (3.8) and (1.7), $u_{n,1}$ and $u_{n,2}$ are bounded on $R^d-B_{R_0+\delta_1}(x_0)$, independently of n. These facts along with (3.14) prove that $\liminf_{n\to\infty}z_n(x,t)>0$, for all $t>0$ and all $x\in R^d-\overline{B}_{R_0+\delta_1}(x_0)$. Since δ_1 may be chosen arbitrarily close to δ, this proves (3.10).

Proof of (3.2). Let ψ_n, ϕ_n and ϕ be as in Theorem 1 with $R=R_0$. Let $v_n(t)=(n/(n-1))(n/(n-1)-e^{-\beta t})^{-1}$ and note that $v_n(0)=n, v_n\ge 1$ and $\beta v_n-\beta v_n^2=(v_n)_t$. Let $f_n(x)=\phi_n(x)+\beta/\alpha$ and define $w_n(x,t)=v_n(t)f_n(x)$. Analogous to (3.7), we have

$$\begin{aligned} Lw_n+\beta w_n-\alpha w_n^2-(w_n)_t&\le\big(v_n-v_n^2\big)\big(\alpha f_n^2-\beta f_n\big)-v_n\psi_n\\ &\le-\psi_n. \end{aligned} \tag{3.15}$$

Let $0 \le h_n \in C_c(R^d)$ satisfy $h_n(x) = (\beta/\alpha)n$ for $|x| \le n, h_n(x) = 0$, for $|x| \ge n+1$, and $0 \le h_n \le (\beta/\alpha)n$. Let $u_n(x,t)$ denote the solution to (1.3) with $\psi = \psi_n$ and $g = h_n$. By Proposition 4, $u_n(x,t) \le w_n(x,t)$. As $n \to \infty, v_n(t)$ increases to $(1-e^{-\beta t})^{-1}$ and $f_n(x)$ increases to $\phi(x) + \beta/\alpha$. Thus,

$$u_n(x,t) \le (1-e^{-\beta t})\left(\phi(x) + \frac{\beta}{\alpha}\right). \tag{3.16}$$

Applying (1.2) and (1.3) with $u = u_n$, $\psi = \psi_n$ and $g = h_n$, and letting $n \to \infty$, we obtain for $t > 0$,

$$\begin{aligned} &P_\mu\Big(X(t,R^d) = 0, X\big(s, B_{R_0}(x_0)\big) = 0, \forall\, s \in [0,t]\Big) \\ &\quad = \lim_{n\to\infty} \exp\left(-\int_{R^d} u_n(x,t)\mu(dx)\right). \end{aligned} \tag{3.17}$$

Now (3.2) follows from (3.16) and (3.17). □

Proof of Theorem 2. By the strong maximum principle, it follows that if $\inf_{x\in R^d - \overline{B}_R(x_0)} \phi(x) = 0$, then $\liminf_{|x|\to\infty} \phi(x) = 0$. Thus, to prove the first part of the theorem, we will assume that $l \equiv \inf_{x \in R^d - \overline{B}_R} \phi(x) \in (0, \beta/\alpha)$ and come to a contradiction. Let $x_0 \in R^d$ satisfy $\phi(x_0) < l + \frac{1}{2}(\beta/\alpha - l)$ and let A denote the connected component of $\{x \in R^d - \overline{B}_R\colon \phi(x) < l + \frac{1}{2}(\beta/\alpha - l)\}$ which contains x_0. Let $\tau_A = \inf\{t \ge 0\colon\ Y(t) \in \partial A\}$. Note that $\phi(x) = l + \frac{1}{2}(\beta/\alpha - l)$, for $x \in \partial A$. Since $L\phi = \alpha\phi^2 - \beta\phi$ in A, we have

$$\phi(x_0) = E_{x_0}\phi(Y(t\wedge\tau_A)) + E_{x_0}\int_0^{t\wedge\tau_A}(\beta\phi - \alpha\phi^2)(Y(s))\,ds. \tag{3.18}$$

Since $l \in (0, \beta/\alpha)$, $\beta\phi - \alpha\phi^2$ is positive and bounded away from zero on A. Therefore, since (3.18) holds for all $t > 0$, it follows that $P_x(\tau_A < \infty) = 1$. Thus, letting $t \to \infty$ in (3.18), we obtain

$$l + \frac{1}{2}\left(\frac{\beta}{\alpha} - l\right) = E_{x_0}\phi(Y(\tau_A)) \le \phi(x_0) < l + \frac{1}{2}\left(\frac{\beta}{\alpha} - l\right),$$

which is a contradiction.

Assume now that (i) holds. We will prove that the support of the supercritical measure-valued diffusion is recurrent. Let $x_0 \in R^d$ and $R > 0$. Recall the notation $Z(t) \equiv X(t, R^d)$ and recall that

$$P_\mu\Big(\lim_{t\to\infty} Z(t) = \infty \mid X(\cdot) \text{ survives}\Big) = 1.$$

Let $T_n = \inf\{t > 0\colon Z(t) \ge n\}$. Then $P_\mu(T_n < \infty \mid X(\cdot) \text{ survives}) = 1$, for $n = 1, 2, \dots$. Using this along with (1.4), (1.9), the strong Markov property applied at time T_n and the fact that $\phi \geq \beta/\alpha$, we have for any n,

$$\begin{aligned}
&P_\mu(X(t, B_R(x_0)) = 0, \text{ for all } t \geq 0 \mid X(\cdot) \text{ survives})\\
&\quad = \frac{1}{1 - \exp((-\beta/\alpha)\mu(R^d))}\\
&\qquad \times P_\mu(X(t, B_R(x_0)) = 0, \text{ for all } t \geq 0 \text{ and } X(\cdot) \text{ survives})\\
&\quad \leq \frac{1}{1 - \exp((-\beta/\alpha)\mu(R^d))} \sup_{\substack{\mu \in \mathscr{M}(R^d)\\ \mu(R^d) \geq n}} P_\mu(X(t, B_R(x_0)) = 0, \text{ for all } t \geq 0)\\
&\quad \leq \frac{\exp(-(\beta/\alpha)n)}{1 - \exp((-\beta/\alpha)\mu(R^d))}.
\end{aligned}$$

Since n is arbitrary, this proves recurrence.

Assume now that (ii) holds. We will prove that the support of the supercritical measure-valued diffusion is transient. For simplicity of notation, we will assume that $d \geq 2$. The proof for $d = 1$ is similar. Let $x_0 \in R^d$, let $R_0 > 0$ and let $0 \not\equiv \mu \in \mathscr{M}(R^d)$ satisfy $\operatorname{supp}(\mu) \cap \overline{B}_{r_0}(x_0) = \varnothing$. Choose $x_1 \in R^d$ and $R_1 > 0$ such that $\overline{B}_{R_0}(x_0) \cap \overline{B}_{R_1}(x_1) = \varnothing$ and such that $\phi(x) \leq \frac{1}{2}(\beta/\alpha)$, for $x \in B_{R_1}(x_1)$. Let

$$\begin{aligned}
A = \Big\{X\big(1, B_{R_1}(x_1)\big) > 0, X\big(1, R^d - B_{R_1}(x_1)\big) = 0,&\\
X\big(s, B_{R_0}(x_0)\big) = 0, \forall s \in [0,1]\Big\}&.
\end{aligned}$$

By Proposition 1,

$$P_\mu(A) > 0. \tag{3.19}$$

By the Markov property, (1.9) and the fact that $\phi(x) \leq \frac{1}{2}(\beta/\alpha)$ on $B_{R_1}(x_1)$, we have

$$\begin{aligned}
&P_\mu\Big(\Big\{X\big(t, B_{R_0}(x_0)\big) = 0, \text{ for all } t \geq 0\Big\} \cap A \mid \mathscr{F}_1\Big)\\
&\qquad \geq 1_A \exp\Big(-\frac{1}{2}\frac{\beta}{\alpha}X(1, R^d)\Big),
\end{aligned} \tag{3.20}$$

where $\mathscr{F}_1 = \sigma(X(s), 0 \leq s \leq 1)$.

By the Markov property and (1.4), we have

$$P_\mu(\{X(t) \text{ survives}\} \cap A \mid \mathscr{F}_1) = 1_A\Big(1 - \exp\Big(-\frac{\beta}{\alpha}X(1, R^d)\Big)\Big). \tag{3.21}$$

From (3.19)–(3.21), it follows that

$$\begin{aligned}
&P_\mu\Big(\Big\{X\big(t, B_{R_0}(x_0)\big) = 0, \text{ for all } t \geq 0\Big\} \cap A\Big)\\
&\qquad + P_\mu(\{X(\cdot) \text{ survives}\} \cap A) > P_\mu(A).
\end{aligned}$$

Thus,

$$P_\mu\Big(X\big(t, B_{R_0}(x_0)\big) = 0, \text{ for all } t \geq 0 \mid X(\cdot) \text{ survives}\Big) > 0.$$

which proves transience. □

PROOF OF THEOREM 3. We will assume that $\inf_{x \in R^d - \bar{B}_R} \phi(x) = 0$ and come to a contradiction. The theorem will then follow from Theorem 2. Let $A = \{x \in R^d\colon\ \phi(x) < \beta/\alpha\}$ and let $\tau_A = \inf\{t \geq 0\colon\ Y(t) \notin A\}$. By assumption, A is not empty. Since $L\phi = \alpha\phi^2 - \beta\phi \leq 0$ on A, we have for $x_0 \in A$, $E_{x_0}\phi(Y(t \wedge \tau_A)) \leq \phi(x_0) < \beta/\alpha$. Since $Y(t)$ is recurrent, $P_{x_0}(\tau_A < \infty) = 1$. Thus, letting $t \to \infty$ above gives $\beta/\alpha = E_{x_0}\phi(Y(\tau_A)) < \beta/\alpha$, which is a contradiction. □

4. Proofs of Theorems 4 and 5 and Proposition 2.

PROOF OF THEOREM 4. In the one-dimensional case, we have $B_R(x_0) = (x_0 - R, x_0 + R)$ and the solution ϕ to (1.8) satisfies $L\phi + \beta\phi - \alpha\phi^2 = 0$ in $(x_0 + R, \infty)$, $\lim_{x \to x_0 + R} \phi(x) = \infty$, $L\phi + \beta\phi - \alpha\phi^2 = 0$ in $(-\infty, x_0 - R)$ and $\lim_{x \to x_0 - R} \phi(x) = \infty$. Without loss of generality, we will consider ϕ on the right half line and let $x_0 + R = 0$. Thus $\phi(x)$ is the minimal positive solution of

$$\begin{aligned} &\tfrac{1}{2}a\phi'' + b\phi' + \beta\phi - \alpha\phi^2 = 0 \quad \text{in } (0,\infty), \\ &\lim_{x \to 0^+} \phi(x) = \infty. \end{aligned} \tag{4.1}$$

By considering the supercritical measure-valued diffusion as the limit of rescaled branching diffusions, as outlined at the beginning of the article, and by applying the Ikeda–Watanabe comparison theorem ([4], Chapter 6, Theorem 1.1) to the individual diffusions in the approximating particle system, it follows that for any $t > 0$, $P_{\delta_x}(X(s, (-\infty, 0]) = 0$, for all $s \in [0, t])$ is a nondecreasing function of $x \in (0, \infty)$, where δ_x denotes the delta measure at x. Letting $t \to \infty$, it follows that $P_{\delta_x}(X(t, (-\infty, 0]) = 0$ for all $t \geq 0)$ is a nondecreasing function of $x \in (0, \infty)$. By (1.9) and the topology of R^1, we have $P_{\delta_x}(X(t, (-\infty, 0]) = 0$, for all $t \geq 0) = e^{-\phi(t)}$. This proves that $\phi(x)$ is nonincreasing for $x \in (0, \infty)$. Thus $\lim_{x \to \infty} \phi(x)$ exists. By Theorem 2, either $\lim_{x \to \infty} \phi(x) = 0$ or $\lim_{x \to \infty} \phi(x) \geq \beta/\alpha$.

If $I^+ = \infty$, then $P_{x_2}(\tau_{x_1} < \infty) = 1$, for $x_1 < x_2$, where $\tau_{x_1} = \inf\{t \geq 0\colon Y(t) = x_1\}$ ([9], Theorem 5.1.1). Using this, the proof that if $I^+ = \infty$, then $\lim_{x \to \infty} \phi(x) \geq \beta/\alpha$ is essentially identical to the proof of Theorem 3; we leave this to the reader.

For the rest of the proof, we will assume that $I^+ < \infty$ and we will utilize the comments and results introduced between Theorems 3 and 4. To prove (i), it suffices to show that if $\lim_{x \to \infty} \phi(x) = 0$, then $\beta \leq -\lambda_{c,+\infty}$. If $\lim_{x \to \infty} \phi(x) = 0$, then for each $\varepsilon > 0$, there exists an n_ε such that

$$(L + \beta - \varepsilon)\phi \leq 0 \quad \text{and} \quad (L + \beta - \varepsilon)\phi \not\equiv 0 \quad \text{for } x > n_\varepsilon.$$

It then follows from the discussion between Theorems 3 and 4 that $L - (-\beta + \varepsilon)$ is subcritical on (n_ε, ∞). Thus, the generalized principal eigenvalue, $\lambda_c((n_\varepsilon, \infty))$, of L on (n_ε, ∞) satisfies

$$\lambda_c((n_\varepsilon, \infty)) \le -\beta + \varepsilon. \tag{4.2}$$

Recalling the definition of $\lambda_{c,+\infty}$, it follows from (4.2) that $\lambda_{c,+\infty} \le -\beta + \varepsilon$; since $\varepsilon > 0$ is arbitrary, we conclude that $\beta \le -\lambda_{c,\infty}$.

We now turn to the proof of (ii). First assume that $\beta < -\lambda_{c,+\infty}$. Choose $n_0 > 0$ such that

$$\beta \le -\lambda_c((n_0, \infty)). \tag{4.3}$$

Then $C_{L+\beta}((n_0, \infty)) \ne \phi$. Let $u > 0$ satisfy

$$Lu + \beta u = 0 \quad \text{in } (n_0, \infty). \tag{4.4}$$

Let $\tau_m = \inf\{t \ge 0\colon Y(t) = m\}$. Since $Lu = -\beta u$ in (n_0, ∞), we have

$$0 \le E_x u\big(Y(t \wedge \tau_{n_0+1})\big) = u(x) - \beta E_x \int_0^{t \wedge \tau_{n_0+1}} u(Y(s))\, ds \qquad \text{for } x > n_0 + 1. \tag{4.5}$$

Letting $t \to \infty$ in (4.5), it follows that

$$E_x\left(\int_0^\infty u(Y(s))\, ds, \tau_{n_0+1} = \infty\right) < \infty \quad \text{for } x > n_0 + 1. \tag{4.6}$$

Since $I^+ < \infty$, it follows from [9], Theorem 5.1.1, that

$$P_x\Big(\tau_{n_0+1} = \infty \text{ and } \lim_{t\to\infty} Y(t) = \infty\Big) > 0 \quad \text{for } x > n_0 + 1. \tag{4.7}$$

From (4.6) and (4.7), we conclude that

$$\liminf_{x\to\infty} u(x) = 0. \tag{4.8}$$

Since ϕ is a minimal positive solution, it follows that $\phi(x) = \lim_{n\to\infty} w_n(x)$, for $x \ge n_0 + 1$, where w_n solves $Lw_n + \beta w_n - \alpha w_n^2 = 0$ in $(n_0 + 1, n_0 + n + 1)$, $w_n(n_0 + 1) = \phi(n_0 + 1)$ and $w_n(n_0 + n + 1) = 0$. The proof of this is similar to the existence proof for ϕ appearing in the proof of Theorem 1. Normalize the function u above by $u(n_0 + 1) = \phi(n_0 + 1)$. Note that

$$Lu + \beta u - \alpha u^2 = -\alpha u^2 < 0 \quad \text{in } (n_0 + 1, \infty). \tag{4.9}$$

From (4.9) and Proposition 3, it follows that $u(x) \ge w_n(x)$, for $x \in [n_0 + 1, n_0 + n + 1]$. Thus $u(x) \ge \phi(x)$, for $x \ge n_0 + 1$. Since $\lim_{x\to\infty} \phi(x)$ exists, it follows from (4.8) that $\lim_{x\to\infty} \phi(x) = 0$.

We now assume that $\beta = -\lambda_{c,+\infty} = -\lambda_c$. Then $\beta = -\lambda_c((n_0, \infty))$, for any $n_0 > 0$. Thus, (4.3) holds and the proof that $\lim_{x\to\infty} \phi(x) = 0$ follows from the proof in the case $\beta < -\lambda_{c,+\infty}$. □

Proof of Theorem 5. If $\beta < -\lambda_{c,\infty}$ or if $\beta = -\lambda_{c,\infty} = -\lambda_c$, then for sufficiently large n_0, $\beta \le -\lambda_c((R^d - \overline{B}_{n_0}))$, where $B_{n_0} = \{|x| < n_0\}$. Thus

$C_{L+\beta}(R^d - \overline{B}_{n_0}) \neq 0$. Let $u > 0$ satisfy

$$Lu + \beta u = 0 \quad \text{in } R^d - \overline{B}_{n_0}. \tag{4.10}$$

Note that (4.10) is analogous to (4.4). The proof is now like the proof of Theorem 4(ii) starting from (4.4), except that, in this case, we do not know that $\lim_{|x|\to\infty} \phi(x)$ exists. Thus we can only conclude that $\liminf_{|x|\to\infty} \phi(x) = 0$. □

PROOF OF PROPOSITION 2. Let $\lambda_c(D)$ denote the generalized principal eigenvalue of L on D. Let $H_n = \{x \in R^d : x_1 < -n\}$, for $n > 0$. Since $L = \frac{1}{2}\Delta$ on H_n, $\lambda_c(H_n)$ coincides with the generalized principal eigenvalue of $\frac{1}{2}\Delta$ on H_n. This latter quantity is zero because a positive Green's function does not exist for $\frac{1}{2}\Delta + \varepsilon$ on H_n, if $\varepsilon > 0$. Thus, $\lambda_c(H_n) = 0$, for all $n > 0$. It then follows from the definition of $\lambda_{c,\infty}$ that $\lambda_{c,\infty} \geq 0$. On the other hand, since $1 \in C_L(D)$, for any domain D, we have $\lambda_c(D) \leq 0$, for any domain D, and thus $\lambda_{c,\infty} \leq 0$. We conclude that $\lambda_{c,\infty} = 0$.

The individual components of the diffusion corresponding to L are independent. From this and the derivation of the supercritical measure-valued diffusion as the weak limit of rescaled branching L-diffusions, it follows that the d marginal measures obtained from $X(t)$ constitute independent measure-valued diffusions. In particular, the first component corresponds to the one-dimensional supercritical measure-valued diffusion associated with the operator $\frac{1}{2}(d^2/dx^2) + b_1(x)(d/dx)$ and the constants α and β. Since $b_1(x) = \gamma$ for $x > 1$, it follows from Example 1 that the support of this one-dimensional supercritical measure-valued diffusion is transient if $\beta \in (0, \gamma^2/2)$. Thus, clearly, the support of the original supercritical measure-valued diffusion is also transient. □

5. Proof of Theorem 6. We first prove that if $\beta \leq -\lambda_c$, then the support of the measure-valued diffusion exhibits local extinction. Let B_R denote the ball of radius R centered at the origin. The argument given in [6], page 207, shows that

$$P_\mu\left(\int_t^\infty X(s, B_R)\, ds = 0\right) = \lim_{n\to\infty} \exp\left(-\int_{R^d} v_n(x,t)\,\mu(dx)\right), \tag{5.1}$$

where v_n is the unique positive solution in $C_0(R^d)$ to the evolution equation

$$\begin{aligned} u_t &= Lu + \beta u - \alpha u^2, \qquad R^d \times [0,\infty), \\ u(x,0) &= \phi_n(x), \qquad x \in R^d, \end{aligned} \tag{5.2}$$

and ϕ_n is the minimal positive solution to (1.5) with $x_0 = 0$. Thus, to demonstrate local extinction, we must show that $\lim_{t\to\infty} \lim_{n\to\infty} v_n(x,t) = 0$.

For $\delta > 0$, the function $k_\delta(x,t) \equiv (\beta/\alpha)(1+\delta - e^{-\beta t})^{-1}$ satisfies $(k_\delta)_t = Lk_\delta + \beta k_\delta - \alpha k_\delta^2$ and $k_\delta(x,0) = \beta/\alpha\delta$. Thus, it follows by Proposition 4 that

$$(5.3) \qquad v_n(x,t) \le \frac{\beta}{\alpha}(1-e^{-\beta t})^{-1} \quad \text{for } x \in R^d, t > 0 \text{ and all } n.$$

By Proposition 4, v_n is monotone nondecreasing in n. Let $v(x,t) \equiv \lim_{n\to\infty} v_n(x,t)$. By (5.3), $v(x,t) < \infty$, for $x \in R^d$ and $t > 0$.

We will now show that v satisfies (5.2) with ϕ_n replaced by ϕ, where $\phi = \lim_{n\to\infty}\phi_n$ is as in (1.6) and (1.8) with $x_0 = 0$. Since $L\phi_n + \beta\phi_n - \alpha\phi_n^2 = -\psi_n \le 0$, it follows that $(v_n)_t(x,0) \le 0$. Using this along with Proposition 4 and the uniqueness of the solution to (5.1), it follows directly that $v_n(x,t)$ is nonincreasing in t for all $t \ge 0$. From this, it follows that $\lim_{t\to 0} v(x,t) = \phi(x)$. Let $\varepsilon \in (0,t)$. Equating expectations at time $s = 0$ and at time $s = t - \varepsilon$ for the local martingale $v_n(Y(s), t-s) - \int_0^s(\alpha v_n^2 - \beta v_n)(Y(r), t-r)\,dr$, $s \in [0,t]$, and letting $n \to \infty$, gives

$$v(x,t) = E_x v(Y(t-\varepsilon),\varepsilon) - E_x\int_0^{t-\varepsilon}(\alpha v^2 - \beta v)(Y(r), t-r)\,dr.$$

This shows that v satisfies $v_t = Lv + \beta v - \alpha v^2$. We conclude that v satisfies (5.2) with ϕ_n replaced by ϕ.

By (5.1), $v(x,t)$ is monotone nonincreasing in t. Let $w(x) \equiv \lim_{t\to\infty} v(x,t)$. Since $v(x,0) = \phi$, we have

$$(5.4) \qquad w \le \phi.$$

The same proof as the one given to show that $\phi_n(x) = \lim_{t\to\infty} u_n(x,t)$ in Theorem 1, shows that w satisfies

$$Lw + \beta w - \alpha w^2 = 0 \quad \text{in } R^d.$$

By the strong maximum principle, either $w > 0$ or $w \equiv 0$. We will show that $w \equiv 0$.

To prove that $w \equiv 0$, we utilize several results concerning subcritical operators. The reader should recall the basic facts noted following the statement of Theorem 3. Fix $n_0 > R + 1$. If $\beta < -\lambda_c$ or if $\beta = -\lambda_c$, but $\lambda_c(R^d - \overline{B}_{n_0}) < \lambda_c$, then clearly, $L + \beta$ on $R^d - \overline{B}_{n_0}$ is subcritical. If $\beta = -\lambda_c = -\lambda_c(R^d - \overline{B}_{n_0})$, then $L + \beta$ on $R^d - \overline{B}_{n_0}$ is subcritical by Theorem 4.4.1(iii) or Corollary 4.4.2 in [9]. Thus, the assumption that $\beta \le -\lambda_c$ guarantees that $L + \beta$ is subcritical on $R^d - \overline{B}_{n_0}$. From the subcriticality of $L + \beta$ on $R^d - \overline{B}_{n_0}$, it follows that

$$(5.5) \qquad E_x\big(\exp(\beta\sigma_{n_0});\, \sigma_{n_0} < \infty\big) < \infty \quad \text{for all } x \in R^d - B_{n_0},$$

where $\sigma_{n_0} = \inf\{t \ge 0\colon |Y(t)| \le n_0\}$ (see [9], Lemma 7.3.4(i)). Since $L + \beta$ is also subcritical on the smaller domain $B_m - \overline{B}_{n_0}$, for $m > n_0$, it follows from [9], Theorems 3.6.6 and 4.3.2, that

$$(5.6) \qquad E_x \exp\big(\beta(\sigma_{n_0} \wedge \tau_m)\big) < \infty \quad \text{for } x \in B_m - B_{n_0},\ m > n_0.$$

where $\tau_m = \inf\{t \ge 0\colon |Y(t)| \ge m\}$.

Recall from the proof of Theorem 1 that $\phi_n(x) = \lim_{m\to\infty} v_m(x)$, where v_m satisfies (2.8). Applying the Feynman–Kac formula to the operator $L + \beta - \alpha v_m$ and using (5.6) and the fact that ψ_n in (2.8) is supported in $\overline{B}_R$, it follows that

$$v_m(x) = E_x\left(\exp\left(\int_0^{\sigma_{n_0}} (\beta - \alpha v_m)(Y(t))\,dt\right) v_m\big(Y(\sigma_{n_0})\big);\ \sigma_{n_0} < \tau_m\right),\qquad x \in B_m - B_{n_0}.$$

Letting $m \to \infty$ and using (5.5) we obtain from the dominated convergence theorem that

$$\phi_n(x) = E_x\left(\exp\left(\int_0^{\sigma_{n_0}} (\beta - \alpha \phi_n)(Y(t))\,dt\right) \phi_n\big(Y(\sigma_{n_0})\big);\ \sigma_{n_0} < \infty\right)\qquad \text{for } x \in R^d - B_{n_0}.$$

Letting $n \to \infty$ and again appealing to (5.5) and the dominated convergence theorem gives

$$\phi(x) = E_x\left(\exp\left(\int_0^{\sigma_{n_0}} (\beta - \alpha \phi)(Y(t))\,dt\right) \phi\big(Y(\sigma_{n_0})\big);\ \sigma_{n_0} < \infty\right)\qquad \text{for } x \in R^d - B_{n_0}. \tag{5.7}$$

The same argument that showed that $L + \beta$ on $R^d - \overline{B}_{n_0}$ is subcritical, of course, also shows that $L + \beta$ is subcritical on $R^d - \overline{B}_{n_0-1}$. Let $G_{\beta-\alpha\phi}(x, y)$ denote the Green's function for $L + \beta - \alpha\phi$ on $R^d - \overline{B}_{n_0-1}$. The representation for ϕ given in (5.7) shows that ϕ is a so-called positive solution of minimal growth at infinity for the operator $L + \beta - \alpha\phi$ (see [9], Theorems 7.3.5 and 7.3.6). For each fixed y, the Green's function $G_{\beta-\alpha\phi}(\cdot, y)$ is also a positive solution of minimal growth at infinity for the operator $L + \beta - \alpha\phi$ ([9], Theorem 7.3.7) and thus, by the maximum principle for such solutions ([9], Theorem 7.3.6), it follows that for any fixed $y_0 \in B_{n_0} - \overline{B}_{n_0-1}$, there exists a constant $c > 0$ such that

$$\phi(x) \le cG_{\beta-\alpha\phi}(x, y_0)\quad \text{for } x \in R^d - B_{n_0}. \tag{5.8}$$

We now assume that $w > 0$ and come to a contradiction. Since $\beta \le -\lambda_c$, the operator $L + \beta$ on R^d is either critical or subcritical. It then follows from Theorem 4.6.3 in [9] that $L + \beta - \alpha w$ on R^d is subcritical. Let $G_{\beta-\alpha w}(x, y)$ denote its Green's function. Since $\beta - \alpha\phi \le \beta - \alpha w$ and $R^d - \overline{B}_{n_0-1} \subset R^d$, it follows that

$$G_{\beta-\alpha\phi}(x, y) \le G_{\beta-\alpha w}(x, y)\quad \text{for } x, y \in R^d - \overline{B}_{n_0-1} \tag{5.9}$$

(see [9], pages 129–130). From (5.4), (5.8) and (5.9), we conclude that

$$w(x) \le cG_{\beta-\alpha w}(x, y_0)\quad \text{for } x \in R^d - B_{n_0}. \tag{5.10}$$

Note that $w \in C_{L+\beta-\alpha w}(R^d)$, the cone of positive harmonic functions on R^d for the operator $L + \beta - \alpha w$. By [9], Theorem 7.3.9, a function $u \in$

$C_{L+\beta-\alpha w}(R^d)$ cannot satisfy $u(x) \le cG_{\beta-\alpha w}(x, y_0)$, for all x in a neighborhood of infinity. Thus, (5.10) constitutes a contradiction and we conclude that $w \equiv 0$.

We now assume that $\beta > -\lambda_c$. Choose a bounded domain $D \subset R^d$ with a smooth boundary such that $\beta > -\lambda_c(D)$. This is possible because if $\{D_n\}_{n=1}^{\infty}$ is an increasing sequence of domains such that $\cup_{n=1}^{\infty} D_n = R^d$, then $\lambda_c \equiv \lambda_c(R^d) = \lim_{n\to\infty} \lambda(D_n)$ ([9], Theorem 4.4.1(i)). Let $\hat{Y}(t)$ denote the diffusion process on D which corresponds to the operator L and which is killed upon exiting D. Let $\hat{X}(t)$ denote the measure-valued diffusion with values in $\mathscr{M}(D)$, constructed from the diffusion process $\hat{Y}(t)$ and with the same branching parameters α and β as for the original measure-valued process $X(t)$. Denote the measures corresponding to $\hat{X}(t)$ by $\hat{P}_\mu$, $\mu \in \mathscr{M}(D)$. Note, of course, that a path $\hat{X}(\cdot)$ becomes extinct if and only if $\hat{X}(t, D) = 0$, for some $t \ge 0$. Let $\hat{T}$ denote the extinction time for $\hat{X}(t)$. By considering P_μ and $\hat{P}_\mu$ as the weak limits of rescaled branching diffusions, and running those branching diffusions with processes $Y(t)$ and $\hat{Y}(t)$ which coincide until reaching $R^d - \overline{D}$, it is clear that, for $0 \le t_1 < t_2 < \infty$ and $\mu \in \mathscr{M}(D)$,

$$P_\mu(X(t, D) = 0, \text{for all } t \in [t_1, t_2]) \le \hat{P}_\mu\big(\hat{X}(t_1, D) = 0\big) = \hat{P}_\mu\big(\hat{T} \le t_1\big).$$

Thus,

$$\lim_{t_1\to\infty} \lim_{t_2\to\infty} P_\mu(X(t, D) = 0, t \in [t_1, t_2]) \le \hat{P}_\mu(\hat{T} < \infty). \tag{5.11}$$

If the left-hand side of (5.11) is not equal to 1, then local extinction does not occur. Thus, to complete the proof, it suffices to show that $\hat{P}_\mu(\hat{T} < \infty) < 1$.

Since D is bounded with a smooth boundary, it follows that $\lambda_c(D)$ is the classical principal eigenvalue for the operator L on D with the Dirichlet boundary condition ([9], Theorem 4.3.2). Let $\phi_0 > 0$ denote the corresponding principal eigenfunction. Let

$$M = \sup_{x\in D} \phi_0(x) \quad \text{and} \quad \varepsilon_0 = \frac{\lambda_c + \beta}{\alpha \mathscr{M}} > 0.$$

Then the function $v^-(x) \equiv \varepsilon_0 \phi_0(x)$ satisfies

$$Lv^- + \beta v^- - \alpha(v^-)^2 \ge 0 \quad \text{in } D.$$

Now define $v^+(x) = N\phi_0^{1/2}(x)$, where $N > 0$ will be fixed later. We have

$$\begin{aligned} Lv^+ + \beta v^+ - \alpha(v^+)^2 &= (L - \lambda_c(D))v^+ + (\beta + \lambda_c(D))v^+ - \alpha(v^+)^2 \\ &= N\phi_0^{1/2}(L - \lambda_c(D))\phi_0 - \tfrac{1}{4}N\phi_0^{-3/2}(\nabla\phi_0 a \nabla\phi_0) \\ &\quad + N(\beta + \lambda_c(D))\phi_0^{1/2} - \alpha N^2\phi_0 \\ &= N\big[-\tfrac{1}{4}\phi_0^{-3/2}(\nabla\phi_0 a \nabla\phi_0) + (\beta + \lambda_c(D))\phi_0^{1/2} - \alpha N\phi_0\big]. \end{aligned} \tag{5.12}$$

By the Hopf maximum principle, $\nabla\phi_0(x) \ne 0$, for $x \in \partial D$. Thus $\lim_{x\to\partial D}(\nabla\phi_0 a \nabla\phi_0/\phi_0^{3/2})(x) = \infty$. From this it follows that the right-hand side of (5.12) is negative for all $x \in D$, if N is chosen sufficiently large.

Choose N sufficiently large so that the right-hand side of (5.12) is negative and so that $v^+(x) \geq v^-(x)$, for $x \in D$.

By the method of upper and lower solutions, it now follows that there exists a function v which satisfies $v^- \leq v \leq v^+$ and solves $Lv + \beta v - \alpha v^2 = 0$ in D. Since $v \geq v^-$, it follows that $v \not\equiv 0$, and since $v \leq v^+$, it follows that $v = 0$ on ∂D. Thus, by the theory of evolution equations [7], $v(x)$ is the unique positive solution to

$$\begin{aligned} u_t &= Lu + \beta u - \alpha u^2 && \text{in } D \times (0, \infty), \\ u(x,t) &= 0, && x \in \partial D, t > 0, \\ u(x,0) &= v(x), && x \in D. \end{aligned}$$

By the log-Laplace equation for $\hat{X}(t)$ analogous to (1.2) and (1.3) for $X(t)$, we then have for $0 \not\equiv \mu \in \mathscr{M}(D)$,

$$\hat{E}_\mu \exp\left(-\langle v, \hat{X}(t)\rangle\right) = \exp\left(-\int_{R^d} v(x)\mu(dx)\right). \tag{5.13}$$

Now, if $\hat{P}_\mu(\hat{T} < \infty) = 1$, then the left-hand side of (5.13) converges to 1 as $t \to \infty$. This is impossible since the right-hand side of (5.13) is independent of t and is smaller than 1. Thus, we conclude that $\hat{P}_\mu(\hat{T} < \infty) < 1$, for $0 \not\equiv \mu \in \mathscr{M}(D)$. □

6. Proof of Theorem 7. Throughout the proof, it is assumed that ψ and g satisfy $0 \lneqq \psi, g \in C_c(R^d)$, as in the statement of the theorem.

Let $u_\lambda(x,t)$ denote the unique positive solution in $C_0(R^d)$ to

$$\begin{aligned} u_t &= Lu + \beta u - \alpha u^2 + \lambda\psi, \qquad (x,t) \in R^d \times (0,\infty), \\ u(x,0) &= 0, \qquad x \in R^d. \end{aligned}$$

The proof of existence for u_λ in [7], Chapter 6, Theorems 1.4 and 1.5, shows that u_λ may be obtained by the method of successive iterations. This method shows easily that $u_\lambda(x,t)$ is continuous in λ, for each $x \in R^d$ and $t \in [0,\infty)$. Define $\hat{u}_{\lambda,h} = (1/h)(u_{\lambda+h} - u_\lambda)$, for $h > 0$. Then $\hat{u}_{\lambda,h}$ satisfies the linear equation

$$\begin{aligned} u_t &= Lu + (\beta - \alpha u_\lambda - \alpha u_{\lambda+h})u + \psi = 0 \quad \text{in } R^d \times (0,\infty), \\ u(x,0) &= 0, \qquad x \in R^d, \\ u(\cdot,t) &\in C_0(R^d). \end{aligned}$$

Let $\hat{u}_\lambda$ denote the solution to the linear equation

$$\begin{aligned} u_t &= Lu + (\beta - 2\alpha u_\lambda)u + \psi = 0 \quad \text{in } R^d \times (0,\infty), \\ u(x,0) &= 0, \qquad x \in R^d, \\ u(\cdot,t) &\in C_0(R^d). \end{aligned}$$

By the Feynman–Kac formula,

$$\hat{u}_{\lambda,h}(x,t) = E_x\int_0^t \psi(Y(s))\exp\left(\int_0^t (\beta - \alpha u_\lambda - \alpha u_{\lambda+h})(Y(r), t-r)\,dr\right) ds \tag{6.1}$$

and

$$\hat{u}_\lambda(x,t) = E_x\int_0^t \psi(Y(s))\exp\left(\int_0^t (\beta - 2\alpha u_\lambda)(Y(r), t-r)\,dr\right) ds. \tag{6.2}$$

Since $u_\lambda(x,t)$ is continuous in λ, it follows from (6.1) and (6.2) that

$$\lim_{h\to 0} \hat{u}_{\lambda,h}(x,t) = \hat{u}_\lambda(x,t). \tag{6.3}$$

Applying (1.2) and (1.3) with u, ψ and g replaced by u_λ, $\lambda\psi$ and 0, then differentiating (1.2) in λ and using (6.3) and then setting $\lambda = 0$ and noting that $u_0 \equiv 0$, we obtain

$$E_\mu\int_0^t \langle \psi, X(s)\rangle\, ds = \int_{R^d} \hat{u}_0(x,t)\,\mu(dx), \tag{6.4}$$

where $\hat{u}_0$ satisfies

$$\begin{aligned} u_t &= Lu + \beta u + \psi \quad \text{in } R^d \times (0,\infty), \\ u(x,0) &= 0. \end{aligned} \tag{6.5}$$

The solution to (6.5) is given by

$$\hat{u}_0(x,t) = \int_{R^d} dy \int_0^t p_{-\beta}(s,x,y)\psi(y)\,ds, \tag{6.6}$$

where $p_{-\beta}(t,x,y)$ is the transition density for $L + \beta$; that is, $p_{-\beta}(t,x,y)$ is the density of the measure $p_{-\beta}(t,x,\cdot) = e^{\beta t}P_x(Y(t) \in \cdot)$.

The facts used in the sequel can be found in [9], Chapter 4. If $L + \beta$ on R^d is subcritical, then $\int_0^\infty p_{-\beta}(t,x,y)\,dt = G_{-\beta}(x,y)$, where $G_{-\beta}(x,y)$ is the Green's function for $L + \beta$ on R^d. In this case then,

$$f(x) \equiv \lim_{t\to\infty} \hat{u}_0(x,t) = \int G_{-\beta}(x,y)\psi(y)\,dy.$$

Thus f is the minimal positive solution to $(L+\beta)f = -\psi$ in R^d. On the other hand, if $L + \beta$ on R^d is not subcritical, then $\lim_{t\to\infty}\int_0^t p_{-\beta}(s,x,y)\,ds = \infty$, for all $x, y \in R^d$. In this case then, $\lim_{t\to\infty} \hat{u}_0(x,t) = \infty$. Part (a) now follows using these facts and (6.4) and recalling that $L + \beta$ is subcritical if $\beta < \lambda_c$ and supercritical if $\beta > -\lambda_c$.

For part (b), let v_λ denote the solution to

$$\begin{aligned} v_t &= Lv + \beta v - \alpha v^2 = 0 \quad \text{in } R^d \times (0,\infty), \\ v(x,0) &= \lambda g(x), \qquad x \in R^d, \\ v(\cdot,t) &\in C_0(R^d). \end{aligned}$$

Define $\hat{v}_{\lambda,h} = (1/h)(v_{\lambda+h} - v_\lambda)$, for $h > 0$. Then $\hat{v}_{\lambda,h}$ satisfies the linear equation

$$v_t = Lv + (\beta - \alpha v_\lambda - \alpha v_{\lambda+h})v = 0 \quad \text{in } R^d \times (0,\infty)$$
$$v(x,0) = g, \qquad x \in R^d.$$

Let $\hat{v}_\lambda$ denote the solution to the linear equation

$$v_t = Lv + (\beta - 2\alpha v_\lambda)v = 0 \quad \text{in } R^d \times (0,\infty),$$
$$v(x,0) = g, \qquad x \in R^d.$$

A proof similar to the proof of (6.3) shows that

$$\hat{v}_\lambda(x,t) = \lim_{h\to 0} \hat{v}_{\lambda,h}(x,t). \tag{6.7}$$

Applying (1.2) and (1.3) with u, ψ and g replaced by v_λ, 0 and λg, then differentiating (1.2) in λ and using (6.7) and then setting $\lambda = 0$ and noting that $v_0 \equiv 0$, we obtain

$$E_\mu\langle g, X(t)\rangle = \int_{R^d} \hat{v}_0(x,t)\mu(dx), \tag{6.8}$$

where $\hat{v}_0$ satisfies

$$v_t = Lv + \beta v \quad \text{in } R^d \times (0,\infty), \qquad v(x,0) = g(x), \qquad x \in R^d. \tag{6.9}$$

The solution to (6.9) is given by

$$\hat{v}_0(x,t) = \int_{R^d} p_{-\beta}(t,x,y)g(y)\,dy. \tag{6.10}$$

Note that the transition density $p_{\lambda_c}(t,x,y)$ for $L - \lambda_c$ satisfies

$$p_{\lambda_c}(t,x,y) = \exp((-\lambda_c - \beta)t)p_{-\beta}(t,x,y). \tag{6.11}$$

Part (b) now follows from (6.8), (6.10), (6.11) and the following result ([9], Theorem 4.9.9):

(i) If $L - \lambda_c$ on R^d is subcritical or if it is critical but not product L^1-critical, then $\lim_{t\to\infty} \int p_{\lambda_c}(t,x,y)g(y)\,dy = 0$.

(ii) If $L - \lambda_c$ on R^d is product L^1-critical, then

$$\lim_{t\to\infty} \int_{R^d} p_{\lambda_c}(t,x,y)g(y)\,dy = \phi_c(x)\int_{R^d} \tilde{\phi}_c(y)g(y)\,dy,$$

where ϕ_c and $\tilde{\phi}_c$ are the ground states for $L - \lambda_c$ and $\tilde{L} - \lambda_c$, normalized by $\int_{R^d} \phi_c \phi_c\,dx = 1$. □

APPENDIX

We elaborate concerning the claims made in the four examples appearing at the end of Section 1. Recall that if $h > 0$, then the h-transform of an operator L is defined by $L^h f = (1/h)L(fh)$. Equivalently,

$$L^h = L + a(\nabla h/h)\cdot\nabla + L^h/h.$$

Subcriticality, criticality, λ_c, λ_c^∞ and product L^1-criticality are all invariant under h-transforms. Also, in the subcritical case, the Green's functions $G(x, y)$ and $G^h(x, y)$ for L and L^h are related by $G^h(x, y) = (1/(h(x)))G(x, y)h(y)$ (see [9], Chapter 4).

EXAMPLE 1. Let $L = \frac{1}{2}(d^2/dx^2) + b_0(d/dx)$, $b_0 \neq 0$. Let $h(x) = \exp(-b_0 x)$. Then $L^h = \frac{1}{2}(d^2/dx^2) - b_0^2/2$. The operator $\frac{1}{2}(d^2/dx^2)$ on R is critical since it corresponds to a recurrent diffusion. Thus, by h-transform invariance, it follows that $\lambda_c = -b_0^2/2$ for L and that $L - \lambda_c$ is critical. One can check that the ground state for $L - \lambda_c$ is given by $\phi_c(x) = h(x)$ and that the ground state for $\tilde{L} - \lambda_c$ is given by $\tilde{\phi}_c(x) = \exp(b_0 x)$. Thus $L - \lambda_c$ is not product L^1-critical.

The Green's function for $\frac{1}{2}(d^2/dx^2) + \varepsilon$ on (m, ∞) exists if and only if $\varepsilon \leq 0$. Thus, by h-transform invariance, $\lambda_{c,+\infty} = -b_0^2/2$ and, by symmetry, $\lambda_{c,-\infty} = -b_0^2/2$.

For $\lambda > 0$, the Green's function $\overline{G}_\lambda$ for $\frac{1}{2}(d^2/dx^2) - \lambda$ is given by

$$\overline{G}_\lambda(x, y) = \int_0^\infty \exp(-\lambda t)\frac{\exp\left(-|x-y|^2/2t\right)}{(2\pi t)^{1/2}}\,dt.$$

By applying a Fourier transform before integration and an inverse Fourier transform after integration, one can calculate that $\overline{G}_\lambda(x, y) = (2\pi/\sqrt{2\lambda}) \times\exp(-\sqrt{2\lambda}|y - x|)$. Since $(L + \beta)^h = \frac{1}{2}(d^2/dx^2) - (b_0^2/2 - \beta)$, it follows that the Green's function $G_{-\beta}(x, y)$, for $L + \beta$, is given by $G_{-\beta}(x, y) = (1/(h(y)))\overline{G}_\lambda(x, y)h(x)$, with $\lambda = b_0^2/2 - \beta$; that is

$$G_{-\beta}(x, y) = \frac{2\pi}{\sqrt{b_0^2 - 2\beta}}\exp\left(-\sqrt{b_0^2 - 2\beta}|y - x| - b_0(x - y)\right).$$

EXAMPLE 2. Let $L = \frac{1}{2}\Delta + kx \cdot \nabla$ on $R^d, d \geq 1, k > 0$. Let $h(x) = \exp(-k|x|^2)$. Then $L^h = \frac{1}{2}\Delta - kx \cdot \nabla - kd$. The operator $\frac{1}{2}\Delta - kx \cdot \nabla$ is critical since it corresponds to a recurrent diffusion. Thus, by h-transform invariance, $\lambda_c = -kd$ for L, and $L - \lambda_c$ is critical. One can check that the ground state of $L - \lambda_c$ is given by $\phi_c(x) = h(x)$. Since $\tilde{L} - \lambda_c = \frac{1}{2}\Delta - kx \cdot \nabla$, it follows that $\tilde{\phi}_c(x) = 1$. Thus, $L - \lambda_c$ is product L^1-critical.

We now show that $\lambda_{c,\infty} = -\infty$, if $d \geq 2$. (The same proof with slightly different notation shows that $\lambda_{c,+\infty} = \lambda_{c,-\infty} = -\infty$, if $d = 1$.) Let $h(x) = \exp(-k|x|^2/2)$. Then $L^h = \frac{1}{2}\Delta - k^2|x|^2/2 - kd/2$. Fix $\lambda_0 \in R$. Let $B_n = \{x \in R^d: |x| < n\}$. Choose n_0 so that $k^2|x|^2/2 + kd/2 + \lambda_0 \geq 0$ on $R^d - \overline{B}_{n_0}$. Since the Green's function for $\frac{1}{2}\Delta - \lambda$ on $R^d - \overline{B}_{n_0}$ exists for all $\lambda \geq 0$, it follows by comparison and the definition of the Green's function that the Green's function for $L^h - \lambda_0$ exists on $R^d - \overline{B}_{n_0}$. Thus, by h-transform invariance, $\lambda_c(R^d - \overline{B}_{n_0}) \leq \lambda_0$ for the operator L. Since λ_0 is arbitrary, it follows that $\lambda_{c,\infty} = -\infty$.

The claims in Examples 3 and 4 are easy to verify.

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