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NO TRIPLE POINT OF PLANAR BROWNIAN MOTION IS ACCESSIBLE

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We show that the boundary of a connected component of the complement of a planar Brownian path on a fixed time interval contains almost surely no triple point of this Brownian path.

1. Introduction. We will say that $z \in \mathbf{R}^2$ is a frontier point (not to be confused with the standard boundary point) of the planar Brownian motion $Z_{[0,T]} = \{Z_t,\ 0 \le t \le T\}$ if z is on the boundary of one of the connected components of the complement of $Z_{[0,T]}$ in the plane. A point z is called a triple point for $Z_{[0,T]}$ if $z = Z(t_1) = Z(t_2) = Z(t_3)$ for some distinct t_1, t_2, $t_3 \in [0, T]$. In the sequel, $T > 0$ is fixed. We will prove the following result.

Theorem 1. *Almost surely, no frontier point of* $Z_{[0,T]}$ *is a triple point.*

Although the cardinality of the set of frontier points is that of the real line, the frontier points are in a sense exceptional; it is easy to see that, for a fixed $t \le T$, Z_t is almost surely not a frontier point of $Z_{[0,T]}$ although all points of $Z_{[0,T]}$ are boundary points. The boundary of the unbounded connected component of the complement of $Z_{[0,T]}$ (which consists exclusively of frontier points) has been called a self-avoiding planar Brownian motion [Mandelbrot (1982)]. It has been conjectured that the Hausdorff dimension of this set (and also of the set of frontier points) is $4/3$ [Mandelbrot (1982)]; see Burdzy and Lawler (1990b) for some rigorous estimates. The geometry of the boundary of a connected component of the complement of a planar Brownian path has been studied in several works [see Burdzy (1989a, b), Burdzy and Lawler (1990b) and Werner (1994)]. Let us mention two recent papers which will not be referred to elsewhere in this paper, but which are somewhat related to it: Burdzy (1995) and Le Gall and Meyre (1992). Le Gall (1991) presents a clear overview of the results derived before 1990.

Let us now recall some known facts about multiple points. For any $T > 0$, $(Z_t,\ 0 \le t \le T)$ has almost surely points of any (even uncountable) multiplicity [for points of finite multiplicity, see Dvoretzky, Erdős and Kakutani (1954) and Adelman and Dvoretzky (1985); about points of infinite multiplicity, see

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Dvoretzky, Erdős and Kakutani (1958), Le Gall (1987a) and Bass, Burdzy and Khoshnevisan (1994); or, alternatively, Le Gall (1991) for an overview]. All these sets of multiple points are dense in $\{Z_t,\ 0 \le t \le T\}$ and the Hausdorff dimension of each one of them is 2 [see Le Gall (1987b) and Bass, Burdzy and Khoshnevisan (1994)].

In his book, Lévy (1965) noticed that the density of double points in the planar Brownian curve implies the existence (and the density) of frontier double points in $Z_{[0,T]}$. Loosely speaking, the boundary of a connected component of the complement of $Z_{[0,T]}$ contains a lot of double points of the Brownian path. Here is a sketch of Lévy's argument. Let L denote the boundary of a connected component of the complement of $Z_{[0,T]}$. Assume that a connected subset $L' \subset L$, which is not a singleton, contains no double point of $Z_{[0,T]}$. It is then easy to see that $L' = Z_{[t_1,t_2]}$ for some $t_1 < t_2$ (otherwise, L' contains a double point of $Z_{[0,T]}$). This can never be the case, since $Z_{[t_1,t_2]}$ contains double points. Hence, double points of Z are dense in L.

The proof of our main theorem relies heavily on intersection exponent estimates just as proofs in Burdzy and Lawler (1990b) do. For this reason we recall here a definition and a few relevant facts [see Burdzy and Lawler (1990a)]. Fix some $n \ge 1$, $p \ge 1$. Let $Y^1, \dots, Y^{p+n}$ be $p+n$ independent planar Brownian motions started from $Y_0^1 = \cdots Y_0^p = (-1,0)$ and $Y_0^{p+1} = \cdots = Y_0^{p+n} = (+1,0)$. We set, for all $R > 0$ and $j \in \{1, \dots, p+n\}$,

$$T_R^j = \inf\{t \ge 0, |Y_t^j| = R\}.$$

The intersection exponent $\xi(p,n)$ is defined by

$$\xi(p,n) = \lim_{R\to\infty} \frac{-\log P\Big(\Big(\bigcup_{1 \le j \le p} Y^j\big(\big[0, T_R^j\big]\big)\Big) \cap \Big(\bigcup_{p+1 \le j \le p+n} Y^j\big(\big[0, T_R^j\big]\big)\Big) = \varnothing\Big)}{\log R}.$$

It is easy to see that this limit exists using a subadditivity argument [see, e.g., Lawler (1991)]. It has been conjectured that these exponents are rational numbers; except for $\xi(2,1) = 2$ [see Lawler (1989), Burdzy and Lawler (1990a) or Lawler (1991)], the exact value of these exponents is not known. See Burdzy and Lawler (1990b) for some estimates. It is greatly satisfying that the only known value [i.e., $\xi(2,1) = 2$] happens to be the one that is needed in our proof. From this, we easily deduce that $\xi(4,2) \ge 4$, but our key bound (Proposition 4) is that $\xi(4,2)$ is strictly bigger than 4. We then derive consequences of this result for disconnection probabilities of six paths (Section 5) and, finally, we prove the theorem in the last section, using the fact, informally speaking, that in the neighborhood of a triple point z, $Z_{[0,T]}$ is similar to six independent Brownian paths started from z.

We would like to stress that we do not prove that the theorem holds for all T's simultaneously. The possibility that some triple points $Z_{t_1} = Z_{t_2} = Z_{t_3}$ (with $t_1 < t_2 < t_3$) may be frontier points of $Z_{[0,t_3]}$ is not ruled out by our theorem, although we conjecture that such points do not exist.

Let us justify the use of the term "accessible" in the title. A point z in a closed set $K \subset \mathbf{R}^2$ is called accessible if there exists a continuous path f: $[0,1) \to \mathbf{R}^2$, such that $f(0) = z$ and $f((0,1)) \cap K = \varnothing$ [see, e.g., Ohtsuka (1970), page 253]. Of course, every accessible point of $Z_{[0,T]}$ is a frontier point of $Z_{[0,T]}$. It is not difficult [see Burdzy and Lawler (1990b), pages 1003–1004] to show that every frontier point of $Z_{[0,T]}$ is in fact an accessible point of $Z_{[0,T]}$. The proof uses only continuity of Z and compactness of $Z_{[0,T]}$.

2. Preliminaries. In this part, we introduce some notation and we recall some facts and tools of various origins (probability, geometrical function theory, potential theory) we will use in this paper.

2.1. *Notation.* We will identify $\mathbf{R}^2$ and $\mathbf{C}$ and we will use both vector and complex notation; $C(x, r)$ and $D(x, r)$ will denote, respectively, the circle and the open disc centered at x with radius r. If X is a random variable, $\sigma(X)$ will denote the sigma field generated by X. If $Y = (Y_t,\ t \geq 0)$ is a process in $\mathbf{R}^d$ and K is a closed set in $\mathbf{R}^d$, we set

$$T_K(Y) = \inf\{t \geq 0, Y_t \in K\}.$$

If $K = \{x\}$, we will write $T_x(Y) = T_K(Y)$. If $I \subset \mathbf{R}_+ \stackrel{\text{df}}{=} [0, \infty)$, we set $Y_I = Y(I) = \{Y_t,\ t \in I\}$. The complement of an event A will be denoted A^c. The boundary of a set $\Omega \subset \mathbf{R}^2$ will be denoted $\partial\Omega$. We also define, for $\rho > 0$,

$$D(\Omega, \rho) = \bigcup_{y \in \Omega} D(y, \rho).$$

2.2. *The three-dimensional Bessel process.* We now recall some well-known facts about three-dimensional Bessel processes, which can be found, for example, in Revuz and Yor (1991). Let $B = (B_t,\ t \geq 0)$ denote a linear Brownian motion started from 0, and $\beta = (\beta_t,\ t \geq 0)$ a three-dimensional Bessel process also started from 0. We set, for every $r > 0$,

$$\tau_r = T_r(B), \qquad \rho_r = T_r(\beta) \quad \text{and} \quad \sigma_r = \sup\{t < \tau_r, B_t = 0\}.$$

Then, we have the following results.

2.2.1. *Williams' decomposition of the Brownian path.* For all $r > 0$ the two processes $(B_{\sigma_r + u},\ u \leq \tau_r - \sigma_r)$ and $(\beta_u,\ u \leq \rho_r)$ have the same law [Williams (1974)].

2.2.2. *The three-dimensional Bessel process as a Brownian motion condition not to hit* 0. For all $0 < r < r'$, $(\beta_{\rho_r + u},\ u \leq \rho_{r'} - \rho_r)$ has the same law as $(B_{\tau_r + u},\ u \leq \tau_{r'} - \tau_r)$ conditional on $\{\inf\{t > \tau_r,\ B_t = 0\} > \tau_{r'}\}$.

2.2.3. *Time-reversal.* The processes $(\beta_{\tau_r} - \beta_{\tau_r - u},\ u \leq \tau_r)$ and $(\beta_u,\ u \leq \tau_r)$ have the same law.

2.3. *Skew-product decomposition.* Planar Brownian motion is invariant under conformal mapping [see, e.g., Le Gall (1991), Chapter II, Theorem 1].

In particular, the analyticity of the exponential mapping implies that [see, e.g., Le Gall (1991), Chapter II, Theorem 3] a planar Brownian motion $(X_t,\, t \geq 0)$ started from 1 may be represented as

$$X_t = \exp\big(B_{A(t)} + i\theta_{A(t)}\big),$$

where B and θ are independent Brownian motions started from 0 and where $A(t) = \int_0^t |X_s|^{-2}\, ds$. As an immediate consequence, if $0 < a < 1 < b$,

$$P\big(T_a(|X|) < T_b(|X|)\big) = P\big(T_{\log a}(B) < T_{\log b}(B)\big) = \frac{\log b}{\log(b/a)}. \tag{1}$$

2.4. *Potential theory*. We will also use (mainly in Section 4) some potential theoretical results. We refer to Doob (1984) for detailed statements and definitions.

2.4.1. *h-processes*. We start with a review of h-processes. The proofs may be found in Doob (1984) and Meyer, Smythe and Walsh (1972). Let $D \subset \mathbf{C}$ be a Greenian domain, and let h be a strictly positive superharmonic function in D. Let $p_i^D(x, y)$ be the transition density for Brownian motion killed at the hitting time of D^c and

$$p_t^h(x, y) = \frac{h(y)}{h(x)} p_t^D(x, y).$$

Any process with the p_t^h-transition densities will be called an h-process (conditioned Brownian motion). Let X be such a process, started from x under the probability measure P_x^h, and let σ be the lifetime of X. Suppose that M is a closed subset of D, and let $L = \sup\{t < \sigma \colon X(t) \in M\}$ be the last exit time from M. Let

$$\begin{aligned}
Y^1(t) &= X(t), & t &\in [0, T(M)),\\
Y^2(t) &= X(T(M) + t), & t &\in [0, \sigma - T(M)),\\
Y^3(t) &= X(t), & t &\in [0, L),\\
Y^4(t) &= X(L + t), & t &\in [0, \sigma - L),\\
Y^5(t) &= X(\sigma - t), & t &\in [0, \sigma).
\end{aligned}$$

Under P_x^h, each process Y^k is an h_k-process in a domain D_k. We have $D_1 = D_4 = D \setminus M$, $D_2 = D_3 = D_5 = D$ and $h_1 = h_2 = h$; h_3 is a potential supported by ∂M; h_4 has the boundary values 0 on ∂M and the same boundary values as h on $\partial D \setminus M$; h_5 is the Green function $G_D(x, \cdot)$ if $x \in D$ or a harmonic function with a pole at x if $x \in \partial D$. The initial distributions of Y^1 and Y^3 are concentrated on $\{x, \Delta\}$, where Δ is the coffin state. For the remaining initial distributions see Doob (1984).

2.4.2. *The Harnack principle*. The Harnack principle says that if h is a strictly positive harmonic function in $D(x, r)$ and $a \in (0, 1)$, then $h(y) < ch(z)$ for all y, $z \in D(x, ar)$, where $c < \infty$ depends only on a.

Here is a version of the boundary Harnack principle we will use. Let $\Omega \subset \mathbf{R}^2$ be an open connected set whose boundary is a finite union of graphs of Lipschitz functions (possibly in different orthonormal coordinate systems). Let V be an open set, K a compact set, $K \subseteq V$. There exists a constant c_1 such that if u and v are strictly positive harmonic functions in Ω that vanish continuously on $(\partial\Omega) \cap V$, then

$$\frac{u(x)}{v(x)} \le c_1 \frac{u(y)}{v(y)} \quad \text{for all } x, y \in K \cap \Omega.$$

See Bass and Burdzy (1991), Bañuelos, Bass and Burdzy (1991) or Bass (1995) for strong versions of this result and for references.

2.5. *Conformal invariance*, *prime ends*. We now recall some facts about conformal mappings, which can be found, for example, in Ahlfors [(1973), Section 4.6], Ohtsuka [(1970), Chapter III] or Pommerenke [(1992), Chapter 2]. For any two simply connected open planar domains Ω_1 and Ω_2 such that each one has more than two boundary points, there exists a conformal one-to-one mapping from Ω_1 onto Ω_2; the mapping has a continuous one-to-one extension to a mapping of $\overline{\Omega}_1$ onto $\overline{\Omega}_2$ if the boundaries of Ω_1 and Ω_2 are sufficiently "nice" (for instance, if they are Jordan curves). If the boundaries are not "nice," then one must use the concept of prime ends introduced by Carathéodory instead of boundary points (see the above-mentioned references for details). For instance, if f is a one-to-one conformal map from Ω onto $D(0,1)$, f induces a one-to-one correspondence between $C(0,1)$ and the prime ends of Ω.

Recall also that, for any three distinct points on the circle $C(0,1)$, there exists an analytic one-to-one mapping from $D(0,1)$ onto $D(0,1)$ which maps these three points onto three other arbitrarily chosen distinct points in $C(0,1)$ which have the same cyclic order. Hence, for any two simply connected open planar domains which have more than two boundary points and any three distinct prime ends a, b, c on the boundary of the first domain, there exists an analytical one-to-one mapping of the first domain onto the other which takes (a, b, c) or (b, a, c) onto three other arbitrarily chosen distinct prime ends of the second domain.

2.6. *A lemma*. We state without a proof an easy lemma which will be useful in the sequel. Let us fix $n \ge 1$. Let $X^1, \dots, X^n$ denote n planar Brownian motions started on the unit circle and independent given their starting points (the starting points may not be independent). Let $y_1, \dots, y_n$ denote n independent uniformly distributed random variables on the circle $C(0,2)$.

LEMMA 1. *There exists a constant $k_n > 1$ such that*, *for any bounded measurable positive function f*: $C(0,2) \to \mathbf{R}$, *and independently of $X_0^1, \dots, X_0^n$*,

$$\begin{aligned}(k_n)^{-1} E(f(y_1, \dots, y_n)) &\le E\big(f\big(X_{T_2(|X^1|)}, \dots, X_{T_2(|X^n|)}\big)\big) \\ &\le k_n E(f(y_1, \dots, y_n)).\end{aligned}$$

This result can be viewed as a direct consequence of the fact that (in the notation of Section 2.3), $\theta_{T_2(|X|)}$ is a Cauchy random variable [see, e.g., Revuz and Yor (1991), Chapter 3, Proposition 3.3].

3. Intersection exponents. Recall the intersection exponents $\xi(p,n)$ defined in the Introduction. In order to prove Theorem 1, we need in fact estimates of nonintersection probabilities of Brownian motions which have random initial distributions. Therefore, we introduce an analogue of $\xi(p,n)$ for Brownian motions with uniformly distributed starting points on the unit circle. Let $Z^1,\dots,Z^{p+n}$ denote $p+n$ independent planar Brownian motions started uniformly and independently on $C(0,1)$. We set, for all $R>0$ and for all $j\in\{1,\dots,p+n\}$,

$$S_R^j = \inf\{t\ge 0,\ |Z_t^j| = R\} = T_R(|Z^j|).$$

The subadditivity argument can be easily adapted [using Proposition 5.2.1 in Lawler (1991)] to show that the following limit exists:

$$\xi_u(p,n) = \lim_{R\to\infty} \frac{-\log P\Big(\big(\bigcup_{1\le j\le p} Z^j_{[0,S_R^j]}\big)\cap\big(\bigcup_{p+1\le j\le p+n} Z^j_{[0,S_R^j]}\big)=\varnothing\Big)}{\log R}.$$

Then we have the following lemma.

LEMMA 2. *For all* $n\ge 1$,

$$\xi_u(1,n) = \xi(1,n).$$

One expects that $\xi_u(p,n)=\xi(p,n)$ for all $n\ge 1$ and $p\ge 1$, but proving it seems to be more difficult.

PROOF OF LEMMA 2. Recall the definitions of Y^j and T_R^j for $1\le j\le n+1$, with $p=1$, from the Introduction. Lemma 1 and scaling yield that, for all $R>2$,

$$\begin{aligned}
&P\left(Y^1_{[0,T_R^1]}\cap\left(\bigcup_{j=2}^{n+1} Y^j_{[0,T_R^j]}\right)=\varnothing\right)\\
&\quad\le P\left(Y^1_{[T_2^1,T_R^1]}\cap\left(\bigcup_{j=2}^{n+1} Y^j_{[T_2^j,T_R^j]}\right)=\varnothing\right)\\
&\quad\le k_{n+1}P\left(Z^1_{[S_2^1,S_R^1]}\cap\left(\bigcup_{j=2}^{n+1} Z^j_{[S_2^j,S_R^j]}\right)=\varnothing\right)\\
&\quad\le k_{n+1}P\left(Z^1_{[0,S_{R/2}^1]}\cap\left(\bigcup_{j=2}^{n+1} Z^j_{[0,S_{R/2}^j]}\right)=\varnothing\right)
\end{aligned}$$

and, consequently, $\xi(n,1)\ge\xi_u(n,1)$. Note that this argument shows in fact that $\xi(p,n)\ge\xi_u(p,n)$ for all $n\ge 1$, $p\ge 1$.

We now turn our attention toward the opposite inequality. We define

$$\tau = \sup\{t \le T_R^1, |Z_t^1| = 1\}$$

and

$$\sigma = \inf\{t \ge \tau, |Z_t^1| = 2\}.$$

As $Z^1_{[\sigma, T_R^1]} \cap D(0,1) = \varnothing$,

$$\begin{aligned} P\left(Z^1_{[0,S_R^1]} \cap \left(\bigcup_{j=2}^{n+1} Z^j_{[0,S_R^j]} \right) = \varnothing \right) &\le P\left(Z^1_{[\sigma,S_R^1]} \cap \left(\bigcup_{j=2}^{n+1} Z^j_{[0,S_R^j]} \right) = \varnothing \right) \\ &\le P\left(Z^1_{[\sigma,S_R^1]} \cap \left(\bigcup_{j=2}^{n+1} X^j_{[0,T_R(|X^j|)]} \right) = \varnothing \right), \end{aligned}$$

where $X^2, \dots, X^{n+1}$ are independent (and independent of Z^1) planar Brownian motions started from 0.

The process $(Z_{\sigma+u}, u \le S_R^1 - \sigma)$ is a planar Brownian motion started with uniform distribution on $C(0,2)$ conditioned to hit $C(0,R)$ before $C(0,1)$, which is an event of probability $\log 2/\log R$ [cf. (1)]. Hence, using a simple symmetry argument, if X^1 denotes a planar Brownian motion started from 2, independent of $X^2, \dots, X^{n+1}$,

$$P\left(Z^1_{[0,S_R^1]} \cap \left(\bigcup_{j=2}^{n+1} Z^j_{[0,S_R^j]} \right) = \varnothing \right) \le \frac{P\left(X^1_{[0,T_R(|X^1|)]} \cap \left(\bigcup_{j=2}^{n+1} X^j_{[0,T_R(|X^j|)]} \right) = \varnothing \right)}{\log 2/\log R}.$$

A simple shift-and-scaling argument now completes the proof. □

We now present a straightforward consequence of the last lemma. See also Theorem 1.2 (i) in Burdzy and Lawler (1990b).

Corollary 3. *We have*

$$\xi_u(2,2) \ge 5/2 \quad \text{and} \quad \xi_u(4,1) \ge 3.$$

Proof. As $\xi_u(2,1) = \xi(2,1) = 2$, these estimates are easy consequences of Beurling's projection theorem on harmonic measure in a disc [see Ahlfors (1973) and Oksendal (1983)]. Trivially,

$$\begin{aligned} &P\left(\left(\bigcup_{j=1,2} Z^j_{[0,S_R^j]} \right) \cap \left(\bigcup_{j=3,4} Z^j_{[0,S_R^j]} \right) = \varnothing \right) \\ &\quad \le P\left(\left(\bigcup_{j=1,2} Z^j_{[0,S_R^j]} \right) \cap Z^3_{[0,S_R^3]} = \varnothing \text{ and } Z^1_{[0,S_R^1]} \cap Z^4_{[0,S_R^4]} = \varnothing \right), \end{aligned}$$

and Beurling's theorem shows that, for every continuous path L connecting the circles $C(0,1)$ and $C(0,R)$,

$$P\Big(L \cap Z^4_{[0,S^4_R]} = \varnothing\Big) \le P\Big([-R,-1] \cap Y^4_{[0,T^4_R]} = \varnothing \Big| Y^4_0 = 1\Big);$$

it is a classical fact that this last quantity is bounded by $(4/\pi)R^{-1/2}$ [see, e.g., Werner (1995)]. Hence, as $\xi_u(2,1) = 2$, $\xi_u(2,2) \ge 2 + 1/2$. Similarly, $\xi_u(4,1) \ge 2 + 2(1/2)$.

4. The key estimate.

4.1. *Probability of making a loop for an h-process.* Before stating and proving Proposition 4, let us first derive three technical estimates for h-processes which will be useful in its proof.

(a) Suppose that h is a positive harmonic function in $D(z,\rho)$. We will show that for any $p < 1$ there is $b > 0$ such that the probability that an h-process starting from z makes a closed loop around $D(z, b\rho)$ before hitting $C(z,\rho)$ is greater than p.

First find $b_1 < 1$ such that the standard Brownian motion starting from a point of $C(x,r)$ has a chance greater than $\sqrt{p}$ of making a closed loop around $D(x, b_1 r)$ before hitting $C(x, r/b_1)$.

Next use the Harnack principle to find $b_2 \in (0,1)$ such that $h(y)/h(x) > \sqrt{p}$ for all $x, y \in D(z, b_2\rho)$.

Let $b = b_2 b_1^2/2$. The strong Markov property applied at the hitting time of $C(z, b\rho/b_1)$ shows that it will suffice to prove that the probability that an h-process starting from a point of $C(z, b\rho/b_1)$ makes a closed loop around $D(z, b\rho)$ before hitting $C(z,\rho)$ is greater than p. By our choice of b_1, the probability that a Brownian motion starting from a point of $C(z, b\rho/b_1)$ makes a closed loop around $D(z, b\rho)$ before hitting $C(z, b\rho/b_1^2)$ is greater than $\sqrt{p}$.

Let P_x and P_x^h denote the distributions of Brownian motion and an h-process, respectively, starting from x, and let X stand for the generic process. If $x \in C(z, b\rho/b_1)$ and $y \in C(z, b\rho/b_1^2)$, then both x and y belong to $D(z, b_2\rho)$. Using the relationship between the hitting densities for an h-process and a Brownian motion starting from $x \in C(z, b\rho/b_1)$ shows that

$$\begin{aligned} P_x^h\Big(X\big(T_{C(z,b\rho/b_1^2)}\big) \in dy\Big) &= \frac{h(y)}{h(x)} P_x\Big(X\big(T_{C(z,b\rho/b_1^2)}\big) \in dy\Big) \\ &\ge \sqrt{p}\, P_x\Big(X\big(T_{C(z,b\rho/b_1^2)}\big) \in dy\Big). \end{aligned}$$

The distribution of an h-process and of a Brownian motion starting from x and stopped at the hitting time of $C(z, b\rho/b_1^2)$ are identical if we condition them to hit the same point of $C(z, b\rho/b_1^2)$. Hence, the last formula shows that the Radon–Nikodym derivative of the distribution of an h-process starting from $x \in C(z, b\rho/b_1)$ and stopped at the hitting time of $C(z, b\rho/b_1^2)$ with respect to the distribution of Brownian motion starting from the same point x and stopped at the hitting time of $C(z, b\rho/b_1^2)$ is greater than $\sqrt{p}$. It follows that an h-process starting from a point of $C(z, b\rho/b_1)$ makes a closed loop around $D(z, b\rho)$ before hitting $C(z, b\rho/b_1^2)$ with probability greater than $\sqrt{p} \cdot \sqrt{p} = p$.

(b) For this part, suppose that $\delta \in (0, 1/4)$ is fixed and we set

$$\tilde{\Delta} = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 < 0, 0 < x_2 < 1\},$$

$$\tilde{\Delta}_1 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 < 0, \delta < x_2 < 1 - \delta\}.$$

Suppose that $z_1 \le -1$ and let

$$\Lambda_1 = \{(x_1, x_2) \in \tilde{\Delta} : x_1 = z_1 - 3/4\},$$

$$\Lambda_2 = \{(x_1, x_2) \in \tilde{\Delta} : x_1 = z_1 - 1/2\},$$

$$\Lambda_3 = \{(x_1, x_2) \in \tilde{\Delta} : x_1 = z_1 - 1/2\},$$

$$K_1 = \{(x_1, x_2) \in \tilde{\Delta} : z_1 - 1/2 < x_1 < z_1 + 1/2\},$$

$$K_2 = \{(x_1, x_2) \in \tilde{\Delta}_1 : z_1 - 1/4 < x_1 < z_1 + 1/4\},$$

$$K_3 = \{(x_1, x_2) \in \tilde{\Delta} : z_1 - 1 < x_1 < z_1 + 1/2\},$$

$$K_4 = \{(x_1, x_2) \in \tilde{\Delta} : z_1 - 1 < x_1 < z_1 + 1\},$$

$$Q = \{(x_1, x_2) \in \tilde{\Delta} : z_1 - 1 < x_1 < z_1 + 1/2\},$$

Suppose that h is a strictly positive harmonic function in K_4 which vanishes on $\partial\tilde{\Delta}$. Let v be the center of Λ_1, and let $\tilde{C}$ be the event that a process hits Λ_2, then makes a closed loop around K_2 inside K_1 (all this before exiting K_3) and finally exits K_3 through Λ_3. The P_v-probability of $\tilde{C}$ is equal to $p > 0$, which depends only on δ.

The Harnack principle implies that $h(y)/h(v) > b > 0$ for all $y \in \Lambda_3$ and so, for $y \in \Lambda_3$, we have

$$P_v^h\big(X(T_{\partial K_3}) \in dy\big) = \frac{h(y)}{h(v)} P_v\big(X(T_{\partial K_3}) \in dy\big) > bP_v\big(X(T_{\partial K_3}) \in dy\big).$$

Just as in part (a) we deduce that the Radon–Nikodym derivative $P_v^h/P_v > b$ on the event $\{X(T_{\partial K_3}) \in \Lambda_3\}$. Thus $P_v^h(\tilde{C}) > bp$.

For a fixed $y \in \Lambda_2$, the function $x \to P_x(X(T_{\partial Q}) \in dy)$ is a harmonic function in Q which vanishes on $\partial\tilde{\Delta} \cap \partial Q$, and the same can be said about h. The boundary Harnack principle (see Section 2.4) implies that, for some $b_1 > 0$ and all $x \in \Lambda_1$,

$$\begin{aligned} P_x^h\big(X(T_{\partial Q}) \in dy\big) &= \frac{h(y)}{h(x)} P_x\big(X(T_{\partial Q}) \in dy\big) \\ &\ge b_1 \frac{h(y)}{h(v)} P_v\big(X(T_{\partial Q}) \in dy\big) \\ &= b_1 P_v^h\big(X(T_{\partial Q}) \in dy\big). \end{aligned}$$

The strong Markov property applied at the exit time from Q now implies that $P_x^h(\tilde{C}) \ge b_1 b p = b_2 > 0$ for all $x \in \Lambda_1$.

(c) Suppose that J, an arc of $C(0,1)$, and $\rho > 0$ are such that $D(J, \rho)$ does not cover $C(0,1)$ [recall that $D(J,\rho) = \bigcup_{y \in J} D(y, \rho)$]. Suppose that h is a positive harmonic function in $D(J, \rho)$. We will show that every h-process starting from a point of $D(J, \rho/2)$ makes a closed loop around $D(J, 3\rho/4)$ before leaving $D(J, \rho)$ with probability greater than $p > 0$ which depends only on ρ.

If $|x_1 - x_2| = r$, then Brownian motion starting from a point of $D(x_1, r)$ can hit $D(x_2, r)$ without leaving $D(x_1, 2r) \cup D(x_2, r)$ with probability $p_1 > 0$. With probability $p_2 > 0$, Brownian motion starting from a point of $D(x, r)$ will make a closed loop around $D(x, r)$ before leaving $D(x, 2r)$. There exists a $k < \infty$ which depends only on ρ and which has the following property: one can find an $r > 0$ and a sequence of points $\{x_1\}_{1 \le j \le k}$ such that $x_1 = x_k$, $|x_j - x_{j+1}| = r$ for all $j < k$, $\bigcup_{1 \le j \le k} D(x_j, 2r) \subset D(J, 7\rho/8) \setminus D(J, 3\rho/4)$ and the discs $D(x_j, r)$ form a closed loop around $D(J, 3\rho/4)$. A repeated application of the strong Markov property at the consecutive hitting times of $D(x_j, r)$'s shows that Brownian motion starting from a point of $D(x_1, r)$ can hit all $D(x_j, r)$'s and finally make a loop around $D(x_1, r)$ before leaving $\bigcup_{1 \le j \le k} D(x_j, 2r) \subset D(J, 7\rho/8) \setminus D(J, 3\rho/4)$ with probability greater than $p_1^k p_2$. Actually, the starting point of the process can be in any of the discs $D(x_j, r)$, so the strong Markov property applied at the first hitting time of $\bigcup_{1 \le j \le k} D(x_j, r)$ implies that Brownian motion starting from a point of $D(J, \rho/2)$ makes a closed loop around $D(J, 3\rho/4)$ before leaving $D(J, 7\rho/8)$ with probability greater than $p_1^k p_2$.

The Harnack principle implies that there exists $b_1 > 0$ which depends only on ρ such that $h(y)/h(x) > b_1$ for all x, $y \in D(J, 15\rho/16)$. A calculation similar to that in parts (a) and (b) of this section gives

$$\begin{aligned} P_x^h\big(X(T_{\partial D(J, 7\rho/8)}) \in dy\big) &= \frac{h(y)}{h(x)} P_x\big(X(T_{\partial D(J, 7\rho/8)}) \in dy\big) \\ &\ge b_1 P_x\big(X(T_{\partial D(J, 7\rho/8)}) \in dy\big), \end{aligned}$$

for all $x \in D(J, \rho/2)$. Hence the Radon–Nikodym derivative $P_x^h / P_x \ge b_1$ on $[0, T_{\partial D(J, 7\rho/8)}]$. Now, an application of the strong Markov property at the hitting time $T_{\partial D(J, 7\rho/8)}$ shows that an h-process starting from a point of $D(J, \rho/2)$ makes a closed loop around $D(J, 3\rho/4)$ before leaving $D(J, \rho)$ with probability greater than $b_1 p_1^k p_2$.

4.2. *The key estimate.* We are now ready to prove our key result, that is, Proposition 4. Let $Z^1, \dots, Z^6$ denote independent planar Brownian motions started uniformly and independently on $C(0,1)$.

PROPOSITION 4. *We have $\xi_u(2,4) > 4$. In other words, for some fixed $c_1 > 0$, $\alpha_1 > 0$ and all $R > 1$,*

$$P\left(\left(\bigcup_{1 \le j \le 2} Z^j_{[0, T_R(|Z^j|)]}\right) \cap \left(\bigcup_{3 \le j \le 6} Z^j_{[0, T_R(|Z^j|)]}\right) = \varnothing\right) \le c_1 R^{-4-\alpha_1}.$$

As the proof of this proposition is long, technical and complicated, we first offer an outline of the general idea of the proof.

OUTLINE OF THE PROOF. Suppose that $\varepsilon = 1/R$. Let X^j, $j = 1, 2$, and Y^j, $j = 1, 2, 3, 4$, be two-dimensional Brownian motions. Suppose that they are jointly independent and that they start from distinct deterministic points $x_0^1, x_0^2, y_0^1, \dots, y_0^4$ on $C(0, \varepsilon)$. The scaling property and a conditioning argument show that it will suffice to prove that

$$P\left(\left(\bigcup_{1 \le j \le 2} X^j_{[0, T_{C(0,1)}(X^j)]}\right) \cap \left(\bigcup_{1 \le j \le 4} Y^j_{[0, T_{C(0,1)}(Y^j)]}\right) = \varnothing\right) \le c_1 \varepsilon^{4+\alpha_1},$$

where c_1 and α_1 are independent of ε and of the starting points $(x_0^1, x_0^2, y_0^1, \dots, y_0^4)$ in $C(0, \varepsilon)^6$.

We know that the probability that the path $X^1([0, T_{C(0,1)}(X^1)])$ is disjoint from the paths $Y^1([0, T_{C(0,1)}(Y^1)])$ and $Y^2([0, T_{C(0,1)}(Y^2)])$ [in short, the probability that $X^1 \cap (Y^1 \cup Y^2) = \varnothing$] is of order ε^2 [because $\xi(2,1) = 2$]. The same is true for the probability that $X^2 \cap (Y^3 \cup Y^4) = \varnothing$. We will argue that, given both these events, there is a significant conditional probability that $X^2 \cap Y^1 \ne \varnothing$ or that $X^1 \cap Y^3 \ne \varnothing$. More precisely, we will show that this conditional probability is of order $1 - \varepsilon^\beta$ for a positive β. In order to achieve this goal, we first observe that the traces of X^1 and X^2 have to differ significantly on the circles $C(0, a^k)$, for some fixed $a > 0$ and many values of k (the opposite event has a small probability). Next we "fix" the paths of X^1 and X^2. If the trace of X^2 is "larger" than that of X^1, then the process Y^1 (which is already conditioned to avoid X^1) will be likely to hit X^2. In the opposite case, Y^3 will be likely to hit X^1.

PROOF OF PROPOSITION 4. We will divide the proof into several steps and reuse the notation introduced in the outline of the proof.

Step 1 (*Notation and definitions*). Let Δ be the connected component of

$$D(0,1) \setminus X^1\big(\big[0, T_{C(0,1)}(X^1)\big]\big)$$

such that $C(0,1) \subset \partial\Delta$. Again let

$$\tilde{\Delta} = \big\{(x_1, x_2) \in \mathbf{R}^2 \colon x_1 < 0, 0 < x_2 < 1\big\}.$$

If $y_0^1 \notin \Delta$, then $X^1_{[0, T_{C(0,1)}(X^1)]} \cap Y^1_{[0, T_{C(0,1)}(Y^1)]} \ne \varnothing$. In view of the inequality we are trying to prove, we may concentrate on the opposite event; that is, we will condition X^1 on

$$E_1 = \big\{y_0^1 \in \Delta\big\}.$$

Let $z_0 = X^1(T_{C(0,1)}(X^1))$. The point z_0 corresponds to two distinct prime ends $\hat{\zeta}_0$ and $\hat{\zeta}_0'$ in Δ. Let ζ_1 and ζ_2 denote the endpoints of the connected component of $\Delta \cap C(0, \varepsilon)$ which contains y_0^1. It is easy to see that, almost surely, $\zeta_1 \ne \zeta_2$. Hence [using, e.g., Proposition 2.14 in Pommerenke (1992)] the prime ends $\hat{\zeta}_1$ and $\hat{\zeta}_2$ corresponding to ζ_1 and ζ_2 are also distinct; $\hat{\zeta}_1$ and $\hat{\zeta}_2$ divide the set of prime ends of Δ into two nonempty parts M_1 and M_2, and as $C(0,1) \subset \partial\Delta$, $\hat{\zeta}_0$ and $\hat{\zeta}_0'$ are on the same side, say, in M_1. Now fix $\hat{\zeta}_3$, a prime end of Δ in M_2.

We choose (cf. Section 2.6) an analytic one-to-one mapping f from Δ onto $\tilde{\Delta}$ such that $f(\{\hat{\zeta}_0, \hat{\zeta}_0'\}) = \{(0,0), (1,0)\}$ and $f(\hat{\zeta}_3) = -\infty$. Then, intuitively speaking, $C(0,1) \subset f^{-1}(\{(x_1, x_2) \in \partial\Delta: x_1 = 0\})$, and $\hat{\zeta}_1$ and $\hat{\zeta}_2$ are mapped onto two points which lie on different half-lines $\partial\tilde{\Delta}^d$ and $\partial\tilde{\Delta}^u$, where

$$\partial\tilde{\Delta}^d \stackrel{\text{df}}{=} \{(x_1, x_2) \in \partial\tilde{\Delta}: x_2 = 0\},$$

$$\partial\tilde{\Delta}^u \stackrel{\text{df}}{=} \{(x_1, x_2) \in \partial\tilde{\Delta}: x_2 = 1\}.$$

Let

$$\tilde{\Delta}_1 = \tilde{\Delta}_1(\delta) = \{(x_1, x_2) \in \tilde{\Delta}: \delta < x_2 < 1 - \delta\},$$

$$\Delta_1 = f^{-1}(\tilde{\Delta}_1).$$

The value of the parameter $\delta \in (0, 1/4)$ will be chosen later in the proof. Suppose that $0 < a < 1/2$ (the value of a will be specified later), and let m_0 be the largest integer k such that $a^k > \varepsilon$. For $k = 1, 2, \dots, m_0$, let J_k^Δ be a connected component of $\Delta \cap C(0, a^k)$ which must be crossed by every continuous path in Δ which connects $C(0, \varepsilon)$ and $C(0,1)$. We may and will assume that J_k^Δ's are chosen so that every continuous path starting from $C(0,\varepsilon)$ and going to $C(0,1)$ in Δ must intersect J_k^Δ before J_{k-1}^Δ for every k. Similarly, let $J_k^{\Delta_1}$ be a connected component of $\Delta_1 \cap C(0, a^k)$ which is a subset of J_k^Δ and which must be crossed by every continuous path in Δ_1 which connects $C(0,\varepsilon)$ and $C(0,1)$. We will write $\tilde{J}_k^\Delta \stackrel{\text{df}}{=} f(J_k^\Delta)$ and $\tilde{J}_k^{\Delta_1} \stackrel{\text{df}}{=} f(J_k^{\Delta_1})$.

Step 2 (*Choosing a value for* a). Suppose that $z = (z_1, 1/2)$ for some $z_1 < -1$ and that Γ is a continuous path which has one endpoint on each set $\partial\tilde{\Delta}^u$ and $\partial\tilde{\Delta}^d$ and intersects $\{(x_1, x_2) \in \tilde{\Delta}: x_1 \in (z_1 - 1, z_1 + 1)\}$. Without loss of generality assume that the intersection point belongs to $K = \{(x_1, x_2): x_1 = z_1 - 1,\ x_2 \le 1/2)\}$. Note that there is a $q_0 > 0$ such that (i) Brownian motion starting from z can exit $\tilde{\Delta}$ through $\{(x_1, x_2) \in \partial\tilde{\Delta}^d: x_1 < z_1 - 1\}$ without hitting K with probability $q_0 > 0$, and (ii) Brownian motion starting from z can exit $\tilde{\Delta}$ through $\{(x_1, x_2) \in \partial\tilde{\Delta}^d: x_1 > z_1 - 1\}$ without hitting K with probability $q_0 > 0$. At least one of these events implies that the trajectory of the process intersects Γ. Hence the harmonic measure of Γ in $\tilde{\Delta}$ with respect to z is greater than q_0.

Suppose that $1 < k < m_0$ and find a point $z = (z_1, 1/2) \in \tilde{J}_k^\Delta$. Let $v = f^{-1}(z)$. We have $|v| = a^k$. Now choose $a > 0$ so small that the probability that a Brownian motion starting from v makes a closed loop around 0 before leaving $D(0, a^{k-1}) \setminus D(0, a^{k+1})$ is greater than $1 - q_0$. If such a closed loop is made, the process hits the boundary of Δ before hitting $J_{k-1}^\Delta \cup J_{k+1}^\Delta$. It follows that the harmonic measure of $J_{k-1}^\Delta \cup J_{k+1}^\Delta$ in Δ with respect to v is less than q_0. By the conformal invariance of the harmonic measure we see that the harmonic measure of $\tilde{J}_{k-1}^\Delta \cup \tilde{J}_{k+1}^\Delta$ in $\tilde{\Delta}$ with respect to z is less than q_0. Hence, the paths $\tilde{J}_{k-1}^\Delta$ and $\tilde{J}_{k+1}^\Delta$ are separated by

$$\{(x_1, x_2) \in \tilde{\Delta}: x_1 \in (z_1 - 1, z_1 + 1)\}.$$

Step 3 (*Comparing the trajectories of* X^1 *and* X^2). Assume now that X^1 is fixed and that E_1 holds. Let L_k be the connected component of $\Delta_1 \cap [D(0, a^{k-1}) \setminus D(0, a^{k+1})]$ which contains $J_k^{\Delta_1}$. Recall that X^2 is a Brownian motion independent of X^1 and starting from a point of $C(0, \varepsilon)$. Let

$$A_k^1 \stackrel{\text{df}}{=} \left\{X_{[0, T_{C(0,1)}(X^2)]} \cap L_k \neq \varnothing\right\},$$

$$A_k^2 \stackrel{\text{df}}{=} \left\{D\left(L_k, r_1 a^{k+6}\right) \cap X^2\left[0, T_{C(0,1)}(X^2)\right] = \varnothing\right\},$$

$$A_k \stackrel{\text{df}}{=} A_k^1 \cup A_k^2,$$

where $r_1 > 0$ is a constant which will be chosen below. We introduce stopping times

$$T_1 = \inf\left\{t \colon X^2(t) \in \bigcup_k C(0, a^{4k+2})\right\},$$

$$T_j = \inf\left\{t > T_{j-1} \colon X^2(t) \in \bigcup_k C(0, a^{4k+2}) \text{ and } |X^2(t)| \neq |X^2(T_{j-1})|\right\}, \qquad j > 1.$$

Let us consider the processes $V_j \stackrel{\text{df}}{=} \{X^2(t),\ t \in [T_j, T_{j+1}]\}$. We will now use a technique we shall reuse several times in this proof. We first condition on the value of the finite sequence

$$\Xi = \left(X^2(T_j)\right)_{j \le 1}.$$

Conditional on Ξ, all the processes V_j are independent h-processes. We then use this independence to obtain estimates of a conditional probability. However, we then finally remove the conditioning, noticing that these estimates are in fact independent of the value of Ξ.

Conditional on Ξ, each process V_j is an h-process in a domain $D(0, a^{4k-2}) \setminus D(0, a^{4k+6})$ starting from a point of $C(0, a^{4k+2})$. The strong Markov property and part (a) of Section 4.1 show that if one of the V_j's starts from $C(0, a^{4k+2})$ and hits $D(L_{4k}, r_1 a^{4k+6})$ at a point z, then it makes a closed loop around $D(z, r_1 a^{4k+6})$ before hitting $C(z, a^{4k+2}/16)$ with probability greater than p_1 (p_1 and the corresponding r_1 will be chosen below; note that the way r_1 is chosen does not depend on Ξ). If such a loop occurs, V_j intersects L_{4k}. A similar argument applies when V_j hits $D(L_{4k+4}, r_1 a^{4k+10})$. This shows that the conditional probability of A_{4k} is greater than p_1, given any value of Ξ. The event A_{4k} is determined by the processes V_j whose paths lie inside $D(0, a^{4k-2}) \setminus D(0, a^{4k+6})$ or $D(0, a^{4k-6}) \setminus D(0, a^{4k+2})$. This implies that, conditional on Ξ, all the events $(A_{4k})_{k \le m_0/4}$ are independent and that $P(A_{4k} | \Xi) > p_1$ for any Ξ.

Let N be the number of integers $k < m_0/4$ such that A_{4k} holds, and let $m_1 = m_0/4$. Conditional on Ξ, we may give a lower bound for N just as we would do it for a sequence of Bernoulli trials. Let $q_1 = 1 - p_1$. Recall that

$$\int_{-\infty}^{y} \exp\left(-\frac{x^2}{2}\right) dx \le \exp\left(-\frac{y^2}{2}\right),$$

for $y \le -1$. We have for large m_0 (i.e., for small $\varepsilon > 0$)

$$\begin{aligned} P\left(N < \frac{m_1 p_1}{3} \middle| \Xi\right) &< \int_{-\infty}^{m_1 p_1/2} \frac{1}{\sqrt{2\pi m_1 p_1 q_1}} \exp\left(-\frac{(u - m_1 p_1)^2}{2 m_1 p_1 q_1}\right) du \\ &= \int_{-\infty}^{(-m_1 p_1/2)/\sqrt{m_1 p_1 q_1}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right) dv \\ &\le \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-m_1 p_1}{8 q_1}\right). \end{aligned}$$

As this last estimate is in fact independent of Ξ, we can remove the conditioning.

Recall that $|\log \varepsilon / \log a| \in [m_1, m_1 + 1]$. We now choose $p_1 < 1$ so large that, for some $\gamma_1 > 0$, $\varepsilon_1 > 0$,

$$P(N < \gamma_1 |\log \varepsilon|) < P\left(N < \frac{m_1 p_1}{3}\right) < \varepsilon^5,$$

for all $\varepsilon < \varepsilon_1$. We also fix some $r_1 > 0$ which corresponds to our choice of p_1 in a way determined in part (a), Section 4.1. Let us stress that r_1 does not depend on δ at all. We set

$$E_2 = \{N \ge \gamma_1 |\log \varepsilon|\},$$

and we finally have

$$P_{X^2}(E_2 | X^1, E_1) < \varepsilon^5, \tag{2}$$

for all $\varepsilon < \varepsilon_1$, where ε_1 is some deterministic constant.

Step 4 (*Intersections of* Y^1 *and* X^2). Assume now that both X^1 and X^2 are fixed, and that E_1 and E_2 hold. By a slight abuse of notation, in this step of the proof, we will use the symbol Y^1 to denote a Brownian motion starting from y_0^1 and conditioned to hit $C(0,1)$ before hitting any other part of $\partial\Delta$. Hence, Y^1 is an h_2-process. Let $\tilde{Y}^1$ denote a process in $\tilde{\Delta}$ obtained by a time-change from $f(Y^1)$ so that the quadratic variation of $\tilde{Y}^1$ on the time interval (t_1, t_2) is equal to $t_2 - t_1$. Then $\tilde{Y}^1$ is an h_3-process in $\tilde{\Delta}$ for some positive harmonic function h_3.

Let $\mathscr{M}$ be the set of m's such that A^1_{4m} holds. For every $m \in \mathscr{M}$, choose a point z^m in $\{X^2_{[0, T_{C(0,1)}(X^2)]} \cap L_m\}$. We will write $\tilde{z}^m = (z_1^m, z_2^m) = f(z^m)$. Let

$$\begin{aligned} \Lambda_2^m &= \left\{(x_1, x_2) \in \tilde{\Delta} : x_1 = z_1^m - 1/2\right\}, \\ \Lambda_4^m &= \left\{(x_1, x_2) \in \tilde{\Delta} : x_1 = z_1^m - 1\right\}, \\ \Lambda_5^m &= \left\{(x_1, x_2) \in \tilde{\Delta} : x_1 = z_1^m + 1\right\} \end{aligned}$$

and

$$K_1^m = \{(x_1, x_2) \in \tilde{\Delta}: z_1^m - 1/2 < x_1 < z_1^m + 1/2\},$$
$$K_4^m = \{(x_1, x_2) \in \tilde{\Delta}: z_1^m - 1 < x_1 < z_1^m + 1\}.$$

Note that if $n > m$ and $n \in \mathscr{M}$, then Λ_5^m lies to the left of Λ_4^n because $\tilde{J}_{4m-1}^{\Delta}$ and $\tilde{J}_{4m-3}^{\Delta}$ are separated by a set $\{(x_1, x_2) \in \tilde{\Delta}: x_1 \in (z_1 - 1, z_1 + 1)\}$ for some z_1.

Let $\Lambda = \bigcup_{m \in \mathscr{M}} (\Lambda_2^m \cup \Lambda_4^m \cup \Lambda_5^m)$. We now repeat a part of the argument from the previous step. First we define stopping times

$$T_1 = \inf\{t: \tilde{Y}^1(t) \in \Lambda\},$$
$$T_j = \inf\{t > T_{j-1}: \tilde{Y}^1(t) \in \Lambda \text{ and } \tilde{Y}_1^1(t) \neq \tilde{Y}_1^1(T_{j-1})\}, \qquad j > 1,$$

where $\tilde{Y}_1^1(t)$ denotes the first coordinate of $\tilde{Y}^1(t)$. Let us consider the processes

$$V_j \overset{df}{=} \{\tilde{Y}^1(t), t \in [T_j, T_{j+1}]\}.$$

Again, conditional on

$$\Xi' = (Y^1(T_j))_{j \geq 1},$$

the processes $(V_j)_{j \geq 0}$ are independent h-processes.

If $m \in \mathscr{M}$, let $\tilde{C}_m$ be the event that the process $\tilde{Y}^1$ makes a closed loop around z^m within K_4^m. By part (b), Section 4.1, conditional on Ξ', if V_j starts from a point of Λ_2^m, it makes a closed loop around K_1^m within K_4^m with probability greater than $q_2 > 0$, where q_2 depends only on δ. Hence, using the independence and unconditioning, the probability of $\bigcap_{m \in \mathscr{M}} \tilde{C}_m^c$ is smaller than $(1 - q_2)^{N_1}$, where N_1 is the number of m's such that A_{4m}^1 holds.

Let us express this statement in terms of Y^1. If $\tilde{C}_m$ holds for $m \in \mathscr{M}$, then Y^1 makes a closed loop around z^m between $C(0, a^{4m+2})$ and $C(0, a^{4m-2})$ and, consequently, the trajectories of Y^1 and X^2 intersect. Thus,

$$(3) \quad P_{Y^1}^{h_2}\left(Y^1_{[0, T_{C(0,1)}(Y^1)]} \cap X^2_{[0, T_{C(0,1)}(X^2)]} = \varnothing \,|\, X^1, X^2, E_1, E_2\right) \leq (1 - q_2)^{N_1}.$$

Step 5 (*Choosing a value for* δ). The parameter δ will depend on r_1 chosen in Step 3. We want to choose $\delta > 0$ so small that, for every k, $D(J_k^{\Delta_1}, r_1 a^{k+6}/2)$ intersects both

$$\partial\Delta^u \overset{\text{df}}{=} f^{-1}(\partial\tilde{\Delta}^u) \quad \text{and} \quad \partial\Delta^d \overset{\text{df}}{=} f^{-1}(\partial\tilde{\Delta}^d).$$

Take any $\delta < 1/2$. If, for all k, $D(J_k^{\Delta_1}, r_1 a^{k+6}/2) \cap \partial\Delta^u \neq \varnothing$ and $D(J_k^{\Delta_1}, r_1 a^{k+6}/2) \cap \partial\Delta^d \neq \varnothing$, then we are done. Suppose that one of these statements, say, the first one, is not true for a certain k. An argument similar to that in part (c), Section 4.1, shows that it is possible to find a $p > 0$ (depending on r_1 but not on k or a), a point $z \in D(J_k^{\Delta_1}, r_1 a^{k+6}/2)$ and a Brownian motion Z starting from z which makes a closed loop around $J_k^{\Delta_1}$ inside $D(J_k^{\Delta_1}, r_1 a^{k+6}/2)$ with probability greater than p. The conformal invariance of Brownian motion implies that with probability greater than p, a certain Brownian motion makes a closed loop around $J_k^{\Delta_1}$ before leaving $f(D(J_k^{\Delta_1}, r_1 a^{k+6}/2))$ and hence before hitting $\tilde{\Delta}^u$. It follows from the definition of $\tilde{J}_k^{\Delta_1}$ that its closure intersects both the upper and lower parts of the boundary $\tilde{\Delta}_1$. Now we can take $\delta > 0$ so small that any Brownian motion starting from any point cannot make a closed loop around $\tilde{J}_k^{\Delta^1}$ without hitting $\partial\tilde{\Delta}^u$ with probability more than $p/2$. With this choice of δ, we must have $D(J_k^{\Delta_1}, r_1 a^{k+6}/2) \cap \partial\Delta^u \neq \emptyset$ and $D(J_k^{\Delta_1}, r_1 a^{k+6}/2) \cap \partial\Delta^d \neq \emptyset$.

Step 6 (*Intersections of* Y^3 *and* X^1). Assume once more that X^1 and X^2 are fixed and that E_1 and E_2 hold. In view of the probability we are going to estimate [i.e., (4)], we can also restrict ourselves to the case where $y_0^3 \in \Delta$, since otherwise, Y^3 intersects necessarily X^1. Consider an m such that A_{4m}^2 holds. In view of the previous step, every continuous path starting from $C(0, \varepsilon) \cap \Delta$ and going to $C(0, 1)$ without hitting the trajectory of X^2 must intersect

$$K_m \stackrel{\text{df}}{=} X^1_{[0, T_{C(0,1)}(X^1)]} \cup D\big(J_{4m}^{\Delta_1}, r_1 a^{4m+6}/2\big).$$

Let us condition the process Y^3 to hit $C(0,1)$ before hitting the path of X^2. Then Y^3 is an h_4-process. Let U_m be the hitting time of K_m by Y^3. By part (c), Section 4.1, an h_4-process, starting from a point of $D(J_{4m}^{\Delta_1}, r_1 a^{4m+6}/2)$, can make a closed loop around $D(J_{4m}^{\Delta_1}, r_1 a^{4m+6}/2)$ before leaving $D(J_{4m}^{\Delta_1}, 3r_1 a^{4m+6}/4)$ with probability greater than some fixed $q_3 > 0$. Making such a loop implies intersecting the path of X^1 in view of the fact, proved in the previous step, that $D(J_{4m}^{\Delta_1}, r_1 a^{4m+6}/2) \cap \partial\Delta^u \neq \emptyset$. The strong Markov property applied at U_m shows that Y^3 intersects the trajectory of X^1 with probability greater than q_3.

Let F_m be the event that Y^3 intersects the path of X^1 within $D(0, a^{4m}) \setminus D(0, a^{4m+1})$. By applying the strong Markov property at the stopping time U_m we see that if A_{4m}^2 holds, then the event F_m happens with probability greater than q_3 given $\bigcap_{j>k} F_j^c$, where the intersection is taken over j such that A_{4j}^2 holds. Let N_2 be the number of m such that A_{4m}^2 holds. Then we obtain, as in (3),

$$(4)\quad P_{Y^3}^{h_4}\Big(Y^3_{[0, T_{C(0,1)}(Y^3)]} \cap X^1_{[0, T_{C(0,1)}(X^1)]} = \emptyset \,|\, X^1, X^2, E_1, E_2\Big) \leq (1 - q_3)^{N_2}.$$

Step 7 (*Combining the estimates*). Let $G_{j_1, j_2, \dots, j_m}^{k_1, k_2, \dots, k_n}$ denote the event

$$\left\{\left(\bigcup_{j = j_1, \dots, j_m} X^j_{[0, T_{C(0,1)}(X^j)]}\right) \cap \left(\bigcup_{j = k_1, \dots, k_n} Y^j_{[0, T_{C(0,1)}(Y^j)]}\right) = \emptyset\right\}.$$

Note that the same argument as in the first part of the proof of Lemma 2 shows that, for sufficiently small $\beta_2 > 0$, there exists $c_0 > 0$ such that, for all $\varepsilon < 1$ and independently of the starting points $x_0^1, \dots, y_0^4$ on $C(0, \varepsilon)$,

$$P\big(G_1^{1,2}\big) \leq k_3 P\left(Z^1_{[0, S_{1/(2\varepsilon)}]} \cap \left(\bigcup_{j=2,3} Z^j_{[0, S^j_{1/(2\varepsilon)}]}\right)\right) \leq c_0 \varepsilon^{2-\beta_2},$$

where we have used the notation (Z^j, S^j) from Section 3. Similarly, $P(G_2^{3,4}) \le c_0 \varepsilon^{2-\beta_2}$.

Let $q_4 = \max(1 - q_2, 1 - q_3)$. Then $q_4 < 1$ and we obtain from (3) and (4) that

$$P_{Y^1}\big(G_{1,2}^1 | X^1, X^2, E_1, E_2\big) \le q_4^{N_1} P_{Y^1}\big(G_1^1 | X^1, X^2, E_1, E_2\big),$$
$$P_{Y^3}\big(G_{1,2}^3 | X^1, X^2, E_1, E_2\big) \le q_4^{N_2} P_{Y^3}\big(G_2^3 | X^1, X^2, E_1, E_2\big).$$

Note that $q_4^{N_1+N_2} = q_4^N \le \varepsilon^{\beta_1}$ for some fixed $\beta_1 > 0$ given the event E_2. Choose $\beta_2, \beta_3 \in (0,1)$ such that $2(2+\beta_2) + \beta_1 > 4 + \beta_3$. Then, for $\varepsilon < \varepsilon_1$,

$$\begin{aligned}
&P\big(G_{1,2}^{1,2,3,4} | E_1, E_2\big)\\
&\quad \le E_{X^1,X^2}\Big(P_{Y^1}\big(G_{1,2}^1 | X^1, X^2\big) P_{Y^3}\big(G_{1,2}^3 | X^1, X^2\big)\\
&\qquad\qquad \times P_{Y^2,Y^4}\big(G_1^2 \cap G_2^4 | X^1, X^2\big) | E_1, E_2\Big)\\
&\quad \le E_{X^1,X^2}\Big(q_4^{N_1+N_2} P_{Y^1,Y^3,Y^3,Y^4}\big(G_1^{1,2} \cap G_2^{3,4} | X^1, X^2\big) | E_1, E_2\Big)\\
&\quad \le \varepsilon^{\beta_1} P\big(G_1^{1,2} \cap G_2^{3,4} | E_1, E_2\big).
\end{aligned}$$

Hence, for all $\varepsilon < \varepsilon_1$,

$$\begin{aligned}
P\big(G_{1,2}^{1,2,3,4}\big) &= P\big(G_{1,2}^{1,2,3,4} \cap E_1\big)\\
&\le P(E_1 \cap E_2) P\big(G_{1,2}^{1,2,3,4} | E_1, E_2\big) + P\big(E_1 \cap (E_2)^c\big)\\
&\le P(E_2 \cap E_1) \varepsilon^{\beta_1} P\big(G_1^{1,2} \cap G_2^{3,4} | E_1, E_2\big)\\
&\qquad + P\big((E_2)^c | E_1\big)\\
&\le \varepsilon^{\beta_1} P\big(G_1^{1,2}\big) P\big(G_2^{3,4}\big) + \varepsilon^5\\
&\le (c_0)^2 \varepsilon^{4+\beta_3} + \varepsilon^5,
\end{aligned}$$

and this completes the proof of Proposition 4. □

5. Estimates of disconnection probabilities.

5.1. *Unconditioned processes*. We are now going to derive some consequences of the results obtained in the previous section. One may note similarities with some parts of Section 8 in Burdzy and Lawler (1990b). If K, K', and K'' are three compact sets in the plane, we will say that K disconnects K' from K'' if every continuous path M: $[0,1] \to \mathbf{C}$ such that $M(0) \in K'$ and $M(1) \in K''$ intersects K. Similarly, we will say that K disconnects K' from ∞ if every continuous path M: $[0,1) \to \mathbf{C}$ such that $M(0) \in K'$ and $\lim_{u \to 1} |M(u)| = \infty$ intersects K.

We fix $(x_1, \dots, x_6) \in C(0,1)^6$ and a compact path-connected set L which contains these points. Now let $X^1, \dots, X^6$ denote six planar Brownian motions which are independent given their starting points $X_0^1 = x_1, \dots, X_0^6 = x_6$.

LEMMA 5. *For some fixed constants $c_2 > 0$ and $\alpha_2 > 0$, which are independent of $(x_1, \dots, x_6)$ and L, for all $R \geq 2$,*

$$P\left(\left(\bigcup_{1\leq j\leq 6} X^j_{[0, T_R(|X^j|)]}\right) \cup L \ \textit{does not disconnect } 0 \textit{ from } \infty\right) \leq c_2 R^{-2-\alpha_2}.$$

PROOF. This lemma is a consequence of Proposition 4 and of the analyticity of the mapping $z \to z^2$. We fix a point $x_0 \in L$, and we define, for every $j \in \{1, \dots, 6\}$, a continuous path $(X^j_u, u \in [-1, 0])$ joining $x_0 = X^j_{-1}$ to $x_j = X^j_0$ in L. Without loss of generality we can assume that $x_0 \in (0, \infty) \subset \mathbf{C}$. Let $(\theta^j_u, u \geq -1)$ denote the continuous determination of the argument of $(X^j_u, u \geq -1)$ such that $\theta^j_{-1} = 0$.

We then define, for all $j \in \{1, \dots, 6\}$, the continuous square root $(\tilde{X}^j_u, u \geq -1)$ of $(X^j_u, u \geq -1)$ such that

$$\tilde{X}^j_{-1} = \begin{cases} \sqrt{x_0}\,, & \text{if } j = 1,2,3,4, \\ -\sqrt{x_0}\,, & \text{if } j = 5,6. \end{cases}$$

Choose any $j \leq 4$ and $k = 5$ or 6. Consider the event that the path of $\tilde{X}^j_u$ intersects the path of $\tilde{X}^k_u$. If $\tilde{X}^j_t = \tilde{X}^k_s$ for some t, s, then we must have $|X^j_t| = |X^k_s|$ and $\theta^j_t - \theta^k_s = 2\pi + 4p\pi$ for some integer p. Hence, as $X^j_{-1} = X^k_{-1} = x_0$, $X^j_{[-1,t]} \cup X^k_{[-1,s]}$ disconnects 0 from ∞. Therefore, for all $R > 4$,

$$\begin{aligned} &P\left(\left(\bigcup_{1\leq j\leq 6} X^j_{[0, T_R(|X^j|)]}\right) \cup L \ \text{does not disconnect } 0 \text{ from } \infty\right) \\ &\quad \leq P\left(\bigcup_{1\leq j\leq 6} X^j_{[-1, T_R(|X^j|)]} \ \text{does not disconnect } 0 \text{ from } \infty\right) \\ &\quad \leq P\left(\left(\bigcup_{1\leq j\leq 4} \tilde{X}^j_{[-1, T_R(|X^j|)]}\right) \cap \left(\bigcup_{j=5,6} \tilde{X}_{[-1, T_R(|X^j|)]}\right) = \varnothing\right) \\ &\quad \leq P\left(\left(\bigcup_{1\leq j\leq 4} \tilde{X}^j_{[T_2(|\tilde{X}^j|), T_{\sqrt{R}}(|\tilde{X}^j|)]}\right) \cap \left(\bigcup_{j=5,6} \tilde{X}^j_{[T_2(|\tilde{X}^j|), T_{\sqrt{R}}(|\tilde{X}^j|)]}\right) = \varnothing\right). \end{aligned}$$

The $(\tilde{X}^j_u,\ u \leq 0)_{1\leq j\leq 6}$ are independent time-changed planar Brownian motions. Hence, Lemma 1 and Proposition 4 imply that this last probability is majorized by $k_6 c_1 (\sqrt{R}/2)^{-4-\alpha_1}$ for all $R > 4$; Lemma 5 follows. □

We now show that Lemma 5 still holds if we replace L by two paths L^1 and L^2 joining, respectively, X^1_0 to X^2_0 and X^3_0 to X^4_0. Let us suppose that L^1 (respectively, L^2) is a continuous path joining x_1 to x_2 (respectively, x_3 to x_4).

LEMMA 6. *For some fixed* $c_2 > 1$ *and* $\alpha_3 > 0$, *which are independent of* $(x_1, \dots, x_6)$, L^1, L^2, *for all* $R \geq 2$,

$$P\left(\left(\bigcup_{1 \leq j \leq 6} X^j_{[0, T_R(|X^j|)]}\right) \cup L^1 \cup L^2 \text{ does not disconnect } 0 \text{ from } \infty\right) \leq c_3 R^{-2-\alpha_3}.$$

PROOF. We first introduce some further notation. For all $n \geq 1$, we set

$$A_n = \left\{\left(\bigcup_{1 \leq j \leq 6} X^j_{[0, T_n(|X^j|)]}\right) \cup L^1 \cup L^2 \text{ does not disconnect } 0 \text{ from } \infty\right\},$$

$$B_n = \left\{\left(\bigcup_{1 \leq j \leq 6} X^j_{[0, T_n(|X^j|)]}\right) \cup L^1 \cup L^2 \text{ is connected}\right\}$$

and

$$Q_{n+1} = B_{n+1} \setminus B_n.$$

If $(\bigcup_{1 \leq j \leq 6} X^j_{[0, T_n(|X^j|)]}) \cup L^1 \cup L^2$ is not connected, then at least one of the three following events occur:

$$B_n^1 = \left\{\left(\bigcup_{j = 1,2} X^j_{[0, T_n(|X^j|)]}\right) \cap \left(\bigcup_{j = 3,4} X^j_{[0, T_n(|X^j|)]}\right) = \varnothing\right\},$$

$$B_n^2 = \left\{X^5_{[0, T_n(|X^5|)]} \cap \left(\bigcup_{1 \leq j \leq 4} X^j_{[0, T_n(|X^j|)]}\right) = \varnothing\right\},$$

$$B_n^3 = \left\{X^6_{[0, T_n(|X^6|)]} \cap \left(\bigcup_{1 \leq j \leq 4} X^j_{[0, T_n(|X^j|)]}\right) = \varnothing\right\}.$$

We deduce from Lemma 1 and Corollary 3 that, for some fixed constant c_4 independent of $(x_1, \dots, x_6)$, L^1 and L^2, for all $n \geq 1$,

$$P\big((B_n)^c\big) \leq P\big(B_n^1\big) + P\big(B_n^2\big) + P\big(B_n^3\big) \leq c_4 n^{-9/4}.$$

Lemma 5 combined with the strong Markov property applied at $T_n(|X^j|)$'s and a scaling argument imply that, for all $1 \leq n < N$,

$$P(A_N | Q_n) \leq c_2 (N/n)^{-2-\alpha_2}$$

and

$$P(A_N \cap B_1) \leq c_2 N^{-2-\alpha_2}.$$

We set $\alpha_4 = \min(1/4, \alpha_2)$. Now, for all $N \geq 3$,

$$P(A_N) \leq P\big((B_N)^c\big) + P(A_N \cap B_1) + \sum_{n=2}^{n=N} P(Q_n \cap A_N)$$

$$\leq c_4 N^{-9/4} + c_2 N^{-2-\alpha_2} + c_2 \sum_{n=2}^{n=N} \left(\frac{N}{n}\right)^{-2-\alpha_4} P(Q_n).$$

However,

$$\begin{aligned}\sum_{n=2}^{n=N} n^{2+\alpha_4} P(Q_n) &\le \sum_{n=2}^{n=N} n^{2+\alpha_4}\Big(P\big((B_{n-1})^c\big) - P\big((B_n)^c\big)\Big)\\ &\le P\big((B_1)^c\big) + \sum_{n=2}^{n=N-1}\Big((n+1)^{2+\alpha_4} - (n)^{2+\alpha_4}\Big)P\big((B_n)^c\big)\\ &\le 1 + \sum_{n=2}^{N-1} c_4(2+\alpha_4)\frac{(1+1/n)^{1+\alpha_4}}{n}\\ &\le 1 + 12c_4\log N.\end{aligned}$$

Finally, for all $N \ge 3$,

$$P(A_N) \le \big(c_4 + c_2(2 + 12c_4 \log N)\big)N^{-2-\alpha_4}.$$

Lemma 6 follows.

5.2. *Conditioned processes.* Let us now define six identically distributed independent processes $W^1, \dots, W^6$ as follows. For all $j \in \{1, \dots, 6\}$, for all $u \ge 0$,

$$W_u^j = \exp\Big(-\beta^j_{A_j(u)} + i\theta^j_{A_j(u)}\Big),$$

where β^j is a three-dimensional Bessel process started from 0, θ^j is an independent linear Brownian motion started with the uniform law on $[0, 2\pi]$ (i.e., W^j starts with uniform distribution on the unit circle) and where $A_j(u) = \int_0^u |W_v^j|^{-2}\, dv$ is the usual time-change. Now we set, for all $\varepsilon < 1$ and for all $j \in \{1, \dots, 6\}$,

$$\hat{T}^j_\varepsilon = \inf\big\{u > 0, |W_u^j| = \varepsilon\big\}.$$

We also put

$$\mathscr{W}(\varepsilon, 6) = \bigcup_{1 \le j \le 6} W^j_{[0, \hat{T}^j_\varepsilon]}.$$

Let L^1 and L^2 be two (random) $\sigma(W_0^1, W_0^2)$- and $\sigma(W_0^3, W_0^4)$-measurable paths joining W_0^1 to W_0^2 and W_0^3 to W_0^4, respectively. Then we have the following analogue of Lemma 6.

LEMMA 7. *For some* $\alpha_5 > 0$, $c_5 > 0$, *for all* L^1, L^2, $\varepsilon < 1$,

$$P\big(L^1 \cup L^2 \cup \mathscr{W}(\varepsilon, 6) \text{ does not disconnect } 0 \text{ from } \infty\big) \le c_5 \varepsilon^{2+\alpha_5}.$$

PROOF. Recall the notation of Section 3. One has to notice, using Section 2.2.2, that, for all $1 \le j \le 6$, the process

$$\Big(2W^j_{\hat{T}_{1/2}+u},\, u \le \hat{T}^j_\varepsilon - \hat{T}^j_{1/2}\Big)$$

has the same law as

$$\big(Z_u^j,\, u \le S^j_{2\varepsilon}\big) \text{ conditional on } \big\{S^j_{2\varepsilon} < S^j_2\big\}.$$

Let $f(z) = z^{-1}$. It is well known that $(uf(Z_u^j),\ u \geq 0)$ has the same distribution as $(Z_u^j, u \geq 0)$, and so

$$P\left(\left(\bigcup_{1 \leq j \leq 6} Z_{[0, T_{1/R}(|Z^j|)]}\right) \cup f(L^1) \cup f(L^2) \text{ does not disconnect 0 from } \infty\right)$$
$$= P\left(\left(\bigcup_{1 \leq j \leq 6} Z_{[0, T_R(|Z^j|)]}\right) \cup L^1 \cup L^2 \text{ does not disconnect 0 from } \infty\right).$$

This, Lemma 6 and the fact that $P(S^j_{2\varepsilon} < S^j_2) = \log 2/|\log \varepsilon|$ [cf. (1)], imply that

$$P\big(L^1 \cup L^2 \cup \mathscr{W}(\varepsilon, 6) \text{ does not disconnect 0 from } \infty\big) \leq c_3 \left(\frac{|\log \varepsilon|}{\log 2}\right)^6 \left(\frac{\varepsilon}{2}\right)^{2+\alpha_3},$$

for all $\varepsilon < 1/2$; Lemma 7 follows immediately. □

6. Proof of Theorem 1. Let $Z = (Z_t,\ t \geq 0)$ denote a planar Brownian motion started from 0. We want to show that, for any fixed T, $Z_{[0,T]}$ has no frontier triple point. Note that almost surely Z_0 and Z_T are not frontier points of $Z_{[0,T]}$.

If z is a frontier triple point for $Z_{[0,T]}$, with $z \neq Z_0$ and $z \neq Z_T$, then, for some $a > 0$, there exist $0 < t_1 < s_1 < t_2 < s_2 < t_3 < T$ such that $z = Z_{t_1} = Z_{t_2} = Z_{t_3}$, $|z| > a$, $|Z_{s_1} - z| > a$, $|Z_{s_2} - z| > a$ and $|Z_T - z| > a$, such that z is on the boundary of a connected component of the complement of $Z_{[0,T]}$, which intersects the circle $C(z, a)$. We will say that such a point is an a-frontier-triple point.

A simple scaling argument shows that it suffices to prove the nonexistence of 2-frontier-triple points. Let us now fix $k > 0$; we are going to prove that $K = [-k, k]^2$ contains almost surely no 2-frontier-triple points.

We first fix $\varepsilon < 1/2$. We cover K with $N_\varepsilon < 4k^2\varepsilon^{-2}$ discs $D_i = D(z_i, \varepsilon)$ of radius ε. We also fix an $i \in \{1, \dots, N_\varepsilon\}$ for a while; we define

$$U_1^i = \inf\{t > 0, |Z_t - z_i| = 1\}$$

and, for all $j \geq 1$,

$$V_j^i = \inf\big\{t > U_j^i, |Z_t - z_i| = \varepsilon\big\},$$
$$U_{j+1}^i = \inf\big\{t > V_j^i, |Z_t - z_i| = 1\big\}.$$

If $z \in D_i$ is a 2-frontier-triple point, then $U_4^i < T$; moreover, as $D(z_i, 1) \subset D(z, 2)$, $Z_{[0,T]}$ does not disconnect D_i from $C(z_i, 1)$. Hence, $Z_{[0, U_4^i]}$ does not disconnect D_i from $C(z_i, 1)$. We are now going to estimate the probability of this last event, using Lemma 7.

We first set, for all $j \ge 1$,

$$\sigma_j^i = \sup\left\{t < V_j^i, |Z_t - z_i| = 1/2\right\},$$

$$\tau_j^i = \inf\left\{t > \sigma_j^i, |Z_t - z_i| = 2\varepsilon\right\},$$

$$\eta_j^i = \sup\left\{t < U_{j+1}^i, |Z_t - z_i| = 2\varepsilon\right\},$$

$$\sigma_j^i = \inf\left\{t > \eta_j^i, |Z_t - z_i| = 1/2\right\}.$$

We also define, for $j = 1, 2, 3$, $s_{2j-1}^i = \tau_j^i - \sigma_j^i$ and, for all $u \le s_{2j-1}^i$,

$$Z_u^{i,2j-1} = 2Z_{\sigma_j^i + u}.$$

We then set, for all $j = 1, 2, 3$, $s_{2j}^i = \rho_j^i - \eta_j^i$ and, for all $u \ne s_{2j}^i$,

$$Z_u^{i,2j} = 2Z_{\rho_j^i - u}.$$

It is easy to see, using the skew-product decomposition of Z and the standard properties of three-dimensional Bessel processes and their relations with Brownian motion recalled in Section 2.2, that the joint law of $(Z_u^{i,j}, u \le s_j^i)$, after the usual time-change, is absolutely continuous (with uniformly bounded density independent of ε, see Lemma 1) with respect to the law of $\mathscr{W}(4\varepsilon, 6)$ as defined in the previous paragraph.

Moreover, $(Z_t,\ \rho_1^i \le u \le \sigma_2^i)$ [respectively, $(Z_t,\ \rho_2^i \le u \le \sigma_3^i)$] connects $Z_0^{i,2} = 2Z_{\rho_1^i}$ to $Z_0^{i,3} = 2Z_{\sigma_2^i}$ (respectively, $Z_0^{i,4}$ to $Z_0^{i,5}$) outside the circle $C(z_i, \varepsilon)$.

Hence, Lemma 7 shows readily that

$$P\left(Z_{[0,U_4^i]} \text{ does not disconnect } D_i \text{ from } C(z,1)\right) \le c_5 k_6 (4\varepsilon)^{2+\alpha_5}.$$

Consequently,

$$\begin{aligned} &P(\exists\, z \in K,\ z \text{ is a 2-frontier-triple point}) \\ &\quad \le \sum_{i=1}^{N_\varepsilon} P\left(Z_{[0,U_4^i]} \text{ does not disconnect } D_i \text{ from } C(z_i,1)\right) \\ &\quad \le N_\varepsilon c_5 k_6 (4\varepsilon)^{2+\alpha_5} \\ &\quad \le 4k^2 c_5 k_6 (4\varepsilon)^{\alpha_2}. \end{aligned}$$

Therefore (as this is true for all $\varepsilon < 1/2$), there are almost surely no 2-frontier-triple points in K. Since this is valid for all K, the theorem follows. $\square$

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