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ON SOME BOUNDARY CROSSING PROBLEMS FOR GAUSSIAN RANDOM WALKS[1]

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We consider random walks with Gaussian distribution of summands. New representations for Wiener–Hopf factorization components are obtained. The factorization method is used to study the distribution of the excess over one-sided and two-sided boundaries. Asymptotic expansions for these distributions and for the expectation of the first exit time are obtained under the assumption that the boundaries tend to infinity.

1. Introduction and main results. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed (i.i.d.) random variables, $S_n = X_1 + \cdots + X_n$, $n \geq 1$. For arbitrary positive numbers a and b, introduce the random variables $N_b = \inf\{n \geq 1\colon S_n \geq b\}$ and $N_{a,b} = \inf\{n \geq 1\colon S_n \notin (-a, b)\}$ which are equal to the first exit times from $(-\infty, b)$ and $(-a, b)$, respectively, for the sequence $\{S_n\}$. We put always $\inf \varnothing = \infty$.

We consider Gaussian random walks, that is,

$$\mathbf{P}(X_1 < y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{(t-\alpha)^2}{2\sigma^2}\right) dt, \tag{1}$$

and obtain for them some new representations of the Wiener–Hopf factorization components as well as the asymptotic expansions for the probabilities

$$\mathbf{P}(S_{N_b} \geq b + x,\ N_b < \infty), \quad \mathbf{P}(S_{N_{a,b}} \geq b + x) \tag{2}$$

and for $\mathbf{E}N_b$, $\mathbf{E}N_{a,b}$ as $a \to \infty$, $b \to \infty$, $x \geq 0$. The method used here can be applied to different types of random walks [see Borovkov (1962), Rogozin (1969), Presman (1971), Lotov (1979, 1987)]. Our choice of the Gaussian model is motivated by the intention to make the statements more definite and clear.

The basis of the approach is complex analysis of the Laplace–Stieltjes transforms of the joint distributions of (N_b, S_{N_b}) and $(N_{a,b}, S_{N_{a,b}})$. To make it possible, these transforms are first expressed by the Laplace–Stieltjes transforms of the distributions of ladder values $(\eta_\pm, \chi_\pm)$. Here

$$\eta_\pm = \inf\{n \geq 1\colon S_n \gtreqless 0\}, \qquad \chi_\pm = S_{\eta_\pm}.$$

This step is quite natural, since, for example, S_{N_b} can be treated as a sum of independent random variables distributed identically as χ_+, and N_b, in its

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turn, is equal to a sum of random variables distributed identically as η_+. At first glance, the representations obtained at this stage seem to be of limited use, since the unknown functions

$$\mathbf{E}\big(z^{N_b}\exp\{\lambda S_{N_b}\};\ N_b<\infty\big),\qquad \mathbf{E}\big(z^{N_{a,b}}\exp\{\lambda S_{N_{a,b}}\}\big) \tag{3}$$

are expressed via other unknown functions

$$\varphi_\pm(z,\lambda)=\mathbf{E}(z^{\eta_\pm}\exp\{\lambda\chi_\pm\};\ \eta_\pm<\infty).$$

Nevertheless, there is the following remarkable fact. The functions

$$r_{z\pm}(\lambda)=1-\varphi_\pm(z,\lambda),\qquad r_z(\lambda)=1-z\mathbf{E}\exp\{\lambda X_1\} \tag{4}$$

satisfy, for $\mathrm{Re}\,\lambda=0$, $|z|\le 1$, the relation

$$r_{z+}(\lambda)r_{z-}(\lambda)=r_z(\lambda) \tag{5}$$

(Wiener–Hopf factorization) and therefore $r_{z\pm}(\lambda)$ are called factorization components. The analytic properties of the function $r_z(\lambda)$ in λ (the presence of zeros and poles, possibilities of analytical continuation) can be studied in an easy way in many cases. Representation (5) allows us in this case to establish similar properties of each factorization component and then analyze functions (3). The subsequent isolation of the dominating singularities of these functions (in λ) enables us to obtain the main terms of the asymptotics of distributions (2) and to estimate the remainder terms.

The arguments above show that it is also desirable to have the factorization components in an explicit form. The following representations are well known: for $|z|<1$ and $\mathrm{Re}\,\lambda=0$,

$$r_{z\pm}(\lambda)=\exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{E}\big(\exp(\lambda S_n);\ S_n\gtreqless 0\big)\right\}. \tag{6}$$

Unfortunately, in many cases it has not been possible to make good use of them. It is known also that the factorization components $r_{z\pm}(\lambda)$ can be explicitly represented by zeros and poles of the function $r_z(\lambda)$ in those cases when either $\mathbf{E}(e^{\lambda X_1};\ X_1<0)$ or $\mathbf{E}(e^{\lambda X_1};\ X_1>0)$ are rational functions (for integer-valued X_1, the rationality property must be required with respect to $t=e^\lambda$). Complete information can be found in Borovkov (1976a). It is evident that the normal distribution does not possess this property. At the same time, numerous problems of sequential testing make it important to consider Gaussian random walks with boundaries. Therefore, everywhere in the sequel we assume that condition (1) holds.

Let us introduce some notation to be used throughout the paper. Put $d=z\exp\{-\alpha^2/(2\sigma^2)\}$. The function

$$r_z\left(\frac{\lambda}{\sigma}-\frac{\alpha}{\sigma^2}\right)\equiv 1-d\exp\left\{\frac{\lambda^2}{2}\right\}$$

has a sequence of zeros for $0 < d \le 1$; we denote them by $\{\lambda_{\pm n}(z),\ n \ge 1\}$. Here $\lambda_{-n}(z) = -\lambda_n(z),\ n \ge 1$, and only two of them, say $\lambda_{\pm 1}(z)$, are real. Simple calculation shows that

$$\tag{7} \begin{aligned} &\lambda_1(z) = \left(2\ln\frac{1}{d}\right)^{1/2}, \qquad \lambda_{2k}(z) = \overline{\lambda_{2k+1}(z)} = x_k(z) + iy_k(z), \qquad k \ge 1;\\ &y_k(z) = \left(-\ln\frac{1}{d} + \left(\ln^2\frac{1}{d} + 4k^2\pi^2\right)^{1/2}\right)^{1/2}, \qquad x_k(z) = \frac{2\pi k}{y_k(z)}. \end{aligned}$$

It can be easily seen that $x_k(z) = \sqrt{2\pi k} + O(1/\sqrt{k}),\ y_k(z) = \sqrt{2\pi k} + O(1/\sqrt{k})$, as $k \to \infty$.

The following theorems are proved in Section 2.

THEOREM 1. *For arbitrary* λ *and* $z \in (0, \exp\{\alpha^2/(2\sigma^2)\})$ *we have*

$$\tag{8} \begin{aligned} r_{z+}\left(\frac{\lambda}{\sigma} - \frac{\alpha}{\sigma^2}\right) &= r_{z-}\left(-\frac{\lambda}{\sigma} - \frac{\alpha}{\sigma^2}\right)\\ &= \sqrt{1-d}\exp\left\{-\lambda\sum_{n=1}^{\infty}\frac{d^n}{\sqrt{2\pi n}} + \frac{\lambda^2}{8}\right\}\\ &\quad\times\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n(z)}\right)\exp\left(\frac{\lambda}{\lambda_n(z)}\right). \end{aligned}$$

THEOREM 2. *If* $\alpha = 0$, *then for every* λ,

$$\tag{9} \begin{aligned} r_{1+}\left(\frac{\lambda}{\sigma}\right) &= r_{1-}\left(-\frac{\lambda}{\sigma}\right)\\ &= -\frac{\lambda}{\sqrt{2}}\exp\left\{\frac{K}{\sqrt{2\pi}}\lambda + \frac{\lambda^2}{8}\right\}\prod_{n=1}^{\infty}\frac{1}{2}\left[\left(1 - \frac{\lambda}{\sqrt{2\pi n}}\right)^2 + 1\right]\exp\left(\frac{\lambda}{\sqrt{2\pi n}}\right), \end{aligned}$$

where the constant $K = 1.460\ldots$ *is defined by*

$$\sum_{m=1}^{n}\frac{1}{\sqrt{m}} = 2\sqrt{n} - K + O\left(\frac{1}{\sqrt{n}}\right).$$

In what follows we also use the notation

$$\mu_n(z) = \frac{\lambda_n(z)}{\sigma} - \frac{\alpha}{\sigma^2}, \qquad |n| \ge 1$$

for zeros of the function $r_z(\lambda)$. In the case $z = 1$ the argument z in $\mu_n(z),\ x_k(z)$ and $r_{z\pm}(\lambda)$ is omitted: $\mu_n = \mu_n(1),\ x_k = x_k(1),\ r_\pm(\lambda) = r_{1\pm}(\lambda)$. For $\alpha = 0$, we have $\mu_1 = 0,\ \mu_{2k} = \overline{\mu_{2k+1}} = \sqrt{2\pi k}/\sigma(1+i),\ k \ge 1$. If $\alpha \ne 0$, then $\mu_1 = |\alpha|/\sigma^2 - \alpha/\sigma^2$, $\operatorname{Re}\mu_{2k} = \operatorname{Re}\mu_{2k+1} = x_k/\sigma^2 - \alpha/\sigma^2 \ge \sqrt{2\pi k}/\sigma - \alpha/\sigma^2$ and

$\operatorname{Re}\mu_{2k+1} > \operatorname{Re}\mu_{2k-1}$, $k \geq 1$. In particular,

$$\begin{aligned}\operatorname{Re}\mu_2 = \operatorname{Re}\mu_3 &= \frac{x_1}{\sigma} - \frac{\alpha}{\sigma^2}\\ &= 2\pi\sigma^{-1}\left(-\frac{\alpha^2}{2\sigma^2} + \left(\frac{\alpha^4}{4\sigma^4} + 4\pi^2\right)^{1/2}\right)^{-1/2} - \frac{\alpha}{\sigma^2} > \mu_1.\end{aligned}$$

In Section 3, the asymptotic representations of the distributions (2) and $\mathbf{E}N_b$, $\mathbf{E}N_{a,b}$ as $a \to \infty$, $b \to \infty$ are established. For some other approximating formulas, see Siegmund (1985).

THEOREM 3. *For every integer $k \geq 1$, $x \geq 0$ and $b \to \infty$,*

(i)
$$\begin{aligned}&\mathbf{P}(S_{N_b} \geq b + x,\ N_b < \infty)\\ &\quad = \sum_{i=1}^{2k-1} F_i(x)\exp(-\mu_i b) + O(\exp(-(b+x)\operatorname{Re}\mu_{2k})),\end{aligned}$$

where

$$F_i(x) = \int_x^\infty f_i(y)\,dy,$$

and the functions f_i are defined by the relations

$$\frac{r_+(\lambda)}{r'_+(\mu_i)(\lambda - \mu_i)} = \int_0^\infty e^{\lambda y} f_i(y)\,dy. \tag{10}$$

(ii) *If $\alpha > 0$, then*

$$\alpha\mathbf{E}N_b = b + \frac{r''_+(0)}{2r'_+(0)} - \sum_{i=2}^{2k+1} \frac{r'_+(0)\exp(-\mu_i b)}{\mu_i r'_+(\mu_i)} + O(\exp(-b\operatorname{Re}\mu_{2k+2})).$$

THEOREM 4. *Let $\alpha = 0$. Then for every $x \geq 0$ and $a \to \infty$, $b \to \infty$ we have the equalities*

$$\begin{aligned}&\mathbf{P}(S_{N_{a,b}} \geq b + x)\\ &\quad = F_1(x)\frac{a + K\sigma/\sqrt{2\pi}}{b + a + 2K\sigma/\sqrt{2\pi}}\\ &\qquad + O\left(\exp\left(-\frac{\sqrt{2\pi}}{\sigma}(b+x)\right)\right) + O\left(\exp\left(-\frac{\sqrt{2\pi}}{\sigma}(a+x)\right)\right),\end{aligned} \tag{11}$$

where the function F_1 is defined as in Theorem 3, with $\mu_1 = 0$;

$$\begin{aligned}\mathbf{E}N_{a,b} &= \frac{ab}{\sigma^2} + \frac{K}{\sigma\sqrt{2\pi}}(a+b) + \frac{K^2}{2\pi} + \frac{1}{4}\\ &\quad + O\left((a^2+b^2)\left(\exp\left(-\frac{\sqrt{2\pi}}{\sigma}a\right) + \exp\left(-\frac{\sqrt{2\pi}}{\sigma}b\right)\right)\right).\end{aligned} \tag{12}$$

THEOREM 5. *Let $\alpha \neq 0$. Then for $x \geq 0$, $a \to \infty$, $b \to \infty$ we have*

$$\mathbf{P}(S_{N_{a,b}} \geq b + x) = \frac{F_1(x)\exp(-\mu_1 b)(1 - q_1 \exp\{-(2|\alpha|/\sigma^2)a\})}{1 - q_1 q_2 \exp\{-2|\alpha|/\sigma^2(a+b)\}} + F_1(x) O\biggl(\exp\biggl(-\mu_1(a+b) - \biggl(\operatorname{Re}\mu_2 + \frac{2\alpha}{\sigma^2}\biggr)a\biggr)\biggr) + O(\exp(-(b+x)\operatorname{Re}\mu_2)),$$

$$q_1 = \frac{\sigma^2 r_-(\mu_1)}{2|\alpha| r'_-(-\mu_1 - 2\alpha/\sigma^2)}, \qquad q_2 = -\frac{\sigma^2 r_+(-\mu_1 - 2\alpha/\sigma^2)}{2|\alpha| r'_+(\mu_1)}.$$

If $\alpha < 0$, then

$$\mathbf{E}N_{a,b} = |\mathbf{E}X_1|^{-1}\biggl(1 - q_1 q_2 \exp\biggl\{-\frac{2|\alpha|}{\sigma^2}(a+b)\biggr\}\biggr)^{-1} \times \biggl[a - \frac{r''_-(0)}{2r'_-(0)} - \exp(-\mu_1 b)\biggl(\biggl(a + b - \frac{r''_-(0)}{2r'_-(0)}\biggr)q_2 - \frac{r'_+(0)}{\mu_1 r'_+(\mu_1)} + \frac{q_2}{\mu_1}\biggr) + \exp(-\mu_1(a+b))q_1\biggl(q_2 b - \frac{r'_+(0)}{\mu_1 r'_+(\mu_1)} + \frac{q_2}{\mu_1}\biggr)\biggr] + O((a+b)\exp(-b\operatorname{Re}\mu_2)) + O(\exp(-a(\operatorname{Re}\mu_2 - \mu_1))).$$

2. Proof of Theorems 1 and 2. We can put $\sigma^2 = 1$ throughout the proofs of theorems without loss of generality. The function $r_z(\lambda - \alpha) \equiv 1 - d\exp\{\lambda^2/2\}$ is entirely of the second order. Applying the Hadamard theorem [see, e.g., Hille (1962), Theorem 14.2.6.] on the representation of an entire function as an infinite product, we get

$$1 - d\exp\biggl\{\frac{\lambda^2}{2}\biggr\} = \exp\{a_0 + a_1\lambda^2\}\prod_{n=1}^{\infty}\biggl(1 - \frac{\lambda^2}{\lambda_n^2(z)}\biggr)\exp\biggl\{\frac{\lambda^2}{\lambda_n^2(z)}\biggr\}.$$

Symmetry of the set $\{\lambda_n(z)\}$ implies convergence of the series $\sum \lambda_n^{-2}(z)$. Therefore, we come to the representation

$$1 - d\exp\biggl\{\frac{\lambda^2}{2}\biggr\} = \exp\{a_0 + a_2\lambda^2\}\prod_{n=1}^{\infty}\biggl(1 - \frac{\lambda^2}{\lambda_n^2(z)}\biggr), \tag{13}$$

with some coefficients a_0, a_2 which we need to determine. It remains to distribute the factors in (13) between factorization components; this is the main idea of the proof.

Return to factorization (5) and suppose that $|z| < 1$. It follows from the definition that the factor $r_{z+}(\lambda)$ is the Laplace–Stieltjes transform (LST) of a function defined on $[0, \infty)$; it is analytic for $\operatorname{Re}\lambda < 0$, continuous up to the boundary $\operatorname{Re}\lambda = 0$, bounded and nonzero for $\operatorname{Re}\lambda \leq 0$. The function $r_{z-}(\lambda)$ has similar properties on the half-plane $\operatorname{Re}\lambda \geq 0$, and $r_{z+}(-\infty) = r_{z-}(\infty) = 1$. The representation of the function $r_z(\lambda)$ as the product of two factors having such properties is unique [see Borovkov (1976b)].

In our case the functions $r_{z\pm}(\lambda)$ can be continued analytically on the whole plane for $|z| < 1$. To prove it, consider first the function

$$R_{z+}(\lambda) = \begin{cases} r_{z+}(\lambda), & \operatorname{Re}\lambda \le 0, \\ \dfrac{r_z(\lambda)}{r_{z-}(\lambda)}, & \operatorname{Re}\lambda \ge 0. \end{cases} \tag{14}$$

This definition is correct for $\operatorname{Re}\lambda = 0$ due to (5). Both the functions $r_z(\lambda)$ and $r_{z-}(\lambda)$ are analytic for $\operatorname{Re}\lambda > 0$, and $r_{z-}(\lambda) \neq 0$. Therefore, the function $R_{z+}(\lambda)$ is entire. After that, we can regard $r_{z+}(\lambda)$ as a result of analytical continuation on the whole plane. The order of $r_{z+}(\lambda)$ is 2. This also follows from (14) since $|r_{z+}(\lambda)|$ is bounded for $\operatorname{Re}\lambda \le 0$ and $|r_{z-}(\lambda)|$ is bounded and isolated from zero for $\operatorname{Re}\lambda > 0$. Therefore, the order of $r_{z+}(\lambda)$ is determined by behavior of $r_z(\lambda)$ on the half-plane $\operatorname{Re}\lambda > 0$. Symmetric arguments give the property of $r_{z-}(\lambda)$ to be the second-order entire function. Thus, the relation (5) is true for every λ.

Next consider the identity

$$r_{z+}(\lambda - \alpha) r_{z-}(\lambda - \alpha) = 1 - d \exp\{\lambda^2/2\}. \tag{15}$$

There are no zeros of $r_z(\lambda)$ between the lines $\operatorname{Re}\lambda = 0$ and $\operatorname{Re}\lambda = -\alpha$; therefore, $r_{z+}(\lambda - \alpha) \neq 0$ for $\operatorname{Re}\lambda \le 0$, and $r_{z-}(\lambda - \alpha) \neq 0$ for $\operatorname{Re}\lambda \ge 0$. Thus, using the uniqueness property, we can consider (15) as the Wiener–Hopf factorization of the function $1 - d\exp\{\lambda^2/2\}$, which corresponds to a random walk with the standard normal distribution of summands. The identity

$$r_{z+}(-\lambda - \alpha) r_{z-}(-\lambda - \alpha) = 1 - d \exp\{\lambda^2/2\}$$

allows us to identify $r_{z+}(\lambda-\alpha)$ and $r_{z-}(-\lambda-\alpha)$. Moreover, factorization components for $1-d\exp\{\lambda^2/2\}$ exist for $d \le 1$. Hence, the same is true for $r_{z\pm}(\lambda-\alpha)$.

Let $0 < d < 1$. The numbers $\{\lambda_{-n}(z),\ n \ge 1\}$ cannot be zeros of $r_{z+}(\lambda - \alpha)$ since $\operatorname{Re}\lambda_{-n}(z) \le 0$. Consequently, they are zeros of $r_{z-}(\lambda - \alpha)$. Similarly, the numbers $\{\lambda_n(z),\ n \ge 1\}$, and only these, are zeros of $r_{z+}(\lambda - \alpha)$. We can apply the Hadamard theorem again, which gives the representation

$$r_{z+}(\lambda - \alpha) = r_{z-}(-\lambda - \alpha) = \exp\{b_0 + b_1\lambda + b_2\lambda^2\} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n(z)}\right) \exp\left\{\frac{\lambda}{\lambda_n(z)}\right\}.$$

Let us specify the coefficients b_i, $i = 0, 1, 2$. We have $r_z(-\alpha) = 1 - d = \exp(2b_0)$. That is, $b_0 = \frac{1}{2}\ln(1 - d)$. To find b_1 and b_2, we use (6):

$$\begin{aligned} &-\sum_{n=1}^{\infty} \frac{d^n}{n} \mathbf{E}(\exp(\lambda Y_n);\ Y_n > 0) \\ &\qquad = b_0 + b_1\lambda + b_2\lambda^2 + \sum_{n=1}^{\infty} \left(\ln\left(1 - \frac{\lambda}{\lambda_n(z)}\right) + \frac{\lambda}{\lambda_n(z)} \right). \end{aligned} \tag{16}$$

Here Y_n are Gaussian random variables with parameters $(0, n)$. Differentiating (16) twice and putting $\lambda = 0$, we obtain

$$b_1 = -\sum_{n=1}^{\infty} \frac{d^n}{n} \frac{1}{\sqrt{2\pi n}} \int_0^{\infty} y \exp\left(\frac{-y^2}{2n}\right) dy = -\sum_{n=1}^{\infty} \frac{d^n}{\sqrt{2\pi n}},$$

$$b_2 = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{d^n}{n} \frac{1}{\sqrt{2\pi n}} \int_0^{\infty} y^2 \exp\left(\frac{-y^2}{2n}\right) dy + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2(z)} \tag{17}$$

$$= \frac{d}{4(d-1)} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2(z)}.$$

The second equality in (17) can also be obtained by differentiating (13) in λ^2 since $a_2 = 2b_2$. Let us calculate the last sum in (17). Denote for brevity $\gamma = \ln(1/d)$. Then for $k \geq 1$ we have

$$\frac{1}{\lambda_{2k}^2} + \frac{1}{\lambda_{2k+1}^2} = \frac{2(x_k^2 - y_k^2)}{(x_k^2 + y_k^2)^2}$$

$$= 2\left(\frac{4\pi^2 k^2}{y_k^2} - y_k^2\right)\left(\frac{4\pi^2 k^2}{y_k^2} + y_k^2\right)^{-2}$$

$$= \frac{2y_k^2(4\pi^2 k^2 - (-\gamma + \sqrt{\gamma^2 + 4\pi^2 k^2})^2)}{(4\pi^2 k^2 + (-\gamma + \sqrt{\gamma^2 + 4\pi^2 k^2})^2)^2}$$

$$= \frac{2y_k^2(-2\gamma^2 + 2\gamma\sqrt{\gamma^2 + 4\pi^2 k^2})}{(8\pi^2 k^2 + 2\gamma^2 - 2\gamma\sqrt{\gamma^2 + 4\pi^2 k^2})^2}.$$

(We omit here the argument z.) Taking into account that $\sqrt{\gamma^2 + 4\pi^2 k^2} = y_k^2 + \gamma$, we obtain

$$\frac{1}{\lambda_{2k}^2} + \frac{1}{\lambda_{2k+1}^2} = \frac{2y_k^2(-2\gamma^2 + 2\gamma y_k^2 + 2\gamma^2)}{[2(y_k^2 + \gamma)^2 - 2\gamma y_k^2 - 2\gamma^2]^2} = \frac{\gamma}{(y_k^2 + \gamma)^2}$$

$$= \frac{\gamma}{\gamma^2 + 4\pi^2 k^2} = \frac{\gamma}{4\pi^2} \frac{1}{m^2 + k^2}, \qquad m^2 = \frac{\gamma^2}{4\pi^2}.$$

This yields

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{1}{2\gamma} + \sum_{n=2}^{\infty} \frac{1}{\lambda_n^2} = \frac{1}{2\gamma} + \frac{\gamma}{4\pi^2} \sum_{k=1}^{\infty} \frac{1}{m^2 + k^2}$$

$$= \frac{1}{2\gamma} + \frac{\gamma}{4\pi^2}\left(\pi \frac{e^{\pi m} + e^{-\pi m}}{e^{\pi m} - e^{-\pi m}} - \frac{1}{m}\right)\frac{1}{2m} = \frac{1}{4} + \frac{d}{2(1-d)}.$$

Therefore,

$$b_2 = -\frac{d}{4(1-d)} + \frac{1}{2}\left(\frac{1}{4} + \frac{d}{2(1-d)}\right) = \frac{1}{8}.$$

This completes the proof of Theorem 1.

Under conditions of Theorem 1, put $\alpha = 0$. We shall prove (9) by passage to the limit in (8) as $z \to 1$, isolating the first factor under the sign of infinite product. It is clear that

$$\lambda_1(z) = \sqrt{2\ln\frac{1}{z}} = \sqrt{2(1-z)}(1+O(1-z)).$$

Therefore,

$$\sqrt{1-d}\left(1-\frac{\lambda}{\lambda_1(z)}\right) \to -\frac{\lambda}{\sqrt{2}}.$$

Let us study asymptotic behavior of the function $S(z) = \sum_{n=1}^{\infty}(z^n/\sqrt{n})$ as $z \to 1$, $z < 1$. Denote $A(n) = \sum_{m=1}^{n} 1/\sqrt{m}$. It follows from the Euler–Maclaurin formula that $A(n) = 2\sqrt{n} - K + O(1/\sqrt{n})$. Applying the Abel transform, we obtain

$$\sum_{m=1}^{n}\frac{z^m}{\sqrt{m}} = A(n)z^n - \int_1^n A(x)z^x \ln z\, dx,$$

where $A(x) = \sum_{1\le m\le x} 1/\sqrt{m}$, $A(x) = 0$ for $0 \le x \le 1$. Letting $n \to \infty$, we have

$$S(z) = -\ln z \int_0^{\infty} A(x)z^x\, dx.$$

Put $\tilde{A}(x) = 2\sqrt{x} - K$. Then

$$\begin{aligned}
\int_0^{\infty}\tilde{A}(x)z^x\, dx &= \int_0^{\infty}(2\sqrt{x}-K)e^{x\ln z}\, dx\\
&= \left(\frac{\sqrt{\pi}}{(1-z)^{3/2}} - \frac{K}{1-z}\right)(1+O(1-z)),\\
|A(x)-\tilde{A}(x)| &= \left|2\sqrt{[x]} - K + O\left(\frac{1}{\sqrt{[x]}}\right) - 2\sqrt{x} + K\right|\\
&= O\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

uniformly in x. Therefore,

$$\begin{aligned}
S(z) &= -\ln z\int_0^{\infty}\tilde{A}(x)z^x\, dx - \ln z\int_0^{\infty}(A(x)-\tilde{A}(x))z^x\, dx\\
&= \frac{\sqrt{\pi}}{\sqrt{1-z}} - K + O(\sqrt{1-z}).
\end{aligned}$$

This implies

$$\sum_{n=1}^{\infty}\frac{d^n}{\sqrt{2\pi n}} - \frac{1}{\lambda_1(z)} = \frac{1}{\sqrt{2(1-z)}} - \frac{K}{\sqrt{2\pi}} - \frac{1}{\sqrt{2(1-z)}} + o(1) \to -\frac{K}{\sqrt{2\pi}}.$$

Finally, observe that for $z=1$, $\alpha=0$ and $k\ge 1$,

$$\Big(1-\frac{1}{\lambda_{2k}}\Big)\Big(1-\frac{1}{\lambda_{2k+1}}\Big)\exp\Big\{\lambda\Big(\frac{1}{\lambda_{2k}}+\frac{1}{\lambda_{2k+1}}\Big)\Big\}$$
$$=\Big(1-\frac{\lambda}{\sqrt{2\pi k}(1+i)}\Big)\Big(1-\frac{\lambda}{\sqrt{2\pi k}(1-i)}\Big)\exp\Big\{\frac{\lambda}{\sqrt{2\pi k}}\Big\}$$
$$=\frac{1}{2}\Big[\Big(1-\frac{\lambda}{\sqrt{2\pi k}}\Big)^2+1\Big]\exp\Big\{\frac{\lambda}{\sqrt{2\pi k}}\Big\}.$$

This completes the proof of Theorem 2.

REMARKS. (i) The representations (8) and (9) are convenient for the calculation of the moments of $\chi_\pm$ since $\mathbf{E}\chi_\pm^k=-r_{1\pm}^{(k)}(0)$. From (9), for $\alpha=0$ we find

$$\mathbf{E}\chi_+=-\mathbf{E}\chi_-=\frac{\sigma}{\sqrt{2}},$$
$$\mathbf{E}\chi_+^2=\frac{K\sigma^2}{\sqrt{\pi}},\qquad \mathbf{E}\chi_+^3=\frac{3\sigma^3}{4\sqrt{2}}+\frac{3K^2\sigma^3}{2\sqrt{2\pi}}. \tag{18}$$

If $\alpha>0$, then we deduce from (8) for $z=1$ $[\lambda_n=\lambda_n(1)]$

$$\mathbf{E}\chi_+=\frac{\sigma^2}{\alpha}\sqrt{1-d}\exp\Big\{1-\frac{\alpha}{\sigma}\sum_{n=1}^{\infty}\frac{d^n}{\sqrt{2\pi n}}+\frac{\alpha^2}{8\sigma^2}\Big\}\prod_{n=2}^{\infty}\Big(1-\frac{\alpha}{\sigma\lambda_n}\Big)\exp\Big\{\frac{\alpha}{\sigma\lambda_n}\Big\},$$
$$\frac{\mathbf{E}\chi_+^2}{2\mathbf{E}\chi_+}=\frac{\sigma^2}{\alpha}+\frac{\alpha}{4}-\sigma\sum_{n=1}^{\infty}\frac{d^n}{\sqrt{2\pi n}}-\frac{1}{\alpha}\sum_{n=2}^{\infty}\frac{1}{\lambda_n^2(1-\alpha\sigma^{-1}\lambda_n^{-1})}.$$

(ii) The method demonstrated here which gives the representations of the factorization components by the zeros of the function $r_z(\lambda)$ can also be used for any other distribution of X_1 with $\mathbf{E}e^{\lambda X_1}$ an entire function. The main difficulty will consist in finding these zeros and in determining the polynomial coefficients in the exponent before the infinite product. Moreover, one can find in this way factorization components in those cases when $r_z(\lambda)$ is a meromorphic function which consequently can be represented as a ratio of two entire functions. Representing the numerator and denominator in this ratio as infinite products, one can determine which factors correspond to each of the factorization components and, therefore, obtain the representations of the factorization components as ratios of two infinite products.

3. Proof of Theorems 3, 4 and 5. Consider first an arbitrary random walk $\{S_n\}$ generated by i.i.d. random variables. Let $r_{z\pm}(\lambda)$ be the factorization components defined in (4) and g be a function of the form

$$g(\lambda)=\int_{-\infty}^{\infty}e^{\lambda y}\,dG(y), \tag{19}$$

where the total variation of G is finite. For $|z| < 1$, $\operatorname{Re}\lambda = 0$, and for arbitrary real t we define the operators

$$\mathscr{A}_t^+ g(z, \lambda) = r_{z+}(\lambda)[r_{z+}^{-1}(\lambda)g(\lambda)]^{[t,\infty)},$$
$$\mathscr{A}_t^- g(z, \lambda) = r_{z-}(\lambda)[r_{z-}^{-1}(\lambda)g(\lambda)]^{(-\infty,t]},$$

denoting everywhere

$$\left[\int_{-\infty}^{\infty} e^{\lambda y}\, dG(y)\right]^D = \int_D e^{\lambda y}\, dG(y), \qquad D \subset R.$$

These operators depend also on z; the function g may depend on z as well.

For an arbitrary stopping time $\tau \ge 0$, introduce on the event $\{\tau < \infty\}$ the random variables

$$\tau_+(t) = \inf\{n \ge \tau \colon S_n \ge t\},$$
$$\tau_-(t) = \inf\{n \ge \tau \colon S_n \le t\}.$$

The following assertion was proved in Lotov (1989).

THEOREM 6. *For any real t, $|z| < 1$ and* $\operatorname{Re}\lambda = 0$, *we have*

$$\mathbf{E}\big(z^{\tau_\pm(t)} \exp\{\lambda S_{\tau_\pm(t)}\};\ \tau_\pm(t) < \infty\big) = \mathscr{A}_t^\pm \mathbf{E}(z^\tau \exp\{\lambda S_\tau\};\ \tau < \infty). \tag{20}$$

Denote

$$Q(z, \lambda) = \mathbf{E}\big(z^{N_b} \exp\{\lambda S_{N_b}\};\ N_b < \infty\big),$$
$$Q_1(z, \lambda) = \mathbf{E}\big(z^{N_{a,b}} \exp\{\lambda S_{N_{a,b}}\};\ S_{N_{a,b}} \le -a\big),$$
$$Q_2(z, \lambda) = \mathbf{E}\big(z^{N_{a,b}} \exp\{\lambda S_{N_{a,b}}\};\ S_{N_{a,b}} \ge b\big).$$

It follows immediately from Theorem 6 that

$$Q(z, \lambda) = \mathscr{A}_b^+ e(z, \lambda)$$

[here $e(z, \lambda) = e(\lambda) \equiv 1$] and also

$$\begin{aligned} Q_1(z, \lambda) &= \mathscr{A}_{-a}^- e(z, \lambda) - \mathscr{A}_{-a}^- Q_2 e(z, \lambda), \\ Q_2(z, \lambda) &= \mathscr{A}_b^+ e(z, \lambda) - \mathscr{A}_b^+ Q_1 e(z, \lambda), \end{aligned} \tag{21}$$

or, in equivalent form,

$$\begin{aligned} Q_2(z, \lambda) &= \mathscr{A}_b^+ e(z, \lambda) - \mathscr{A}_b^+ \mathscr{A}_{-a}^- e(z, \lambda) + \mathscr{A}_b^+ \mathscr{A}_{-a}^- Q_2(z, \lambda), \\ Q_1(z, \lambda) &= \mathscr{A}_{-a}^- e(z, \lambda) - \mathscr{A}_{-a}^- \mathscr{A}_b^+ e(z, \lambda) + \mathscr{A}_{-a}^- \mathscr{A}_b^+ Q_1(z, \lambda). \end{aligned} \tag{22}$$

The identities (21) were found by Kemperman [(1963), formula (3.20)] in a different way without defining the operators $\mathscr{A}_t^\pm$. From (22) we can also obtain a representation for Q_i as a series containing compositions of the operators $\mathscr{A}_b^+$ and $\mathscr{A}_{-a}^-$. To do this, one should use the identity for Q_i for recurrent substitutions instead of Q_i in the right-hand side. The relation (20) makes clear the probabilistic sense of all summands in such expansions.

Now we shall study the asymptotic behavior of expressions of the types $\mathscr{A}_b^+ g$ and $\mathscr{A}_{-a}^- g$ as $a \to \infty$ and $b \to \infty$. Suppose that condition (1) holds and $z \in (0, 1)$. Then the function $r_{z+}^{-1}(\lambda)$ has the prime poles $\mu_k(z) = \lambda_k(z) - \alpha$, $k = 1, 2, \ldots$. Consider, for some $k \geq 0$, the function

$$w_z(\lambda) = r_{z+}^{-1}(\lambda) - \sum_{i=1}^{2k+1} \frac{A_i(z)}{\lambda - \mu_i(z)}, \qquad A_i(z) = (r'_{z+}(\mu_i(z)))^{-1}.$$

This function is evidently analytic at any point λ of the half-plane $\operatorname{Re}\lambda < \operatorname{Re}\mu_{2k+2}(z) = x_{k+1}(z) - \alpha$. Let $\beta \in (\operatorname{Re}\mu_{2k+1}(z), \operatorname{Re}\mu_{2k+2}(z))$. Denote by $V(\beta)$ the set of all functions g of the type

$$g(\lambda) = \int_{-\infty}^{\infty} e^{\lambda y}\, dG(y), \qquad \int_{-\infty}^{\infty} e^{\beta y}\, |dG(y)| < \infty.$$

It is clear that $r_z(\lambda) \equiv 1 - z\exp\left\{\lambda\alpha + \lambda^2/2\right\} \in V(\beta)$ as well as $r_{z-}(\lambda) \in V(\beta)$. We also conclude from (4) that

$$\inf_{\operatorname{Re}\lambda=\beta} |r_{z-}(\lambda)| > 0,$$

and, in addition, the function H_z in the representation

$$r_{z-}(\lambda) = \int_{(-\infty,\, 0]} e^{\lambda y}\, dH_z(y)$$

has no singular component. Thus we can apply Theorem 6 from Borovkov [(1976a), Appendix 2], which states that $r_{z-}^{-1}(\lambda) \in V(\beta)$. Therefore, $r_{z+}(\lambda) = r_z(\lambda) r_{z-}^{-1}(\lambda) \in V(\beta)$. The function $r_{z+}(\lambda)$ also satisfies the conditions of Borovkov's Theorem 6. In fact, $1 - r_{z+}(\lambda)$ is a LST of an absolutely continuous function and, therefore, by the Riemann–Lebesgue lemma $1 - r_{z+}(\beta + iy) \to 1$ as $|y| \to \infty$ and, in addition, $\inf_{\operatorname{Re}\lambda=\beta} |r_{z+}(\lambda)| > 0$. Since $(\lambda - \mu_i(z))^{-1} \in V(\beta)$ for all i, we have $w_z(\lambda) \in V(\beta)$.

For arbitrary $\varepsilon > 0$, the functions $r_{z+}^{-1}(\lambda)$, $A_i(z)(\lambda - \mu_i(z))^{-1}$ and $w_z(\lambda)$ belong to $V(-\varepsilon)$. All of them are equal to the LST of the functions defined on the nonnegative half-line. Therefore, the same is true for $w_z(\lambda)$, $\operatorname{Re}\lambda \leq \beta$. Moreover, for $\operatorname{Re}\lambda \leq -\varepsilon$ the functions $r_{z+}^{-1}(\lambda) - 1$, $A_i(z)(\lambda - \mu_i(z))^{-1}$, $i = 1, 2, \ldots$, equal the LST of absolutely continuous functions. Thus, for the function $w_z(\lambda)$ we have the representation

$$w_z(\lambda) = 1 + \int_0^{\infty} e^{\lambda y} h_z(y)\, dy, \qquad \operatorname{Re}\lambda \leq \beta,$$

where

$$\left| \int_x^{\infty} h_z(y)\, dy \right| \leq C(\beta) e^{-\beta x}; \tag{23}$$

$C(\beta)$ denotes a constant independent of z. The estimate (23) follows from the inclusion $w_z(\lambda) \in V(\beta)$ and Theorem 1 in Borovkov [(1976a), Appendix 2]. Hence, we have proved the following lemma.

LEMMA 1. *Suppose that the condition* (1) *holds. Then, for every integer* $k \geq 0$, $z \in (0,1)$, $\beta \in (\operatorname{Re}\mu_{2k+1}(z), \operatorname{Re}\mu_{2k+2}(z))$ *and* λ, $\operatorname{Re}\lambda \leq \beta$, *we have the representation*

$$r_{z+}^{-1}(\lambda) = 1 + \sum_{i=1}^{2k+1} \frac{A_i(z)}{\lambda - \mu_i(z)} + \int_0^\infty e^{\lambda y} h_z(y)\,dy, \tag{24}$$

where the function h_z *satisfies* (23).

Suppose that the function g satisfies (19) and $g = [g]^{(-\infty,0]}$. Then, evidently, $g \in V(\beta)$ and in the representation

$$r_{z+}(\lambda)[w_z(\lambda)g(\lambda)]^{[b,\infty)} = \int_b^\infty e^{\lambda y}\varphi_z(y)\,dy$$

we have the estimate

$$\left|\int_x^\infty \varphi_z(y)\,dy\right| \leq C(\beta)e^{-\beta x}, \qquad x \geq b.$$

Therefore,

$$\begin{aligned}
\mathscr{A}_b^+ g(z,\lambda) &= r_{z+}(\lambda)\left[\sum_{i=1}^{2k+1} \frac{A_i(z)}{\lambda - \mu_i(z)} \int_{-\infty}^{0+} \exp(\lambda y)\,dG(y)\right]^{[b,\infty)} \\
&\quad + \int_b^\infty \exp(\lambda y)\varphi_z(y)\,dy \\
&= -r_{z+}(\lambda)\sum_{i=1}^{2k+1} A_i(z)\int_b^\infty \exp(\lambda y)\int_{-\infty}^{0+} \exp(-\mu_i(z)(y-t))\,dG(t)\,dy \\
&\quad + \int_b^\infty \exp(\lambda y)\varphi_z(y)\,dy \\
&= r_{z+}(\lambda)\sum_{i=1}^{2k+1} \frac{A_i(z)g(\mu_i(z))\exp((\lambda - \mu_i(z))b)}{\lambda - \mu_i(z)} \\
&\quad + \int_b^\infty \exp(\lambda y)\varphi_z(y)\,dy.
\end{aligned} \tag{25}$$

Letting $z \to 1$ in (25) for every $k \geq 0$ we obtain

$$\begin{aligned}
\mathbf{E}(\exp(\lambda S_{N_b});\ N_b < \infty) = \lim_{z \to 1} \mathscr{A}_b^+ e(z,\lambda) \\
= \sum_{i=1}^{2k+1} \frac{r_+(\lambda)\exp((\lambda - \mu_i)b)A_i(1)}{\lambda - \mu_i} \\
+ \int_b^\infty \exp(\lambda y)h(y)\,dy,
\end{aligned}$$

where

$$\left|\int_{x+b}^\infty h(y)\,dy\right| = O\big(\exp(-(b+x)\operatorname{Re}\mu_{2k})\big), \qquad x \geq 0,\ b \to \infty.$$

This yields the statement of Theorem 3.

The expansion (25) and its analog for $\mathscr{A}_{-a}^{-} g(z, \lambda)$ together with formulas (22) can be used for deriving the asymptotic representations for the distribution of $S_{N_{a,b}}$ with the remainder term of order $O(e^{-\beta b}) + O(e^{-\gamma a})$, for arbitrary large β and γ [see Lotov (1987)]. To avoid cumbersome calculations, we shall restrict ourselves to the expansions with the remainder term of $O(\exp(-\operatorname{Re}\mu_2 b)) + O(\exp(-(\operatorname{Re}\mu_2 + 2\alpha)a))$. To do this, we rewrite (25) in a slightly modified form:

$$\mathscr{A}_b^{+} g(z, \lambda) = \frac{v_z(\lambda)\exp((\lambda - \mu_1(z))b) g(\mu_1(z))}{v_z(\mu_1(z))} + (\lambda - \mu_1(z)) \int_b^{\infty} \exp(\lambda y)\varphi_z(y)\, dy. \tag{26}$$

A similar representation for the operator $\mathscr{A}_{-a}^{-}$ is given by

$$\mathscr{A}_{-a}^{-} g_1(z, \lambda) = \frac{u_z(\lambda)\exp(-(\lambda - \mu_{-1}(z))a) g_1(\mu_{-1}(z))}{u_z(\mu_{-1}(z))} + (\lambda - \mu_{-1}(z)) \int_{-\infty}^{-a} \exp(\lambda y)\psi_z(y)\, dy. \tag{27}$$

Here $g = [g]^{(-\infty, 0]}$, $g_1 = [g_1]^{[0, \infty)}$, $\mu_{-k}(z) = \lambda_{-k}(z) - \alpha = -\mu_k(z) - 2\alpha$, $v_z(\lambda) = (r_{z+}(\lambda))/(\lambda - \mu_1(z))$, $u_z(\lambda) = (r_{z-}(\lambda))/(\lambda - \mu_{-1}(z))$ and the functions φ_z and ψ_z satisfy the estimates

$$\begin{aligned} \left| \int_{x+b}^{\infty} \varphi_z(y)\, dy \right| &= O\big(\exp(-\operatorname{Re}\mu_2(z)(x+b))\big), \\ \left| \int_{-\infty}^{-x-a} \psi_z(y)\, dy \right| &= O\big(\exp(\operatorname{Re}\mu_{-2}(z)(x+a))\big). \end{aligned} \tag{28}$$

For the further considerations, it was necessary to isolate the factors $\lambda - \mu_1(z)$ and $\lambda - \mu_{-1}(z)$ on the right-hand sides of (26) and (27). This does not change the order of the estimates for φ_z and ψ_z.

Using (26) and (27), we write the asymptotic representations for $\mathscr{A}_b^{+} e(z, \lambda)$, $\mathscr{A}_b^{+}\mathscr{A}_{-a}^{-} e(z, \lambda)$, $\mathscr{A}_b^{+}\mathscr{A}_{-a}^{-} Q_2(z, \lambda)$ and substitute them in the first of the identities (22). Denoting

$$h_1(z) = \frac{u_z(\mu_1(z))}{u_z(\mu_{-1}(z))}, \qquad h_2(z) = \frac{v_z(\mu_{-1}(z))}{v_z(\mu_1(z))}, \qquad t(z) = \exp\{\mu_{-1}(z) - \mu_1(z)\},$$

we obtain the identity

$$
\begin{aligned}
Q_2(z,\lambda) = {} & \frac{v_z(\lambda)\exp((\lambda-\mu_1(z))b)}{v_z(\mu_1(z))} \\
& \times \{1 - h_1(z)t^a(z) + h_1(z)Q_2(z,\mu_{-1}(z))t^a(z)\} \\
& + (\lambda-\mu_1(z))\int_b^\infty \exp(\lambda y)\varphi_z^{(1)}(y)\,dy \\
& - \frac{v_z(\lambda)\exp((\lambda-\mu_1(z))b)}{v_z(\mu_1(z))}(\mu_1(z)-\mu_{-1}(z)) \\
& \times \int_{-\infty}^{-a}\exp(\mu_1(z)y)\psi_z^{(1)}(y)\,dy.
\end{aligned}
\tag{29}
$$

Here the functions $\varphi_z^{(1)}$ and $\psi_z^{(1)}$ satisfy the estimators (28). Put $\lambda = \mu_{-1}(z)$ in (29). This gives us the equation from which we find

$$
\begin{aligned}
Q_2(z,\mu_{-1}(z)) = {} & \frac{h_2(z)t^b(z)(1-h_1(z)t^a(z))}{1-h_1(z)h_2(z)t^{a+b}(z)} \\
& + \frac{\mu_{-1}(z)-\mu_1(z)}{1-h_1(z)h_2(z)t^{a+b}(z)}\int_b^\infty \exp(\mu_{-1}(z)y)\varphi_z^{(1)}(y)\,dy \\
& - \frac{h_2(z)(\mu_{-1}(z)-\mu_1(z))t^b(z)}{1-h_1(z)h_2(z)t^{a+b}(z)}\int_{-\infty}^{-a}\exp(\mu_1(z)y)\psi_z^{(1)}(y)\,dy.
\end{aligned}
\tag{30}
$$

For integrals in the right-hand side of (30), we have estimates

$$
\begin{aligned}
\left|\int_b^\infty \exp(\mu_{-1}(z)y)\varphi_z^{(1)}(y)\,dy\right| &= \Bigg|-\exp(\mu_{-1}(z)y)\int_y^\infty \varphi_z^{(1)}(t)\,dt\Bigg|_b^\infty \\
&\qquad + \mu_1(z)\int_b^\infty \exp(\mu_{-1}(z)y)\int_y^\infty \varphi_z^{(1)}(t)\,dt\Bigg| \\
&= O\big(\exp((\mu_{-1}(z)-\operatorname{Re}\mu_2(z))b)\big),
\end{aligned}
$$

and also

$$
\left|\int_{-\infty}^{-a}\exp(\mu_1(z)y)\psi_z^{(1)}(y)\,dy\right| = O\big(\exp((\operatorname{Re}\mu_{-2}(z)-\mu_1(z))a)\big).
$$

Thus, the sum of the last two summands in (30) equals

$$
\begin{aligned}
\Delta(z) = {} & O\big(\exp((\mu_{-1}(z)-\operatorname{Re}\mu_2(z))b)\big) \\
& + O\big(\exp((\operatorname{Re}\mu_{-2}(z)-\mu_1(z))a + (\mu_{-1}(z)-\mu_1(z))b)\big).
\end{aligned}
$$

The factors

$$
\frac{\mu_{-1}(z)-\mu_1(z)}{1-h_1(z)h_2(z)t^{a+b}(z)}
$$

in (30) are bounded uniformly in z for sufficiently large values of a and b. This is obvious for $\alpha \neq 0$, since $|t(z)| < \delta < 1$ in this case. The proof of the uniform in z boundedness of these factors for $\alpha = 0$ can be found in Lotov (1979).

Substituting (30) to (29), we finally obtain

$$
\begin{aligned}
Q_2(z,\lambda) &= \frac{v_z(\lambda)\exp((\lambda-\mu_1(z))b)}{v_z(\mu_1(z))}\\
&\quad\times\left[\frac{1-h_1(z)t^a(z)}{1-h_1(z)h_2(z)t^{a+b}(z)}+\Delta(z)O(|t|^a(z))\right.\\
&\qquad\left.+\,O\big(\exp((\operatorname{Re}\mu_{-2}(z)-\mu_1(z))a)\big)\right]\\
&\quad+(\lambda-\mu_1(z))\int_b^\infty \exp(\lambda y)\varphi_z^{(1)}(y)\,dy \qquad (31)\\
&=\frac{v_z(\lambda)\exp((\lambda-\mu_1(z))b)}{v_z(\mu_1(z))}\\
&\quad\times\left[\frac{1-h_1(z)t^a(z)}{1-h_1(z)h_2(z)t^{a+b}(z)}+O\big(\exp((\operatorname{Re}\mu_{-2}(z)-\mu_1(z))a)\big)\right]\\
&\quad+\int_b^\infty \exp(\lambda y)\,d\rho_z(y),
\end{aligned}
$$

$$
\left|\int_{x+b}^\infty d\rho_z(y)\right| = O\big(\exp(-\operatorname{Re}\mu_2(z)(b+x))\big). \qquad (32)
$$

This representation can be used for deriving the complete asymptotic expansions of the distribution $\mathbf{P}(N_{a,b}=n,\ S_{N_{a,b}}\in A)$ as $n\to\infty$, $a=a(n)\to\infty$, $b=b(n)\to\infty$ [see Lotov (1979)]. However, this problem is rather complicated and we shall study, as in the case $a=\infty$, only the distribution of $S_{N_{a,b}}$. Denote

$$
\Pi(z,a,b)=\frac{1-h_1(z)t^a(z)}{1-h_1(z)h_2(z)t^{a+b}(z)}
$$

and find $\lim_{z\to 1}\Pi(z,a,b)$. Suppose first that $\alpha=0$. Then, for $z\to 1$,

$$
\begin{aligned}
h_1(z)&=1+\frac{r''_{z-}(\mu_{-1}(z))}{2r'_{z-}(\mu_{-1}(z))}(\mu_1(z)-\mu_{-1}(z))+O((\mu_1(z)-\mu_{-1}(z))^2),\\
h_2(z)&=1+\frac{r''_{z+}(\mu_1(z))}{2r'_{z+}(\mu_1(z))}(\mu_{-1}(z)-\mu_1(z))+O((\mu_1(z)-\mu_{-1}(z))^2),\\
t(z)&=1+(\mu_1(z)-\mu_{-1}(z))+O((\mu_{-1}(z)-\mu_1(z))^2)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\lim_{z\to 1}\Pi(z,a,b)&=\left(a-\frac{\mathbf{E}\chi_-^2}{2\mathbf{E}\chi_-}\right)\left(a+b+\frac{\mathbf{E}\chi_+^2}{2\mathbf{E}\chi_+}-\frac{\mathbf{E}\chi_-^2}{2\mathbf{E}\chi_-}\right)^{-1}\\
&=\left(a+\frac{K}{\sqrt{2\pi}}\right)\left(a+b+\frac{2K}{\sqrt{2\pi}}\right)^{-1}.
\end{aligned}
$$

Thus letting $z\to 1$, we obtain from (31) the relation which is equivalent to (11). To prove (12), one can use the Wald identity $\mathbf{E}N_{a,b}\mathbf{E}X_1^2=\mathbf{E}S_{N_{a,b}}^2$. Denote for brevity $\kappa = K/\sqrt{2\pi}$; then

$$\mathbf{E}S^2_{N_{a,b}} = \frac{a+\kappa}{a+b+2\kappa}\int_0^\infty (b+y)^2 f_1(y)\,dy + \frac{b+\kappa}{a+b+2\kappa}\int_0^\infty (a+y)^2 f_1(y)\,dy + O\big((a^2+b^2)(\exp(-\sqrt{2\pi}b)+\exp(-\sqrt{2\pi}a))\big).$$

Taking into account that

$$\int_0^\infty y f_1(y)\,dy = \kappa, \qquad \int_0^\infty y^2 f_1(y)\,dy = \frac{\mathbf{E}\chi_+^3}{3\mathbf{E}\chi_+} = \frac{1}{3}\left(\frac{1}{4}+\frac{K^2}{2\pi}\right)$$

[see (18)], we obtain (12) after simple calculations.

Let $\alpha \neq 0$. Then, as $z \to 1$, we have

$$\mu_1(z) \to \mu_1 = |\alpha| - \alpha, \qquad \mu_{-1}(z) \to \mu_{-1} = -|\alpha| - \alpha,$$
$$t(z) \to e^{-2|\alpha|}, \qquad h_i(z) \to q_i, \qquad i = 1, 2,$$
$$v_z(\mu_1(z)) \to r'_+(\mu_1).$$

All these relations make (31) and (32) equivalent to the first statement of Theorem 5. The second one (asymptotic expansion for $\mathbf{E}N_{a,b}$ in the case $\alpha < 0$) follows from the Wald identity $\mathbf{E}N_{a,b}\mathbf{E}X_1 = \mathbf{E}S_{N_{a,b}}$ by obvious computations.

4. Remarks. 1. Equalities (20), (26) and (27) provide an instrument to study various models of random walks related to successive attainment of straight-line boundaries. Such an example is given by the so-called oscillating (or controlled) random walks, that is, random walks which change distribution of jumps when the trajectories reach certain levels. Another application consists in studying the number of crossings of a strip by trajectories of a random walk. To make it clear, consider the sequence of stopping times (which may be improper):

$$\tau_1 = \inf\{n \geq 1\colon S_n \leq -a\},$$
$$\tau_2 = \inf\{n \geq \tau_1\colon S_n \geq b\},$$
$$\tau_3 = \inf\{n \geq \tau_2\colon S_n \leq -a\}$$

and so on. We put $\inf\varnothing = \infty$ as before. Denote by ξ the number of upcrossings of the strip $\{(x, y),\ -a < y < b\}$ by the sequence $S_1, S_2, \dots$. We have, evidently, $\mathbf{P}(\xi < \infty) = 1$ for $\mathbf{E}X_1 \neq 0$ and

$$\begin{aligned}\mathbf{P}(\xi \geq k) &= \mathbf{P}(\tau_{2k} < \infty)\\ &= \mathbf{E}(z^{\tau_{2k}}\exp\{\lambda S_{\tau_{2k}}\};\ \tau_{2k} < \infty)_{z=1,\,\lambda=0}\\ &= \lim_{z\to 1}\big(\mathscr{A}_b^+\mathscr{A}_{-a}^-\big)^k e(z, 0).\end{aligned}$$

The last expression can be calculated in an explicit form for certain random walks. In other cases, one can use asymptotic representations for $\mathscr{A}_b^+ g$ and

TABLE 1

$(-a, b)$	P_1	$\hat{P}_1$	E_1	$\hat{E}_1$
$(-1, 1)$	0.50001062*	0.50000000	2.78286001*	2.75456221
$(-1, 2)$	0.38112069*	0.37995664	4.35408708*	4.33714309
$(-1, 3)$	0.30724286*	0.30639522	5.94286132*	5.91972396
$(-2, 5)$	0.31617122*	0.31629267	14.66237597*	14.66746657
$(-3, 2)$	0.58120470	0.58110087	9.49885470	9.50230483
$(-3, 6)$	0.35252750	0.35243717	23.82515440	23.83262832
$(-5, 9)$	0.36804470	0.36811878	53.73355240	53.74553268

$\mathscr{A}_{-a}^{-} g$ as $a \to \infty$, $b \to \infty$ [see (26) and (27)] for obtaining corresponding asymptotic expansions of $\mathbf{P}(\xi \geq k)$.

2. It is clear from Section 3 that condition (1) is mainly used to specify coefficients of asymptotic expansions and remainder terms. The method can be applied in the general case as well. For example, the proof of Theorem 4 remains valid if we require that $\mathbf{E}X_1 = 0$, $|\mathbf{E}\exp\{\lambda X_1\}| < \infty$ for $|\operatorname{Re}\lambda| < \varepsilon$, $\varepsilon > 0$, and X_1 is lattice-valued or its distribution has an absolutely continuous component. In the latter case, the constant $K\sigma/\sqrt{2\pi}$ in the numerator of the right-hand side of (11) should be replaced by $-\mathbf{E}\chi_-^2/2\mathbf{E}\chi_-$ and the constant $2K\sigma/\sqrt{2\pi}$ in the denominator should be replaced by $\mathbf{E}\chi_+^2/2\mathbf{E}\chi_+ - \mathbf{E}\chi_-^2/2\mathbf{E}\chi_-$. The constants in the exponents of the remainder terms will be determined by the width of a strip $\delta_1 < \operatorname{Re}\lambda < \delta_2$ containing no complex zeros of the function $1 - \mathbf{E}\exp\{\lambda X_1\}$.

3. As Monte Carlo experiments show, the approximation formulas presented in Theorems 3–5 provide high accuracy even for small values of a and b.

Table 1 contains $P_1 = \mathbf{P}(S_{N_{a,b}} \geq b)$ and $E_1 = \mathbf{E}N_{a,b}$ computed using Monte Carlo simulations for $\mathbf{E}X_1 = 0$. For calculation of each quantity, 10^7 trajectories were simulated (10^8 trajectories in the cases marked by an asterisk). Approximations $\hat{P}_1$ and $\hat{E}_1$ for P_1 and E_1, respectively, are given by (11) and (12) taken without remainder terms. Here $K = 1.460313687$, $\sigma^2 = 1$.

Probabilities $P_2 = \mathbf{P}(S_{N_{a,b}} \geq b)$ in Table 2 correspond to the case $\mathbf{E}X_1 = -0.5$; $\hat{P}_2$ is given by the main term of the first expansion of Theorem 5. Here

TABLE 2

$(-a, b)$	P_2	$\hat{P}_2$
$(-1, 1)$	0.16909570	0.17091482
$(-1, 2)$	0.06153590	0.06116015
$(-1, 3)$	0.02240640	0.02227592
$(-2, 5)$	0.00350190	0.00349039
$(-3, 2)$	0.07393190	0.07387836
$(-3, 6)$	0.00135840	0.00135031
$(-5, 9)$	0.00007330	0.00006889

also $\sigma^2 = 1$, and $q_1 = q_2 = 0.56037023$ [see Lotov (1987) for calculation of these quantities].

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