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BROWNIAN MOTION IN A WEDGE WITH VARIABLE REFLECTION: EXISTENCE AND UNIQUENESS

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Existence and uniqueness in law of reflecting Brownian motion in a wedge is proved. The direction of reflection along the sides of the wedge varies in a reasonable fashion, except perhaps at the corner.

1. Introduction. In this paper we prove existence and uniqueness (in law) of reflecting Brownian motion (RBM) in a wedge, where the direction of reflection along the sides varies in a reasonable fashion, except perhaps at the corner. Varadhan and Williams (1985) have completely solved the problem when the direction of reflection is constant on each side of the wedge. Under various geometric conditions, Dupuis and Ishii (1991, 1993) have considered similar questions for more general domains and higher dimensions. In fact, their results are crucial for the present work. Other authors on this subject include Skorokhod (1961, 1962), McKean (1963, 1969), Ikeda and Watanabe (1989), Watanabe (1971), El Karoui (1975), El Karoui and Chaleyat-Maurel (1978), El Karoui, Chaleyat-Maurel and Marchal (1980), Stroock and Varadhan (1971), Bensoussan and Lions (1982), Tanaka (1979), Lions and Sznitman (1984), Harrison and Reiman (1981), Saisho (1987), Tsuchiya (1976, 1980) and Kwon (1992). Special cases of their results overlap with certain cases of ours, but we include new results. Rogers (1990, 1991) and Burdzy and Marshall (1992) have studied the question of whether or not the corner of the wedge is hit by RBM started away from the corner. While we do not use their results, we make use of some of their techniques.

Let $S = S(\xi) = S_\xi = \{(r, \theta): r \ge 0,\ 0 \le \theta \le \xi\}$ be the wedge of angle $\xi \in (0, 2\pi)$, where (r, θ) are polar coordinates. Let

$$\partial S_1 = \{(r, \theta): r \ge 0,\ \theta = 0\},$$
$$\partial S_2 = \{(r, \theta): r \ge 0,\ \theta = \xi\}$$

be the sides of the wedge. Denote by $\theta(x) \in (-\pi/2, \pi/2)$, $x \in \partial S \setminus \{0\}$, the *angle of reflection at* x, where $\theta(x)$ is measured clockwise from the inward normal at x.

A nondecreasing continuous function $\mathscr{K}: [0,\infty) \to [0,\infty)$ is a *Dini modulus of continuity* if, for each $t > 0$,

$$\int_0^t \frac{\mathscr{K}(u)}{u}\, du < \infty. \tag{1.1}$$

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We say $\theta(\cdot) \in C^{1,\mathscr{K}(\cdot)}_{\mathrm{loc}}(\partial S_i \setminus \{0\})$, $i = 1, 2$, if $\theta(\cdot) \in C^1(\partial S_i \setminus \{0\})$, and for each compact set $K \subseteq \partial S_i \setminus \{0\}$ there is some $C_K > 0$ depending on K such that

$$|D\theta(x_1) - D\theta(x_2)| \le C_K \mathscr{K}(|x_1 - x_2|), \qquad x_1, x_2 \in K,$$

where D is any first-order partial derivative.

The principal result is existence and uniqueness of RBM in an arbitrary wedge S. The main feature is that $\theta(x)$ need not have limiting values as $x \to 0$ in $\partial S_1 \setminus \{0\}$ or as $x \to 0$ in $\partial S_2 \setminus \{0\}$. Here and throughout this paper, $\theta(\cdot)$ is taken to be bounded away from $\pm\pi/2$. This corresponds to "nontangential reflection." Also, our interest is primarily in behavior near 0, so we will assume that far away the reflection is normal.

THEOREM 1.1. *For the wedge $S = S_\xi$, $0 < \xi < 2\pi$, suppose $\theta(x) = 0$ for $|x| \ge 1$,*

$$\theta(\cdot) \in C^{1,\mathscr{K}(\cdot)}_{\mathrm{loc}}(\partial S_i \setminus \{0\}), \qquad i = 1, 2, \tag{1.2}$$

$$\sup_{\partial S \setminus \{0\}} |\theta| < \frac{\pi}{2} \tag{1.3}$$

and, for some $\varepsilon > 0$,

$$\Bigg[\sup_{\substack{|x| \le \varepsilon \\ x \in \partial S_2 \setminus \{0\}}} \theta(x)\Bigg] - \Bigg[\inf_{\substack{|x| \le \varepsilon \\ x \in \partial S_1 \setminus \{0\}}} \theta(x)\Bigg] < 2\xi. \tag{1.4}$$

Then RBM in S exists uniquely in law.

REMARK. Notice in the case of $\xi \ge \pi/2$ that condition (1.4) is a consequence of (1.3).

Let us outline the key ideas of this paper. Following the approach of Varadhan and Williams (1985), we characterize RBM in S as a solution of a submartingale problem. Uniqueness in law of RBM in S is equivalent to having a unique solution of the submartingale problem. Commonly in this type of problem, uniqueness rather than existence is the most difficult part to prove. Aside from technicalities, we reduce consideration to the upper half-space via conformal invariance of RBM. Another conformal transformation, due to Rogers (1990, 1991), reduces the problem to consideration of a Lipschitz domain with constant vertical direction of reflection. Here our assumptions on the angle of reflection $\theta(\cdot)$ come into play. To prove uniqueness in this setting, we use results of Dupuis and Ishii (1993) on pathwise (or strong) uniqueness of stochastic differential equations with reflection (SDER) in nonsmooth domains. Our key idea is to show that pathwise uniqueness for a related stochastic differential equation implies uniqueness for the submartingale problem. A general theorem of this type, "pathwise uniqueness for SDE with reflection implies uniqueness for the corresponding submartingale problem," seems quite difficult to prove in a setting with nonsmooth state space or nonsmooth reflection. For this reason we are unable to use the very general existence and uniqueness results of Tsuchiya (1976, 1980) on diffusions with reflection in the upper half-space. Finally, note that the methods of Kwon (1992) can be used to obtain results when $\theta(\cdot)$ has limiting values along the sides of S.

The paper is organized as follows. In Section 2 we give formulations of RBM as a solution of a submartingale problem and also as a solution of a stochastic differential equation with reflection. The main result of this section is that pathwise uniqueness for a certain stochastic differential equation with reflection implies uniqueness for the corresponding submartingale problem. The proof is deferred to Section 5. In Section 3 we present some results on hitting times of RBM crucial to later developments. Section 4 extends the submartingale property to a certain class of functions. This is essential for the proof of the main theorem in Section 2 concerning pathwise uniqueness and uniqueness in law. In Section 5 we show that a solution of the submartingale problem, under appropriate hypotheses, is the solution of a stochastic differential equation. With this, we prove the main theorem of Section 2. Section 6 presents conformal invariance of RBM under certain technical hypotheses. These hypotheses are required because of the nonsmooth nature of the state space and reflection field at the origin. In Section 7 we prove the half-space case by using Rogers' conformal transformation. This transformation gives rise to a setting in which the results of Dupuis and Ishii on pathwise uniqueness apply. Finally, in Section 8 we handle the general wedge.

2. Formulations of RBM. Let $D \subseteq \mathbb{R}^2$ be a domain such that $0 \in \partial D$, ∂D is Lipschitz and $\partial D \setminus \{0\}$ is C^2. More precisely, ∂D is the graph of a Lipschitz function $y = f(x)$, $x \in \mathbb{R}$, and f is C^2 on $\mathbb{R} \setminus \{0\}$. We also assume D lies above the graph of f. Let $\theta(x) \in (-\pi/2, \pi/2)$, $x \in \partial D \setminus \{0\}$, denote the *angle of reflection*, measured positive in the clockwise sense with respect to the inward unit normal $n(x)$. Denote by $\gamma(x)$ the vector making angle $\theta(x)$ with $n(x)$, and call it the *direction of reflection*. Any normalization is allowed (this will be apparent below), and in the sequel we will take either $\gamma \cdot n = 1$ or $|\gamma| = 1$, as the situation demands.

Let $\Omega_{\overline{D}} = C([0,\infty), \overline{D})$ be the space of continuous paths in $\overline{D}$, and let $Z_t(\omega) = \omega(t)$, $t \in [0,\infty)$, be the coordinate map. Set $\mathscr{M}_t = \sigma(\omega(s)\colon s \le t\}$ and $\mathscr{M} = \sigma\{\omega(s)\colon s \ge 0\}$. A *solution of the submartingale problem on D, starting from $x \in \overline{D}$*, is a probability measure P_x on $(\Omega_{\overline{D}}, \mathscr{M})$ such that

$$P_x(\omega(0) = x) = 1; \tag{2.1}$$

for each $f \in C_b^2(\overline{D})$ with f constant near $\{0\}$ and $\gamma \cdot \nabla f \ge 0$ on $\partial D \setminus \{0\}$,

$$f(\omega(t)) - \tfrac{1}{2}\int_0^t (\Delta f)(\omega(s))\, ds \tag{2.2}$$

is a P_x-submartingale;

$$E^{P_x}\left[\int_0^\infty I_{\{0\}}(\omega(s))\,ds\right] = 0. \tag{2.3}$$

A family $\{P_x\colon x \in \overline{D}\}$ is a *solution of the submartingale problem on* D if, for each $x \in \overline{D}$, P_x solves the submartingale problem on D, starting from x. We say $Z(\cdot)$ together with $\{P_x\colon x \in \overline{D}\}$ is an RBM in D and $Z(\cdot)$ under P_x is an RBM in D starting from x.

The statement "RBM in S exists uniquely in law" now has precise meaning: there is exactly one solution of the submartingale problem on D.

For a stopping time τ, we will consider RBM in D stopped at time τ. This is just the law of the coordinate process $Z(\cdot)$ under P_x, where P_x satisfies (2.1)–(2.3) except t in (2.2) is replaced by $t \wedge \tau$ and the ∞ in (2.3) is replaced by τ. In this case we say P_x solves the submartingale problem stopped at time τ.

For uniqueness, we make use of some special cases of results in Dupuis and Ishii (1993). For this it is necessary to introduce certain stochastic differential equations with reflection. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, let $\{\mathscr{F}_t\colon t \ge 0\}$ be a right-continuous complete filtration and let $\{B_t\colon t \ge 0\}$ be standard $\{\mathscr{F}_t\}$-Brownian motion in $\mathbb{R}^2$.

Definition 2.1 (SDER). A continuous $\{\mathscr{F}_t\}$-adapted process $X(t)$ is a solution to the SDER for D, with direction of reflection $\gamma(\cdot)$, initial condition $x \in \overline{D}$ and Brownian motion $\{B_t\colon t \ge 0\}$ if

$$P\big(X(t) \in \overline{D}, \forall t \ge 0\big) = 1, \tag{2.4}$$

and

$$X(t) = x + B(t) + Y(t), \tag{2.5}$$

where

$$|Y|(t) = \int_{(0,t]} I(X(s) \in \partial D)|dY|(s) < \infty \quad \text{a.s.} \tag{2.6}$$

and

$$Y(t) = \int_{(0,t]} \gamma(X(s))|dY|(s) \quad \text{a.s.} \tag{2.7}$$

Lemma 2.2. *Let $X(\cdot)$ be a solution to the SDER for D with direction of reflection $\gamma(\cdot)$ and initial condition $x \in \overline{D}$. Then the law of $X(\cdot)$ on $(\Omega_{\overline{D}}, \mathscr{M})$ solves the submartingale problem on D starting from x.*

Proof. Let P_x be the law of $X(\cdot)$ on $(\Omega_{\overline{D}}, \mathscr{M})$. It is routine to check that P_x satisfies (2.1) and (2.2). For (2.3), by the occupation time formula for semimartingales [Revuz and Yor (1991), Corollary VI.1.6, page 209],

$$\int_0^t I_{\{0\}}(X_1(s))\,ds = \int_{-\infty}^\infty I_{\{0\}}(a) L_t^a\,da = 0 \quad \text{for } t > 0,$$

where $L_\cdot^a$ is the local time for the semimartingale $X_1(\cdot)$ at a. □

There is a natural type of uniqueness associated with SDER: *pathwise uniqueness*. This means if $X(\cdot)$ and $X'(\cdot)$ are solutions to SDER (on the same filtered probability space with the same Brownian motion and initial condition), then $X = X'$ a.s.

We will also consider *SDER stopped at time* τ, where τ is a stopping time. This just means replace t in (2.4)–(2.7) by $t \wedge \tau$. The meaning of the statement "pathwise uniqueness for SDER stopped at time τ" is clear:

$$X(\cdot \wedge \tau) = X'(\cdot \wedge \tau) \quad \text{a.s.}$$

Uniqueness in law for SDER holds if whenever X and X' are solutions to SDER (with the same initial condition), possibly on different filtered probability spaces or possibly with different Brownian motions, then the laws of X and X' on $(\Omega_{\overline{D}}, \mathscr{M})$ coincide.

Assume pathwise uniqueness holds for SDER for D. Then the usual proof of uniqueness in law for SDER for D goes through. [See Ikeda and Watanabe (1989), corollary on page 166.] Hence to show uniqueness in law holds for RBM in D, it suffices to show a solution P_x of the submartingale problem in D starting from x has a realization as a solution to the SDER for D. More precisely, there is a continuous stochastic process $\{X(t)\colon t \geq 0\}$ defined on some filtered probability space $(E, \mathscr{G}, \{\mathscr{G}_t\}_{t \geq 0}, Q)$ such that the following hold:

(i) there exists a two-dimensional $\{\mathscr{G}_t\}$-Brownian motion B with $B(0) = 0$ a.s.;

(ii) X is a two-dimensional process adapted to $\{\mathscr{G}_t\colon t \geq 0\}$;

(iii) there exists a two-dimensional continuous $\{\mathscr{G}_t\colon t \geq 0\}$-adapted process $\{Y(t)\colon t \geq 0\}$, with $Y(0) = 0$, such that

$$|Y|(t) = \int_0^t I(X(s) \in \partial D)|dY|(s) < \infty \quad \text{a.s. } Q$$

and

$$Y(t) = \int_0^t \gamma(X(s))|dY|(s) \quad \text{a.s. } Q;$$

(iv) $X(t) = x + B(t) + Y(t)$;

(v) the law of X on $(\Omega_{\overline{D}}, \mathscr{M})$ under Q coincides with P_x.

Then it is clear that, to prove the law of RBM in D stopped at time

$$\sigma_\delta(\omega) = \inf\{t \geq 0\colon |\omega(t)| = \delta\} \tag{2.8}$$

is unique, it is enough to know pathwise uniqueness for SDER stopped at time σ_δ holds and there is a realization $X(\cdot)$ satisfying (i)–(v) up to time $\sigma_\delta(X)$.

We will do this below. A general theorem that pathwise uniqueness for SDER for D implies uniqueness in law for RBM in D seems quite difficult to prove. We content ourselves by proving a special case directly applicable to our needs. The next result will be proved in Section 5.

THEOREM 2.3. *Assume D is a domain such that $0 \in \partial D$, ∂D is Lipschitz, $\partial D \setminus \{0\}$ is C^2,*

$$\tilde{\theta} = \sup|\theta(x)| < \frac{\pi}{2}$$

and the inner and outer cones for D at $\{0\}$ are, respectively,

$$C_{\mathrm{IN}} = \left\{(y_1, y_2) \colon y_2 \geq |y_1| \tan \tilde{\theta}\right\} \subseteq D,$$
$$C_{\mathrm{OUT}} = \left\{(y_1, y_2) \colon y_2 \leq -|y_1| \tan \tilde{\theta}\right\} \subseteq D^c.$$

If the direction of reflection is $\gamma \equiv (0,1)$, then for some $\delta > 0$ pathwise uniqueness for SDER for D stopped at time σ_δ implies uniqueness in law for RBM in D stopped at time σ_δ.

The explicit nature above of the inner and outer cones is crucial to the proof.

3. Auxiliary results on stopping times. As in Section 2, D is a domain with $0 \in \partial D$, ∂D Lipschitz and $\partial D \setminus \{0\}$ is C^2. Recall, for $r \geq 0$,

$$\sigma_r(\omega) = \inf\{t \geq 0 \colon |\omega(t)| = r\}.$$

LEMMA 3.1. *Let $0 < a < \delta < b$. Then for each $t > 0$ there is a $p(a, \delta, b) < 1$ such that, for any family of solutions $\{P_z \colon z \in \overline{D}\}$ of the submartingale problem on D,*

$$\sup_{\substack{|x| = \delta \\ x \in \overline{D}}} P_z(\sigma_a \wedge \sigma_b \leq t) < p(a, \delta, b).$$

PROOF. For $|z| = \delta$, P_z is a probability measure on $\Omega_\delta := \{\omega \in \Omega_{\overline{D}} \colon |\omega(0)| = \delta\}$ and the set $\{\sigma_a \wedge \sigma_b \leq t\}$ is a closed subset of Ω_δ. Since $a > 0$ and $\partial D \setminus \{0\}$ is C^2, RBM in D starting from z is uniquely determined up to time $\sigma_a \wedge \sigma_b$. Consequently the law of $Z(\cdot \wedge \sigma_a \wedge \sigma_b)$ under P_{z_n} converges weakly to the law of $Z(\cdot \wedge \sigma_a \wedge \sigma_b)$ under P_z whenever $z_n \to z$ with $z_n \in \overline{D}$ and $|z_n| = \delta$ [see Stroock and Varadhan (1971)]. In particular, the function

$$P_z(\sigma_a \wedge \sigma_b \leq t), \qquad z \in \overline{D},\ |z| = \delta,$$

is upper semicontinuous, and so it achieves its supremum at some $z_0 \in \overline{D}$ with $|z_0| = \delta$. Hence to prove the lemma it is enough to show that, for each $z \in \overline{D}$ with $|z| = \delta$,

$$P_z(\sigma_a \wedge \sigma_b > t) > 0. \tag{3.1}$$

Choose $\omega_0 \in \Omega_\delta$ such that $\omega_0(0) = z$ and

$$a < \inf_{s \leq t} |\omega_0(s)| \leq \sup_{s \leq t} |\omega_0(s)| < b.$$

Then choose $\gamma > 0$ such that, for any $\omega \in \{\omega \in \Omega_\delta\colon \sup_{s \le t} |\omega(s) - \omega_0(s)| \le \gamma\}$,

$$a < \inf_{s \le t} |\omega(s)| \le \sup_{s \le t} |\omega(s)| < b.$$

Then

$$P_z(\sigma_a \wedge \sigma_b > t) \ge P_z\Big(\sup_{s \le t} |Z(s) - \omega_0(s)| \le \gamma\Big).$$

The lower probability is positive by the support theorem [cf. Kwon and Williams (1991), Theorem 3.1]. Thus (3.1) holds. □

REMARK. Suppose K is a closed set satisfying $K \subseteq \overline{B_a(0)}^c \cap B_b(0)$. Then, for some $p_K(a, b) < 1$,

$$\sup_{z \in K \cap \overline{D}} P_z(\sigma_a \wedge \sigma_b \le t) < p_K(a, b),$$

where P_z is any solution of the submartingale problem on D starting from z. The proof is similar to that of Lemma 3.1.

The following lemma is from Varadhan and Williams [(1985), Lemma 3.1]. Their proof is valid in the present context.

LEMMA 3.2. *Suppose $x \in \overline{D}$ and P is a solution of the submartingale problem starting from x. Let τ be a stopping time, and let $\{P^\tau_\omega\}$ be a regular conditional probability distribution (r.c.p.d.) of $P|\mathscr{M}_\tau$. For each $\omega \in \{\tau < \infty\}$, define $\hat{P}^\tau_\omega$ on $(\Omega_{\overline{D}}, \mathscr{M})$ by*

$$\hat{P}^\tau_\omega(\mathscr{A}) = P^\tau_\omega(Z(\cdot + \tau(\omega)) \in \mathscr{A}), \qquad \mathscr{A} \in \mathscr{M}.$$

Then there is a P-null set $N \in \mathscr{M}_\tau$ such that, for $\omega \notin N \cup \{\tau = \infty\}$, $\hat{P}^\tau_\omega$ is a solution of the submartingale problem starting from $\tau(\omega)$.

LEMMA 3.3. *Let P_x be a solution of the submartingale problem on D starting from x. Then for the sequence of stopping times T_n, $n \ge 0$, defined by*

$$\begin{aligned} T_0 &= 0, \\ T_{2n+1} &= \inf\{t \ge T_{2n}\colon |\omega(t)| = \delta\}, \\ T_{2n+2} &= \inf\{t \ge T_{2n+1}\colon |\omega(t)| = \delta/2\}, \\ &\vdots \end{aligned}$$

we have $T_n \to \infty$ a.s. P_x as $n \to \infty$.

PROOF. By Lemmas 3.1 and 3.2,

$$\begin{aligned} P_x(T_{2n} \le t) &= E^{P_x}\Big[I(T_{2n-1} \le t)\hat{P}^{T_{2n-1}}_\omega\big[\sigma_{\delta/2} \le t - T_{2n-1}(\omega)\big]\Big] \\ &\le E^{P_x}\Big[I(T_{2n-1} \le t)\hat{P}^{T_{2n-1}}_\omega\big[\sigma_{\delta/2} \le t\big]\Big] \\ &\le E^{P_x}\Big[I(T_{2n-1} \le t)\hat{P}^{T_{2n-1}}_\omega\big[\sigma_{2\delta} \wedge \sigma_{\delta/2} \le t\big]\Big] \\ &\le p(\delta/2, \delta, 2\delta)P_x[T_{2n-1} \le t] \\ &\le p(\delta/2, \delta, 2\delta)P_x(T_{2n-2} \le t). \end{aligned}$$

Hence, iterating,

$$P_x(T_{2n} \le t) \le p(\delta/2, \delta, 2\delta)^n.$$

Since $p(\delta/2, \delta, 2\delta) < 1$, $P_x(\lim_{n\to\infty} T_n \le t) = 0$. It follows that

$$P_x\Big(\lim_{n\to\infty} T_n = \infty\Big) = 1. \qquad \square$$

LEMMA 3.4. *Suppose for some bounded open set B, with $0 \in \overline{B}$, that*

$$h \in C(\overline{D} \cap \overline{B}) \cap C^2\big(\overline{D} \cap \overline{B} \setminus \{0\}\big),$$

with $\gamma \cdot \Delta h \ge 0$ on $\partial D \cap \overline{B} \setminus \{0\}$, $h(0) = 0$ and $\Delta h \ge 0$ on $\overline{D} \cap \overline{B} \setminus \{0\}$. Then, for $\tau_B = \inf\{t \ge 0: \omega(t) \notin B\}$ and for any solution P_z of the submartingale problem on D starting from $x \in \overline{D} \cap B$,

$$E^{P_x}\left[\int_0^{\tau_B} \Delta h(Z_s) I\big(h(Z_s) > 0\big)\, ds\right] \le 2 \sup_{\overline{B}} |h|.$$

PROOF. With a change of notation, this is a consequence of the proof of Theorem 4.3 in DeBlassie and Toby (1993). $\square$

LEMMA 3.5. *Let P_x be a solution of the submartingale problem on $S(\xi)$, $0 < \xi < 2\pi$, starting from x, with angle of reflection $\theta(x)$, $x \in \partial S(\xi)$. Suppose $\sup|\theta| < \pi/2$, the functions $\theta_1 = \theta|_{\partial S_1 \setminus \{0\}}$, $\theta_2 = \theta|_{\partial S_2 \setminus \{0\}}$ are continuous and, for some $\varepsilon > 0$,*

$$\alpha := \frac{\sup_{|x| \le \varepsilon} \theta_2(x) - \inf_{|x| \le \varepsilon} \theta_1(x)}{\xi} < 2$$

(supremum on ∂S_2, infimum on ∂S_1). Then, for $|x| < \varepsilon$,

$$P_x(\sigma_\varepsilon < \infty) = 1.$$

PROOF. Let $\delta > 0$ be so small that if

$$\tilde{\alpha} = \begin{cases} \alpha, & \text{for } \alpha \ne 0, \\ \alpha + \delta, & \text{for } \alpha = 0, \end{cases}$$

then (taking suprema on ∂S_2) the following hold:

$$0 \ne \tilde{\alpha} < 2; \tag{3.2}$$

$$\sup_{|x| \le \varepsilon} \theta_2(x) + \delta\xi < \frac{\pi}{2}; \tag{3.3}$$

$$\sup_{|x| \le \varepsilon} \left[\sup_{|y| \le \varepsilon} \theta_2(y) - \theta_2(x) + \delta\xi\right] < \pi. \tag{3.4}$$

The last two are possible because $\sup_z |\theta(z)| < \pi/2$. Define

$$\beta = \inf_{|x| \le \varepsilon} \theta_1(x) \quad (\text{infimum on } \partial S_1),$$

and for $z = r\exp(i\theta) \in S(\xi) \setminus \{0\}$ set

$$\Phi(z) = r^{\tilde{\alpha}} \cos(\tilde{\alpha}\theta + \beta).$$

Then $\Phi \in C^2(S(\xi) \setminus \{0\})$, $\Delta\Phi = 0$ on $S(\xi) \setminus \{0\}$,

$$\inf_{0 \le \theta \le \xi} \cos(\tilde{\alpha}\theta + \beta) > 0 \tag{3.5}$$

$$\left[\text{by (3.3) and that } \beta = \sup_{|x| \le \varepsilon} \theta_2(x) \text{ if } \alpha = 0\right]$$

and, in polar coordinates $\langle r, \theta \rangle$,

$$\begin{aligned} \nabla\Phi &= \tilde{\alpha} r^{\tilde{\alpha}-1} \langle \cos(\tilde{\alpha}\theta + \beta), -\sin(\tilde{\alpha}\theta + \beta)\rangle, \\ &= \tilde{\alpha}\Phi^{1-1/\tilde{\alpha}}[\cos(\tilde{\alpha}\theta + \beta)]^{-(1-1/\tilde{\alpha})}\langle \cos(\tilde{\alpha}\theta + \beta), -\sin(\tilde{\alpha}\theta + \beta)\rangle. \end{aligned} \tag{3.6}$$

Here and below, the notation $\langle\ ,\ \rangle$ denotes the components of a vector relative to the rotating orthogonal unit vectors $\mathbf{e_r} = \nabla r$ and $\mathbf{e_\theta} = r\,\nabla\theta$, in that order. Moreover (normalizing $|\gamma| = 1$),

$$\gamma(x) = \begin{cases} \langle \sin\theta_1(x), \cos\theta_1(x)\rangle, & x \in \partial S_1 \setminus \{0\}, \\ \langle -\sin\theta_2(x), -\cos\theta_2(x)\rangle, & x \in \partial S_2 \setminus \{0\}. \end{cases}$$

By choice of β and $\tilde{\alpha}$,

$$\begin{cases} \theta_1(z) - \beta \in [0, \pi), & z \in \partial S_1 \setminus \{0\}, |z| \le \varepsilon, \\ \tilde{\alpha}\xi + \beta - \theta_2(z) \in [0, \pi), & z \in \partial S_2 \setminus \{0\}, |z| \le \varepsilon. \end{cases} \tag{3.7}$$

Hence $\tilde{\alpha}\gamma \cdot \nabla\Phi \ge 0$ on $\partial S(\xi) \cap \overline{B_\varepsilon(0)} \setminus \{0\}$. If

$$g(y) = \frac{\tilde{\alpha}^2}{2(2-\tilde{\alpha})} y^{2/\tilde{\alpha}}, \qquad y \ne 0,$$

then $g \circ \Phi \in C(S(\xi)) \cap C^2(S(\xi) \setminus \{0\})$ and on $\partial S(\xi) \cap \overline{B_\varepsilon(0)} \setminus \{0\}$, by (3.2), $\gamma \cdot \nabla[g \circ \Phi] \ge 0$. Moreover, on $S(\xi) \setminus \{0\}$,

$$\begin{aligned} \Delta[g \circ \Phi] &= [g'' \circ \Phi]|\nabla\Phi|^2 \\ &= \Phi^{-(2-2/\tilde{\alpha})}\left[\tilde{\alpha}^2\Phi^{2-2/\tilde{\alpha}}(\cos(\tilde{\alpha}\theta + \beta))^{-(2-2/\tilde{\alpha})}\right] \\ &= \tilde{\alpha}^2[\cos(\tilde{\alpha}\theta + \partial)]^{-2+2/\tilde{\alpha}} \\ &\ge C > 0 \quad [\text{by (3.5)}]. \end{aligned}$$

Applying Lemma 3.4 with $D = S(\xi)^\circ$ and $h = g \circ \Phi$,

$$CE^{P_x}\left[\int_0^{\sigma_\varepsilon} I(g \circ \Phi(Z_s) > 0)\,ds\right] < \infty, \qquad |x| < \varepsilon.$$

However,

$$\begin{aligned} P_x(g \circ \Phi(Z_s) > 0, s \le \sigma_\varepsilon) &= P_x(Z_s \ne 0, s \le \sigma_\varepsilon) \\ &= P_x(s \le \sigma_\varepsilon), \quad \text{a.e.}\,(ds), \end{aligned}$$

by property (2.3) of a solution of the submartingale problem. Hence

$$E^{P_x}[\sigma_\varepsilon] < \infty \quad \text{as desired.} \qquad \square$$

LEMMA 3.6. *With the same hypotheses and notation as in Lemma 3.5, for any $p > \alpha \vee 0$ and $|x| < \varepsilon$,*

$$E^{P_x}\left[\int_0^{\sigma_\varepsilon} |Z_u|^{p-2} I(Z_u \neq 0)\, du\right] < \infty.$$

PROOF. We use the notation in the proof of Lemma 3.5, with δ chosen as there and with the additional requirement

$$p > \tilde{\alpha} \vee 0.$$

Define

$$h(y) = y^{p/\tilde{\alpha}}, \qquad y \neq 0.$$

Then $h \circ \Phi \in C(S(\xi)) \cap C^2(S(\xi) \setminus \{0\})$,

$$\gamma \cdot \nabla[h \circ \Phi] \geq 0 \quad \text{on } \partial S(\xi) \cap \overline{B_\varepsilon(0)} \setminus \{0\},$$

and, for $z = r\exp(i\theta) \in S(\xi) \cap \overline{B_\varepsilon(0)} \setminus \{0\}$,

$$\begin{aligned}
\Delta[h \circ \Phi] &= \frac{p(p-\tilde{\alpha})}{\tilde{\alpha}^2} \Phi^{-2+p/\tilde{\alpha}}\left[\tilde{\alpha}^2 \Phi^{2-2/\tilde{\alpha}}[\cos(\tilde{\alpha}\theta + \beta)]^{-2+2/\tilde{\alpha}}\right] \\
&= p(p-\tilde{\alpha})\Phi^{(p-2)/\tilde{\alpha}}[\cos(\tilde{\alpha}\theta + \beta)]^{-2+2/\tilde{\alpha}} \\
&\geq cr^{p-2}, \quad \text{where } c > 0 \text{ by } (3.5).
\end{aligned}$$

By Lemma 3.4 applied to $h \circ \Phi$,

$$E^{P_z}\left[\int_0^{\sigma_\varepsilon} |Z_s|^{p-2} I(Z_s \neq 0)\, ds\right] < \infty,$$

as desired. $\square$

4. Extension of the submartingale property to other functions. Throughout this section, P_z will be a solution of the submartingale problem on D starting from z. We will assume the hypotheses of Theorem 2.3 are in effect.

For $\tilde{\theta} = \sup_x |\theta(x)|$, choose

$$\frac{1}{2} < p < \frac{\pi}{2\tilde{\theta} + \pi}. \tag{4.1}$$

This is possible since $\tilde{\theta} < \pi/2$. Next choose

$$\beta \in \left(-\frac{\pi}{2} + p\tilde{\theta}, \frac{\pi}{2} - p(\pi + \tilde{\theta})\right). \tag{4.2}$$

Note that this interval is nonvoid since $p < \pi/(2\tilde{\theta} + \pi)$. Define

$$\Psi_\beta(z) = r^p \cos(p\theta + \beta), \qquad z = r\exp(i\theta), \qquad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

We now show that

$$\inf\left\{\cos(p\theta + \beta) : -\tilde{\theta} \le \theta \le \pi + \tilde{\theta}\right\} > 0 \tag{4.3}$$

and

$$\gamma \cdot \nabla \Psi_\beta \ge 0 \quad \text{on } \partial D \setminus \{0\}. \tag{4.4}$$

Since $p > \frac{1}{2}$,

$$\left(-\frac{\pi}{2} + p\tilde{\theta}, \frac{\pi}{2} - p(\pi + \tilde{\theta})\right) \subseteq \left((1-p)\tilde{\theta} - p\pi, -(1-p)\tilde{\theta}\right). \tag{4.5}$$

Set

$$\partial D_1 = \left\{r\exp(i\theta) \in \partial D \setminus \{0\} : -\tilde{\theta} \le \theta \le \tilde{\theta}\right\},$$
$$\partial D_2 = \left\{r\exp(i\theta) \in \partial D \setminus \{0\} : \pi - \tilde{\theta} \le \theta \le \pi + \tilde{\theta}\right\}.$$

Since $C_{\mathrm{OUT}} \subseteq D^c$ and $C_{\mathrm{IN}} \subseteq D$,

$$\partial D \setminus \{0\} = \partial D_1 \cup \partial D_2.$$

Expressing $\gamma(z) \equiv (0,1)$ in polar coordinates,

$$\gamma(z) = \langle \sin\theta, \cos\theta \rangle, \qquad z = r\exp(i\theta) \in \partial D_1 \cup \partial D_2,$$

where $\langle a, b \rangle = a\mathbf{e}_\mathbf{r} + b\mathbf{e}_\theta$. Then since

$$\nabla \Psi_\beta = pr^{p-1}\langle \cos(p\theta + \beta), -\sin(p\theta + \beta)\rangle,$$

we end up with

$$\gamma \cdot \nabla \Phi_\beta = pr^{p-1}\sin[(1-p)\theta - \beta], \qquad z = r\exp(i\theta) \in \partial D_1 \cup \partial D_2.$$

If $r\exp(i\theta) \in \partial D_1$, then $-\tilde{\theta} \le \theta \le \tilde{\theta}$ and so

$$\begin{aligned}(1-p)\theta - \beta &\in \left[-(1-p)\tilde{\theta} - \beta, (1-p)\tilde{\theta} - \beta\right] \\ &\subseteq (0, p\pi) \quad [\text{by (4.2) and (4.5)}] \\ &\subseteq [0, \pi] \qquad (\text{since } p < 1).\end{aligned}$$

Hence

$$\gamma \cdot \nabla \Psi_\beta \ge 0 \quad \text{on } \partial D_1.$$

If $r\exp(i\theta) \in \partial D_2$, then $\pi - \tilde{\theta} \le \theta \le \pi + \tilde{\theta}$ and

$$\begin{aligned}(1-p)\theta - \beta &\in \left[(1-p)(\pi - \tilde{\theta}) - \beta, (1-p)(\pi + \tilde{\theta}) - \beta\right] \\ &\subseteq (\pi(1-p), \pi) \quad [\text{by (4.2) and (4.5)}] \\ &\subseteq [0, \pi],\end{aligned}$$

and so we have

$$\gamma \cdot \Psi_\beta \ge 0 \quad \text{on } \partial D_2.$$

For each $K \geq 0$, set

$$\tau_K = \inf\{t \geq 0 \colon \Psi_\beta(\omega(t)) = K\}.$$

The next result extends the submartingale property to a certain class of functions.

THEOREM 4.1. *Let p be from* (4.1). *Then, for some $K > 0$, if $B = \{x \in D \colon \Psi_\beta(x) < K\}$ and $h \in C(\overline{D} \cap \overline{B})$ and $f \in C(\overline{D} \cap \overline{B}) \cap C^2(\overline{D} \cap \overline{B} \setminus \{0\})$ are such that*

$$\begin{aligned} \tfrac{1}{2}\Delta f &= h && \text{in } \overline{D} \cap \overline{B} \setminus \{0\}, \\ \gamma \cdot \nabla f &\geq 0 && \text{on } \partial D \cap \overline{B} \setminus \{0\}, \\ |f(x) - f(0)| &= o(|x|^p) && \text{as } x \to 0. \end{aligned}$$

Then

$$f(\omega(t \wedge \tau_K)) - \int_0^{t \wedge \tau_K} (I_D h)(\omega(s))\, ds$$

is a P_z-submartingale, $z \in \overline{B}$.

The proof will be given at the end of this section. For it, we need the following result.

THEOREM 4.2. *For some $K > 0$, if $B = \{x \in D \colon \Psi_\beta(x) < K\}$ and $z \in \overline{D} \cap \overline{B} \setminus \{0\}$, then for $\tau_{0K} = \tau_0 \wedge \tau_K$,*

$$a_1 \Psi_\beta(z)\left[K^{-1+2/p} - \Psi_\beta(z)^{-1+2/p}\right] \leq E^{P_z}[\tau_{0K}] \leq a_2 K^{2/p}, \tag{4.6}$$

where a_1 and a_2 are independent of z.

PROOF. Define

$$g_1(y) = \begin{cases} 0, & \text{for } y = 0, \\ y\left[K^{-1+2/p} - y^{-1+2/p}\right]/(2-p), & \text{for } 0 < y \leq K, \\ p^{-1}K^{2/p}[1 - y/K], & \text{for } y \geq K. \end{cases}$$

First we prove that

$$E^{P_z}[\tau_{0K}] \geq a_1 g_1(\Psi_\beta(z)), \tag{4.7}$$

which is the desired lower inequality. Since

$$\Delta \Psi_\beta = 0 \quad \text{on } \overline{D} \setminus \{0\}$$

and

$$|\nabla \Psi_\beta(z)|^2 = p^2\left(\Psi_\beta(z)/\cos(p\theta + \beta)\right)^{2-2/p}, \qquad z = r\exp(i\theta) \neq 0,$$

using (4.3)–(4.4) we can proceed as in the proof of Theorem 2.3 in Varadhan and Williams (1985), to obtain (4.7). This yields the lower inequality in (4.6).

To prove the upper inequality, it is enough to show that

$$E^{P_z}(\tau_{0K}) \le a_2 K^{2/p}, \qquad z \in \overline{D} \cap \overline{B} \setminus \{0\}. \tag{4.8}$$

Define

$$g_2(y) = y^{2/p}.$$

Then $g_2 \circ \Psi_\beta \in C(\overline{D}) \cap C^2(\overline{D} \setminus \{0\})$ and, by (4.4), $\gamma \cdot \nabla[g_2 \circ \Psi_\beta] \ge 0$ on $\partial D \setminus \{0\}$. Moreover, on $\overline{D} \setminus \{0\}$ for some constant $a_3 > 0$,

$$\begin{aligned}
\Delta[g_2 \circ \Psi_\beta] &= [g_2'' \circ \Psi_\beta]|\nabla \Psi_\beta|^2 \\
&= \frac{2}{p}\left(\frac{2-p}{p}\right)\Psi_\beta^{-2+2/p}\left[p^2 \Psi_\beta^{2-2/p}(\cos(p\theta+\beta))^{-2+2/p}\right] \\
&= 2(2-p)[\cos(p\theta+\beta)]^{-2+2/p} \\
&\ge a_3
\end{aligned}$$

by (4.3) and the fact that $2 - p > 0$.

By Lemma 3.4 with $h = g_2 \circ \Psi_\beta$,

$$E^{P_z}\left[\int_0^{\tau_K} a_3 I\big(g_2 \circ \Psi_\beta(Z_s) > 0\big)\, ds\right] \le 2 \sup_{\overline{B}} |g_2 \circ \Psi_\beta|,$$

which is the same as

$$E^{P_z}\left[\int_0^{\tau_K} a_3 I(Z_s \ne 0)\, ds\right] \le 2K^{2/p}.$$

Then, using property (2.3) of the submartingale problem,

$$E^{P_z}[\tau_K] \le 2a_3^{-1}K^{2/p}.$$

Since $E^{P_z}[\tau_{0K}] \le E^{P_z}[\tau_K]$, (4.8) holds with $a_2 = 2a_3^{-1}$. □

Now we can prove Theorem 4.1. Let $0 < \eta < K$, where K is from Theorem 4.2, and define the following for $n = 0, 1, 2, \dots$:

$$\begin{aligned}
T_0 &= 0; \\
T_{2n+1} &= \inf\{t \ge T_{2n} \colon \omega(t) = 0\}; \\
T_{2n+2} &= \inf\{t \ge T_{2n+1} \colon \Psi_\beta(\omega(t)) = \eta\}; \\
&\vdots
\end{aligned}$$

By Theorem 4.2, once we show $T_n \to \infty$ a.s. P_z as $n \to \infty$, we can modify the proof of Theorem 3.5 in Varadhan and Williams (1985) in the following manner, to yield the desired conclusion.

First replace t by $t \wedge \tau_K$ and ε by K throughout their proof.

Their expression (3.32) remains valid if the initial "$t =$ " is replaced by "$t \ge E^p[t \wedge \tau_K] =$ " and the T_{2n+1} appearing in the last equality in the expression is replaced by $T_{2n+1} \wedge \tau_K$. In (3.33)–(3.35) replace T_{2n+1} by $T_{2n+1} \wedge \tau_K$ and τ_0 by τ_{0K}.

To show $T_n \to \infty$ a.s. P_z, choose δ and M, both positive, such that

$$B_\delta(0) \cap \overline{D} \subseteq \left\{ z : \Psi_\beta(z) \le \frac{\eta}{2} \right\} \subseteq \left\{ z : \Psi_\beta(z) \le \frac{3\eta}{2} \right\} \subseteq B_M(0) \cap \overline{D}.$$

Then, by Lemma 3.2 and the remark after Lemma 3.1,

$$\begin{aligned} P_z(T_{2n+1} \le t) &\le E^{P_z}\Big[I(T_{2n} \le t) \hat{P}_\omega^{T_{2n}}(\sigma_0 \le t) \Big] \\ &\le E^{P_z}\Big[I(T_{2n} \le t) \hat{P}_\omega^{T_{2n}}(\sigma_\delta \wedge \sigma_M \le t) \Big] \\ &\le P_z(T_{2n-1} \le t) p_S(\delta, M), \end{aligned}$$

where $S = \{z \in \overline{D}: \Psi_\beta(z) = \eta\}$. Then $P_z(T_{2n+1} \le t) \le p_S(\delta, M)^n \to 0$ as $n \to \infty$. Hence $P_z(\lim_{n\to\infty} T_n \le t) = 0$, giving $T_n \to \infty$ a.s. P_z as $n \to \infty$.

5. Identification of the martingale part of a submartingale and proof of Theorem 2.3. Throughout this section, P_x will be a solution of the submartingale problem on D starting from x. As in Section 2, D is a domain such that $0 \in \partial D$, ∂D is Lipschitz and $\partial D \setminus \{0\}$ is C^2.

LEMMA 5.1. *Suppose, for some* $\varepsilon > 0$, $f \in C^2(\overline{D} \cap \overline{B_\varepsilon(0)})$ *is such that* $f(Z_{t\wedge\sigma_\varepsilon}) - \frac{1}{2}\int_0^{t\wedge\sigma_\varepsilon} \Delta f(Z_s)\, ds$ *can be written as a* P_x*-submartingale* $M_f(t) + A_f(t)$, *where* M_f *is a* P_x*-martingale and* A_f *is a continuous adapted increasing process. Then* A_f *can change only when* $Z(\cdot) \in \partial D$:

$$\int_0^{t\wedge\sigma\varepsilon} I_D(Z(s))\, dA_f(s) = 0 \quad a.s.\ P_x.$$

The proof can be modeled on the arguments in Stroock and Varadhan [(1971), pages 161–162].

In what follows, we use the notation and terminology of Ikeda and Watanabe (1989). A stochastic process $X(\cdot)$ on $(\Omega, \mathscr{F}, P)$ is called a *local* $\{\mathscr{F}_t\}$*-martingale* if it is adapted to a filtration $\{\mathscr{F}_t\}$ and if there exists a sequence of $\{\mathscr{F}_t\}$-stopping times $\sigma_n < \infty$ such that $\sigma_n \uparrow \infty$ and $X_n(t) = X(t \wedge \sigma_n)$ is an $\{\mathscr{F}_t\}$-martingale for each n. If X_n is a square integrable martingale for each n, then X is called a *locally square integrable* $\{\mathscr{F}_t\}$*-martingale*. Write $\mathscr{M}_2^{c,\mathrm{loc}}$ for the space of a.s. continuous locally square integrable martingales with $X(0) = 0$ a.s., and $\mathscr{M}_2^c$ for the space of a.s. continuous square integrable martingales.

LEMMA 5.2. *Suppose, for some* $\delta > 0$, $h \in C^2(\overline{D} \cap \overline{B_\delta(0)})$ *with* $\gamma \cdot \nabla h = 0$ *on* $\overline{B_\delta(0)} \cap \partial D \setminus \{0\}$, $h(0) = 0$ *and* $\Delta h \ge 0$. *If* $h(Z_{t\wedge\sigma_\delta}) - \frac{1}{2}\int_0^{t\wedge\sigma_\delta} \Delta h(Z_s)\, ds$ *is a* P_x*-submartingale with square integrable martingale part* M_h, *then*

$$\int_0^{t\wedge\sigma_\delta} I(Z(s) \in \partial D)\, d\langle M_h, M_h\rangle(s) = 0.$$

PROOF. For each integer $n \ge 1$ there is a continuous function $k_n: \mathbb{R} \to [0, \infty)$ [see the proof of Lemma 2.5 in DeBlassie (1990)] such that $k_n \equiv 0$ in a neighborhood of 0, $0 \le k_n' \le 1$, $k_n'' \ge 0$, $k_n'(t) \to I_{(0,\infty)}(t)$ as $n \to \infty$ and, for some $C > 0$,

$$|k_n(t) - t \vee 0| \le Cn^{-1}.$$

Write

$$G(t) = h(Z_{t \wedge \sigma_\delta}) = h(x) + M_h(t) + A_h(t) + \tfrac{1}{2}\int_0^{t \wedge \sigma_\delta} \Delta h(Z_s)\, ds, \tag{5.1}$$

where $M_h \in \mathscr{M}_2^c$ and $A_h(t)$ is a continuous adapted increasing process. Then, for $n \ge 1$,

$$\begin{aligned} k_n(G(t)) &= k_n(G(0)) + \int_0^t k_n'(G)[dM_h + dA_h] \\ &\quad + \tfrac{1}{2}\int_0^{t \wedge \sigma_\delta} k_n'(G_s)\, \Delta h(Z_s)\, ds + \tfrac{1}{2}\int_0^t k_n''(G)\, d\langle M_h, M_h\rangle. \end{aligned} \tag{5.2}$$

By dominated convergence and the Burkholder–Davis–Gundy inequalities, as $n \to \infty$ the martingale part in (5.2) converges in L^2, uniformly for t in compact sets to $\int_0^t I(G > 0)\, dM_h$. By dominated convergence, the dA_h and ds parts converge in L^1, uniformly for t in compact sets, to

$$\int_0^t I(G > 0)\, dA_h \quad \text{and} \quad \tfrac{1}{2}\int_0^{t \wedge \sigma_\delta} I(G > 0)\, \Delta h(Z_s)\, ds,$$

respectively. The remaining integrand on the right-hand side of (5.2) is nonnegative and, as $n \to \infty$, $k_n(G_t)$ converges in L^1, uniformly for t in compact sets, to $0 \vee G(t)$. Then the last term on the right-hand side of (5.2) converges in L^1 to a continuous adapted increasing process $\mathscr{A}_h^+$. Thus (5.2) becomes

$$\begin{aligned} 0 \vee h(Z_{t \wedge \sigma_\delta}) &= 0 \vee G(t) \\ &= 0 \vee h(x) \\ &\quad + \int_0^{t \wedge \sigma_g} I(G > 0)\left[dM_h + dA_h + \tfrac{1}{2}\, \Delta h(Z_s)\, ds\right] + \mathscr{A}_h^+. \end{aligned} \tag{5.3}$$

This argument also works for $k_n(-G)$, giving

$$\begin{aligned} &0 \vee \left(-h(Z_{t \wedge \sigma_\delta})\right) \\ &\quad = 0 \vee (-h(x)) - \int_0^{t \wedge \sigma_\delta} I(G < 0)\left[dM_h + dA_h + \tfrac{1}{2}\, \Delta h(Z_s)\, ds\right] + \mathscr{A}_h^-, \end{aligned}$$

where $\mathscr{A}_h^-$ is a continuous adapted increasing process. Subtracting this from (5.3) gives

$$\begin{aligned} h(Z_{t \wedge \sigma_\delta}) = h(x) &+ \int_0^{t \wedge \sigma_\delta} I(G \ne 0)\left[dM_h + dA_h + \tfrac{1}{2}\, \Delta h(Z_s)\, ds\right] \\ &+ (\mathscr{A}_h^+ - \mathscr{A}_h^-)(t). \end{aligned}$$

Comparing martingale parts with (5.1), we see that

$$\int_0^{t\wedge\sigma_\delta} I(h(Z_s)=0)\,d\langle M_h, M_h\rangle(s) = 0. \tag{5.4}$$

Given $\gamma > 0$, define a sequence of stopping times ($n = 0, 1, 2, \dots$)

$$T_0 = 0$$
$$T_{2n+1} = \inf\{t \ge T_{2n} : |Z_t| = \gamma\},$$
$$T_{2n+2} = \inf\{t \ge T_{2n+1} : |Z_t| = \gamma/2\}.$$

Now, on $[T_{2n-1}, T_{2n}]$, $Z(\cdot)$ stays away from $\overline{D} \cap B_{\gamma/2}(0)$, so the theory of Stroock and Varadhan [(1971), Theorem 2.5] applies and yields

$$\begin{aligned}\int_{\sigma_\delta\wedge T_{2n-1}}^{\sigma_\delta\wedge T_{2n}} & I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)\,d\langle M_h, M_h\rangle(s)\\ &= \int_{\sigma_\delta\wedge T_{2n-1}}^{\sigma_\delta\wedge T_{2n}} I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)|\nabla h(Z_s)|^2\,ds\\ &= 0\end{aligned}$$

and

$$\int_{\sigma_\delta\wedge T_{2n-1}}^{\sigma_\delta\wedge T_{2n}} I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)\,ds = 0$$

(using 2 and 5 of that theorem). Also, since $|Z_s| \le \gamma$ for $s \in [T_{2n}, T_{2n+1}]$, we have

$$\int_{\sigma_\delta\wedge T_{2n}}^{\sigma_\delta\wedge T_{2n+1}} I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)\,d\langle M_h, M_h\rangle(s) = 0,$$

$$\int_{\sigma_\delta\wedge T_{2n}}^{\sigma_\delta\wedge T_{2n+1}} I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)\,ds = 0.$$

Thus, by Lemma 3.3,

$$\int_0^{t\wedge\sigma_\delta} I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)\,d\langle M_h, M_h\rangle(s) = 0,$$

$$\int_0^{t\wedge\sigma_\delta} I\big(Z(s)\in \overline{B_\gamma(0)}^c \cap \partial D\big)\,ds = 0;$$

and upon letting $\gamma \to 0$ we obtain

$$\int_0^{t\wedge\sigma_\delta} I(Z(s)\in \partial D\setminus\{0\})\,d\langle M_h, M_h\rangle(s) = 0,$$

$$\int_0^{t\wedge\sigma_\delta} I(Z(s)\in \partial D\setminus\{0\})\,ds = 0.$$

Combined with (5.4) [using $h(0) = 0$] and (2.3) we get

$$\int_0^{t\wedge\sigma_\delta} I(Z(s)\in \partial D)\,d\langle M_h, M_h\rangle(s) \equiv 0,$$

$$\int_0^{t\wedge\sigma_\delta} I_{\partial D}(Z_s)\,ds = 0. \qquad \square$$

COROLLARY 5.3.

(i) *With h as in Lemma* 5.2,

$$\int_0^{t\wedge\sigma_\delta} I\big(h(Z_s)=0\big)\,d\langle M_h, M_h\rangle(s)=0.$$

(ii) *We have* $\int_0^{t\wedge\sigma_\delta} I_{\partial D}(Z_s)\,ds=0$.

THEOREM 5.4. *Let P_x be a solution of the submartingale problem in D starting from x.*

(i) *Suppose, for some $\delta>0$, $h\in C^2(\overline{D}\cap\overline{B_\delta(0)})$ with $\gamma\cdot\nabla h\ge 0$ on $\overline{B_\delta(0)}\cap\partial D\setminus\{0\}$, $h(0)=0$ and $\Delta h\ge 0$. If $h(Z_{t\wedge\sigma_\delta})-\frac{1}{2}\int_0^{t\wedge\sigma_\delta}\Delta h(Z_s)\,ds$ is a submartingale with square integrable martingale part M_h, then*

$$\langle M_h, M_h\rangle(t\wedge\sigma_\delta)=\int_0^{t\wedge\sigma_\delta}|\nabla h(Z_s)|^2\,ds.$$

(ii) *Suppose, for the same δ, $g\in C^2(\overline{D}\cap\overline{B_\delta(0)})$ with $\gamma\cdot\nabla g\ge 0$ on $\overline{B_\delta(0)}\cap\partial D\setminus\{0\}$, $g(0)=0$ and $\Delta g=0$. If h is as above with $\gamma\cdot\nabla h=0$ on $\overline{B_\delta(0)}\cap\partial D\setminus\{0\}$ and $\Delta h=0$ and if $g(Z_{t\wedge\sigma_\delta})$ is a submartingale with square integrable martingale part M_g, then*

$$\langle M_h, M_g\rangle(t\wedge\sigma_\delta)=\int_0^{t\wedge\sigma_\delta}(\nabla h\cdot\nabla g)(Z_s)\,ds.$$

PROOF. By the submartingale property applied to $k_n\circ h$ and $k_n\circ(-h)$,

$$\begin{aligned}k_n\circ h\big(Z_{t\wedge\sigma_\delta}\big)&=k_n\circ h(x)+M_h^{(n)}(t)+A_h^{(n)}(t)\\&\qquad+\tfrac{1}{2}\int_0^{t\wedge\sigma_\delta}\Big[[k_n''\circ h]|\nabla h|^2+[k_n'\circ h]\,\Delta h\Big](Z_s)\,ds\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}&k_n\circ(-h)\big(Z_{t\wedge\sigma_\delta}\big)\\&\quad=k_n\circ(-h)(x)+M_{-h}^{(n)}(t)-A_{-h}^{(n)}(t)\\&\qquad+\tfrac{1}{2}\int_0^{t\wedge\sigma_\delta}\Big[[k_n''\circ(-h)]|\nabla h|^2-[k_n'\circ(-h)]\,\Delta h\Big](Z_s)\,ds,\end{aligned}\tag{5.6}$$

where $M_{\pm h}^{(n)}\in\mathscr{M}_2^c$ and $A_{\pm h}^{(n)}$ are continuous, adapted, increasing processes. By Lemma 5.1,

$$\int_0^{t\wedge\sigma_\delta} I_D(Z(s))\,dA_{\pm h}^{(n)}(s)=0.\tag{5.7}$$

Step 1. *Computation of $\langle M_h^{(n)}, M_h^{(n)}\rangle$ and $\langle M_{-h}^{(n)}, M_{-h}^{(n)}\rangle$*. An application of Itô's formula using (5.5) gives

$$\begin{aligned}k_n^2\circ h\big(Z_{t\wedge\sigma_\delta}\big)&=k_n^2\circ h(x)+\int_0^{t\wedge\sigma_\delta}2k_n\circ h(Z(s))\big[dM_h^{(n)}(s)+dA_h^{(n)}(s)\big]\\&\quad+\int_0^{t\wedge\sigma_\delta}\Big[[k_n\circ h]\big\{[k_n''\circ h]|\nabla h|^2+[k_n'\circ h]\,\Delta h\big\}\Big](Z_s)\,ds\\&\quad+\langle M_h^{(n)}, M_h^{(n)}\rangle(t\wedge\sigma_\delta).\end{aligned}\tag{5.8}$$

On the other hand, by the submartingale property applied to $f = k_n^2 \circ h$,

$$k_n^2 \circ h(Z_{t \wedge \sigma_\delta}) = k_n^2 \circ h(x) + \tilde{M}_h^{(n)}(t) + \tilde{A}_h^{(n)}(t)$$
$$+ \int_0^{t \wedge \sigma_\delta} \Big\{ \big[[k_n' \circ h]^2 + [k_n \circ h][k_n'' \circ h] \big] |\nabla h|^2 \tag{5.9}$$
$$+ [k_n \circ h][k_n' \circ h] \Delta h \Big\}(Z_s)\, ds$$

where $\tilde{M}_h^{(n)} \in \mathscr{M}_2^c$ and $\tilde{A}_h^{(n)}$ is a continuous, adapted, increasing process. Moreover, by Lemma 5.1,

$$\int_0^{t \wedge \sigma_\delta} I_D(Z(s))\, d\tilde{A}_h^{(n)}(s) = 0. \tag{5.10}$$

Comparing finite variation parts of (5.8) and (5.9),

$$\begin{aligned} &\int_0^{t \wedge \sigma_\delta} 2k_n \circ h(Z(s))\, dA_h^{(n)}(s) + \langle M_h^{(n)}, M_h^{(n)} \rangle (t \wedge \sigma_\delta) \\ &\qquad = \tilde{A}_h^{(n)}(t \wedge \sigma_\delta) + \int_0^{t \wedge \sigma_\delta} [k_n' \circ h]^2 |\nabla h|^2 (Z_s)\, ds. \end{aligned} \tag{5.11}$$

By (5.7) and (5.10), the $A_h^{(n)}$ and $\tilde{A}_h^{(n)}$ pieces are supported on $\{s: Z_s \in \partial D\}$. By Lemma 5.2 and Corollary 5.3(ii), the ds and $\langle M_h^{(n)}, M_h^{(n)} \rangle$ parts do not change on this set. Hence by (5.11),

$$\langle M_h^{(n)}, M_h^{(n)} \rangle (t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} [k_n' \circ h]^2 |\nabla h|^2 (Z_s)\, ds. \tag{5.12}$$

A similar argument shows

$$\langle M_{-h}^{(n)}, M_{-h}^{(n)} \rangle (t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} [k_n' \circ (-h)]^2 |\nabla h|^2 (Z_s)\, ds. \tag{5.13}$$

This completes Step 1.

Step 2. *Proof of part* (i). Since $h(Z_{t \wedge \sigma_\delta}) - \int_0^{t \wedge \sigma_\delta} \frac{1}{2} \Delta h(Z_s)\, ds$ is a submartingale, we can write

$$h(Z_{t \wedge \sigma_\delta}) = h(x) + M_h(t \wedge \sigma_\delta) + A_h(t \wedge \sigma_\delta) + \tfrac{1}{2} \int_0^{t \wedge \sigma_\delta} \Delta h(Z_s)\, ds, \tag{5.14}$$

where $M_h \in \mathscr{M}_2^c$ and A_h is a continuous adapted increasing process. By Itô's formula,

$$k_n \circ h(Z_{t \wedge \sigma_\delta}) = k_n \circ h(x)$$
$$+ \int_0^{t \wedge \sigma_\delta} k_n' \circ h(z(s)) \big[dM_h(s) + dA_h(s) + \tfrac{1}{2} \Delta h(Z_s)\, ds \big]$$
$$+ \tfrac{1}{2} \int_0^{t \wedge \sigma_\delta} k_n'' \circ h(Z(s))\, d\langle M_h, M_h \rangle (s).$$

Comparing martingale parts with (5.5),

$$M_h^{(n)}(t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} k_n' \circ h(Z(s))\, dM_h(s).$$

By (5.12) this gives

$$\int_0^{t \wedge \sigma_\delta} [k_n' \circ h]^2 (Z(s))\, d\langle M_h, M_h \rangle(s) = \int_0^{t \wedge \sigma_\delta} [k_n' \circ h]^2 |\nabla h|^2 (Z(s))\, ds.$$

Letting $n \to \infty$ yields

$$\int_0^{t \wedge \sigma_\delta} I(h(Z(s)) > 0)\, d\langle M_h, M_h \rangle(s) = \int_0^{t \wedge \sigma_\delta} I(h(Z(s)) > 0) |\nabla h(Z_s)|^2\, ds.$$

Similarly,

$$\int_0^{t \wedge \sigma_\delta} I(h(Z(s)) < 0)\, d\langle M_h, M_h \rangle(s) = \int_0^{t \wedge \sigma_\delta} I(h(Z(s)) < 0) |\nabla h(Z(s))|^2\, ds.$$

Adding,

$$\int_0^{t \wedge \sigma_\delta} I(h(Z(s)) \neq 0)\, d\langle M_h, M_h \rangle(s) = \int_0^{t \wedge \sigma_\delta} I(h(Z(s)) \neq 0) |\nabla h(Z(s))|^2\, ds.$$

By Corollary 5.3(i), this is the same as

$$\langle M_h, M_h \rangle(t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} I(h(Z(s)) \neq 0) |\nabla h(Z(s))|^2\, ds.$$

Much like the proof of Lemma 5.2, we can show [also using (2.3)] that

$$\int_0^{t \wedge \sigma_\delta} I(h(Z(s)) = 0) |\nabla h(Z(s))|^2\, ds = 0. \tag{5.15}$$

Hence

$$\langle M_h, M_h \rangle(t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} |\nabla h(Z(s))|^2\, ds,$$

completing the proof of part (i).

Step 3. *Computation of* $\langle M_{\pm h}^{(n)}, M_{\pm g}^{(n)} \rangle (t \wedge \sigma_\delta)$. For the rest of the proof, we assume the hypotheses of part (ii) are in force. Write

$$g(Z_{t \wedge \sigma_\delta}) = g(x) + M_g(t) + A_g(t), \tag{5.16}$$

where $M_g \in \mathscr{M}_2^c$ and A_g is a continuous adapted increasing process with (using Lemma 5.1)

$$\int_0^{t \wedge \sigma_\delta} I_D(Z(s))\, dA_g(s) = 0. \tag{5.17}$$

By Itô's formula and (5.14)–(5.16),

$$
\begin{aligned}
&[k_n \circ h][k_n \circ g](Z_{t \wedge \sigma_\delta}) \\
&\quad = [k_n \circ h][k_n \circ g](x) \\
&\qquad + \int_0^{t \wedge \sigma_\delta} [k_n \circ g][k_n' \circ h](Z(s))[dM_h(s) + dA_h(s)] \\
&\qquad + \tfrac{1}{2}\int_0^{t \wedge \sigma_\delta} [k_n \circ g][k_n'' \circ h](Z(s))\, d\langle M_h, M_h\rangle(s) \\
&\qquad + \int_0^{t \wedge \sigma_\delta} [k_n \circ h][k_n' \circ g](Z(s))[dM_g(s) + dA_g(s)] \\
&\qquad + \tfrac{1}{2}\int_0^{t \wedge \sigma_\delta} [k_n \circ h][k_n'' \circ g](Z(s))\, d\langle M_g, M_g\rangle(s) \\
&\qquad + \int_0^{t \wedge \sigma_\delta} [k_n' \circ h][k_n' \circ g](Z(s))\, d\langle M_h, M_g\rangle(s) \qquad (5.18)\\
&\quad = [k_n \circ h][k_n \circ g](x) \\
&\qquad + \int_0^{t \wedge \sigma_\delta} [k_n \circ g][k_n' \circ h](Z(s))[dM_h(s) + dA_h(s)] \\
&\qquad + \int_0^{t \wedge \sigma_\delta} [k_n \circ h][k_n' \circ g](Z(s))[dM_g(s) + dA_g(s)] \\
&\qquad + \tfrac{1}{2}\int_0^{t \wedge \sigma_\delta} \{[k_n \circ g][k_n'' \circ h]|\nabla h|^2 \\
&\qquad\qquad + [k_n \circ h][k_n'' \circ g]|\nabla g|^2\}(Z(s))\, ds \\
&\qquad + \int_0^{t \wedge \sigma_\delta} [k_n' \circ h][k_n' \circ g](Z(s))\, d\langle M_h, M_g\rangle(s),
\end{aligned}
$$

where we have used the identities

$$
\langle M_h, M_h\rangle(t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} |\nabla h(Z(s))|^2\, ds,
$$

$$
\langle M_g, M_g\rangle(t \wedge \sigma_\delta) = \int_0^{t \wedge \sigma_\delta} |\nabla g(Z(s))|^2\, ds
$$

from part (i).

On the other hand, on ∂D,

$$
\begin{aligned}
&\gamma \cdot \nabla\{[k_n \circ h][k_n \circ g]\} \\
&\quad = \gamma \cdot \{[k_n \circ h][k_n' \circ g]\,\nabla g + [k_n' \circ h][k_n \circ g]\,\nabla h\} \qquad (5.19)\\
&\quad = [k_n \circ h][k_n' \circ g]\,\gamma \cdot \nabla g \geq 0
\end{aligned}
$$

since $\gamma \cdot \nabla h = 0$ on ∂D. By the submartingale property applied to $[k_n \circ h][k_n \circ g](Z_t)$,

$$
\begin{aligned}
&[k_n \circ h][k_n \circ g](Z_{t\wedge\sigma_\delta}) \\
&\qquad = [k_n \circ h][k_n \circ g](x) + \tilde{M}^{(n)}_{h,g}(t\wedge\sigma_\delta) + \tilde{A}^{(n)}_{h,g}(t\wedge\sigma_\delta) \\
&\qquad\qquad + \frac{1}{2}\int_0^{t\wedge\sigma_\delta}\Big\{[k_n'' \circ h][k_n \circ g]|\nabla h|^2 + [k_n'' \circ g][k_n \circ h]|\nabla g|^2 \\
&\qquad\qquad\qquad + 2[k_n' \circ h][k_n' \circ g]\,\nabla h \cdot \nabla g\Big\}(Z(s))\,ds,
\end{aligned}
$$

where $\tilde{M}^{(n)}_{h,g} \in \mathscr{M}^c_2$ and $\tilde{A}^{(n)}_{h,g}$ is a continuous adapted increasing process with

$$
\int_0^{t\wedge\sigma_\delta} I_D(Z(s))\,d\tilde{A}_{h,g}(s) = 0, \tag{5.20}
$$

by Lemma 5.1.

Comparing finite variation parts with (5.18),

$$
\begin{aligned}
&\tilde{A}^{(n)}_{h,g}(t\wedge\sigma_\delta) + \int_0^{t\wedge\sigma_\delta}\{[k_n' \circ h][k_n' \circ g]\,\nabla h\cdot\nabla g\}(Z(s))\,ds \\
&\qquad = \int_0^{t\wedge\sigma_\delta}[k_n \circ g][k_n' \circ h](Z(s))\,dA_h(s) \\
&\qquad\quad + \int_0^{t\wedge\sigma_\delta}[k_n' \circ g][k_n \circ h](Z(s))\,dA_g(s) \\
&\qquad\quad + \int_0^{t\wedge\sigma_\delta}[k_n' \circ h][k_n' \circ g](Z(s))\,d\langle M_h, M_g\rangle(s).
\end{aligned}
$$

By (5.17) and its analogue for h and (5.20), the $\tilde{A}^{(n)}_{h,g}$, A_g and A_h parts are supported on $\{s\colon Z(s) \in \partial D\}$. By Corollary 5.3(ii), Lemma 5.2 and the Kunita–Watanabe inequalities, the remaining finite variation parts do not change on $\{s\colon Z(s) \in \partial D\}$. Thus

$$
\begin{aligned}
&\int_0^{t\wedge\sigma_\delta}[k_n' \circ h][k_n' \circ g]\,\nabla h\cdot\nabla g(Z(s))\,ds \\
&\qquad = \int_0^{t\wedge\sigma_\delta}[k_n' \circ h][k_n' \circ g]\,d\langle M_h, M_g\rangle(s).
\end{aligned}
$$

Let $n \to \infty$, to get

$$
\begin{aligned}
&\int_0^{t\wedge\sigma_\delta} I(h(Z_s) > 0) I(g(Z_s) > 0)\,\nabla h\cdot\nabla g(Z_s)\,ds \\
&\qquad = \int_0^{t\wedge\sigma_\delta} I(h(Z_s) > 0) I(g(Z_s) > 0)\,d\langle M_h, M_g\rangle(s).
\end{aligned} \tag{5.21}
$$

We can repeat this argument for $[k_n \circ h][k_n \circ (-g)]$, $[k_n \circ (-h)] \times [k_n \circ (-g)]$ and $[k_n \circ (-h)][k_n \circ g]$; the key is that in any of these cases, the analogue of (5.19) is always nonnegative or always nonpositive. In any event, we can show the following:

$$\int_0^{t\wedge\sigma_\delta} I(h(Z_s) > 0)I(g(Z_s) < 0)\,\nabla h\cdot\nabla g(Z_s)\,ds = \int_0^{t\wedge\sigma_\delta} I(h(Z_s) > 0)I(g(Z_s) < 0)\,d\langle M_h, M_g\rangle(s);$$

$$\int_0^{t\wedge\sigma_\delta} I(h(Z_s) < 0)I(g(Z_s) < 0)\,\nabla h\cdot\nabla g(Z_s)\,ds = \int_0^{t\wedge\sigma_\delta} I(h(Z_s) < 0)I(g(Z_s) < 0)\,d\langle M_h, M_g\rangle(s);$$

$$\int_0^{t\wedge\sigma_\delta} I(h(Z_s) < 0)I(g(Z_s) > 0)\,\nabla h\cdot\nabla g(Z_s)\,ds = \int_0^{t\wedge\sigma_\delta} I(h(Z_s) < 0)I(g(Z_s) > 0)\,d\langle M_h, M_g\rangle(s).$$

Adding these to (5.21), we get

$$\int_0^{t\wedge\sigma_\delta} I(h(Z_s) \neq 0)I(g(Z_s) \neq 0)\,\nabla h\cdot\nabla g(Z(s))\,ds = \int_0^{t\wedge\sigma_\delta} I(h(Z_s) \neq 0)I(g(Z_s) \neq 0)\,d\langle M_h, M_g\rangle(s).$$

By Corollary 5.3(i) and (5.15) and its analogue for g, this is the same as

$$\int_0^{t\wedge\sigma_\delta} \nabla h\cdot\nabla g(Z(s))\,ds = \langle M_h, M_g\rangle(t\wedge\sigma_\delta).$$

This gives the conclusion of part (ii), and the proof of Theorem 5.4 is complete. □

Now we can easily prove Theorem 2.3. Assume the hypotheses of Section 2 and Theorem 2.3 are in effect. Let $h(y_1, y_2) = y_1$ and $g(y_1, y_2) = y_2$. Let K be from Theorem 4.1, and let β be as in Section 4. Choose δ so small that $\{x \in \overline{D}: |x| \le \delta\} \subseteq \{x \in \overline{D}: \Psi_\beta(x) < K\}$. Then $\sigma_\delta \le \tau_K$ and so, by Theorems 4.1 and 5.4,

$$\begin{aligned} Z(t\wedge\sigma_\delta) &= \big(h(Z(t\wedge\sigma_\delta)), g(Z(t\wedge\sigma_\delta))\big) \\ &= z + \big(M_h(t\wedge\sigma_\delta), M_g(t\wedge\sigma_\delta)\big) + \big(0, A_g(t\wedge\sigma_\delta)\big), \end{aligned} \tag{5.22}$$

where $(M_h(t\wedge\sigma_\delta), M_g(t\wedge\sigma_\delta))$ is two-dimensional Brownian motion stopped at time σ_δ and A_g is a continuous adapted increasing process. Moreover, by Lemma 5.1, A_g changes only on $\{s: Z_s \in \partial D\}$.

As pointed out in Section 2, this is enough to imply Theorem 2.3.

THEOREM 5.5. *Suppose $h \in C(\overline{D}) \cap C^2(\overline{D} \setminus \{0\})$ with $h(0) = 0$, $\gamma\cdot\nabla h \ge 0$ on $\partial D \setminus \{0\}$ and Δh bounded and nonnegative on $\overline{D} \setminus \{0\}$. Then, for each*

$m \geq |x|$ *with* $\sigma_m = \inf\{t \geq 0\colon |\omega(t)| = m\}$,

$$\int_0^{\sigma_m} |\nabla h(Z_u)|^2 I\big(h(Z_u) > 0\big)\, du < \infty \quad a.s.\ P_x.$$

PROOF. By the submartingale property,

$$\begin{aligned} k_n \circ h(Z_t) &= k_n \circ h(x) + M_n(t) + A_n(t) \\ &\quad + \tfrac{1}{2}\int_0^t \Big[[k_n'' \circ h]|\nabla h|^2 + [k_n' \circ h]\,\Delta h\Big](Z_u)\, du, \end{aligned}$$

where M_n is a P_x-local martingale and $A_n(t)$ is a continuous adapted increasing process. By Theorem 5.4, for each $m \geq |x|$,

$$\langle M_n, M_n\rangle(t \wedge \sigma_m) = \int_0^{t \wedge \sigma_m} \big|[k_n' \circ h]\,\nabla h(Z_s)\big|^2\, ds.$$

By Itô's formula,

$$\begin{aligned} &E^{P_x}\Big[[k_n \circ h]^2(Z_{t\wedge\sigma_m})\Big] \\ &\quad = [k_n \circ h]^2(x) \\ &\qquad + E^{P_x}\bigg[\int_0^{t\wedge\sigma_m} 2[k_n \circ h](Z_u)\Big\{dA_n(u) + \tfrac{1}{2}\Big[[k_n'' \circ h]|\nabla h|^2 \\ &\qquad\qquad + [k_n' \circ h]\,\Delta h\Big](Z_u)\, du\Big\}\bigg] \\ &\qquad + E^{P_x}\bigg[\int_0^{t\wedge\sigma_m} \big|[k_n' \circ h]|\nabla h|(Z_u)\big|^2\, du\bigg] \\ &\quad \geq E^{P_x}\bigg[\int_0^{t\wedge\sigma_m} \big|[k_n' \circ h]|\nabla h|^2(Z_u)\big|^2\, du\bigg]. \end{aligned}$$

By dominated convergence on the left and Fatou's lemma on the right,

$$\infty > E^{P_x}\Big[\big[0 \vee h(Z_{\sigma_m})\big]^2\Big] \geq E^{P_x}\bigg[\int_0^{\sigma_m} I\big(h(Z_u) > 0\big)|\nabla h(Z_u)|^2\, du\bigg].$$

Thus

$$\int_0^{\sigma_m} I\big(h(Z_u) > 0\big)|\nabla h(Z_u)|^2\, du < \infty \quad \text{a.s. } P_x,$$

giving the desired conclusion. □

6. Conformal invariance of RBM. Let D and $\mathscr{D}$ be domains in $\mathbb{R}^2$ with $0 \in \partial D \cap \partial\mathscr{D}$. Suppose $F\colon \overline{D} \to \overline{\mathscr{D}}$ is a homeomorphism such that $F(0) = 0$, $F\colon \overline{D} \setminus \{0\} \to \overline{\mathscr{D}} \setminus \{0\}$ is a C^2 diffeomorphism and $F\colon D \to \mathscr{D}$ is holomorphic with a holomorphic inverse. Denote by $Z(\cdot)$ and $\mathscr{Z}(\cdot)$ the coordinate processes on $\Omega_{\overline{D}}$ and $\Omega_{\overline{\mathscr{D}}}$, respectively.

Given an angle of reflection $\theta(x)$, $x \in \partial D \setminus \{0\}$, F induces an angle of reflection θ_F on $\partial \mathscr{D} \setminus \{0\}$ via

$$\theta_F(y) = \theta \circ F^{-1}(y), \qquad y \in \partial \mathscr{D} \setminus \{0\}. \tag{6.1}$$

Let $|x| \le \delta$ and write

$$\sigma_\delta = \sigma_\delta(Z) = \inf\{t \ge 0 \colon |Z(t)| = \delta\}.$$

Suppose P_x is a solution of the submartingale problem on D stopped at time σ_δ, starting from x. Define

$$A^Z(t) = \int_0^{t \wedge \sigma_\delta} |F'(Z_u)|^2 I(Z_u \ne 0)\, du. \tag{6.2}$$

If

$$P_x(\sigma_\delta(Z) < \infty) = 1 \tag{6.3}$$

and

$$P_x\big(A^Z(\sigma_\delta(Z)) < \infty\big) = 1, \tag{6.4}$$

then, since F is conformal away from 0 and $Z(\cdot)$ spends 0 Lebesgue time at 0 a.s. P_x, A^Z is a.s. continuous and strictly increasing on $[0, \sigma_\delta(Z)]$ with an a.s. continuous strictly increasing inverse a^Z on $[0, A^Z(\sigma_\delta)]$. Setting

$$\mathscr{Z}^Z(t) = F\big(Z\big(a^Z\big(t \wedge A^Z(\sigma_\delta)\big)\big)\big), \qquad t \ge 0, \tag{6.5}$$

it is not hard to show that the law of $\mathscr{Z}^Z$ on $(\Omega_{\overline{\mathscr{D}}}, \mathscr{M})$ solves the submartingale problem on $\mathscr{D}$ stopped at time $A^Z(\sigma_\delta)$, starting from $F(x)$. Note that here the angle of reflection is θ_F.

Conversely, for $\mathscr{F} = F^{-1}$ and

$$\eta_\delta = \eta_\delta(\mathscr{Z}) = \inf\{t \ge 0 \colon |\mathscr{F}(\mathscr{Z}_t)| = \delta\},$$

given a solution $\mathscr{P}_{F(x)}$ of the submartingale problem on $\mathscr{D}$ (with angle of reflection θ_F) stopped at time η_ε, starting from $F(x)$, set

$$R^{\mathscr{Z}}(t) = \int_0^{t \wedge \eta_\delta} |\mathscr{F}'(\mathscr{Z}_u)|^2 I(\mathscr{Z}_u \ne 0)\, du. \tag{6.6}$$

If

$$\mathscr{P}_{F(x)}\big(\eta_\delta(\mathscr{Z}) < \infty\big) = 1 \tag{6.7}$$

and

$$\mathscr{P}_{F(x)}\big(R^{\mathscr{Z}}(\eta_\delta(\mathscr{Z})) < \infty\big) = 1, \tag{6.8}$$

then $R^{\mathscr{Z}}$ is a.s. continuous and strictly increasing on $[0, \eta_\delta(\mathscr{Z})]$ with an a.s. continuous and strictly increasing inverse $r^{\mathscr{Z}}$ on $[0, R^{\mathscr{Z}}(\eta_\delta)]$. Moreover,

$$Z^{\mathscr{Z}}(t) = \mathscr{F}\big(\mathscr{Z}\big(r^{\mathscr{Z}}\big(t \wedge R^{\mathscr{Z}}(\eta_\delta)\big)\big)\big), \qquad t \ge 0, \tag{6.9}$$

solves the submartingale problem on D stopped at time $R^{\mathscr{Z}}(\eta_\delta)$, starting from x.

THEOREM 6.1. *The mappings* (6.5) *and* (6.9) *give rise to a one-to-one correspondence between solutions P_x of the submartingale problem on D stopped at time σ_δ satisfying* (6.3)–(6.4) *and solutions $\mathscr{P}_{F(x)}$ of the submartin-*

gale problem on $\mathscr{D}$ stopped at time η_δ satisfying (6.7)–(6.8). *The angles of reflection for D and $\mathscr{D}$ are $\theta(\cdot)$ and $\theta_F(\cdot)$, respectively.*

The proof is left to the reader [cf. DeBlassie and Toby (1993), Proposition 4.1].

7. Half-space: $S = S_\pi$. In this section we prove Theorem 1.1 for $\xi = \pi$. For $S_\pi^0 = \{z \in S_\pi\colon \operatorname{Im} z > 0\}$ define

$$(7.1)\qquad \psi(z) = \exp\left\{\int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\left[\frac{1}{x-z} - \frac{x}{1+x^2}\right]dx\right\}, \qquad z \in S_\pi^0.$$

Then $i\psi$ is known as a *Pick function* [Donoghue (1974)]: it is analytic from S_π^0 into S_π^0.

LEMMA 7.1. *The function ψ has an extension in $C^1(S_\pi \setminus \{0\})$ and* $\operatorname{Re}\psi > 0$ *on $S_\pi \setminus \{0\}$.*

PROOF. The proof of smoothness is almost exactly like the first part of Step 2 in the proof of Theorem 1.1 in Burdzy and Marshall (1992). The major difference is that they use local Hölder continuity of the first partials of $\theta(\cdot)$ on ∂S_1 and on ∂S_2, whereas we require local Dini continuity of those partials. This causes no trouble because the theorem of Zygmund [(1979), page 54] they use applies in this context too.

The last assertion of the lemma is a consequence of our hypotheses on θ. □

LEMMA 7.2. *For $x \in \partial S_\pi \setminus \{0\}$*, $\arg\psi(x) = \theta(x)$. *Moreover, for some positive C_1, C_2, ε and p, with $p < 1$,*

$$C_1\eta^p \le |\psi(z)| \le C_2\eta^{-p}, \qquad z = \zeta + i\eta \in S_\pi \cap B_\varepsilon(0) \setminus \{0\}.$$

PROOF. This follows from (1.3) and is left to the reader. □

A *smooth path in S_π from 0 to $z \in S_\pi$* is a C^∞-mapping $\ell\colon [0,a] \to S_\pi$ with $\ell(0) = 0$, $\ell(a) = z$ and $|\ell'(t)| > 0$ for $t \in [0,a]$. If $\ell'(0)$ is not horizontal, then we say ℓ is *nontangential at* 0. Define

$$\Psi(z) = \int_0^z \psi(w)\,dw, \qquad z \in S_\pi,$$

where the integral is over any simple smooth path in S_π from 0 to z that is nontangential at 0.

LEMMA 7.3. *The function Ψ is well defined on S_π, a homeomorphism onto its image, a C^2-diffeomorphism from $S_\pi \setminus \{0\}$ onto $\Psi(S_\pi \setminus \{0\})$, holomorphic on $S_\pi \setminus \{0\}$ with a holomorphic inverse on $\Psi(S_\pi \setminus \{0\})$, and* $\Psi(0) = 0$. *Moreover, for $x \in \partial S_\pi \setminus \{0\}$*, $\arg\Psi'(x) = \arg\psi(x) = \theta(x)$.

PROOF. Let ℓ be any simple smooth path in S_π from 0 to $z \in S_\pi \setminus \{0\}$ that is nontangential at 0. It is no loss to assume ℓ is parameterized by arc length s, say, $\ell\colon [0, L] \to S_\pi$, $\ell = (\ell_1, \ell_2)$. Since ℓ is nontangential at 0, $\ell_2'(0) \neq 0$ and so, for ε from Lemma 7.2, there exist $\delta > 0$ and $C > 0$ such that

$$\begin{aligned} &|\ell(s)| < \varepsilon \quad \text{for } s \le \delta,\\ &|\ell(\delta)| < |z|,\\ &|\ell_2(s)| \ge Cs \quad \text{for } s \le \delta. \end{aligned}$$

Now ℓ is simple, so the image of $\ell|_{[\delta, L]}$ is contained in $S_\pi \setminus \{0\}$. Then since $\psi \in C^1(S_\pi \setminus \{0\})$ and ψ is holomorphic on S_π^0,

$$\int_{\ell|_{[\delta,L]}} \psi(w)\,dw = \int_\gamma \psi(w)\,dw,$$

for any smooth path γ in $S_\pi \cap B_{|\ell(\delta)|}(0)^c$ connecting $\ell(\delta)$ to z. From this it is clear that

$$\left| \int_{\ell|_{[\delta,L]}} \psi(w)\,dw \right| < \infty.$$

As for the other part of ℓ,

$$\begin{aligned} \left| \int_{\ell|_{[0,\delta]}} \psi(w)\,dw \right| &= \left| \int_0^\delta \psi(\ell(s))\ell'(s)\,ds \right|\\ &\le \int_0^\delta |\psi(\ell(s))|\,ds\\ &\le \int_0^\delta C_2 |\ell_2(s)|^{-p}\,ds \quad \text{(by Lemma 7.2)}\\ &\le C_2 \int_0^\delta (Cs)^{-p}\,ds < \infty, \end{aligned}$$

since $p < 1$. Thus,

$$\left| \int_\ell \psi(w)\,dw \right| < \infty, \tag{7.2}$$

and so $\int_\ell \psi(w)\,dw$ exists.

To show Ψ is well defined, let m be a path satisfying the same properties as ℓ and let $\delta > 0$ be small. Now $B_\delta(0)$ will cut each of ℓ and m into two pieces: the parts $\ell^{(1)}$ and $m^{(1)}$ starting at 0 until $\partial B_\delta(0)$ is hit, and the parts $\ell^{(2)}$ and $m^{(2)}$ from then on. Then the closed path determined by $\ell^{(2)}$, $m^{(2)}$ and the arc γ of $\partial B_\delta(0)$ determined by the first intersections of ℓ and m with $\partial B_\delta(0)$ is contained in $S_\pi \setminus \{0\}$, since ℓ and m are simple. However, ψ is holomorphic on S_π^0 and in $C^1(S_\pi \setminus \{0\})$, so the integral of $\psi(w)$ along this path must vanish. Hence, to prove Ψ is well defined, it suffices to show, as $\delta \to 0$,

$$\int_{\ell^{(1)}} \psi(w)\,dw \to 0,$$

$$\int_{m^{(1)}} \psi(w)\,dw \to 0$$

and

$$\int_{\gamma} \psi(w)\,dw \to 0.$$

The first two are immediate consequences of (7.2). For the last, observe for $p < 1$ from Lemma 7.2, for some a, $b \in [0, \pi]$,

$$\begin{aligned}\left|\int_{\gamma} \psi(w)\,dw\right| &= \left|\int_a^b \psi(\delta \exp(i\theta))\, i\delta \exp(i\theta)\,d\theta\right| \\ &\le \delta \int_0^{\pi} C_2 (\delta \sin\theta)^{-p}\,d\theta \\ &\le C\delta^{1-p} \\ &\to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

Thus Ψ is well defined on $S_\pi \setminus \{0\}$, holomorphic on S_π^0 and, by setting $\Psi(0) = 0$, continuous on S_π. By Lemma 7.1 and arguments from Burdzy and Marshall (1992), $\Psi \in C^2(S_\pi \setminus \{0\})$ and $\Psi'(z) = \psi(z)$. Since $\operatorname{Re}\psi > 0$ on $S_\pi \setminus \{0\}$, Ψ is one-to-one on $S_\pi \setminus \{0\}$ [cf. Rogers (1991), Lemma 1]. To show Ψ is one-to-one on S_π, it suffices to show $\Psi(z) = 0$ only if $z = 0$. So suppose $\Psi(z) = 0$. If $z \in S_\pi^0$, then taking $\ell(t) = tz$, $t \in [0, 1]$,

$$0 = \Psi(z) = z\int_0^1 \psi(tz)\,dt \ne 0, \quad \text{a contradiction.}$$

If $z \in \partial S_\pi \setminus \{0\}$, then taking ℓ to be the path consisting of the line segments from 0 to $i\delta$, $i\delta$ to ε, then ε to z (where $\delta > 0$, $\varepsilon \in \mathbb{R} \setminus \{0\}$),

$$0 = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\ell} \psi(w)\,dw.$$

By the estimates in Lemma 7.2, the integrals over the first two pieces of ℓ converge to 0, so we get

$$0 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{z} \psi(t)\,dt.$$

Hence

$$0 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{z} \operatorname{Re}\psi(t)\,dt \ne 0, \quad \text{a contradiction.}$$

Thus $z = 0$, as desired.

It is now easy to see that $\Psi\colon S_\pi \setminus \{0\} \to S_\pi \setminus \{0\}$ is a C^2-diffeomorphism, that Ψ on S_π^0 has a holomorphic inverse on $\Psi(S_\pi^0)$ and that Ψ is a homeomorphism from S_π onto $\Psi(S_\pi)$. $\square$

The mapping Ψ was used in Rogers (1991) because it twists around the direction of reflection in S_π so that the reflection in $\Psi(S_\pi)$ is always upward.

LEMMA 7.4. *For each* $\delta > 0$,

$$\sup\{|\Psi'(z)|: z \in S_\pi, |z| \geq \delta\} < \infty,$$
$$\sup\{|(\Psi^{-1})'(w)|: w \in \Psi(S_\pi), |w| \geq \delta\} < \infty.$$

PROOF. Choose $\varepsilon \in (0, \delta)$ such that

$$|z| < \varepsilon \quad \Rightarrow \quad |\Psi(z)| < \delta.$$

Then

$$\begin{aligned}
&\sup\{|(\Psi^{-1})'(w)|: w \in \Psi(S_\pi), |w| \geq \delta\} \\
&\quad = \sup\{|\Psi'(\Psi^{-1}(w))|^{-1}: w \in \Psi(S_\pi), |w| \geq \delta\} \\
&\quad \leq \sup\{|\Psi'(z)|^{-1}: z \in S_\pi, |z| \geq \varepsilon\}.
\end{aligned}$$

Consequently it is enough to show that, for each $a > 0$,

$$0 < \inf_{|z| \geq a} |\Psi'(z)| \leq \sup_{|z| \geq a} |\Psi'(z)| < \infty.$$

Since, for $z = \xi + i\eta \in S_\pi^0$,

$$\begin{aligned}
|\Psi'(z)| &= |\psi(z)| \\
&= \exp\left\{\int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\left[\frac{x-\xi}{(x-\xi)^2+\eta^2} - \frac{x}{1+x^2}\right] dx\right\}
\end{aligned}$$

and since $\Psi \in C^2(S_\pi \setminus \{0\})$, it suffices to show that

$$\sup\left\{\left|\int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\left[\frac{x-\xi}{(x-\xi)^2+\eta^2} - \frac{x}{1+x^2}\right] dx\right| : |\xi + i\eta| \geq 2, \eta > 0\right\} < \infty.$$

However, $\theta(x) = 0$ if $|x| \geq 1$, so

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} \frac{\theta(x)}{\pi}\left[\frac{x-\xi}{(x-\xi)^2+\eta^2} - \frac{x}{1+x^2}\right] dx\right| \\
&\quad = \left|\int_{-1}^{1}\right| \leq \int_{-1}^{1}\left|\frac{\theta(x)}{\pi}\,\frac{x-\xi}{(x-\xi)^2+\eta^2}\right| dx + \int_{-1}^{1}\left|\frac{\theta(x)}{\pi}\,\frac{x}{1+x^2}\right| dx.
\end{aligned}$$

Thus it is enough to prove that

$$\sup\left\{\int_{-1}^{1}\left|\frac{\theta(x)}{\pi}\,\frac{x-\xi}{(x-\xi)^2+\eta^2}\right| dx : |\xi + i\eta| \geq 2, \eta > 0\right\} < \infty.$$

If $|\xi + i\eta| \geq 2$, then

$$\begin{aligned}\int_{-1}^{1}\left|\frac{\theta(x)}{\pi}\frac{x-\xi}{(x-\xi)^2+\eta^2}\right|dx &\leq \int_{-1}^{1}\frac{|\theta(x)|}{\pi}4\,dx\\ &\leq \int_{-1}^{1}\frac{\pi/2}{\pi}\cdot 4\,dx\\ &= 4.\end{aligned}$$

This does the trick. □

LEMMA 7.5. *The set* $\Psi(\partial S_\pi)$ *is the graph of a Lipschitz function and* $\Psi(S_\pi^0)$ *is above this graph. Moreover, for* $\tilde{\theta} = \sup|\theta(x)|$, *the inner and outer cones for* $\Psi(S_\pi^0)$ *at* $\{0\}$ *are, respectively,*

$$\begin{aligned}C_{\mathrm{IN}} &= \{(y_1, y_2)\colon\ y_2 \geq |y_1|\tan\tilde{\theta}\}\\ C_{\mathrm{OUT}} &= \{(y_1, y_2)\colon\ y_2 \leq -|y_1|\tan\tilde{\theta}\}.\end{aligned}$$

PROOF. Note that $\operatorname{Re}\Psi\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is increasing because $\operatorname{Re}\Psi'(t) = \operatorname{Re}\psi(t) > 0$, for $t \neq 0$. Hence $\Psi(\mathbb{R})$ is the graph of a function, say, $y = f(x)$. Writing

$$x(t) = \operatorname{Re}\Psi(t), \qquad y(t) = \operatorname{Im}\Psi(t), \qquad t\in\mathbb{R},$$

gives a parametric representation of $\Psi(\mathbb{R})$, and

$$f'(x) = \frac{\operatorname{Im}\Psi'(t)}{\operatorname{Re}\Psi'(t)} = \frac{\operatorname{Im}\psi(t)}{\operatorname{Re}\psi(t)}$$

exists and is continuous for $x \neq 0$. Hence, to prove f is Lipschitz, it is enough to show that $[\operatorname{Im}\psi(t)]/[\operatorname{Re}\psi(t)]$ is bounded for $t\in\mathbb{R}\setminus\{0\}$. For $z = \zeta + i\eta \in S_\pi^0$,

$$\frac{\operatorname{Im}\psi(z)}{\operatorname{Re}\psi(z)} = \tan\left[\int_{-\infty}^{\infty}\frac{\theta(x)}{\pi}\frac{\eta}{(x-\zeta)^2+\eta^2}\,dx\right].$$

Upon letting $\zeta = t\in\mathbb{R}\setminus\{0\}$ and $\eta\downarrow 0$, since

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{\theta(x)}{\pi}\frac{\eta}{(x-t)^2+\eta^2}\,dx = \theta(t),$$

we have

$$\frac{\operatorname{Im}\psi(t)}{\operatorname{Re}\psi(t)} = \tan\theta(t).$$

This is bounded since $\sup|\theta| < \pi/2$, by hypothesis. The last equality also proves the statement about inner and outer cones at $\{0\}$. □

Now we apply the results of Dupuis and Ishii (1993) to $\Psi(S_\pi)$.

THEOREM 7.6. *Let*

$$\eta_\delta(\omega) = \inf\{t \geq 0: |\Psi^{-1}(\omega_t)| = \delta\}.$$

There is a pathwise unique solution to the SDER up to time η_δ *for* $\Psi(S_\pi^0)$ *with direction of reflection* (0,1), *initial condition* $\Psi(x) \in \Psi(S_\pi)$ *and Brownian motion* $\{B_t: t \geq 0\}$.

PROOF. We verify the Case 1 hypotheses of Dupuis and Ishii (1993). Once this is done, we get the desired conclusion from their Corollary 5.2.

Since the direction of reflection is (0,1), the Case 1 hypotheses reduce to finding $b \in (0,1)$ such that

$$\bigcup_{0 \leq t \leq b} \{y: |x - t\gamma(x) - y| < tb\} \subseteq \Psi(S_\pi^0)^c \quad \text{for } x \in \Psi(\mathbb{R}),$$

which is equivalent to finding a truncated cone

$$C = \{z = (z_1, z_2): z_2 \leq -m|z_1|, |z| \leq 1\}$$

for some $m > 0$, such that, for each $x \in \Psi(\mathbb{R})$,

$$x + C \subseteq \Psi(S_\pi^0)^c.$$

This is an immediate consequence of the fact that $[\operatorname{Im} \psi(t)]/[\operatorname{Re} \psi(t)] = \tan \theta(t)$. $\square$

Combined with Theorem 2.3 and Lemma 7.5, this yields the following result.

THEOREM 7.7. *Reflecting Brownian motion in* $\Psi(S_\pi)$ *stopped at time* η_δ *exists uniquely in law*. $\square$

Now we prove Theorem 1.1 for $\xi = \pi$. First we show uniqueness for RBM stopped upon exiting a small neighborhood of the origin. More precisely, there is some $\varepsilon > 0$ such that there is exactly one solution of the submartingale problem on S_π^0 stopped at σ_ε, starting from $|x| \leq \varepsilon$. This follows from Theorem 6.1 (with $D = S_\pi^0$ and $F = \Psi$) and Theorem 7.7 once we find $\varepsilon > 0$ such that, for $|x| \leq \varepsilon$ and any solutions P_x and $\mathscr{P}_{\Psi(x)}$ of the submartingale problem on S_π^0 and $\Psi(S_\pi^0)$, respectively, stopped at times σ_ε and η_ε, respectively,

$$P_x(\sigma_\varepsilon < \infty) = 1, \tag{7.3}$$

$$\mathscr{P}_{\Psi(x)}(\eta_\varepsilon < \infty) = 1, \tag{7.4}$$

$$P_x\left(\int_0^{\sigma_\varepsilon} |\Psi'(Z_u)|^2 I(Z_u \neq 0)\, du < \infty\right) = 1, \tag{7.5}$$

and

$$\mathscr{P}_{\Psi(x)}\left(\int_0^{\eta_\varepsilon} |(\Psi^{-1})'(\mathscr{Z}_u)|^2 I(\mathscr{Z}_u \neq 0)\, du < \infty\right) = 1. \tag{7.6}$$

With ε from Lemma 3.5, (7.3) holds. Make this ε smaller (if necessary) so that it is smaller than the δ from Theorem 2.3. By the proof of Theorem 2.3, the law of the first component of the coordinate process $\mathscr{Z}(\cdot)$ stopped at η_ε under $\mathscr{P}_{\Psi(x)}$ is one-dimensional Brownian motion stopped at time η_ε. Then since η_γ is the first exit time from a bounded neighborhood of 0, it is clear that, for some $\gamma < \varepsilon$,

$$\mathscr{P}_{\Psi(y)}(\eta_\gamma < \infty) = 1 \quad \text{for } |y| \le \gamma.$$

Then replace ε by this γ to get (7.4).

Next we verify (7.5)–(7.6). By Lemma 7.3,

$$\Psi'(z) = |\Psi'(z)|\exp[i\theta(z)], \qquad z \in \partial S_\pi \setminus \{0\},$$

$$(\Psi^{-1})'(w) = |(\Psi^{-1})'(w)|\exp[-i\theta \circ \Psi^{-1}(w)], \qquad w \in \partial\Psi(S_\pi) \setminus \{0\}.$$

Then, for $h = \operatorname{Re}\Psi$ and $g = \operatorname{Im}\Psi^{-1}$, we have the following:

$$h \in C(S_\pi) \cap C^2(S_\pi \setminus \{0\}), \qquad h(0) = 0,$$

$$g \in C(\Psi(S_\pi)) \cap C^2(\Psi(S_\pi) \setminus \{0\}), \qquad g(0) = 0;$$

$$\nabla h(z) = |\Psi'(z)|(\cos\theta(z), -\sin\theta(z)), \qquad z \in \partial S_\pi \setminus \{0\},$$

$$\nabla g(w) = |(\Psi^{-1})'(w)|(-\sin\theta \circ \Psi^{-1}(w), \cos\theta \circ \Psi^{-1}(w)),$$
$$w \in \partial\Psi(S_\pi) \setminus \{0\},$$

$$\Delta h = 0 \quad \text{on } S_\pi \setminus \{0\},$$

$$\Delta g = 0 \quad \text{on } \Psi(S_\pi) \setminus \{0\}.$$

Hence

$$(\gamma \cdot \nabla h)(z) = (\sin\theta(z), \cos(z)) \cdot \nabla h(z) = 0 \quad \text{on } \partial S_\pi \setminus \{0\},$$

$$(0,1) \cdot \nabla g(w) \ge 0 \quad \text{on } \partial\Psi(S_\pi) \setminus \{0\}.$$

Apply Theorem 5.5 to h, $-h$ and g. Then, since $g \ge 0$,

$$\int_0^{\sigma_\varepsilon} |\nabla h(Z_u)|^2 I(h(Z_u) \ne 0)\, du < \infty \quad \text{a.s. } P_x, \tag{7.7}$$

$$\int_0^{\eta_\varepsilon} |\nabla g(\mathscr{Z}_u)|^2 I(g(\mathscr{Z}_u) \ne 0)\, du < \infty \quad \text{a.s. } P_{\Psi(x)}. \tag{7.8}$$

However, by Lemma 7.5, $\{(y_1, y_2)\colon y_1 = 0,\ y_2 > 0\} \subseteq \Psi(S_\pi^0)$. In particular, under P_x, $Z(\cdot)$ is Brownian motion inside S_π^0 and spends 0 Lebesgue time on $\{u\colon h(Z_u) = 0,\ Z_u \ne 0\}$. Then (7.7) becomes

$$\int_0^{\sigma_\varepsilon} |\nabla h(Z_u)|^2 I(Z_u \ne 0)\, du < \infty \quad \text{a.s. } P_x,$$

and, since $|\Psi'| = |\nabla h|$ on $S_\pi \setminus \{0\}$, (7.5) holds.

Note $\{w \in \Psi(S_\pi) \setminus \{0\}\colon g(w) = 0\} = \partial\Psi(S_\pi) \setminus \{0\}$. Then, under $\mathscr{P}_{\Psi(x)}$, $\mathscr{Z}(\cdot)$ spends 0 Lebesgue time there, and so (7.8) becomes

$$\int_0^{\eta_\varepsilon} |\nabla g(\mathscr{Z}_u)|^2 I(\mathscr{Z}_u \ne 0)\, du < \infty \quad \text{a.s. } P_{\Psi(x)}.$$

Since $|\nabla g| = |(\Psi^{-1})'|$ on $\Psi(S_\pi \setminus 0)$, (7.6) holds.

Hence we have verified (7.3)–(7.6), and so RBM stopped upon exiting a small neighborhood of the origin exists uniquely. Then, by localization, RBM in S_π^0 stopped at the first exit time from any bounded set exists uniquely. To complete the proof, it is necessary to show tht no explosion occurs. However, the process cannot explode because of the results of Varadhan and Williams (1885) and the fact that the reflection is normal outside $B_1(x)$. □

8. Arbitrary wedges: existence and uniqueness. Given $\xi \in (0, 2\pi)$, define

$$F(z) = z^{\xi/\pi}, \qquad z \in S_\pi,$$

so that $F\colon S_\pi \to S_\xi$, $D = S_\pi$ and $\mathscr{D} = S_\xi$ satisfy all the hypotheses in Section 6.

Also, just as in Section 7, we need only prove existence and uniqueness for the submartingale problem stopped upon exiting a small neighborhood of the origin. Then, by Theorem 6.1, it is enough to find $\varepsilon_1 > 0$ such that, for $|x| < \varepsilon_1$,

$$P_x(\sigma_{\varepsilon_1} < \infty) = 1, \tag{8.1}$$

$$\mathscr{P}_{F(x)}(\eta_{\varepsilon_1} < \infty) = 1, \tag{8.2}$$

$$P_x\left(\int_0^{\sigma_{\varepsilon_1}} |Z_u|^{2(\xi/\pi - 1)} I(Z_u \neq 0)\, du < \infty\right) = 1, \tag{8.3}$$

$$\mathscr{P}_{F(x)}\left(\int_0^{\eta_{\varepsilon_1}} |\mathscr{Z}_u|^{2(\pi/\xi - 1)} I(\mathscr{Z}_u \neq 0)\, du < \infty\right) = 1, \tag{8.4}$$

for any solutions P_x and $\mathscr{P}_{F(x)}$ of the submartingale problem on S_π and S_ξ, respectively. Recall that the angle of reflection in S_ξ is $\theta(\cdot)$ and the angle of reflection in S_π is $\theta \circ F$.

With ε from (1.4), for some $\varepsilon_1 > 0$,

$$\pi\alpha_\pi = \xi\alpha_\xi,$$

where

$$\alpha_\pi = \frac{1}{\pi}\big[\sup\{\theta \circ F(w)\colon w \in \partial S_2(\pi) \setminus \{0\}, |w| \le \varepsilon_1\}$$
$$-\inf\{\theta \circ F(w)\colon w \in \partial S_1(\pi) \setminus \{0\}, |w| \le \varepsilon_1\}\big]$$
$$\alpha_\xi = \frac{1}{\xi}\big[\sup\{\theta(z)\colon z \in \partial S_2(\xi) \setminus \{0\}, |z| \le \varepsilon\}$$
$$-\inf\{\theta(z)\colon z \in \partial S_1(\xi) \setminus \{0\}, |z| \le \varepsilon\}\big].$$

Then

$$\alpha_\pi < 2$$

and, by (14),

$$\frac{2\xi}{\pi} > \frac{\xi\alpha_\xi}{\pi} = \alpha_\pi.$$

Applying Lemmas 3.5 and 3.6 to P_x on S_π gives (8.1) and (8.3). Also, since $2\pi/\xi > \pi\alpha_\pi/\xi = \alpha_\xi$, by (1.4) and Lemmas 3.5 and 3.6 applied to $\mathscr{P}_{F(x)}$ on S_ξ, (8.2) and (8.4) hold.

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