The Annals of Probability
1996, Vol. 24, No. 1, 35–48

EXTREMAL PROPERTIES OF HALF-SPACES FOR LOG-CONCAVE DISTRIBUTIONS

BY S. BOBKOV

Syktyvkar University

The isoperimetric problem for log-concave product measures in $\mathbb{R}^n$ equipped with the uniform distance is considered. Necessary and sufficient conditions under which standard half-spaces are extremal are presented.

1. Introduction. Let μ be a probability measure on the real line $\mathbb{R}$. Consider the value

$$r_h^{(n)}(p) = \inf \mu^n(A + hD_n), \tag{1.1}$$

where $\mu^n = \mu \times \cdots \times \mu$ is the product measure in $\mathbb{R}^n$, $D_n = [-1,1]^n$ is the n-dimensional cube in $\mathbb{R}^n$ and the infimum is then over all Borel-measurable sets $A \subset \mathbb{R}^n$ of measure $\mu^n(A) \geq p$; $0 < p < 1$, $h > 0$.

We are searching for necessary and suffícent conditions under which the infimum in (1.1) is attained at the half-spaces $A = \{x \in \mathbb{R}^n\colon\ x_1 \leq c\}$ of measure p, for all p and h (the half-spaces of this form will be called standard). In these notes, we give such conditions in the case where μ is log-concave.

There exists the following probabilistic consequence of the above property. If $\zeta_1, \dots, \zeta_n$ are independent random variables on some probability space $(\Omega, \Im, \mathbb{P})$, with common distribution μ, consider a family of random variables $S(t)$, indexed by a set T and formed by linear combinations,

$$S(t) = \sum_{i=1}^{n} a_i(t)\zeta_i,$$

where the coefficients a_i are arbitrary functions on T such that

$$\sigma_1 = \sup_t \sum_{i=1}^{n} |a_i(t)| \leq 1.$$

Set $M = \sup_t S(t)$. Then the extremal property of the standard half-spaces in the isoperimetric problem (1.1) implies that, for all $p \in (0,1)$ and $h > 0$,

$$\mathbb{P}\big\{M - m_p(M) > h\big\} \leq \mathbb{P}\big\{\zeta_1 - m_p(\zeta_1) > h\big\}, \tag{1.2}$$

where $m_p(\cdot)$ denotes quantile of order p of a random variable. For $h < 0$, the converse inequality should be written in (1.2). In other words, the deviations

Received July 1993; revised March 1995.

[1]This research was partially supported by the NSF, Air Force Office of Scientific Research Grant 91-0030 and Army Research Office Grant DAAL 03-92-G-0008.

AMS 1991 *subject classification*. 60G70

Key words and phrases. Isoperimetry

of M from its quantiles are not larger than those of the original random variable ζ_1; this is also equivalent to existence of a Lipschitz function φ from $\mathbb{R}$ to $\mathbb{R}$ such that M and $\varphi(\zeta_1)$ are identically distributed.

In terms of the isoperimetric problem, property (1.2) can be equivalently expressed as follows: (1.1) attains a minimum at the standard half-spaces within the class of all convex subsets of $\mathbb{R}^n$. We will prove, however, that the last implies the extremal property of these half-spaces within the class of all Borel-measurable subsets of $\mathbb{R}^n$. Moreover, in order to obtain such an extremal property with respect to the supremum distance, we will prove that it only suffices to assume that (1.2) is fulfilled for two random variables (in the case $n \geq 2$): $M = -\zeta_1$ and $M = \max\{\zeta_1, \zeta_2\}$.

In the case where μ is a Gaussian measure on the real line, the half-spaces possess a much more intrinsic property [6, 4]; they are extremal in (1.1) even if D_n is replaced by the unit Euclidean ball in $\mathbb{R}^n$. Likewise, (1.2) holds under the weaker assumption $\sigma_2^2 = \sup_t \sum_{i=1}^n |a_i(t)|^2 \leq 1$. It can be shown that such a strong property (when $n \geq 2$) characterizes Gaussian measures in the class of all probability distributions μ. Talagrand [7] considered an enlargement of sets A which is, in a certain sense, even smaller than the Euclidean h-neighborhood of A. An inequality he proved for the two-sided exponential distribution does not express any extremal property but states a rather strong variant of the so-called concentration-of-measure phenomenon. Inequalities (e.g., as in [1] and [2]) for the uniform enlargement defined by the supremum distance state the weakest variant of this phenomenon.

Denote by F the distribution function of the measure μ: $F(x) = \mu((-\infty, x])$, $x \in \mathbb{R}$. By definition, μ is log-concave if it has a density f such that the function $\log(f)$ is concave on the interval (a_F, b_F), where

$$a_F = \inf\{x \in \mathbb{R}: F(x) > 0\}, \qquad b_F = \sup\{x \in \mathbb{R}: F(x) < 1\}.$$

In general, $-\infty \leq a_F < b_F \leq +\infty$. Necessarily, f is continuous and positive on (a_F, b_F), F increases on (a_F, b_F), and so an inverse $F^{-1}\colon (0,1) \to (a_F, b_F)$ exists.

In this paper, we present the following statement on log-concave measures.

THEOREM 1.1. *Let $n \geq 2$. The standard half-spaces are extremal for all $h > 0$ and $p \in (0,1)$ in the isoperimetric problem* (1.1) *if and only if the following hold*:

(a) *μ is symmetric around a point* (*which is the median of F*);
(b) *the support of μ is the real line $\mathbb{R}$* (*i.e., $a_F = -\infty$, $b_F = +\infty$*);
(c) *for any p, $q \in (0,1)$,*

$$\frac{f(F^{-1}(pq))}{pq} \leq \frac{f(F^{-1}(p))}{p} + \frac{f(F^{-1}(q))}{q}. \tag{1.3}$$

This statement does not depend on the dimension. Note that the equality $I_F(p) = f(F^{-1}(p))$ defines 1–1 correspondence between the family of all distribution functions F which have an even positive continuous density f on (a_F, b_F) $(a_F = -b_F)$ and the family of all positive continuous functions I on $(0, 1)$ symmetric around $\frac{1}{2}$. In addition,

$$b_F = \int_{1/2}^{1} dp/I_F(p). \tag{1.4}$$

The log-concavity of f is equivalent to concavity of I_F (see Proposition 6.1), so, under assumptions (a) and (b), (1.3) determines a certain subset of the set of all concave, positive, symmetric around $\frac{1}{2}$ functions on $(0, 1)$ for which the integral (1.4) is infinite.

Often, it is not easy to check (1.3). We will give the following weaker condition: the function $\log(f/F)$ is concave. For the standard Gaussian measure, the last property [i.e., the second derivative of $\log(f/F)$ is never positive] follows from the inequality $1 - F(x) \geq xf(x)/(1 + x^2)$, $x \geq 0$.

Consider another example. Let $F(x) = 1/(1 + \exp(-x))$, $x \in \mathbb{R}^n$ (so-called logistic distribution function). Then, the function $\log(f(x)/F(x)) = -\log(1 + e^x)$ is also concave. In this case, $f(F^{-1}(p)) = p(1-p)$, and (1.3) is easy to verify.

The two-sided exponential distribution F, of density $f(x) = \exp(-|x|)/2$, $x \in \mathbb{R}$, is very close to the above mentioned distribution, but (1.3) is not fulfilled for F (one can take $p = q = 0.7$). In this case, $f(F^{-1}(p)) = \min\{p, 1-p\}$. Thus, one can say that the extremal property of half-spaces is not determined by the tail behavior of F; it means a quality of some different and delicate nature.

2. Description of the proof. For the reader's convenience, we first note several steps which will be performed to prove Theorem 1.1. The first step is to solve the one-dimensional isoperimetric problem. A special case where μ has density $f(x) = \exp(-|x|)/2$, $x \in \mathbb{R}$, has been studied in [7]. In Section 3 we will prove by similar methods the following statements.

PROPOSITION 2.1. *Let μ be a log-concave measure on the real line $\mathbb{R}$. Then for all $p \in (0, 1)$, $h > 0$, the value $\mu(A + [-h, h])$ is minimal on the class of all Borel-measurable sets $A \subset \mathbb{R}$ of measure $\mu(A) \geq p$, if A is the interval $(-\infty, a]$ or the interval $[b, +\infty)$ of μ-measure p (in both cases).*

PROPOSITION 2.2. *For a log-concave measure μ on the real line $\mathbb{R}$, the intervals $(-\infty, a]$ are extremal for all $p \in (0, 1)$, $h > 0$, if and only if μ is symmetric around its median.*

Since in Proposition 2.1, $a = F^{-1}(p)$, $b = F^{-1}(1-p)$, the minimal value of $\mu(A + [-h, h])$ under the assumption $\mu(A) \geq p$ is equal to

$$R_h(p) = \min\{F(F^{-1}(p) + h), 1 - F(F^{-1}(1-p) - h)\}. \tag{2.1}$$

When the measure μ is symmetric around its median, the expression (2.1) is simplified:

$$R_h(p) = F(F^{-1}(p) + h). \tag{2.2}$$

By Proposition 2.2, property (a) in Theorem 1.1 is necessary for extremality of the standard half-spaces (this holds even in the case $n = 1$). Now we explain how to prove necessity of (b) and (c).

Note that the standard half-spaces $A_p = \{x \in \mathbb{R}^n \colon x_1 \le F^{-1}(p)\}$ are extremal in the isoperimetric problem (1.1) if and only if

$$r_h^{(n)}(p) = F\big(F^{-1}(p) + h\big);$$

that is, $r_h^{(n)}(p) = R_h(p)$, for any $0 < p < 1$, $h > 0$ [in the following, R_h is defined by (2.2)]. Suppose that the last identity holds. Given $p, q \in (0,1)$, consider two half-spaces A_p and $B_q = \{x \in \mathbb{R}^n \colon x_2 \le F^{-1}(q)\}$ of μ-measure p and q, respectively. Then the cube $A_p \cap B_q$ has measure pq and, therefore, since (1.1) is attained at the half-space A_{pq}, we get

$$\mu^n\big(A_{pq} + hD_n\big) \le \mu^n\big(\big(A_p \cap B_q\big) + hD_n\big); \tag{2.3}$$

that is

$$R_h(pq) \le R_h(p) R_h(q) \quad \text{for all } p, q \in (0,1),\ h > 0. \tag{2.4}$$

Therefore, in order to prove the necessity of (b) and (c) in Theorem 1.1, it suffices to establish the following lemma (see Section 4).

LEMMA 2.3. *Property* (2.4) *and* (a) *imply* (b) *and* (c).

To prove the sufficiency part in Theorem 1.1, we will show the following in Section 5.

LEMMA 2.4. *Properties* (a) *and* (c) *imply* (2.4).

We will also show (Lemma 5.2) why (c) may be replaced in this lemma by the assumption of log-concavity of f/F.

To complete the proof of Theorem 1.1, the problem now is how to derive the identity $r_h^{(n)} = R_h$ from (2.4) and property (a). For this purpose, we use a statement from [1] (Theorems 1.1 and 1.2, which are formulated here together for the space $X = \mathbb{R}$ with the canonical enlargement).

Let μ be a probability Borel measure on $\mathbb{R}$. For any $n \ge 1$, $h > 0$, define the function of $p \in (0,1)$,

$$\mathfrak{R}_h^{(n)} = \inf \mu^n\big(A + h(-1,1)^n\big),$$

where the infimum is over all Borel-measurable $A \subset \mathbb{R}^n$ of measure $\mu^n(A) \ge p$, and where $(-1,1)^n$ is the open cube in $\mathbb{R}^n$. Clearly, for absolutely continuous probability distributions, $\mathfrak{R}_h^{(n)} = r_h^{(n)}$.

THEOREM 2.5 [1]. *Suppose that the function $\mathfrak{R}_h^{(1)}$ is concave and increasing on* $(0,1)$. *Then $\mathfrak{R}_h^{\infty} = \inf_n \mathfrak{R}_h^{(n)}$ is the maximal function among all increasing bijections $R \le \mathfrak{R}_h^{(1)}$ in* $(0,1)$ *such that, for any $p, q \in (0,1)$,*

$$R(pq) \le R(p)R(q), \tag{2.5}$$

$$S(pq) \ge S(p)S(q), \tag{2.6}$$

where $S(p) = 1 - R(1-p)$. *In particular*, $\mathfrak{R}_h^{\infty} = \mathfrak{R}_h^{(1)}$ *if and only if the function* $R = \mathfrak{R}_h^{(1)}$ *satisfies* (2.5) *and* (2.6).

In this statement, the parameter h is fixed.

First let us check that Theorem 2.5 may be applied to log-concave measures μ satisfying (a) and (b). By Proposition 2.2, $\mathfrak{R}_h^{(1)} = R_h$, and thus we need to show that R_h is concave and increasing on $(0,1)$. Since F has a continuous positive density f on the whole real line, R_h increases (strictly). Since $\log(f)$ is concave, we have that, whenever $h > 0$, the increment $\log(f(x+h)) - \log(f(x)) = \log(f(x+h)/f(x))$ represents a nonincreasing function on $\mathbb{R}$. Consequently, the derivative $R_h'(p) = f(F^{-1}(p)+h)/f(F^{-1}(p))$ does not increase, either. This implies concavity of R_h (see also Proposition 6.1 about a converse statement which shows that Theorem 2.5 cannot be applied to a more general class of probability distributions μ).

Thus, assuming (a) and (b) in Theorem 1.1, one can apply Theorem 2.5 to μ. By Lemma 2.4, where (c) is also assumed, property (2.4) is fulfilled; that is, (2.5) holds for $R = \mathfrak{R}_h^{(1)} \equiv R_h$. It is obvious that the function $S(p) = 1 - R_h(1-p)$ is the inverse of R_h. Then (2.6) may be written as

$$R^{-1}(pq) \geq R^{-1}(p) R^{-1}(q) \quad \text{for all } p, q \in (0,1),$$

which is equivalent to (2.4). Consequently, Theorem 2.5 allows us to conclude that $\mathfrak{R}_h^{\infty} = \mathfrak{R}_h^{(1)}$; that is, for all $n \geq 1$, $\mathfrak{R}_h^{(n)} = \mathfrak{R}_h^{(1)}$, that is, $r_h^{(n)} = R_h$. Thus, the sufficiency part in Theorem 1.1 will be proved.

Now we show that if (1.2) is fulfilled for $M_1 = -\zeta_1$ and $M_2 = \max\{\zeta_1, \zeta_2\}$, then the standard half-spaces are extremal for μ. Recall that ζ_1 and ζ_2 are independent random variables with common log-concave law μ. Inequality (1.2) with $M = M_1$ says that, for all $h > 0$, $p \in (0,1)$,

$$\mu\big((-\infty, a]^h\big) \leq \mu\big([b, +\infty)^h\big),$$

where $a = F^{-1}(p)$, $b = F^{-1}(1-p)$. Hence, by Proposition 2.1, the intervals $(-\infty, a]$ are extremal in the one-dimensional isoperimetric problem. By Proposition 2.2, μ is symmetric around a point, and thus property (a) in Theorem 1.1 is fulfilled. If (1.2) is true for M_2, we obtain (2.3) for dimension $n = 2$. Hence, (2.4) is valid, too, and by Lemma 2.3 properties (b) and (c) in Theorem 1.1 are also fulfilled. Applying this theorem to the product of μ concludes the argument.

3. One-dimensional isoperimetric problem for log-concave measures.

First, we explain why it suffices to prove Proposition 2.1 under the following additional assumption:

(3.1) $f(x) > 0$ for all $x \in \mathbb{R}$ (i.e., $a_F = -\infty$, $b_F = +\infty$), the derivative of $\log(f)$ exists everywhere and represents a continuous decreasing function.

Indeed, the function $u = -\log(f)$ is convex on (a_F, b_F); therefore, there exists a sequence u_n of convex functions on $\mathbb{R}$ such that the following hold:

1. The derivative of u_n is a continuous increasing function;
2. u_n converges pointwise to u on (a_F, b_F) and to $+\infty$ on $\mathbb{R} \setminus [a_F, b_F]$;
3. $f_n = \exp(-u_n)$ is the density of some probability measure μ_n.

Then f_n are log-concave and, moreover, satisfy (3.1). In addition, $\mu_n(A) \to \mu(A)$, for all measurable A. In particular, $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$, $F_n^{-1}(p) \to F^{-1}(p)$ for all $p \in (0,1)$, where F_n is the distribution function of μ_n and F_n^{-1} is the inverse of F_n. Now, if one may apply Proposition 2.1 to μ_n, then, according to (2.1), for all measurable $A \subset \mathbb{R}$, $p \in (0,1)$, $h > 0$, we have

$$\mu_n(A + hD_n) \geq \min\big\{F_n\big(F_n^{-1}(p) + h\big), 1 - F_n\big(F_n^{-1}(1-p) - h\big)\big\}. \tag{3.2}$$

Taking the limit in (3.2), we obtain

$$\mu(A + hD_n) \geq \min\big\{F\big(F^{-1}(p) + h\big), 1 - F\big(F^{-1}(1-p) - h\big)\big\}. \tag{3.3}$$

Since (3.3) provides an equality at the interval $A = (-\infty, F^{-1}(p)]$ or at $A = [F^{-1}(1-p), +\infty)$, Proposition 2.1 follows from (3.3).

Now we may assume (3.1). For convenience, it is better to consider the extended line $[-\infty, +\infty]$ and subsets of it. For any nonempty set $A \subset [-\infty, +\infty]$ and $h > 0$, we use the notation $A^h = A + [-h, h]$. Let $A_p(a)$ denote the interval $[a, b] \subset [-\infty, +\infty]$ of μ-measure p, $-\infty \leq a < b \leq +\infty$. The variable a may vary in the interval $[-\infty, F^{-1}(1-p)]$, and $b = b(a)$ is the function of a defined by the equality

$$F(b) - F(a) = p. \tag{3.4}$$

Note that $B_p(a) = (-\infty, a] \cup [b, +\infty)$, *where* $b = b(a)$, *has measure* $1 - p$.

LEMMA 3.1. *For any* $p \in (0,1)$, *there exists* $a_0 = a_0(p)$ *such that* $\mu(A_p(a)^h)$ *increases in* $[-\infty, a_0]$ *and decreases in* $[a_0, F^{-1}(1-p)]$. *The same holds for* $\mu(B_p(a)^h)$ *and some point* $a_0'(p)$.

PROOF. Clearly, the function b increases, $b(-\infty) = F^{-1}(p)$, $b(F^{-1}(1-p)) = +\infty$. Differentiation of (3.4) gives $f(b)b' - f(a) = 0$. Differentiating the function $\varphi(a) = \mu(A_p(a)^h) \equiv F(b+h) - F(a-h)$, we obtain

$$\begin{aligned}\varphi'(a) &= f(b+h)b' - f(a-h) = f(b+h)\frac{f(a)}{f(b)} - f(a-h)\\ &= f(a)\left[\frac{f(b+h)}{f(b)} - \frac{f(a-h)}{f(a)}\right].\end{aligned}$$

By assumption (3.1), the increment $u(x+h) - u(x)$ of the function $u(x) = -\log(f(x))$ represents an increasing function of $x \in \mathbb{R}$. Therefore, for each

$h > 0$, the function $U(a) = f(a - h)/f(a)$ decreases and $V(a) = f(b + h)/f(b)$ increases (as a composition of two increasing functions). Thus, the function

$$\frac{\varphi'(a)}{f(a)} = V(a) - U(a)$$

is continuous and increasing on $(-\infty, F^{-1}(1 - p))$. Consequently, one of the following occurs:

(a) there exists a point $a_0 \in (-\infty, F^{-1}(1 - p))$ such that $\varphi' < 0$ on $(-\infty, a_0)$ and $\varphi' > 0$ on $(a_0, F^{-1}(1 - p))$;
(b) $\varphi' > 0$ on $(-\infty, F^{-1}(1 - p))$;
(c) $\varphi' < 0$ on $(-\infty, F^{-1}(1 - p))$.

In case (a), the first statement of Lemma 3.1 holds with $a_0(p) = a_0$; in cases (b) and (c), one should set $a_0(p) = F^{-1}(1 - p)$ and $a_0 = -\infty$, respectively. The proof of the second part of Lemma 3.1 does not differ from the above argument.

So, we have proved Proposition 2.1 within the class of all closed intervals $[a, b] \subset [-\infty, +\infty]$ of μ-measure greater than or equal to p, and the next task is to extend this proposition to disjoint unions A formed by closed intervals. Let $\mathscr{I}_p$ denote the family of such sets A of μ-measure greater than or equal to p; for each $A \in \mathscr{I}_p$, let rank (A) denote the number of intervals which form A. In fact, because of approximation argument and absolute continuity of μ, it is sufficient to establish Proposition 2.1 for the class $\mathscr{I}_p$ only.

LEMMA 3.2. *For any* $0 < p < 1$, *Proposition* 2.1 *holds within the family* $\mathscr{I}_p$.

PROOF. Certainly, the parameter $h > 0$ is supposed to be arbitrary, too. Let $\mathscr{I}_p(h)$ denote the family of all sets A from $\mathscr{I}_p$ that are formed by intervals $\Delta_i = [a_i, b_i]$, $-\infty < a_1 < b_1 < \cdots < a_n < b_n \le +\infty$, whose closed h-neighborhoods $\Delta_i^h = [a_i - h, b_i + h]$ do not intersect. Now it suffices to prove the lemma within $\mathscr{I}_p(h)$. Indeed, if $\Delta_i^h \cap \Delta_j^h \ne \varnothing$ for some $i < j$, then $\Delta_i^h \cup \Delta_j^h = [a_i, b_j]^h$ and, therefore, two intervals Δ_i and Δ_j can be replaced by one interval $[a_i, b_j]$ which has a larger measure. Thus, for any set A from $\mathscr{I}_p$, there exists a set A' from $\mathscr{I}_p(h)$ with the same h-neighborhood.

Let $A \in \mathscr{I}_p(h)$ be formed by the intervals Δ_i, $1 \le i \le n$, as above. We will move these intervals, keeping their measures $p_i = \mu(\Delta_i)$ constant. In addition, the interiors of moving intervals should not intersect so that the measure of their union remains constant, too $[= \mu(A)]$. The right ends $b_i = b(a_i)$ should depend on the appropriate left ends a_i as in Lemma 3.1. According to this lemma, in the case $a_i \le a_0(p_i)$, we may move a_i to the left until $a_i = b_{i-1}$ if $i > 1$, or $a_i = -\infty$ if $i = 1$. Respectively, in the case $a_i > a_0(p_i)$, we move a_i to the right until $b_i = a_{i+1}$ if $i < n$, or $b_i = +\infty$ if $i = n$. Applying such a procedure to any of the middle intervals Δ_i, $1 < i < n$, $n \ge 3$, we obtain a new interval Δ'_i of measure p_i and a new set $A_1 = \bigcup \Delta'_j$, where $1 \le j \le n$, $\Delta'_j = \Delta_j$ for $j \ne i$ and $\Delta'_j = \Delta_i$ for $j = i$ (only one interval Δ_i in A has been changed). Then $\mu(A_1) = \mu(A)$. By Lemma 3.1,

$$\mu\left(A_1^h\right) \le \sum \mu\left(\left(\Delta'_j\right)^h\right) \le \sum \mu\left(\Delta_j^h\right) = \mu\left(A^h\right), \tag{3.5}$$

since $A_1^h = \bigcup(\Delta'_j)^h$ and since Δ_j^h are disjoint. By the definition of A_1, rank $(A_1) = n - 1$; that is, A_1 consists of $n - 1$ intervals. One can continue this process and construct $A_2, \dots, A_{n-2}$. The last set will consist of two intervals. Again, applying the above procedure to these two intervals, we obtain a set of B of the following three types: $B = [-\infty, a]$; $B = [b, +\infty]$; $B = [-\infty, a] \cup [b, +\infty]$. In all the cases, $\mu(B) = \mu(A)$ and $\mu(B^h) \le \mu(A^h)$, since (3.5) was used at each step. To exclude the sets of the third type, it remains to apply the second part of Lemma 3.1. □

REMARK 3.3. In order to prove Proposition 2.1, log-concavity was used in Lemma 3.1. Nevertheless, the statement holds for some other distributions, too. In particular, the reasoning in the proof of this lemma can be applied to an arbitrary distribution F which is concentrated on an interval where F has a monotonic density.

PROOF OF PROPOSITION 2.2. Due to Proposition 2.1, there is nothing to prove for symmetric measures. Conversely, assume the intervals A of the form $A = (-\infty, a]$ represent the sets for which the value $\mu(A^h)$ is minimal provided $\mu(A) \ge p$. Consequently, for sets $A = (-\infty, a]$ and $B = [b, +\infty)$ of μ-measure $p \in (0,1)$, one has $\mu(A^h) \le \mu(B^h)$, for all $h > 0$; that is,

$$F\left(F^{-1}(p) + h\right) \le 1 - F\left(F^{-1}(1-p) - h\right). \tag{3.6}$$

For $h = 0$, (3.6) turns into an equality, and we obtain the inequality for derivatives of both parts of (3.6) at $h = 0$: $f(F^{-1}(p)) \le f(F^{-1}(1-p))$. Replacing p with $1 - p$, we get a converse inequality. Thus, for all $p \in (0,1)$,

$$f\left(F^{-1}(p)\right) = f\left(F^{-1}(1-p)\right). \tag{3.7}$$

The function $F^{-1}\colon (0,1) \to (a_F, b_F)$ is smooth, and $(F^{-1}(p))' = 1/f(F^{-1}(p))$. So, by (3.7), $(F^{-1}(p))' = (-F^{-1}(1-p))'$, for all $p \in (0,1)$. Therefore, for some constant m, $F^{-1}(p) + F^{-1}(1-p) = 2m$, for all $p \in (0,1)$. However, this means that m is the median of F, and F is symmetric around m. The proof is complete. □

4. Necessity.

PROOF OF LEMMA 2.3. Suppose that, for all $p \in (0,1)$, $h > 0$,

$$R_h(pq) \le R_h(p) R_h(q), \tag{4.1}$$

where $R_h(p) = F(F^{-1}(p) + h)$. For $h = 0$, (4.1) turns into an equality, and we obtain the inequality for derivatives of both parts of (4.1) at $h = 0$:

$$f\left(F^{-1}(pq)\right) \le f\left(F^{-1}(p)\right)q + f\left(F^{-1}(q)\right)p.$$

This inequality coincides with (1.3), and necessity of (c) in Theorem 1.1 has been proved. To prove (b), note that, for any function R: $(0,1) \to (0,1]$ satisfying (4.1), $R(p^n) \le R(p)^n$, for all $p \in (0,1)$; therefore, $R(p) \to 0$, as $p \to 0$, if $R \not\equiv 1$. In the case $R = R_h$, we have $R(p) \to F(a_F + h)$, as $p \to 0$ [recall that $a_F = F^{-1}(0+)$]. If $a_F > -\infty$, then, for $h < b_h - a_h$, $R(0+) > 0$; consequently R_h does not satisfy (4.1). Thus, $a_F = -\infty$ is necessary. Since F is symmetric around its median, we obtain $b_F = F^{-1}(1-) = +\infty$, and the lemma has been established. □

5. Sufficiency. We prove Lemma 2.4 without assumption about log-concavity of measure μ. Let a distribution function F have the continuous positive density f. As usual, $R_h(p) = F(F^{-1}(p) + h)$, where F^{-1} is the inverse of F.

LEMMA 5.1. *If, for any* $p, q \in (0,1)$,

$$\frac{f(F^{-1}(pq))}{pq} \le \frac{f(F^{-1}(p))}{p} + \frac{f(F^{-1}(q))}{q}, \tag{5.1}$$

then, for all $p, q \in (0,1)$, $h \ge 0$,

$$R_h(pq) \le R_h(p) R_h(q). \tag{5.2}$$

PROOF. *Step* 1. Assume that inequality (5.1) is strict for all p, q.

The family R_h, $h \in \mathbb{R}$, forms a one-parameter group of increasing bijections in $(0,1)$: for any $h, h' \in \mathbb{R}$, $R_{h+h'}$ is the composition of R_h and $R_{h'}$, and R_{-h} is the inverse of R_h. So, if (5.2) holds for h and h', then it holds also for $h + h'$. Fix $p, q \in (0,1)$ and set

$$J(p,q) = \{h \ge 0\colon (5.2) \text{ with } p \text{ and } q \text{ is valid for all } 0 \le h' \le h\}.$$

We need to show that $J(p,q) = [0, +\infty)$, for all $p, q \in (0,1)$. The set $J(p,q)$ is closed in $[0,+\infty)$, since R_h is a continuous function of h. Hence, it is sufficient to show that if $h \in J(p,q)$, then $h + \varepsilon \in J(p,q)$, for all ε small enough. Set $p' = R_h(p)$ and $q' = R_h(q)$, and define $I(t) = f(F^{-1}(t))$, $0 < t < 1$. Then, for all $t \in (0,1)$, $R_\varepsilon(t) = t + I(t)\varepsilon + o(\varepsilon)$, as $\varepsilon \to 0$, by Taylor expansion. In particular,

$$\begin{aligned} R_{h+\varepsilon}(p) &= R_\varepsilon(R_h(p)) = p' + I(p')\varepsilon + o(\varepsilon),\\ R_{h+\varepsilon}(q) &= R_\varepsilon(R_h(q)) = q' + I(q')\varepsilon + o(\varepsilon),\\ R_{h+\varepsilon}(pq) &= R_\varepsilon(R_h(pq)) + I(R_h(pq))\varepsilon + o(\varepsilon). \end{aligned} \tag{5.3}$$

The first two expansions give

$$R_{h+\varepsilon}(p) R_{h+\varepsilon}(q) = p'q' + (p'I(q') + q'I(p'))\varepsilon + o(\varepsilon). \tag{5.4}$$

Since $h \in J(p,q)$, we have $R_h(pq) \le R_h(p)R_h(q) = p'q'$. If $R_h(pq) < p'q'$, then, from (5.3) and (5.4), we obviously get $R_{h+\varepsilon}(pq) < R_{h+\varepsilon}(p)R_{h+\varepsilon}(q)$, for all ε small enough. If $R_h(pq) = p'q'$, then the last inequality holds because

$I(p'q') < p'I(q') + q'I(p')$, which follows from the strict variant of (5.1). In both cases, we obtain $h + \varepsilon \in J(p,q)$.

Step 2 (General case). Let $F_T(x) = F(T(x))$, where T is a smooth convex function from $\mathbb{R}$ to $\mathbb{R}$ such that the derivative T' is increasing and positive on $\mathbb{R}$, $T(-\infty) = -\infty$, $T(+\infty) = +\infty$. Let T^{-1} and F_T^{-1} be the inverse of T and the inverse of F_T, respectively. We first show that F_T satisfies the assumption of Step 1. Indeed,

$$f_T(x) \equiv F_T'(x) = f(T(x))T'(x), \qquad F_T^{-1}(p) = T^{-1}\big(F^{-1}(p)\big);$$

therefore,

$$I_T(p) \equiv f_T\big(F_T^{-1}(p)\big) = f\big(F^{-1}(p)\big)T'\big(T^{-1}\big(F^{-1}(p)\big)\big) \equiv I(p)\,\alpha(p),$$

where $\alpha(p) = T'(T^{-1}(F^{-1}(p)))$. The function α increases in $(0,1)$ as it is a composition of increasing functions; hence, for all $p,q \in (0,1)$, $\alpha(pq) < \min\{\alpha(p), \alpha(q)\}$. Using (5.1) via the function I, we have, for all $p,q \in (0,1)$,

$$\begin{aligned}\frac{I_T(pq)}{pq} &= \frac{I(pq)}{pq}\alpha(pq) \le \frac{I(p)}{p}\alpha(pq) + \frac{I(q)}{q}\alpha(pq)\\ &< \frac{I(p)}{p}\alpha(p) + \frac{I(q)}{q}\alpha(q) = \frac{I_T(p)}{p} + \frac{I_T(q)}{q}.\end{aligned}$$

Thus, F_T satisfies the assumption of Step 1, and one may conclude that, for all $p,q \in (0,1)$, $h > 0$,

$$F_T\big(F_T^{-1}(pq) + h\big) \le F_T\big(F_T^{-1}(p) + h\big)F_T\big(F_T^{-1}(q) + h\big). \tag{5.5}$$

Now, if $T(x) = T_n(x) \to x$, as $n \to \infty$, pointwise, then $F_{T_n}(x) \to F(x)$ and $F_{T_n}^{-1}(p) \to F^{-1}(p)$, for all $x \in \mathbb{R}$, $p \in (0,1)$. For example, one may set

$$T_n(x) = x + (|x| - \log(1 + |x|))/n.$$

Consequently, (5.2) follows from (5.5) by taking the limit as $n \to \infty$, and Lemma 5.1 is proved. □

As we have already seen (Lemma 2.3), (5.1) is also necessary for (5.2). Now we present one sufficient (but not necessary) condition which allows us to conclude (5.2).

LEMMA 5.2. *If f is a continuous and positive density of F on the real line such that the function* $\log(f/F)$ *is concave, then* (5.2) *is fulfilled for all $h > 0$ and $p,q \in (0,1)$.*

PROOF. The function

$$\varphi_h(x) = -\log\big(R_h(\exp(-x))\big) = -\log\big(F\big(F^{-1}(\exp(-x)) + h\big)\big)$$

is positive, continuous and increasing on $(0,+\infty)$, $\varphi_h(0+) = 0$ and (5.2) means that, for all $x, y \ge 0$,

$$\varphi_h(x + y) \ge \varphi_h(x) + \varphi_h(y). \tag{5.6}$$

It is well known that every increasing, positive and convex function φ on $(0, +\infty)$ such that $\varphi(0+) = 0$ satisfies (5.6). Hence, to conclude (5.2), it suffices to show that φ_h is convex. Note that φ_h is differentiable, and its derivative

$$\begin{aligned}\varphi_h'(x) &= \frac{R_h'(\exp(-x))}{R_h(\exp(-x))}\exp(-x)\\ &= \frac{f(F^{-1}(\exp(-x)) + h)}{f(F^{-1}(\exp(-x)))}\,\frac{\exp(-x)}{F(F^{-1}(\exp(-x)) + h)}\end{aligned}$$

does not decrease on $(0, +\infty)$ if and only if the function

$$\frac{f(F^{-1}(p) + h)}{f(F^{-1}(p))}\,\frac{p}{F(F^{-1}(p) + h)}$$

does not increase on $(0, 1)$, or [using the change of variable $p = F(x)$] the function

$$\frac{f(x+h)/F(x+h)}{f(x)/F(x)} \tag{5.7}$$

does not increase on $\mathbb{R}$. This is true, since, for any $h > 0$, (5.7) is the increment of $\log(f/F)$ on the interval $[x, x+h]$. The proof is complete. □

APPENDIX

A probability Borel measure μ on the locally convex space E is called log-concave if, for all nonempty Borel sets $A, B \subset E$ and $\lambda \in (0, 1)$,

$$\mu_*(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda},$$

where $\lambda A + (1-\lambda)B = \{\lambda a + (1-\lambda)b\colon a \in A,\ b \in B\}$, and where μ_* denotes the inner measure. A full description of log-concave measures was given by Borell [3]. In the case $E = \mathbb{R}$, and if μ is not a unit mass δ_x at some point $x \in \mathbb{R}$, the above definition is reduced to that mentioned in Section 1: μ has a density f such that the function $\log(f)$ is concave on $\mathbb{R}$ (as a function with values in $[-\infty, +\infty)$).

In this section, we give other equivalent definitions of log-concavity for measures on the real line. Let μ be a nonatomic probability measure with (continuous) distribution function $F(x) = \mu((-\infty, x])$, $x \in \mathbb{R}$. Set

$$a = \inf\{x \in \mathbb{R}\colon F(x) > 0\}, \qquad b = \sup\{x \in \mathbb{R}\colon F(x) < 1\}.$$

Assume that F strictly increases on (a, b), and let $F^{-1}\colon (a, b) \to (0, 1)$ denote the inverse of F restricted to (a, b).

PROPOSITION A.1. *Under the above assumptions, the following properties are equivalent*:

(a) *μ is log-concave*;

(b) *for all $h > 0$, the function $R_h(p) = F(F^{-1}(p) + h)$ is concave on* $(0, 1)$;

(c) μ *has a continuous, positive density* f *on* (a, b), *and, moreover, the function* $I(p) = f(F^{-1}(p))$ *is concave on* $(0, 1)$.

PROOF. [(a) ⇒ (b)] This was shown in Section 2.

[(a) ⇒ (c)] By assumption, μ has a positive, continuous density f on (a, b) such that $\log(f)$ is concave. Hence, f is absolutely continuous, and, moreover, there exists a Radon–Nikodym derivative f' for f on (a, b) such that the function f'/f is a nonincreasing Radon–Nikodym derivative of $\log(f)$ on (a, b). Then since F is continuously differentiable, the function

$$I'(p) = \frac{f'(F^{-1}(p))}{f(F^{-1}(p))}$$

represents a Radon–Nikodym derivative of I on $(0, 1)$. Clearly, I' does not increase; hence, I is concave.

[(c) ⇒ (a)] By assumption, I is positive and concave on $(0, 1)$; hence, there exists a nonincreasing Radon–Nikodym derivative I'. Since F is continuously differentiable on (a, b), the function $I'(F(x))f(x)$ represents a Radon–Nikodym derivative of $I(F(x))$. However, $I(F(x)) = f(x)$, for all $x \in (a, b)$; hence, f is absolutely continuous and, moreover, $f' \equiv I'(F(x))f(x)$ is a Radon–Nikodym derivative of $f(x)$. Therefore, $I'(F) = f'/f$ represents a Radon–Nikodym derivative of $\log(f)$. Since $I'(F)$ does not increase, $\log(f)$ is concave on (a, b).

[(b) ⇒ (a)] For simplicity, we assume $a = -\infty$, $b = +\infty$ (minor changes should be made in cases $a > -\infty$ and/or $b = +\infty$). For any $h > 0$, R_h is concave; hence, there exists a nonincreasing Radon–Nikodym derivative L_h of R_h which can also be chosen to be continuous from the right. We also have $L_h(p) > 0$, for all $p \in (0, 1)$ [since, in addition to concavity, R_h increases and $R_h(p) \to 1$, as $p \to 1-$]. Using $R_h(F(x)) = F(x + h)$ one can write

$$F(y + h) - F(x + h) = \int_{F(x)}^{F(y)} L_h(t)\, dt,$$

which holds for all $x, y \in \mathbb{R}$ and $h > 0$. Letting $y \to x$, we get the following:

$$F(y + h) - F(x + h) = L_h(F(x))(F(y) - F(x)) + o(F(y) - F(x)), \quad y \to x+;$$

$$F(y + h) - F(x + h) = L_h(F(x)-)(F(y) - F(x)) + o(F(y) - F(x)), \quad y \to x-.$$

In particular, if F is differentiable at x, then there exist limits

$$f_r(x + h) \equiv \lim_{y \to x+} \frac{F(y + h) - F(x + h)}{y - x} = L_h(F(x))F'(x),$$

$$f_l(x + h) \equiv \lim_{y \to x-} \frac{F(y + h) - F(x + h)}{y - x} = L_h(F(x)-)F'(x).$$

Since $h > 0$ is arbitrary and since, by the Lebesgue theorem, F is differentiable at almost all $x \in \mathbb{R}$, the value of $x + h$ can be arbitrary; so we conclude that the values $f_r(x)$ and $f_l(x)$ are well defined for all $x \in \mathbb{R}$ and that f_r and f_l represent the right and left derivatives of F on the whole real line. In addition, these functions satisfy

$$f_r(x+h) = L_h(F(x)) f_r(x), \tag{A.1}$$

$$f_l(x+h) = L_h(F(x)-) f_l(x), \tag{A.2}$$

for every $x \in \mathbb{R}$ and $h > 0$. Using (A.2), note that if $f_l(x_0) = 0$ at some point x_0 then $f_l = 0$ everywhere on $[x_0, +\infty)$. However, $f_r = f_l = F'$ almost everywhere; hence, we would have $f_l(x_1) = 0$ at some point $x_1 > x_0$ and, by (A.1), $f_r = 0$ everywhere on $[x_1, +\infty)$. As a result, $F' = 0$; that is, F is constant on $[x_1, +\infty)$. The last is impossible; consequently, $f_l(x) > 0$, for all x. By the same argument, for all x, $f_r(x) > 0$.

Now, one may introduce the functions $g_r = \log(f_r)$ and $g_l = \log(f_l)$ and may rewrite (A.1) and (A.2) as

$$g_r(x+h) - g_r(x) = L_h(F(x)), \qquad g_l(x+h) - g_l(x) = L_h(F(x)-).$$

We observe that increments of g_r and g_l represent nonincreasing functions. Therefore, g_r and g_l are concave, if they are continuous. For every $x, y \in \mathbb{R}$, $x \le y$, and every $h > 0$, we have

$$g_r(y+h) - g_r(x+h) \le g_r(y) - g_r(x). \tag{A.3}$$

Fixing x and letting $y \to x+$ in (A.3), we get

$$g_r((x+h)+) \le g_r(x+), \tag{A.4}$$

and fixing y and letting $x \to y-$ in (A.3), we get

$$g_r((y+h)-) \ge g_r(y-). \tag{A.5}$$

By (A.4), if $g_r(x_0+) < 0$ at some point x_0, then $g_r(x+) < 0$, for all $x > x_0$; and, by (A.5) if $g_r(y_0+) > 0$ at some point y_0, then $g_r(y+) > 0$, for all $y > y_0$. Hence, if g_r is discontinuous at a point x, then g_r is discontinuous at each point in $(x, +\infty)$. However, g_r is continuous at every point from a set of the second category. Indeed, the function f_r is a pointwise limit

$$f_r(x) = \lim_{n \to \infty} n\left(F\left(x + \frac{1}{n}\right) - F(x)\right)$$

of the sequence of continuous functions, and the same concerns the function g_r. Hence, g_r belongs to the first class of Baire, and, by Baire's theorem (see, e.g., [5], Section 31), all the points where g_r is discontinuous form a set of the first category. Thus, for all x_0, $g_r(x_0+) = g_r(x_0-) = 0$; that is, g_r and (by the same argument) g_l are continuous on the whole real line. Consequently, these functions are concave.

We also obtain that the functions f_r and f_l are continuous, and since $f_r = f_l$ almost everywhere, we conclude that $f_r(x) = f_l(x) = F'(x)$, for all x. Therefore, the derivative $f = F'$ represents a positive density of μ such that the function $\log(f)$ is concave.

The proof is complete. □

Acknowledgments. The author is grateful to the Department of Statistics of the University of North Carolina at Chapel Hill for the invitation to visit The Center for Stochastic Processes. The author thanks the referee for careful reading and useful comments.

REFERENCES

[1] BOBKOV, S. G. (1993). Isoperimetric problem for uniform enlargement. Technical Report 394, Dept. Statistics, Univ. North Carolina.

[2] BOBKOV, S. G. (1994). Isoperimetric inequalities for distributions of exponential type. *Ann. Probab.* **22** 978–994.

[3] BORELL, C. (1974). Convex measures on locally convex spaces. *Ark. Mat.* **12** 239–252.

[4] BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216.

[5] KURATOWSKI, A. (1966). *Topology* **1**. Academic Press, New York.

[6] SUDAKOV, V. N. and TSIREL'SON, B. S. (1978). Extremal properties of half-spaces for spherically invariant measures. *J. Soviet Math.* **9** 9–18. [Translated from *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **41** (1974) 14–24 (in Russian).]

[7] TALAGRAND, M. (1991). A new isoperimetric inequality and the concentration of measure phenomenon. *Israel Seminar* (*GAFA*). *Lecture Notes in Math.* **1469** 94–124. Springer, Berlin.

DEPARTMENT OF MATHEMATICS
SYKTYVKAR UNIVERSITY
167001 SYKTYVKAR
RUSSIA
E-MAIL: root@sucnit.komi.su