The Annals of Probability
1996, Vol. 24, No. 2, 932–960

PASSAGE-TIME MOMENTS FOR NONNEGATIVE STOCHASTIC PROCESSES AND AN APPLICATION TO REFLECTED RANDOM WALKS IN A QUADRANT

BY S. ASPANDIIAROV, R. IASNOGORODSKI AND M. MENSHIKOV

Université Paris VI, Université d'Orléans and Moscow State University

In this paper we get some sufficient conditions for the finiteness or nonfiniteness of the passage-time moments for nonnegative discrete parameter processes. The developed criteria are closely connected with the well-known results of Foster for the ergodicity of Markov chains and are given in terms of sub(super)martingales. Then, as an application of the obtained results, we get explicit conditions for the finiteness or nonfiniteness of passage-time moments for reflected random walks in a quadrant with zero drift in the interior.

1. Introduction. This paper is divided into two parts. In the first (Part 1), we consider nonnegative discrete parameter stochastic processes with asymptotically small drifts. The problem which we take up here is to find effective criteria for the finiteness or nonfiniteness of the moments of passage times of these processes (here passage time means the first hitting time of a compact set containing the origin). This question is particularly important for the Markov chains in the case $p = 1$. In this situation it is equivalent under suitable conditions to the question of determining whether a given Markov chain is ergodic or not and in the latter setting has been thoroughly investigated by many authors [see, e.g., Doob (1953), Foster (1953), Chung (1967)]. One of the possible techniques for studying the problem was initiated, as far as we know, in Foster (1953) and is now called the method of Lyapunov functions [for more details and references, see Fayolle, Malyshev and Menshikov (1994)]. The quintessence of this method consists in constructing a suitable function, called a Lyapunov or test function, such that this function applied to the original stochastic process is a sub(super)martingale. In fact in this "stochastic" setting, the notion of Lyapunov function is close to the well-known one for ordinary differential equations.

As was shown in Lamperti (1963), the method of Lyapunov functions also works in the case of an integer $p > 0$ and permits us to obtain sufficient conditions for the existence of the pth moments of passage times of Markov chains in terms of their drifts. In our paper, we extend these results to cover the case of non-Markovian processes and all positive real $p > 0$, and relax

Received September 1994; revised June 1995.
AMS 1991 *subject classifications*. Primary 60G42, 60J10; secondary 60J60.
Key words and phrases. Passage-time, recurrence classification, reflected random walks.

the hypothesis of Lamperti. Roughly speaking, our results state the following: Under appropriate conditions, the pth moment of the passage time τ_A of a nonnegative process $\{X_n,\ n \geq 0\}$ into the interval $[0, A]$ $(A > 0)$ is finite (resp., infinite) if the process $\{X^{2p}_{n \wedge \tau_A},\ n \geq 0\}$ is a supermartingale (resp., submartingale). In terms of Lyapunov functions, this result can be stated as follows: the function x^{2p} is a suitable Lyapunov function for the problem of existence of the pth moments of passage times. It should be mentioned that the "martingale" methodology of the proofs of this part has been much influenced by papers of Lamperti [see Lamperti (1960, 1963)]. Let us also notice that although the development of the criteria in Part 1 was motivated by the application to the reflected random walks studied in Part 2, we feel these criteria are natural and may be usefully applied beyond this application.

In the second part of the paper (Part 2), we get explicit conditions for the finiteness or nonfiniteness of means of passage times for reflected random walks in a quadrant with zero drift in the interior. The question is resolved in terms of a real parameter α which depends on the geometric data of the problem. To demonstrate this, we first construct nonnegative stochastic processes such that the existence (resp., nonexistence) of the pth moments of their passage times will be equivalent to the existence (resp., nonexistence) of the pth moments of the passage times for reflected random walks in a quadrant. Then we apply the criteria of Part 1 to these one-dimensional processes to get the desired result. An important role in the construction of these processes is played by the properties of certain functions that were used in Varadhan and Williams (1985) to prove the existence and uniqueness of reflected Brownian motion in a wedge.

The results of Part 2 leave open the question of existence of the passage-time moments of the critical order $p = \alpha/2$. This question and refinements of the results of Part 2 are treated in a forthcoming paper by Aspandiiarov and Iasnogorodski (1994). In this paper we also obtain general results on existence and nonexistence of means of functions of first passage times and partial analogues of the results of Part 2 for reflected Brownian motion.

Part 1. Passage-time moments for one-dimensional stochastic processes.

2. Notation. Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_n\}_{n \geq 0}$. Let x and y be some positive reals and let $\{X_n,\ n \geq 0\}$ and $\{Y_n,\ n \geq 0\}$ be discrete-time $\{\mathscr{F}_n\}$-adapted nonnegative stochastic processes such that $X_0 = x$ and $Y_0 = y$. For any positive real number A, we will denote by τ_A and σ_A the following first passage times in the interval $[0, A]$:

$$\tau_A \equiv \tau^X_{x,A} = \inf\{n \geq 0;\ X_n \leq A\},$$
$$\sigma_A \equiv \sigma^Y_{y,A} = \inf\{n \geq 0;\ Y_n \leq A\}.$$

Next, for any real a and b, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and $[a]$ stands for the integer part of a. The symbols $(a)^+$ and $(a)^-$ denote the positive and negative parts of a. We also adopt the following convention: the sign of 0 is equal to 0.

3. Finite moments. Our sufficient condition for existence of the pth moments of τ_A, which will be used in the sequel, is stated as follows:

THEOREM 1. *Let A be some positive real number. Suppose that we are given an $\{\mathscr{F}_n\}$-adapted stochastic process $\{X_n,\ n \geq 0\}$ taking values in an unbounded subset of $\mathbb{R}_+$. Assume that there exist $\lambda > 0$, $p_0 > 0$ such that for any n, $X_n^{2p_0}$ is integrable and*

$$E\left(X_{n+1}^{2p_0} - X_n^{2p_0} \middle| \mathscr{F}_n\right) \leq -\lambda X_n^{2p_0-2} \quad \text{on } \{\tau_A > n\}. \tag{1}$$

Then, for any positive $p < p_0$, there exists a positive constant $\tilde{c} = \tilde{c}(\lambda, p, p_0)$ such that for all $x \geq 0$ whenever $X_0 = x$ with probability 1,

$$E\tau_A^p \leq \tilde{c} x^{2p_0}.$$

Moreover, in the case when $p_0 \geq 1$, the last inequality also holds for $p = p_0$ and for any $p \leq p_0$,

$$E\tau_A^p \leq \tilde{c} x^{2p}.$$

PROOF. Obviously, we need only to consider the case when $X_0 = x > A$. We treat separately two subcases: $p_0 \geq 1$ and $p_0 < 1$.

Subcase 1. Let $p_0 \geq 1$. Using (1) and the simple inequality

$$(1 - y)^p \geq 1 - py \quad \text{for } p \geq 1 \text{ and } 0 < y < 1,$$

it can be easily seen that

$$E\left(X_{n+1}^{2p_0} \middle| \mathscr{F}_n\right) \leq \left(X_n^2 - \frac{\lambda}{p_0}\right)^{p_0} \quad \text{on } \{\tau_A > n\}. \tag{2}$$

For each positive $p \leq p_0$ and $n \geq 0$ we set

$$U_n^{(p)} = \left(X_{n \wedge \tau_A}^2 + \frac{\lambda}{p_0}(n \wedge \tau_A)\right)^p.$$

Now we will show that the process $\{U_n^{(p_0)},\ n \geq 0\}$ is a supermartingale. In fact, we have

$$\begin{aligned} E\left(U_{n+1}^{(p_0)} \middle| \mathscr{F}_n\right) &= E\left(U_{n+1}^{(p_0)} \middle| \mathscr{F}_n\right)\left(1_{(\tau_A > n)} + 1_{(\tau_A \leq n)}\right) \\ &= E\left(\left(X_{n+1}^2 + \frac{\lambda}{p_0}(n+1)\right)^{p_0} \middle| \mathscr{F}_n\right) 1_{(\tau_A > n)} \\ &\quad + \left(X_{\tau_A}^2 + \frac{\lambda}{p_0}\tau_A\right)^{p_0} 1_{(\tau_A \leq n)}. \end{aligned} \tag{3}$$

Next, using the L^{p_0} version of Minkowski's inequality for conditional expectations and (2), we get on $\{\tau_A > n\}$,

$$E\left(\left(X_{n+1}^2 + \frac{\lambda}{p_0}(n+1)\right)^{p_0}\middle|\mathscr{F}_n\right) \le \left(E\left(X_{n+1}^{2p_0}|\mathscr{F}_n\right)^{1/p_0} + \frac{\lambda}{p_0}(n+1)\right)^{p_0}$$
$$\le \left(X_n^2 + \frac{\lambda}{p_0}n\right)^{p_0}.$$

This and (3) imply that

$$E\left(U_{n+1}^{(p_0)}|\mathscr{F}_n\right) \le \left(X_n^2 + \frac{\lambda}{p_0}n\right)^{p_0} 1_{(\tau_A > n)} + \left(X_{\tau_A}^2 + \frac{\lambda}{p_0}\tau_A\right)^{p_0} 1_{(\tau_A \le n)} = U_n^{(p_0)},$$

as was to be shown. Next, we recall the following easy consequence of Jensen's inequality:

LEMMA 1. *Suppose* $\{U_k,\ k \ge 0\}$ *is a supermartingale and* f *is concave and nondecreasing. Then* $\{f(U_k),\ k \ge 0\}$ *is a supermartingale.*

From this lemma and the fact that $\{U_n^{(p_0)},\ n \ge 0\}$ is a supermartingale, it follows that for each positive $p < p_0$, the process $\{U_n^{(p)},\ n \ge 0\}$ is also a supermartingale. Hence, for each positive $p \le p_0$,

$$\left(\frac{\lambda}{p_0}\right)^p E(n \wedge \tau_A)^p \le E\left(U_n^{(p)}\right) \le E\left(U_0^{(p)}\right) = x^{2p},$$

and applying the monotone convergence theorem, we get that

$$E(\tau_A)^p \le \left(\frac{p_0}{\lambda}\right)^p x^{2p}.$$

Subcase 2. Let us turn to the case $p_0 < 1$. We fix some positive $p < p_0$. For any $n \ge 0$, we set

$$U_n = X_{n \wedge \tau_A}^{2p_0} + \frac{\lambda}{p}(n \wedge \tau_A)^p.$$

Let v be such that $(1-p)/(2-2p_0) > v > p/2p_0$. Then, for all n such that $n^v > A$, we have on $\{\tau_A > n\}$,

$$E\left(X_{n+1 \wedge \tau_A}^{2p_0} - X_{n \wedge \tau_A}^{2p_0}|\mathscr{F}_n\right) = E\left\{\left(X_{n+1}^{2p_0} - X_n^{2p_0}\right)1_{(X_{n \wedge \tau_A} > A)}|\mathscr{F}_n\right\}$$
$$= E\left\{\left(X_{n+1}^{2p_0} - X_n^{2p_0}\right)1_{(X_{n \wedge \tau_A} \in (A, n^v))}|\mathscr{F}_n\right\}$$
$$+ E\left\{\left(X_{n+1}^{2p_0} - X_n^{2p_0}\right)1_{(X_{n \wedge \tau_A} \in [n^v, \infty))}|\mathscr{F}_n\right\}.$$

Next, for all n such that $n^v > A$, we have from (1),

$$1_{(X_{n \wedge \tau_A} \in [n^v, \infty))}E\left(X_{n+1}^{2p_0} - X_n^{2p_0}|\mathscr{F}_n\right) \le 0.$$

So, using the simple inequality

$$(n+1)^p - n^p \le pn^{p-1}, \qquad \forall\, n \ge 1,$$

we obtain

$$E(U_{n+1} - U_n) \le E\Big(1_{(X_{n\wedge\tau_A} \in (A, n^v))}\big\{E\big(X_{n+1}^{2p_0} - X_n^{2p_0} | \mathscr{F}_n\big) + \lambda n^{p-1}\big\}\Big) + \lambda n^{p-1} P\big(X_{n\wedge\tau_A} \ge n^v\big).$$

Again, from (1) we have on $\{X_{n\wedge\tau_A} \in (A, n^v)\}$,

$$E\big(X_{n+1}^{2p_0} - X_n^{2p_0} | \mathscr{F}_n\big) \le -\lambda X_n^{2p_0-2}.$$

Hence, for all n such that $n^v > A$,

$$E(U_{n+1} - U_n) \le E\Big(1_{(X_{n\wedge\tau_A} \in [A, n^v))}\big\{-\lambda X_n^{2p_0-2} + \lambda n^{p-1}\big\}\Big) + \lambda n^{p-1} P\big(X_{n\wedge\tau} \ge n^v\big) = \mathrm{I} + \mathrm{II}. \tag{4}$$

Using the choice of v, $v(2-2p_0) < 1-p$, we obtain that for all n such that $n^v > A$,

$$\mathrm{I} \le E\Big(1_{(X_{n\wedge\tau_A} \in (A, n^v))}\big\{-\lambda n^{-v(2-2p_0)} + \lambda n^{p-1}\big\}\Big) \le 0. \tag{5}$$

In order to estimate the second term in the RHS of (4) we notice that (1) and Chebyshev's inequality readily imply that

$$P\big(X_{n\wedge\tau_A} \ge n^v\big) \le \frac{E\big(X_{n\wedge\tau_A}^{2p_0}\big)}{n^{2p_0 v}} \le \frac{x^{2p_0}}{n^{2p_0 v}}, \qquad \forall\, n \ge 1. \tag{6}$$

Finally, joining together (4)–(6), we obtain that for all n such that $n^v > A$,

$$E(U_{n+1} - U_n) \le \frac{\lambda x^{2p_0}}{n^{2p_0 v - p + 1}}.$$

Using again our choice of v (namely, $2p_0 v - p > 0$) and $x > A$, it can be easily deduced from the last bound that there exists $\tilde{c} = \tilde{c}(\lambda, p, p_0)$ such that for all n,

$$E(U_n) \le \tilde{c} x^{2p_0}.$$

Similarly to the case $p_0 \ge 1$ the application of monotone convergence arguments completes the proof. □

REMARK 1. As we will see in the next section, condition (1) is in some sense necessary for the finiteness of pth moments of τ_A.

4. Infinite moments. We start this section with a result that plays a key role in the investigation of the nonexistence of pth moments of the passage times.

LEMMA 2. *Let* $\{Y_n,\ n \ge 0\}$ *be an* $\{\mathscr{F}_n\}$*-adapted stochastic process taking values in an unbounded subset of* $\mathbb{R}_+$. *Suppose there exist positive constants* A, C *and* D *such that for any* n,

$$E\left(Y_{n+1}^2 - Y_n^2 | \mathscr{F}_n\right) \ge -C \quad on\ \{\sigma_A > n\} \tag{7}$$

and, for some $r > 1$,

$$E\left(Y_{n+1}^{2r} - Y_n^{2r} | \mathscr{F}_n\right) \le D Y_n^{2r-2} \quad on\ \{\sigma_A > n\}. \tag{8}$$

Then, for any $\nu \in (0,1)$, *there exist positive* ε *and* δ *that do not depend on* A *such that for any* n,

$$P\left(\sigma_A > n + \varepsilon Y_{n \wedge \sigma_A}^2 | \mathscr{F}_n\right) \ge 1 - \nu \quad on\ \left\{Y_{n \wedge \sigma_A} > A(1+\delta)\right\}. \tag{9}$$

REMARK 2. This lemma is an improvement of Lemma 3.1 in Lamperti (1963).

PROOF. Let us set

$$\sigma' = 1_{(\sigma_A > n)}(\sigma_A - n), \qquad Y_k' = Y_{n+k}^2 \qquad \mathscr{F}_k' = \mathscr{F}_{n+k}.$$

Obviously, $\{Y_k'\}$ is $\{\mathscr{F}_k'\}$-adapted and σ' is a stopping time w.r.t. the filtration $\{\mathscr{F}_k'\}$. It is easy to see that to prove our lemma it suffices to show that for any $\nu \in (0,1)$, there exist $\varepsilon, \delta > 0$ such that on $\{\sigma' > 0\} \cap \{Y_0' > (A(1+\delta))^2\}$ the following holds:

$$P\left(\sigma' > \varepsilon Y_0' | \mathscr{F}_0'\right) \ge 1 - \nu.$$

Let us proceed. On $\{\sigma' > 0\} \cap \{Y_0' > (A(1+\delta))^2\}$,

$$P\left(\sigma' > \varepsilon Y_0' | \mathscr{F}_0'\right) = P\left(\sigma' > [\varepsilon Y_0'] | \mathscr{F}_0'\right) = P\left(Y'_{[\varepsilon Y_0'] \wedge \sigma'} > A^2 | \mathscr{F}_0'\right).$$

Let $U = Y'_{[\varepsilon Y_0'] \wedge \sigma'}$. It can be easily obtained that for any $r > 1$,

$$E\left(U | \mathscr{F}_0'\right) \le A^2 + \left(E\left(U^r | \mathscr{F}_0'\right)\right)^{1/r} \left(P\left(U > A^2 | \mathscr{F}_0'\right)\right)^{1-1/r},$$

whence

$$\begin{aligned} P\left(\sigma' > \varepsilon Y_0' | \mathscr{F}_0'\right) &= P\left(U > A^2 | \mathscr{F}_0'\right) \\ &\ge \left\{ \frac{\left(E\left(Y'_{[\varepsilon Y_0'] \wedge \sigma'} | \mathscr{F}_0'\right) - A^2\right) \vee 0}{\left(E\left(Y'^{r}_{[\varepsilon Y_0'] \wedge \sigma'} | \mathscr{F}_0'\right)\right)^{1/r}} \right\}^{r/(r-1)}. \end{aligned} \tag{10}$$

We first estimate the numerator of the quotient in (10). We have

$$E\left(Y'_{[\varepsilon Y_0'] \wedge \sigma'} - Y_0' | \mathscr{F}_0'\right) = E\left\{ \sum_{k=0}^{[\varepsilon Y_0'] - 1} E\left(Y'_{(k+1) \wedge \sigma'} - Y'_{k \wedge \sigma'} | \mathscr{F}_k'\right) | \mathscr{F}_0' \right\}.$$

Then (7) implies that

$$E\left(Y'_{(k+1) \wedge \sigma'} - Y'_{k \wedge \sigma'} | \mathscr{F}_k'\right) = E\left(Y^2_{(k+1+n) \wedge \sigma_A} - Y^2_{(k+n) \wedge \sigma_A} | \mathscr{F}_{k+n}\right) \ge -C.$$

Therefore,

$$
\begin{aligned}
&E\left(Y'_{[\varepsilon Y'_0]\wedge\sigma'} - Y'_0 | \mathscr{F}'_0\right) \ge -C[\varepsilon Y'_0] \ge -C\varepsilon Y'_0,\\
&E\left(Y'_{[\varepsilon Y'_0]\wedge\sigma'} | \mathscr{F}'_0\right) - A^2 \ge Y'_0 - C\varepsilon Y'_0 - A^2.
\end{aligned} \tag{11}
$$

Next we bound the denominator. By assumption (8) we have that $\forall\ k \ge 0$,

$$
\begin{aligned}
E\left(\left(Y'_{(k+1)\wedge\sigma'}\right)^r | \mathscr{F}'_k\right) &\le D\left(Y'_{k\wedge\sigma'}\right)^{r-1} 1_{(\sigma'>k)} + \left(Y'_{k\wedge\sigma'}\right)^r\\
&= \left(Y'_{k\wedge\sigma'}\right)^r \left(1 + \frac{D 1_{(\sigma'>k)}}{Y'_{k\wedge\sigma'}}\right)\\
&\ge \left(Y'_{k\wedge\sigma'}\right)^r \left(1 + \frac{D}{r}\frac{1_{(\sigma'>k)}}{Y'_{k\wedge\sigma'}}\right)^r\\
&= \left(Y'_{k\wedge\sigma'} + \frac{D}{r} 1_{(\sigma'>k)}\right)^r.
\end{aligned} \tag{12}
$$

[Here we use the elementary inequality $(1+y) \le (1+y/r)^r$ for all $r > 1$ and all $y > 0$.] Set $\tilde{c} = D/r$. As will now be shown, (12) implies that for any fixed $m > 0$ the function defined by

$$
f_m(k) = E\left(\left\{Y'_{k\wedge\sigma'} + \tilde{c}m - \tilde{c}(k\wedge\sigma')\right\}^r | \mathscr{F}'_0\right) \tag{13}
$$

is nonincreasing on $[0, m] \cap \mathbb{Z}$.

Let us fix any positive integer m. We first notice that for any integer $l \in [0, m-1]$, the quantity $m - ((l+1)\wedge\sigma')$ is positive and $\mathscr{F}'_l$-measurable. Then, using Minkowski's inequality, (12) and the last remark, we obtain that for any integer $l \in [0, m-1]$,

$$
\begin{aligned}
f_m(l+1) &= E\left\{E\left(\left(Y'_{(l+1)\wedge\sigma'} + \tilde{c}m - \tilde{c}((l+1)\wedge\sigma')\right)^r | \mathscr{F}'_l\right) | \mathscr{F}'_0\right\}\\
&\le E\left\{\left(Y'_{l\wedge\sigma'} + \tilde{c}1_{(\sigma'>l)} + \tilde{c}m - \tilde{c}((l+1)\wedge\sigma')\right)^r | \mathscr{F}'_0\right\}\\
&= E\left\{\left(Y'_{l\wedge\sigma'} + \tilde{c}m - \tilde{c}(l\wedge\sigma')\right)^r | \mathscr{F}'_0\right\} = f_m(l).
\end{aligned}
$$

Therefore, the function f_m defined in (13) is nonincreasing on $[0, m] \cap Z$. Hence, for any $m > 0$,

$$
\begin{aligned}
E\left\{\left(Y'_{m\wedge\sigma'}\right)^r | \mathscr{F}'_0\right\} &\le E\left\{\left(Y'_{m\wedge\sigma'} + \tilde{c}m - \tilde{c}(m\wedge\sigma')\right)^r | \mathscr{F}'_0\right\}\\
&= f_m(m) \le f_m(0) = \left(Y'_0 + \tilde{c}m\right)^r.
\end{aligned}
$$

Putting together the last bound with $m = [\varepsilon Y_0']$, (11) and (10) we finally obtain on $\{\sigma' > 0\} \cap \{Y_0' > (A(1+\delta))^2\}$,

$$\begin{aligned} P(\sigma' > \varepsilon Y_0' | \mathscr{F}_0') &\geq \left\{ \frac{\left(Y_0' - C[\varepsilon Y_0'] - A^2\right) \vee 0}{Y_0' + \tilde{c}[\varepsilon Y_0']} \right\}^{r/(r-1)} \\ &\geq \left\{ \frac{\left(Y_0' - C[\varepsilon Y_0'] - A^2\right) \vee 0}{Y_0' + \tilde{c}\varepsilon Y_0'} \right\}^{r/(r-1)} \\ &\geq \left(\left(1 - C\varepsilon - \frac{A^2}{Y_0'}\right) \vee 0 \right)^{r/(r-1)} (1 + \tilde{c}\varepsilon)^{-r/(r-1)} \\ &\geq \left(\left(1 - C\varepsilon - \frac{1}{(1+\delta)^2}\right) \vee 0 \right)^{r/(r-1)} (1 + \tilde{c}\varepsilon)^{-r/(r-1)}. \end{aligned}$$

We conclude the proof of Lemma 2 by choosing ε and δ in such a way that

$$\left(1 - C\varepsilon - \frac{1}{(1+\delta)^2}\right)^{r/(r-1)} (1 + \tilde{c}\varepsilon)^{-r/(r-1)} \geq 1 - \nu$$

or, equivalently,

$$0 < \varepsilon \leq \frac{1 - 1/(1+\delta)^2 - (1-\nu)^{(r-1)/r}}{C + \tilde{c}(1-\nu)^{(r-1)/r}}.$$

We can, for example, take $\delta = N$, $\varepsilon = 1/(1+N)^2$ for some sufficiently large $N = N(\nu, r, C, D)$. □

REMARK 3. As can be seen from the last lines of the proof, the assertion of Lemma 2 can be stated in the following form. For each $\delta \in (0, \infty)$ let us set $\nu_\delta = 1 - (1 - (1+\delta)^{-2})^{r/r-1}$. Then, for any $\delta > 0$ and for any $\nu \in (\nu_\delta, 1)$ there exists $\varepsilon > 0$ such that the inequality (9) holds.

LEMMA 3. *Let $\{X_n,\ n \geq 0\}$ and $\{Y_n,\ n \geq 0\}$ be discrete-time $\{\mathscr{F}_n\}$-adapted nonnegative stochastic processes. Suppose that there exists a positive constant B such that for any $n \geq 0$, $X_n \leq BY_n$. Suppose in addition that there exist positive constants A and K such that for any $n \geq 0$, $EY^p_{n \wedge \sigma_A} \leq K$. Under these conditions there exists a constant $K' > 0$ such that for any $n \geq 0$,*

$$EX^p_{n \wedge \tau_{AB}} \leq K'. \tag{14}$$

PROOF. Using the simple inclusion $\{\tau_{AB} > n\} = \{\forall\ \ l \leq n,\ \ X_l > AB\} \subseteq \{\forall\ l \leq n, Y_l > A\} = \{\sigma_A > n\}$, we have

$$\begin{aligned} EX^p_{n \wedge \tau_{AB}} &= EX^p_{n \wedge \tau_{AB}}\left(1_{(\tau_{AB} > n)} + 1_{(\tau_{AB} \leq n)}\right) \leq E\left(X^p_n 1_{(\tau_{AB} > n)}\right) + (AB)^p \\ &\leq B^p E\left(Y^p_{n \wedge \sigma_A}\right) + (AB)^p \leq B^p K + (AB)^p = K', \end{aligned}$$

as was to be shown. □

The main result of this subsection is the following theorem:

THEOREM 2. *Let* $\{X_n,\ n \geq 0\}$ *and* $\{Y_n,\ n \geq 0\}$ *be discrete-time* $\{\mathscr{F}_n\}$*-adapted stochastic processes taking values in an unbounded subset of* $\mathbb{R}_+$. *Suppose that* $\{Y_n,\ n \geq 0\}$ *satisfies the conditions of Lemma* 2 *with some constants* r, A, C *and* D *and suppose that* $Y_0 = y > A$. *Suppose in addition that there exists* $B > 0$ *such that*:

(i) $AB < X_0 = x \leq By$.
(ii) $\forall\ n \geq 0,\ X_n \leq BY_n$.

If for some positive p_0 *the process* $\{X^{2p_0}_{n \wedge \tau_{AB}},\ n \geq 0\}$ *is a submartingale*, *then for any* $p > p_0$,

$$E\sigma_A^p \text{ is infinite}.$$

COROLLARY 1. *Suppose that* $\{X_n,\ n \geq 0\}$ *satisfies the conditions of Lemma* 2 *with some constants* r, A, C *and* D *and suppose that* $X_0 = x > A$. *If for some positive* p_0 *the process* $\{X^{2p_0}_{n \wedge \tau_A},\ n \geq 0\}$ *is a submartingale*, *then for any* $p > p_0$,

$$E\tau_A^p \text{ is infinite}.$$

PROOF OF THEOREM 2. We need to separate two cases.

(a) If $P(\tau_{AB} = \infty) > 0$, then by our assumption (ii), $P(\sigma_A = \infty) > 0$, which obviously implies that $E\sigma_A^p$ is infinite.
(b) Let P-a.s. $\tau_{AB} < \infty$.

Suppose on the contrary that there exists some $p > p_0$ some that $E\sigma_A^p$ is finite. The statement of Lemma 2 with $\nu = \frac{1}{2}$ yields the existence of positive ε and δ such that for all n,

$$\begin{aligned} E\sigma_A^p &\geq E\Big\{\sigma_A^p 1_{(Y_{n \wedge \sigma_A} > A(1+\delta))}\Big\} \geq \tfrac{1}{2}E\Big\{\Big(n + \varepsilon Y^2_{n \wedge \sigma_A}\Big)^p 1_{(Y_{n \wedge \sigma_A} > A(1+\delta))}\Big\} \\ &\geq \frac{\varepsilon^p}{2}E\Big\{Y^{2p}_{n \wedge \sigma_A} 1_{(Y_{n \wedge \sigma_A} > A(1+\delta))}\Big\} \geq \frac{\varepsilon^p}{2}E\Big\{Y^{2p}_{n \wedge \sigma_A}\Big\} - \frac{\varepsilon^p}{2}(A(1+\delta))^{2p}. \end{aligned}$$

Using the assumption on the finiteness of $E\sigma_A^p$, we get from the last inequality the existence of some positive $K = K(A, C, D, p)$ such that for all $n \geq 1$, $EY^{2p}_{n \wedge \sigma_A} \leq K$. By the statement of Lemma 3 there exists some $K' = K'(A, B, C, D, p)$ such that for all $n \geq 1$, $EX^{2p}_{n \wedge \tau_{AB}} \leq K'$. It then follows that the family $\{X^{2p_0}_{n \wedge \tau_{AB}},\ n \geq 0\}$ is uniformly integrable and as $n \to \infty$,

$$E\Big(X^{2p_0}_{n \wedge \tau_{AB}}\Big) \to E\Big(X^{2p_0}_{\tau_{AB}}\Big) \qquad \Big[\leq (AB)^{2p_0}\Big]. \tag{15}$$

On the other hand, since $\{X^{2p_0}_{n \wedge \tau_{AB}},\ n \geq 0\}$ is a submartingale, we have

$$E\Big(X^{2p_0}_{n \wedge \tau_{AB}}\Big) \geq E\Big(X_0^{2p_0}\Big) = x^{2p_0}. \tag{16}$$

The desired contradiction follows now from (15), (16) and the choice of x, $x > AB$. □

REMARK 4. In the Appendix we will show that Theorems 1 and 2 of this section improve the results of Lamperti [cf. Theorems 2.1, 2.2, 3.1 and 3.2 in Lamperti (1963)].

Part 2. Passage-time moments for two-dimensional Markov chains in wedges with boundary reflections.

5. Formulation of the problem and statement of the main results. In the sequel $\tilde{G}$ is the quadrant given by $\tilde{G} = \{(x, y) \in \mathbb{R}^2;\ x \geq 0,\ y \geq 0\}$. The two sides of the quadrant are denoted by $\partial\tilde{G}_1$ and $\partial\tilde{G}_2$, where $\partial\tilde{G}_1 = \{(x, y) \in \tilde{G},\ x \neq 0,\ y = 0\}$ and $\partial\tilde{G}_2 = \{(x, y) \in \tilde{G};\ y \neq 0,\ x = 0\}$. The interior of $\tilde{G}$ is referred to as $\tilde{G}^0$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_n\}_{n \geq 0}$. We are dealing with a discrete-time homogeneous irreducible aperiodic $\{\mathscr{F}_n\}$-adapted Markov chain $\{\tilde{X}_n, n \geq 0\}$ defined on (Ω, F, P), with values in $\mathbb{Z}_+^2$. Its transition mechanism is given as follows. The Markov chain starting from the point $\tilde{z} = (x, y)$ of $\mathbb{Z}_+^2$ jumps to the point $(x+i, y+j)$, $i, j \in Z$, $i, j \geq -1$, with the probability $p_{i,j}^0$, $i, j \in \mathbb{Z}$, $i, j \geq -1$ (respectively, $p_{i,j}^1$, $p_{i,j}^2$, $p_{i,j}^3$) according as $(x, y) \in \tilde{G}^0$ [respectively, $\partial\tilde{G}^1$, $\partial\tilde{G}^2$, $\partial\tilde{G}^3 \equiv (0,0)$].

Regarding the transition probabilities we assume the following moment conditions:

1. For any i, j, $p_{i,-1}^1 = p_{-1,j}^2 = p_{i,-1}^3 = p_{-1,j}^3 = 0$. Next, for any $k = 0, 1, 2$ we set

$$\gamma_k = \sup\left\{\gamma \geq 0;\ \sum_{i,j} (|i|^\gamma + |j|^\gamma)\, p_{i,j}^k < \infty\right\}. \tag{17}$$

We suppose that $\gamma_0 > 2$, $\gamma_1 > 1$, $\gamma_2 > 1$ and denote $\gamma = \min(\gamma_0, \gamma_1, \gamma_2)$.

2. In the interior $\tilde{G}^0$ we assume that

$$\sum_{i,j} i p_{i,j}^0 = \sum_{i,j} j p_{i,j}^0 = 0. \tag{18}$$

Let us define $\tilde{\lambda}_x^0$, $\tilde{\lambda}_y^0$ and $\tilde{R}^0$ by

$$\tilde{\lambda}_x^0 = \sum_{i,j} i^2 p_{i,j}^0, \qquad \tilde{R}^0 = \sum_{i,j} ij p_{i,j}^0, \qquad \tilde{\lambda}_y^0 = \sum_{i,j} j^2 p_{i,j}^0. \tag{19}$$

We assume that the matrix $\begin{pmatrix} \tilde{\lambda}_x^0 & \tilde{R}^0 \\ \tilde{R}^0 & \tilde{\lambda}_y^0 \end{pmatrix}$ is positive definite.

3. On the boundary $\partial\tilde{G}$ let

$$\tilde{p}_1 = \sum_{i,j} i p_{i,j}^1, \qquad \tilde{p}_2 = \sum_{i,j} j p_{i,j}^1, \qquad \tilde{q}_1 = \sum_{i,j} i p_{i,j}^2, \qquad \tilde{q}_2 = \sum_{i,j} j p_{i,j}^2. \tag{20}$$

We suppose that $\tilde{p}_2 \neq 0$, $\tilde{q}_1 \neq 0$ and define $\tilde{\varphi}_1 \in (0, \pi)$ and $\tilde{\varphi}_2 \in (0, \pi)$ by

$$\tan\tilde{\varphi}_1 = -\frac{\tilde{p}_2}{\tilde{p}_1}, \qquad \tan\tilde{\varphi}_2 = -\frac{\tilde{q}_1}{\tilde{q}_2}. \tag{21}$$

DEFINITION 1. Let F be any Borel subset of R^2. For any $\tilde{z} \in \mathbb{Z}_+^2$, the first passage time in F of the Markov chain $\{\tilde{Z}_n, n \geq 0\}$ with $\tilde{Z}_0 = \tilde{z}$ is defined by

$$\tilde{T}_{\{F\}} \equiv \tilde{T}^{\tilde{Z}}_{\tilde{z},\{F\}} = \inf\{n \geq 0; \tilde{Z}_n \in F\}.$$

In particular, if $F_A = \{z; |z| \leq A\}$, we define $\tilde{T}_A$ by

$$\tilde{T}_A = \tilde{T}_{\{F_A\}} = \inf\{n \geq 0; |\tilde{Z}_n| \leq A\}.$$

The main goal of the rest of the paper is to establish effective criteria for the existence or nonexistence of $E\tilde{T}_A^p$ for positive p. In order to state our principal results it will be quite useful to introduce a linear transformation Φ of $\tilde{G}$ which opens it up to a wedge G of a different angle ξ and turns the differential operator

$$Lf = \frac{1}{2}\left(\tilde{\lambda}_x^0 \frac{\partial^2}{\partial x^2} + 2\tilde{R}^0 \frac{\partial^2}{\partial x\,\partial y} + \tilde{\lambda}_y^0 \frac{\partial^2}{\partial y^2}\right) f, \qquad f \in C_b^2(\tilde{G}) \tag{22}$$

into the Laplacian Δ, in the sense that, for any $g \in C_b^2(\tilde{G})$, $L(g \circ \Phi) = \frac{1}{2}\Delta g \circ \Phi$. We can define such a mapping Φ by

$$\begin{aligned} u &= (bx - ry)a, \\ v &= \sqrt{1-r^2}\,ya, \end{aligned} \tag{23}$$

where

$$b = \sqrt{\frac{\tilde{\lambda}_y^0}{\tilde{\lambda}_x^0}}, \qquad r = \frac{\tilde{R}^0}{\sqrt{\tilde{\lambda}_x^0 \tilde{\lambda}_y^0}}, \qquad a = \frac{1}{\sqrt{\tilde{\lambda}_y^0(1-r^2)}}. \tag{24}$$

Then

$$\xi = \arccos(-r), \qquad \xi \in (0, \pi). \tag{25}$$

Another effect of this coordinate change is that it transforms the angles of boundary reflection $\tilde{\varphi}_1, \tilde{\varphi}_2$ to new angles φ_1 and φ_2, the values of which will depend on old angles of reflection and coefficients of the operator L. More precisely, they will be given by

$$\varphi_1 = \frac{\pi}{2} - \alpha_1, \qquad \varphi_2 = \frac{\pi}{2} - \alpha_2,$$

where the angles $\alpha_1, \alpha_2 \in (-\pi/2, \pi/2)$ are defined by

$$\begin{aligned} \alpha_1 &= \arctan\left\{\frac{1}{\sqrt{1-r^2}}(r + b\cot\tilde{\varphi}_1)\right\}, \\ \alpha_2 &= \arctan\left\{\frac{1}{\sqrt{1-r^2}}\left(r + \frac{1}{b}\cot\tilde{\varphi}_2\right)\right\}. \end{aligned} \tag{26}$$

Let

$$\begin{aligned} G_4 = \Phi(\mathbb{Z}_+^2) = \Big\{(u,v) \in R^2;\ & u = a(bx - ry), \\ & v = ay\sqrt{1-r^2},\ (x,y) \in \mathbb{Z}_+^2\Big\} \end{aligned} \tag{27}$$

and

$$(28)\quad G = \Phi(\tilde{G}), \qquad G^0 = \Phi(\tilde{G}^0), \qquad \partial G_1 = \Phi\big(\partial \tilde{G}_1\big), \qquad \partial G_2 = \Phi\big(\partial \tilde{G}_2\big).$$

Fix any $\tilde{z} \in \mathbb{Z}_+^2$. Then applying the mapping Φ to the Markov chain $\{\tilde{Z}_n, n \geq 0\}$ with $\tilde{Z}_0 = \tilde{z}$ we obtain a new Markov chain with state space G_4 and initial value $z = \Phi(\tilde{z})$. It will be denoted $\{Z_n, n \geq 0\}$. Next, we define a characteristic of the Markov chains $\{Z_n, n \geq 0\}$ and $\{\tilde{Z}_n, n \geq 0\}$ which will play the key role in our investigation. Namely, set

$$(29)\qquad \alpha = \frac{\alpha_1 + \alpha_2}{\xi}.$$

We are in a position to state the main results of the study:

THEOREM 3. *Let $\{\tilde{Z}_n, n \geq 0\}$ be the Markov chain defined above. Suppose $\gamma > \alpha > 0$ and $\gamma > 2$. If $\alpha > 2$, then for any $p < \alpha/2$ there exist positive $\tilde{c}_0, \tilde{A}_0$ such that for any $\tilde{A} \geq \tilde{A}_0$, whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{A}$,*

$$(30)\qquad E\tilde{T}_{\tilde{A}}^p \leq \tilde{c}_0 |\tilde{z}|^{2p}.$$

If $\alpha \leq 2$, then for any $p < \alpha/2$ and for any $\nu \in (0, \alpha/2 - p)$ there exist positive $\tilde{c}_0, \tilde{A}_0$ such that for any $\tilde{A} \geq \tilde{A}_0$, whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{A}$,

$$(31)\qquad E\tilde{T}_{\tilde{A}}^p \leq \tilde{c}_0 |\tilde{z}|^{2p+2\nu}.$$

REMARK 5. In fact, these results can be extended to the cases $\gamma \leq 2$ and $\alpha \geq \gamma$. Namely, the following result can be proved [see Theorem 4 in Chapter B, Aspandiiarov (1994)]. Let $\{\tilde{Z}_n, n \geq 0\}$ be the Markov chain defined above. Suppose $\alpha > 0$. If $\min(\alpha, \gamma) > 2$, then for any $p \leq (\min(\alpha, \gamma))/2$ there exist positive $\tilde{c}_0, \tilde{A}_0$ such that for any $\tilde{A} \geq \tilde{A}_0$, whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{A}$,

$$(32)\qquad E\tilde{T}_{\tilde{A}}^p \leq \tilde{c}_0 |\tilde{z}|^{2p}.$$

If $\min(\alpha, \gamma) \leq 2$, then for any $p < (\min(\alpha, \gamma)/2$ and for any $\nu \in (0, (\min(\alpha, \gamma))/2 - p)$ there exist positive $\tilde{c}_0, \tilde{Z}_0$ such that for any $\tilde{A} \geq \tilde{A}_0$, whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| \geq \tilde{A}$,

$$(33)\qquad E\tilde{T}_{\tilde{A}}^p \leq \tilde{c}_0 |\tilde{z}|^{2p+2\nu}.$$

The converse statement is given by the following theorem.

THEOREM 4. *Let $\{\tilde{Z}_n, n \geq 0\}$ be the Markov chain defined above. Suppose $0 < \alpha < \gamma$ and $\gamma > 2$. Then there exist positive constants $\tilde{C}_1, \tilde{A}_1$ such that for any $\tilde{A} \geq \tilde{A}_1$ and for any $p > \alpha/2$, whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{C}_1 \tilde{A}$,*

$$(34)\qquad E\tilde{T}_{\tilde{A}}^p \text{ is infinite}.$$

REMARK 6. In fact, it can be shown that in the case $0 < \gamma \le \alpha$, if $\gamma > 2$, then there exist positive constants $\tilde{C}_1$, $\tilde{A}_1$ such that for any $\tilde{A} \ge \tilde{A}_1$ for any $p > \gamma_0/2$ (resp. $p > \gamma_1/2$, $p > \gamma_2/2$) whenever $\tilde{Z}_0 = \tilde{z} \in \tilde{G}^0 \cap Z_+^2$ (resp. $\in \partial\tilde{G}_1 \cap Z_+^2$, $\in \partial\tilde{G}_2 \cap Z_+^2$) satisfies $|\tilde{z}| > \tilde{C}_1\tilde{A}$,

$$E\tilde{T}_{\tilde{A}}^p \text{ is infinite.} \tag{35}$$

REMARK 7. As the reader might have noticed, these results do not treat the critical case $p = \alpha/2$. Notice also that Theorems 3 and 4 generalize results in Fayolle, Malyshev and Menshikov (1992) on ergodicity of $\{Z_n, n \ge 0\}$.

Looking at the statements of Theorems 3 and 4, a natural question arises. What will happen if we drop the assumption $\alpha > 0$? A satisfactory answer has been obtained in Asymont, Fayolle and Menshikov (1994), where the following result is proved under the condition $\gamma > 2$.

THEOREM 5. *Let* $\{\tilde{Z}_n, n \ge 0\}$ *be the Markov chain defined above.*

(i) *If* $\alpha \ge 0$, *then it is recurrent.*
(ii) *If* $\alpha < 0$, *then it is transient.*

We will remove the condition $\gamma > 2$ and will give below a simple proof of this result in the case $\alpha \ne 0$.

REMARK 8. In fact, in Asymont, Fayolle and Menshikov (1994) the recurrence (resp. transience) of $\{\tilde{Z}_n, n \ge 0\}$ has been obtained under the condition

$$\tilde{\lambda}_x^0 \cot \tilde{\varphi}_2 + \tilde{\lambda}_y^0 \cot \tilde{\varphi}_1 + 2\tilde{R}^0 \ge 0 \quad (\text{resp.}, < 0).$$

However, it can be easily seen that this condition is equivalent to $\alpha \ge 0$ (resp., < 0).

Let us discuss briefly the proof of these results. First, we establish the analogues of Theorems 3 and 4 for the "transformed" Markov chains $\{Z_n, n \ge 0\}$. To this end we use some properties of "good" Lyapunov functions ψ_β: $G \to [0, \infty)$ and the results of Part 1. Thereby, we reduce the two-dimensional problem of existence of pth moments of first passage times to a one-dimensional one. We will be able then to obtain the desired results on the process $\{\tilde{Z}_n, n \ge 0\}$ by means of easy geometrical arguments. Using the same ideas as in the proofs of Theorems 3 and 4, we also prove Theorem 5.

6. Proof of the main results.

6.1. *"Transformed" setting.* Let us detail the Markov chain $\{Z_n, n \ge 0\}$. It takes values in G_4 and is governed by the following transition mechanism. For all integers $i, j \ge -1$, the Markov chain jumps from point $(u, v) \in G_4$ to

$(u + a(bi - rj),\ v + aj\sqrt{1 - r^2})$ with probabilities $p^0_{i,j}$, $i, j \in \mathbb{Z}$, $i, j \geq -1$ (resp., $p^1_{i,j}$, $p^2_{i,j}$, $p^3_{i,j}$) according as $(x, y) \in G^0$ [resp., ∂G^1, ∂G^2, $\partial G^3 \equiv (0,0)$]. These transition probabilities $p^k_{i,j}$, $i, j \in \mathbb{Z}$, $i, j \geq -1$, $k = 0, 1, 2, 3$, satisfy conditions 1–3 [cf. (17)–(21)] of Section 5. Next, we immediately see that conditions 2 and 3 of Section 5 imply that:

2′. In the interior G^0,

$$\sum_{i,j} (bi - rj) p^0_{i,j} = \sum_{i,j} j p^0_{i,j} = 0, \tag{36}$$

$$\begin{aligned} &\sum_{i,j} a^2 (bi - rj)^2 p^0_{i,j} = 1, \qquad \sum_{i,j} (bi - rj) j p^0_{i,j} = 0, \\ &\sum_{i,j} a^2 (1 - r^2) j^2 p^0_{i,j} = 1. \end{aligned} \tag{37}$$

3′. On the boundary ∂G let

$$\begin{aligned} p_1 &= \sum_{i,j} a(bi - rj) p^1_{i,j}, \qquad p_2 = \sum_{i,j} a\sqrt{1 - r^2}\, j p^1_{i,j} \\ q_1 &= \sum_{i,j} a(bi - rj) p^2_{i,j}, \qquad q_2 = \sum_{i,j} a\sqrt{1 - r^2}\, j p^2_{i,j}. \end{aligned} \tag{38}$$

Then,

$$-\frac{p_1}{p_2} = \tan \alpha_1, \qquad -\frac{q_1}{q_2} = 23T(\xi - \alpha_2), \tag{39}$$

where α_1, α_2, ξ were defined in (25) and (26). We denote $\mathbf{P} = (p_1, p_2)$ and $\mathbf{Q} = (q_1, q_2)$.

Definition 2. Let F be any Borel subset of $\mathbb{R}^2$. For any $z \in G_4$, the first passage time of the Markov chain $\{Z_n, n \geq 0\}$ with $Z_0 = z$ in F is defined by

$$T_{\{F\}} \equiv T^Z_{z,\{F\}} = \inf\{n \geq 0; Z_n \in F\}.$$

In particular, if $F_A = \{z; |z| \leq A\}$, we denote

$$T_A = T_{\{F_A\}} = \inf\{n \geq 0; |Z_n| \leq A\}.$$

The first part of the proof of Theorems 3 and 4 will consist in proving Theorems 6 and 7:

Theorem 6. *Let $\{Z_n, n \geq 0\}$ be the family of Markov chains defined previously. Suppose $\gamma > \alpha > 0$ and $\gamma > 2$. If $\alpha > 2$, then for any $p < \alpha/2$ there exist positive c_0, A_0 such that for any $A \geq A_0$, whenever $Z_0 = z \in G_4$ satisfies $|z| > A$,*

$$ET^p_A \leq c_0 |z|^{2p}. \tag{40}$$

If $\alpha \le 2$, *then for any* $p < \alpha/2$ *and for any* $\nu \in (0, \alpha/2 - p)$ *there exist positive* c_0, A *such that for any* $A \ge A_0$, *whenever* $Z_0 = z \in G_4$ *satisfies* $|z| > A$,

$$ET_A^p \le c_0 |z|^{2p+2\nu}. \tag{41}$$

THEOREM 7. *Let* $\{Z_n, n \ge 0\}$ *be the Markov chain defined previously. Suppose* $0 < \alpha < \gamma$ *and* $\gamma > 2$. *Then for any* $p > \alpha/2$ *there exist positive constants* C_1 A_1 *such that for any* $A \ge A_1$, *whenever* $Z_0 = z \in G_4$ *satisfies* $|z| > C_1 A$,

$$ET_A^p \text{ is infinite}. \tag{42}$$

The idea of the proofs of Theorems 6 *and* 7. In view of the results of Part 1 it suffices to construct some positive $\{\mathscr{F}_n\}$-adapted stochastic processes such that they will satisfy the conditions of Theorems 1 and 2 and, roughly speaking, the existence (resp., nonexistence) of the pth moments of passage times corresponding to these processes will be equivalent to the existence (resp., nonexistence) of pth moments of the passage times T_A corresponding to the process $\{Z_n, n \ge 0\}$ (the precise meaning of this statement will become clear during the proofs).

6.2. *Some technical results*. In order to prove the main results, we need to introduce a family $\{\psi_{\beta_1,\beta_2}, \beta_1, \beta_2 \in (-\pi/2, \pi/2)\}$ of nonnegative functions on G which are defined in polar coordinates (r, θ) as follows. For any $\beta_1, \beta_2 \in (-\pi/2, \pi/2)$ such that $\beta_1 + \beta_2 \ne 0$ we set $\beta = (\beta_1 + \beta_2)/\xi$ and

$$\psi_{\beta_1,\beta_2}(r,\theta) = \begin{cases} r^\beta \cos(\beta\theta - \beta_1), & r > 0, \ \theta \in [0, \xi], \\ 0, & r = 0. \end{cases}$$

CONVENTION.

1. Whenever β_1 and β_2 are fixed, the function ψ_{β_1,β_2} will simply be denoted ψ_β.
2. From now on we will only consider the functions ψ_β with $\beta \ne 0$.
3. For any positive functions f and h defined on G, the equality

   $f(z) = h(r, \theta)$ should be understood as $f(z) = h(r_z, \theta_z)$,

   where (r_z, θ_z) are the standard polar coordinates of z.

Let us fix β_1 and $\beta_2 \in (-\pi/2, \pi/2)$ such that $\beta_1 + \beta_2 \ne 0$. The following properties of the functions ψ_β will play a crucial role in the proofs of Theorems 6 and 7:

1. ψ_β is a harmonic function in $G \setminus (0,0)$.
2. The "monotonicity" property. For any $z \in G \setminus (0,0)$,

$$\begin{aligned} &\cos(\beta\theta_z - \beta_1) \ge \cos(|\beta_1| \vee |\beta_2|), \\ &|z|^\beta \ge \psi_\beta(z) \ge |z|^\beta \cos(|\beta_1| \vee |\beta_2|). \end{aligned} \tag{43}$$

3. If we are given the inward pointing nondegenerate vector field $\{\mathbf{v}(z), z \in \partial G_1 \cup \partial G_2\}$ defined by

$$\mathbf{v}(z) = \begin{cases} \mathbf{v}_1 = (-\sin \alpha_1, \cos \alpha_1), & \text{if } z \in \partial G_1, \\ \mathbf{v}_2 = (\sin(\xi - \alpha_2), -\cos(\xi - \alpha_2)), & \text{if } z \in \partial G_2, \end{cases} \tag{44}$$

then

$$(\mathbf{v}(z), \nabla\psi_\beta(z)) = \begin{cases} \beta r^{\beta-1} \sin(\beta_1 - \alpha_1), & \text{if } z \in \partial G_1, \\ \beta r^{\beta-1} \sin(\beta_2 - \alpha_2), & \text{if } z \in \partial G_2. \end{cases} \tag{45}$$

The last property of the function ψ_β is given by Lemma 4.

LEMMA 4. *For any integers i, k such that $0 \le i \le k$ and for any $s \ne 0$ there exists a positive constant $c = c(\beta_1, \beta_2, s, k)$ such that for any $z \in G$,*

$$\left| \frac{\partial^k (\psi_\beta)^s(z)}{\partial u^i \, \partial v^{k-i}} \right| \le c r^{s\beta - k}. \tag{46}$$

The proof can be carried out by induction and is easy. We omit it here.

In the next few paragraphs we will study the asymptotic behavior of $\psi_\beta^s(Z_{n+1}) - \psi_\beta^s(Z_n)$ with $s \ne 0$ as $|Z_n| \to \infty$. This will be one of the components of the proofs of our main results. We need to introduce some further notation.

NOTATION. For any $n \ge 0$ we define $\Delta_n = Z_{n+1} - Z_n$. For any $f \in C^2(G \setminus (0,0))$, $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathbb{R}^2$ we write

$$D^2 f(z, \boldsymbol{\eta}, \boldsymbol{\eta}) = \left(\eta_1^2 \frac{\partial^2}{\partial u^2} + 2\eta_1\eta_2 \frac{\partial^2}{\partial u \, \partial v} + \eta_2^2 \frac{\partial^2}{\partial v^2} \right) f(z).$$

For any $\mathscr{F}_{n+1}$-measurable function F the symbol $E_{Z_n} F$ stands for $E(F|Z_n)$. For $\beta_1, \beta_2 \in (-\pi/2, \pi/2)$ such that $\beta_1 + \beta_2 \ne 0$, $\xi \in (0, \pi)$ and $A > 0$, we denote

$$F_{A, \beta_1, \beta_2, \xi} = \left\{ z \in G, \ \psi_\beta^{1/\beta}(z) \le A \right\}, \tag{47}$$

$$c(\beta) = \cos^{-1/\beta}(|\beta_1| \vee |\beta_2|). \tag{48}$$

Let s be a fixed nonzero real number. By Taylor's formula applied to $\psi_\beta^s(Z_{n+1}) - \psi_\beta^s(Z_n)$ we have the first-order Taylor expansion

$$\psi_\beta^s(Z_{n+1}) - \psi_\beta^s(Z_n) = s\psi_\beta^{s-1}(Z_n) (\nabla\psi_\beta(Z_n), \Delta_n) + R_n(Z_n, \Delta_n, s, \beta, 1) \tag{49}$$

and the second-order Taylor expansion

$$\begin{aligned} \psi_\beta^s(Z_{n+1}) &- \psi_\beta^s(Z_n) \\ &= s\psi_\beta^{s-1}(Z_n) (\nabla\psi_\beta(Z_n), \Delta_n) \\ &\quad + \frac{s}{2}(s-1) \psi_\beta^{s-2}(Z_n) (\nabla\psi_\beta(Z_n), \Delta_n)^2 \\ &\quad + \frac{s}{2} \psi_\beta^{s-1}(Z_n) D^2\psi_\beta(Z_n, \Delta_n, \Delta_n) + R_n(Z_n, \Delta_n, s, \beta, 2), \end{aligned} \tag{50}$$

where the remainders $R_n(Z_n, \Delta_n, s, \beta, k)$ for $k = 1, 2$ can be written in the form

$$R_n(Z_n, \Delta_n, s, \beta, k) = \frac{1}{k!}\int_0^1 \frac{d^{k+1}}{dt^{k+1}}\{\psi_\beta^s(Z_n + t\Delta_n)\}(1-t)^k\,dt, \qquad k = 1, 2. \tag{51}$$

We show first that under suitable conditions on the means of the increments $Z_{n+1} - Z_n$ the asymptotic impact of the remainders $R_n(Z_n, \Delta_n, s, \beta, k)$ is negligible compared to that of the principal terms in the Taylor expansions (49) and (50).

LEMMA 5. *Suppose that for some* $\eta \in (0,1)$ *and* $k \in \{1, 2\}$ *there exist a positive constant* $c = c(k, \eta)$ *and an unbounded subset* $\mathscr{A}_k$ *of G such that for all n,* $E_{Z_n}(|\Delta_n|^{s\beta \vee (k+\eta)}) \le c$ *on* $\{Z_n \in \mathscr{A}_k\}$. *Then there exists a positive constant* $\tilde{c} = \tilde{c}(k, \eta)$ *such that for all n, if* $Z_n \in \mathscr{A}_k$ *and if* $|Z_n|$ *is large enough, then the following estimate holds*:

$$E_{Z_n}(|R_n(Z_n, \Delta_n, s, \beta, k)|) \le \tilde{c}|Z_n|^{s\beta - k - \eta}. \tag{52}$$

PROOF. Let n be any fixed nonnegative integer and let k be any fixed number from $\{1, 2\}$. We will split the term $E_{Z_n}R_n(Z_n, \Delta_n, s, \beta, k)$ into two terms handling, respectively, big and small jumps Δ_n. More precisely, let $\delta \in (0, 1)$ be a fixed real number. Then

$$\begin{aligned} &E_{Z_n}R_n(Z_n, \Delta_n, s, \beta, k)\\ &\quad = E_{Z_n}\big(R_n(Z_n, \Delta_n, s, \beta, k)\{1_{(|\Delta_n| \le \delta|Z_n|)} + 1_{(|\Delta_n| > \delta|Z_n|)}\}\big)\\ &\quad = \mathrm{I} + \mathrm{II}. \end{aligned} \tag{53}$$

To estimate the first term in (53) we notice that Lemma 4 implies the existence of a positive constant c_1 such that

$$\begin{aligned} |\mathrm{I}| &= |E_{Z_n}\big(R_n(Z_n, \Delta_n, s, \beta, k)1_{(|\Delta_n| \le \delta|Z_n|)}\big)|\\ &\le c_1 E_{Z_n}\left\{\left(|\Delta_n|^{k+1}\int_0^1 |Z_n + t\Delta_n|^{s\beta - k - 1}(1-t)^k\,dt\right)1_{(|\Delta_n| \le \delta|Z_n|)}\right\}. \end{aligned}$$

Next, easy geometrical arguments show that for any $t \in [0, 1]$,

$$\begin{aligned} |Z_n| + |\Delta_n| \ge |Z_n + t\Delta_n| &\ge |Z_n + t(\Delta_n)^+| - |t(\Delta_n)^-|\\ &\ge \inf_{\Delta \in R_+^2} |Z_n + \Delta| - |(\Delta_n)^-|\\ &\ge \sin\left(\frac{\pi}{2} \vee \xi\right)|Z_n| - |(\Delta_n)^-|, \end{aligned} \tag{54}$$

where $(\Delta_n)^+ = ((\Delta_n^x)^+, (\Delta_n^y)^+)$ and $(\Delta_n)^- = ((\Delta_n^x)^-, (\Delta_n^y)^-)$. Since the jumps of the Markov chain $\{\tilde{Z}_n, n \ge 0\}$ toward the origin are bounded from below, then the last inequalities imply that for any $t \in [0, 1]$ and for all large enough $|Z_n|$,

$$|Z_n| + |\Delta_n| \ge |Z_n + t\Delta_n| \ge \sin\left(\frac{\pi}{2} \vee \xi\right)|Z_n|/2. \tag{55}$$

These inequalities yield that for any $t \in [0,1]$ we have on $\{|\Delta_n| \le \delta|Z_n|\}$,

$$(56)\quad |Z_n + t\Delta_n|^{s\beta-k-1} \le \begin{cases} (2|Z_n|)^{s\beta-k-1}, & \text{if } s\beta \ge k+1,\\ \left(\sin\left(\frac{\pi}{2}\vee\xi\right)|Z_n|/2\right)^{s\beta-k-1}, & \text{if } s\beta < k+1.\end{cases}$$

Therefore, there exist positive constants c_2, c_3 such that

$$(57)\quad |\mathrm{I}| \le c_2 E_{Z_n}\left\{|\Delta_n|^{k+1}|Z_n|^{s\beta-k-1}1_{(|\Delta_n|\le\delta|Z_n|)}\right\} \le c_3|Z_n|^{s\beta-k-\eta}E_{Z_n}\left(|\Delta_n|^{k+\eta}\right),$$

and by our assumption on η there exists a positive c_4 such that for all large enough $|Z_n|$ with $Z_n \in \mathscr{A}_k$,

$$(58)\quad |\mathrm{I}| \le c_4|Z_n|^{s\beta-k-\eta}.$$

Let us bound the second term in (53) dealing with the big jumps. To this end instead of working with the integral form of $R_n(Z_n, \Delta_n, s, \beta, k)$, $k = 1, 2$, given by (51), we will look directly at their expressions that follow from (49) and (50). Namely, applying again Lemma 4, it can be easily deduced from (49) and (50) that there exists a positive constant c_5 such that

$$(59)\quad |\mathrm{II}| \le c_5 E_{Z_n}\left(\left(|Z_n+\Delta_n|^{s\beta} + \sum_{i=0}^{k}|Z_n|^{s\beta-i}|\Delta_n|^i\right)1_{(|\Delta_n|>\delta|Z_n|)}\right).$$

Let us consider the terms $E_{Z_n}(|\Delta_n|^i 1_{(|\Delta_n|>\delta|Z_n|)})$, $i = 0, 1, \ldots, k$. By the assumption on η we get that there exists a positive constant c_6 such that for any $i \in [0, k]$ and for all large enough $|Z_n|$ such that $Z_n \in \mathscr{A}_k$,

$$(60)\quad E_{Z_n}\left(|\Delta_n|^i 1_{(|\Delta_n|>\delta|Z_n|)}\right) \le E_{Z_n}\left(|\Delta_n|^{i-k-\eta}|\Delta_n|^{k+\eta}1_{(|\Delta_n|>\delta|Z_n|)}\right) \le c_6|Z_n|^{i-k-\eta}.$$

Next, let us investigate the term in (59) including $|Z_n+\Delta_n|^{s\beta}$:

(a) If $s\beta > 0$, then obviously on $\{|\Delta_n| > \delta|Z_n|\}$ we have

$$|Z_n+\Delta_n|^{s\beta} \le (|Z_n| + |\Delta_n|)^{s\beta} \le \left(\frac{1+\delta}{\delta}\right)^{s\beta}|\Delta_n|^{s\beta},$$

and by the assumption on η there exist positive c_7, c_8 such that for all large enough $|Z_n|$ such that $Z_n \in \mathscr{A}_k$,

$$(61)\quad E_{Z_n}\left(|\Delta_n|^{s\beta}1_{(|\Delta_n|\ge\delta|Z_n|)}\right) \le \begin{cases} c_7 \le c_7|Z_n|^{s\beta-k-\eta}, & \text{if } s\beta > k+\eta\\ E_{Z_n}\left(|\Delta_n|^{s\beta-k-\eta}|\Delta_n|^{k+\eta}1_{(|\Delta_n|\ge\delta|Z_n|)}\right) & \\ \qquad \le c_8|Z_n|^{s\beta-k-\eta}, & \text{if } s\beta \le k+\eta.\end{cases}$$

(b) Let $s\beta < 0$. Then, the inequality (55) shows that there exist positive c_9, c_{10} such that for all large enough $|Z_n|$ such that $Z_n \in \mathscr{A}_k$,

$$(62)\quad E_{Z_n}\left(|Z_n+\Delta_n|^{s\beta}1_{(|\Delta_n|>\delta|Z_n|)}\right) \le c_9 E_{Z_n}\left(|Z_n|^{s\beta}1_{(|\Delta_n|>\delta|Z_n|)}\right) \le c_{10}|Z_n|^{s\beta-k-\eta}.$$

Formulae (59)–(62) readily imply the existence of positive constant c_{11} such that for all large enough $|Z_n|$ such that $Z_n \in \mathscr{A}_k$,

$$|\mathrm{II}| \le c_{11}|Z_n|^{s\beta - k - \eta}.$$

Bringing together this, (53) and (58) we obtain the desired result (52). □

The last lemma permits us to prove the following result which plays a crucial role in the proofs of our main results.

LEMMA 6. *Let p_0 be any fixed real number such that $2p_0 < \gamma$. There exist positive constants c, A such that for any n the following two statements hold*:

(a) *On* $\{Z_n \in G^0\} \cap \{|Z_n| > A\}$,

$$\begin{aligned}(63)\qquad &\operatorname{sgn}(p_0(2p_0-\beta))E_{Z_n}\big(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\big)\\ &\quad\ge c|p_0(2p_0-\beta)|\psi_\beta^{(2p_0-2)/\beta}(Z_n).\end{aligned}$$

(b) *For each $i = 1, 2$ we have on* $\{Z_n \in \partial G_i\} \cap \{|Z_n| > A\}$,

$$\begin{aligned}(64)\qquad &\operatorname{sgn}(p_0 \sin(\beta_i - \alpha_i))E_{Z_n}\big(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\big)\\ &\quad\ge c|p_0 \sin(\beta_i - \alpha_i)|\psi_\beta^{(2p_0-1)/\beta}(Z_n).\end{aligned}$$

PROOF. (a) If $p_0(2p_0 - \beta) = 0$, then (63) is trivial. So let us suppose that $p_0(2p_0 - \beta) \neq 0$. From the moment conditions (36) and (37) and the expression for the gradient of the function ψ_β we easily obtain that on $\{Z_n \in G^0\}$,

$$\begin{aligned}&E_{Z_n}\big(\nabla\psi_\beta(Z_n), \Delta_n\big) = 0,\\ (65)\qquad &E_{Z_n}\big(\nabla\psi_\beta(Z_n), \Delta_n\big)^2 = |\nabla\psi_\beta(Z_n)|^2 = \beta^2|Z_n|^{2\beta-2},\\ &E_{Z_n}\big(D^2\psi_\beta(Z_n, \Delta_n, \Delta_n)\big) = \Delta\psi_\beta(Z_n) = 0.\end{aligned}$$

Therefore, from the second-order Taylor expansion (50) with $s = 2p_0/\beta$ we have on $\{Z_n \in G^0\}$,

$$\begin{aligned}(66)\qquad &E_{Z_n}\big(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\big)\\ &\quad= p_0(2p_0-\beta)\psi_\beta^{2p_0/\beta-2}(Z_n)|Z_n|^{2\beta-2}\\ &\qquad+ E_{Z_n}R_n\left(Z_n, \Delta_n, \frac{2p_0}{\beta}, \beta, 2\right).\end{aligned}$$

Set $c_0 = ((c(\beta))^{-\beta+2} \wedge 1)/2$. By the "monotonicity" property of ψ_β,

$$(67)\quad 2C_0\psi_\beta^{(2p_0-2)/\beta}(Z_n) \ge \psi_\beta^{2p_0/\beta-2}(Z_n)|Z_n|^{2\beta-2} \ge 2c_0\psi_\beta^{(2p_0-2)/\beta}(Z_n).$$

On the other hand, we recall that the statement of Lemma 5 applied with $\mathscr{A}_2 = G^0$, $\eta = ((\gamma_0 - 2)/2 \wedge 1)$ and $s = 2p_0/\beta$ ensures the existence of positive $\tilde{c}_0$, $\tilde{A}_0$ such that

$$\left| E_{Z_n} R_n\left(Z_n, \Delta_n, \frac{2p_0}{\beta}, \beta, 2\right)\right| \le \tilde{c}_0 |Z_n|^{2p_0 - 2 - \eta} \tag{68}$$

$$\text{on } \left\{|Z_n| > \tilde{A}_0\right\} \cap \left\{|Z_n| \in G^0\right\}.$$

Then, bringing together (66)–(68) we see that there exists a positive A_0 such that on $\{Z_n \in G^0\} \cap \{|Z_n| > A_0\}$ the desired estimate (63) holds with $A = A_0$ and $c = c_0$.

(b) We suppose that $p_0 \sin(\beta_i - \alpha_i) \neq 0$. If not, (64) is obvious. First, from (38), (39) and property 3 of ψ_β, we have on $\{Z_n \in \partial G_i\}$,

$$E_{Z_n}\left(\nabla\psi_\beta(Z_n), \Delta_n\right) = \tilde{d}_i \beta |Z_n|^{\beta - 1} \sin(\beta_i - \alpha_i), \tag{69}$$

where $\tilde{d}_1 = |\mathbf{P}|$ and $\tilde{d}_2 = |\mathbf{Q}|$. Hence, we get from the first-order Taylor expansion (49) with $s = 2p_0/\beta$ that on $\{Z_n \in \partial G_i\}$,

$$\begin{aligned} &E_{Z_n}\left(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\right) \\ &\quad = 2p_0 \tilde{d}_i \sin(\beta_i - \alpha_i)|Z_n|^{\beta-1}\psi_\beta^{2p_0/\beta - 1}(Z_n) \\ &\qquad + E_{Z_n} R_n\left(Z_n, \Delta_n, \frac{2p_0}{\beta}, \beta, 1\right). \end{aligned} \tag{70}$$

Set $c_1 = (\tilde{d}_1 \wedge \tilde{d}_2)((c(\beta))^{-\beta+1} \wedge 1)$. By the "monotonicity" property of ψ_β,

$$\tilde{d}_i \psi_\beta^{2p_0/\beta - 1}(Z_n)|Z_n|^{\beta - 1} \ge c_1 \psi_\beta^{(2p_0 - 1)/\beta}(Z_n). \tag{71}$$

Next, applying the statement of Lemma 5 with $\mathscr{A}_1 = \partial G_i$, $\eta = ((\gamma_i - 1)/2 \wedge 1)$ and $s = 2p_0/\beta$, we get the existence of positive $\tilde{c}_i$, $\tilde{A}_i$ such that

$$\left| E_{Z_n} R_n\left(Z_n, \Delta_n, \frac{2p_0}{\beta}, \beta, 1\right)\right| \le \tilde{c}_i |Z_n|^{2p_0 - 1 - \eta} \tag{72}$$

$$\text{on } \left\{|Z_n| > \tilde{A}_i\right\} \cap \left\{Z_n \in \partial G_i\right\}.$$

The inequalities (70)–(72) obviously imply that there exists a positive constant A_i such that on $\{Z_n \in \partial G_i\} \cap \{|Z_n| > A_i\}$:

$$\begin{aligned} &\operatorname{sgn}(p_0 \sin(\beta_i - \alpha_i)) E_{Z_n}\left(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\right) \\ &\quad \ge c_1 |p_0 \sin(\beta_i - \alpha_i)| \psi_\beta^{(2p_0 - 1)/\beta}(Z_n). \end{aligned}$$

Therefore, letting $A = \max(A_0, \tilde{A}_0, A_1, A_2)$ and $c = c_0 \wedge c_1$ we complete the proof of Lemma 6. □

CONVENTION. From now on by function ψ_α we mean the function $\psi_{\alpha_1, \alpha_2}$ with the angles α_1, α_2 defined in (26).

LEMMA 7. *If* $\gamma > 2$ *and* $\alpha \neq 0$, *then for any* $r \in [1, \gamma/2)$ *there exist positive* $A_{(r)}, C_{(r)}$ *such that for any* n *on* $\{|Z_n| > A_{(r)}\}$,

$$|E_{Z_n}\{\psi_\alpha^{2r/\alpha}(Z_{n+1}) - \psi_\alpha^{2r/\alpha}(Z_n)\}| \le C_{(r)}\psi_\alpha^{(2r-2)/\alpha}(Z_n). \tag{73}$$

PROOF. Let us take any fixed $r \in [1, \gamma/2)$. First, recalling the formulae (65) and (69), we obtain from the second-order Taylor expansion (50) with $s = 2r/\alpha$, $\beta_i = \alpha_i$, $i = 1, 2$,

$$E_{Z_n}\big(\psi_\alpha^{2r/\alpha}(Z_{n+1}) - \psi_\alpha^{2r/\alpha}(Z_n)\big)$$
$$= r(2r-\alpha)\psi_\alpha^{2r/\alpha-2}(Z_n)|Z_n|^{2\alpha-2} + E_{Z_n}R_n\left(Z_n, \Delta_n, \frac{2r}{\alpha}, \alpha, 2\right).$$

Second, by our assumption $2r < \gamma$, $2 < \gamma$. Therefore Lemma 5 is applicable with some $\eta < \gamma - 2$, and the proof is terminated by referring to the "monotonicity" property of ψ_α. □

Let us define the process $\{Y_n,\ n \ge 0\}$ by

$$Y_n = \psi_\alpha^{1/\alpha}(Z_n), \qquad \forall\, n \ge 0.$$

COROLLARY 2. *If* $\gamma > 2$ *and* $\alpha \neq 0$, *then there exist positive constants* $\tilde{A}, r, C, D$ *such that for any* $A \ge \tilde{A}$ *the process* $\{Y_n,\ n \ge 0\}$ *satisfies the conditions* (7) *and* (8) *of Lemma* 2.

PROOF. Let us take any fixed real number $r \in (1, \gamma/2)$. We set $\tilde{A} = \max(A_{(1)}, A_{(r)})$, $C = C_{(1)}$ and $D = C_{(r)}$. Then (73) implies that for any $A \ge \tilde{A}$ and for such r, C, D the statements (7) and (8) hold. □

6.3. *Proof of Theorems* 3 *and* 6.

PROOF OF THEOREM 6. Let us take any fixed positive p such that $p < \alpha/2$. Let ν be any fixed positive number such that $\nu \in (0, \alpha/2 - p)$. It is easy to see that there exist p_0, $\beta_1 = \beta_1(p, \alpha_1, \alpha_2, \xi)$ and $\beta_2 = \beta_2(p, \alpha_1, \alpha_2, \xi)$, such that:

(i) $\beta_i \in (-\pi/2, \pi/2)$ and $\beta_i < \alpha_i$, for $i = 1, 2$.
(ii) $p < p_0 < (\beta_1 + \beta_2)/2\xi < \alpha/2$.
(iii) $p_0 \ge 1$ (resp. $p_0 = p + \nu$) if $\alpha > 2$ (resp. $\alpha \le 2$).

Let us fix such p_0, β_1, β_2. Set $\beta = (\beta_1 + \beta_2)/\xi$. For each $n \ge 0$ we also set

$$X_n = \psi_\beta^{1/\beta}(Z_n).$$

We claim that the process $\{X_n,\ n \ge 0\}$ satisfies the conditions of Theorem 1. To verify this claim we have to show that there exist $\lambda > 0$ and $A > 0$ such that for any n,

$$E\big(X_{n+1}^{2p_0} - X_n^{2p_0}|\mathscr{F}_n\big) \le -\lambda X_n^{2p_0-2} \quad \text{on } \{\tau_A > n\}. \tag{74}$$

Obviously,

$$E\left(X_{n+1}^{2p_0} - X_n^{2p_0} | \mathscr{F}_n\right) = E_{Z_n}\left(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\right).$$

Recall that because of the choice of β_1 and β_2 we have

$$\sin(\alpha_1 - \beta_1) > 0, \qquad \sin(\alpha_2 - \beta_2) > 0, \qquad 2p_0 < \beta < \gamma.$$

We set $\lambda = cp_0 \min((\beta - 2p_0), \sin(\alpha_1 - \beta_1), \sin(\alpha_2 - \beta_2))$, with the constant c defined in Lemma 6. Then Lemma 6 shows that there exists a positive constant A such that

$$\begin{aligned} &E_{Z_n}\left(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\right) \\ &\quad \le -\lambda \psi_\beta^{(2p_0-2)/\beta}(Z_n) \quad \text{on } \{Z_n \in G^0\} \cap \{|Z_n| > A\} \\ &\quad \le -\lambda \psi_\beta^{(2p_0-1)/\beta}(Z_n) \quad \text{on } \{Z_n \in (\partial G_1 \cup \partial G_2)\} \cap \{|Z_n| > A\}. \end{aligned}$$

However, obviously there exists a positive A_1 such that on $\{|Z_n| > A_1\}$, $\psi_\beta^{1/\beta}(Z_n) \ge 1$. Therefore, setting $A_2 = A \vee A_1$, we obtain

$$E_{Z_n}\left(\psi_\beta^{2p_0/\beta}(Z_{n+1}) - \psi_\beta^{2p_0/\beta}(Z_n)\right) \le -\lambda \psi_\beta^{(2p_0-2)/\beta}(Z_n) \text{ on } \{|Z_n| > A_2\}. \tag{75}$$

Rewriting the last inequality in terms of $\{X_n,\ n \ge 0\}$ and using the obvious inclusions $\{z \in G;\ \psi_\beta^{1/\beta}(z) > A\} \subset \{z \in G;\ |z| > A\}$ and $\{\tau_A > n\} \subset \{X_n > A\}$ we finish the verification of (74).

Then, by Theorem 1 we know that there exists a positive constant $\tilde{c} = \tilde{c}(\lambda, p, p_0)$ such that whenever $X_0 = x$ satisfies $x > A_2$,

$$E\tau_{A_2}^p \le \begin{cases} \tilde{c}x^{2p_0}, & \text{if } p_0 < 1, \\ \tilde{c}x^{2p}, & \text{if } p_0 \ge 1. \end{cases}$$

In other words, there exists a positive constant $\tilde{c} = \tilde{c}(\lambda, p, \nu)$ such that whenever $Z_0 = z \in G_4$ satisfies $|\psi_\beta^{1/\beta}(z)| > A_2$,

$$ET_{F_{A_2,\beta_1,\beta_2,\xi}}^p \le \begin{cases} \tilde{c}\psi_\beta^{(2p+2\nu)/\beta}(z), & \text{if } \alpha \le 2, \\ \tilde{c}\psi_\beta^{2p/\beta}(z), & \text{if } \alpha > 2. \end{cases} \tag{76}$$

(Recall that we defined $F_{A,\beta_1,\beta_2,\xi}$ as $F_{A,\beta_1,\beta_2,\xi} = \{z \in G,\ \psi_\beta^{1/\beta}(z) \le A\}$.)

The "monotonicity" property of ψ_β and (76) imply that for any $A \ge A_2 c(\beta)$ whenever $Z_0 = z \in G_4$ satisfies $|z| > A$,

$$ET_A^p \le ET_{A_2c(\beta)}^p \le ET_{F_{A_2,\beta_1,\beta_2,\xi}}^p \le \begin{cases} \tilde{c}|z|^{2p+2\nu}, & \text{if } \alpha \le 2, \\ \tilde{c}|z|^{2p}, & \text{if } \alpha > 2. \end{cases}$$

We complete the proof of Theorem 6 by setting $A_0 = A_2 c(\beta)$, $c_0 = \tilde{c}$. □

Now the proof of Theorem 3 is obvious.

PROOF OF THEOREM 3. We only treat the case $\alpha > 2$. The same arguments will work in the other case. Let $p < \alpha/2$. Then by Theorem 6 there exist positive constants c_0, A_0 such that for any $A \geq A_0$, whenever $Z_0 = z \in G_4$ satisfies $|z| > A$,

$$ET_A^p \leq c_0 |z|^{2p}. \tag{77}$$

Recall that the Markov chain $\{Z_n,\ n \geq 0\}$ was introduced as the image of the Markov chain $\{\tilde{Z}_n,\ n \geq 0\}$ under the linear isomorphism $\Phi\colon \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined in (23)–(25). This implies that for any $A \geq A_0$ whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\Phi(\tilde{z})| > A$,

$$E\tilde{T}_{R(A)}^p \leq c_0 |\Phi(\tilde{z})|^{2p},$$

where $F(A) = \{\tilde{z} \in \mathbb{R}_+^2;\ |\Phi(\tilde{z})| \leq A\}$. Next, it is easy to see that there exist constants $\tilde{c}_\Phi, \tilde{C}_\Phi$ such that for any positive A, $\{\tilde{z} \in \mathbb{R}_+^2; |\tilde{z}| \leq A\tilde{c}_\Phi\} \subset \{\tilde{z} \in \mathbb{R}_+^2; |\Phi(\tilde{z})| \leq A\} \subset \{\tilde{z} \in \mathbb{R}_+^2 : |\tilde{z}| \leq A\tilde{C}_\Phi\}$. Hence for any $A \geq A_0$, whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > A\tilde{C}_\Phi$,

$$E\tilde{T}_{A\tilde{C}_\Phi}^p \leq c_0 \tilde{C}_\Phi^2 |\tilde{z}|^{2p}.$$

This is the desired result with $\tilde{A}_0 = A_0 \tilde{C}_\Phi$ and $\tilde{c}_0 = c_0 \tilde{C}_\Phi^2$. □

6.4. *Proof of Theorems* 4 *and* 7.

PROOF OF THEOREM 7. Let $p > \alpha/2$. We fix any p_0 such that $\alpha/2 < p_0 < (p \wedge \gamma/2)$. Then there exist $\beta_1 = \beta_1(p_0, \alpha_1, \alpha_2, \xi, \gamma)$ and $\beta_2 = \beta_2(p_0, \alpha_1, \alpha_2, \xi, \gamma)$ such that:

(a) $\beta_i \in (-\pi/2, \pi/2)$ and $\beta_i > \alpha_i$, for $i = 1, 2$.
(b) $\alpha/2 < (\beta_1 + \beta_2)/2\xi < p_0 < \gamma/2$.

Set $\beta = (\beta_1 + \beta_2)/\xi$. We define the processes $\{X_n,\ n \geq 0\}$ and $\{Y_n,\ n \geq 0\}$ by

$$X_n = \psi_\beta^{1/\beta}(Z_n), \qquad \forall\ n \geq 0,$$

$$Y_n = \psi_\alpha^{1/\alpha}(Z_n), \qquad \forall\ n \geq 0.$$

Obviously, the processes $\{X_n,\ n \geq 0\}$ and $\{Y_n,\ n \geq 0\}$ are $\{\mathscr{F}_n\}$-adapted. Moreover, by the "monotonicity" property of ψ_α we have for any $n \geq 0$,

$$X_n \leq c(\alpha) Y_n. \tag{78}$$

The proof is based on the two following facts:

(i) By Corollary 2 there exist constants $\tilde{A}, r, C, D$ such that for all $A \geq \tilde{A}$ the process $\{Y_n,\ n \geq 0\}$ satisfies the conditions of Lemma 2.

(ii) There exists a positive $\tilde{A}_2$ such that for any $A \geq \tilde{A}_2$ whenever $X_0 = x$ satisfies $x > Ac(\alpha)$ the process $\{X_{n \wedge \tau_{Ac(\alpha)}}^{2p_0},\ n \geq 0\}$ is a submartingale.

(To prove the latter fact it suffices to notice that by the choice of β_1 and β_2,

$$\sin(\beta_1 - \alpha_1) > 0, \qquad \sin(\beta_2 - \alpha_2) > 0, \qquad \beta < 2p_0 < \gamma,$$

and by Lemma 6 there exists a positive $\tilde{A}_2$ such that for any $A \geq \tilde{A}_2$, whenever $X_0 = x$ satisfies $x > Ac(\alpha)$, we have for any n,

$$E\left(X_{n+1}^{2p_0} - X_n^{2p_0} | \mathscr{F}_n\right) \geq 0 \quad \text{on } \{\tau_{Ac(\alpha)} > n\}.\big)$$

Set $A_3 = (\tilde{A} \vee \tilde{A}_2)$. Let $A \geq A_2$. Bringing together the facts (i), (ii) and (78) we see that the processes $\{X_n,\ n \geq 0\}$ and $\{Y_n,\ n \geq 0\}$ satisfy the conditions of Theorem 2 with the constants $A, B = c(\alpha), C, D, r, p_0$. Therefore, applying the statement of Theorem 2, we obtain that for any $q > p_0$ whenever $Y_0 = y$ satisfies $y > A$,

$$E\sigma_A^q \text{ is infinite.}$$

It then follows that whenever $Z_0 = z \in G_4$ satisfies $\psi_\alpha^{1/\alpha}(z) > A$,

$$ET_{F_{A,\alpha_1,\alpha_2,\xi}} \text{ is infinite.}$$

Plainly, this implies that whenever $Z_0 = z \in G_4$ satisfies $|z| > AC(\alpha)$,

$$ET_A^p \text{ is finite,}$$

as was to be shown. □

The proof of Theorem 4 can be carried over similarly to that of Theorem 3 and is omitted.

6.5. *Proof of Theorem* 5. We notice first that the transience (resp., recurrence) of the Markov chain $\{\tilde{Z}_n,\ n \geq 0\}$ implies the transience (resp., recurrence) of the Markov chain $\{Z_n,\ n \geq 0\}$ and vice versa.

To treat the latter question we are going to use the following criteria for transience and recurrence of the arbitrary discrete-time irreducible aperiodic Markov chain $\{U_n,\ n \geq 0\}$ with countable state space $\mathscr{U}$:

LEMMA 8 [Proposition 5.3, Asmussen (1987)]. *The Markov chain* $\{U_n, n \geq 0\}$ *is recurrent, if there exist a function* f *defined on* $\mathscr{U}$ *and a finite set* $K \subset \mathscr{U}$, *such that for any* m,

$$E\{f(U_{m+1}) - f(U_m) | U_m = a\} \leq 0, \qquad \forall\, a \notin K, \tag{79}$$

and the set $\{a \in \mathscr{U};\ f(a) < A\}$ *is finite for each* A.

The next result is a slight modification of Proposition 5.4 in Asmussen (1987). In order to formulate it we need to introduce the following definition:

DEFINITION 3. For any subset K of $\mathscr{U}$ let

$$\partial K = \left\{z \in K;\ \exists\ x \in \mathscr{U} \setminus K \text{ such that } P_x\left(U_{\tau_K} = z\right) > 0\right\},$$

where as usual $\tau_K = \inf\{n \geq 0;\ U_n \in K\}$.

LEMMA 9. *The Markov chain* $\{U_n,\ n \geq 0\}$ *is transient if there exist a bounded function* $f(a)$, $a \in \mathscr{U}$, *and a set* $K \subset \mathscr{U}$, *such that for any* m,

$$E\{f(U_{m+1}) - f(U_m)|U_m = a\} \leq 0, \qquad \forall\, a \notin K, \tag{80}$$

and $f(\tilde{a}) < \inf_{a \in \partial K} f(a)$ *for at least one* $\tilde{a} \notin K$.

Let us check the conditions (79) and (80) for the Markov chain $\{Z_n,\ n \geq 0\}$: We treat both cases $\alpha > 0$ and $\alpha < 0$ simultaneously. Suppose that $\alpha > 0$ (resp. $\alpha < 0$). Then there exist $\beta_1 = \beta_1(\alpha_1, \alpha_2, \xi)$ and $\beta_2 = \beta_2(\alpha_1, \alpha_2, \xi)$ such that:

1. $\beta_i \in (-\pi/2, \pi/2)$ and $\beta_i < \alpha_i$ (resp. $\beta_i > \alpha_i$) for $i = 1, 2$.
2. $0 < (\beta_1 + \beta_2)/\xi < \alpha$ (resp. $\alpha < (\beta_1 + \beta_2)/\xi < 0$).

Set $\beta = (\beta_1 + \beta_2)/\xi$. Let p_0 be any fixed real number of the same sign as β such that $|p_0| < (|\beta| \wedge \gamma)/2$. We define the function $f\colon G \to \mathbb{R}_+$ by

$$f(z) = \psi_\beta^{2p_0/\beta}(z), \qquad z \in G.$$

Obviously, the set $\{z \in G_4;\ f(z) < A\}$ is finite for each A [resp. $f(z)$ is bounded, the origin of the wedge does not belong to $\partial\{z \in G_4;\ f(z) < A\}$ for all sufficiently large A, $\inf_{\partial\{z \in G_4;\ f(z) < A\}} f > 0$ and $f(z) \to 0$ as $|z| \to \infty$]. Next, (79) [resp. (80)] follows from Lemma 6 and the choice of p_0, β_1, β_2. Therefore, applying the statements of Lemma 8 (resp. Lemma 9), we obtain the desired result.

REMARK 9. In fact, the recurrence statement of Theorem 5 in the case $\alpha > 0$ follows from Theorem 3. To see this we notice that Theorem 3 implies that for any $\tilde{Z}_0 = \tilde{z}$ such that $|\tilde{z}| > A$ with large enough A, $\tilde{T}_A < \infty$ with probability 1. On the other hand, if $\{\tilde{Z}_n,\ n \geq 0\}$ is not recurrent, then it is transient and for any $\tilde{Z}_0 = \tilde{z}$ such that $|\tilde{z}| > A$, $\tilde{T}_A = \infty$ with positive probability, which contradicts the former statement.

APPENDIX

A.1. Generalization of results of Lamperti. In this Appendix we will show that the results of Lamperti on the existence and nonexistence of the means of the passage times obtained in Lamperti (1963) follow from our theorems of Part 1. We will also extend the results of Lamperti [see Theorems 2.1, 2.2, 3.1 and 3.2 in Lamperti (1963)] to cover the case of the means of arbitrary (not necessarily integer) positive order.

Let $\{X_n,\ n \geq 0\}$ be a discrete-time nonnegative Markov chain. We will write

$$\mu_r(x) = \begin{cases} E\big((X_{n+1} - X_n)|X_n = x\big), & \text{if } r = 1. \\ E\big(|X_{n+1} - X_n|^r|X_n = x\big), & \text{if } r \neq 1. \end{cases} \tag{81}$$

PROPOSITION 1. *Let $p > 0$ and let $\tilde{A} > 0$. Suppose that for all $x \geq \tilde{A}$, $\mu_2(x)$ exists and*

$$2x\mu_1(x) + (2p - 1)\mu_2(x) \leq -\varepsilon \tag{82}$$

for some $\varepsilon > 0$ and $\mu_2(x) = O(1)$, as $x \to \infty$.

(a) *If $p \leq 1$, suppose in addition that for some $q > 1$,*

$$\mu_{2q}(x) = o(x^{2q-2}) \quad \text{as } x \to \infty. \tag{83}$$

(b) *If $p > 1$, we also suppose that*

$$\mu_{2p}(x) = o(x^{2p-2}) \quad \text{as } x \to \infty. \tag{84}$$

Then for any $\delta > 0$ there exists A_0 such that for all $A \geq A_0$ we have

$$E(\tau_A^p) = \begin{cases} O(X_0^{2p+\delta}), & \text{if } p < 1, \\ O(X_0^{2p}), & \text{if } p \geq 1. \end{cases} \tag{85}$$

PROPOSITION 2. *Let $p > 0$ and let $\tilde{A} > 0$. Suppose that for all $x \geq \tilde{A}$,*

$$2x\mu_1(x) + (2p - 1)\mu_2(x) \geq \varepsilon > 0 \tag{86}$$

for some $\varepsilon > 0$ and

$$\mu_1(x) = O(x^{-1}), \qquad \mu_2(x) = O(1) \quad \text{as } x \to \infty. \tag{87}$$

(a) *If $p \leq 1$, suppose in addition that for some $q > 1$,*

$$\mu_{2q}(x) = o(x^{2q-2}) \quad \text{as } x \to \infty. \tag{88}$$

(b) *If $p > 1$, suppose in addition that*

$$\mu_{2p}(x) = o(x^{2p-2}) \quad \text{as } x \to \infty. \tag{89}$$

Then there exists A_1 such that for all $A \geq A_1$, whenever $X_0 > A$,

$$E(\tau_A^p) \text{ is infinite.} \tag{90}$$

REMARK 10. In fact, Proposition 2 was proved in Lamperti (1963) under the additional assumption $q = 2$.

To prove Propositions 1 and 2 we need to establish one preliminary result. Let r be any positive real number. Let $\Delta_n = X_{n+1} - X_n$. By Taylor's formula,

$$X_{n+1}^{2r} - X_n^{2r} = rX_n^{2r-2}(2X_n\Delta_n + (2r-1)\Delta_n^2) + R_n(X_n, \Delta_n, r), \tag{91}$$

where the remainder $R_n(X_n, \Delta_n, r)$ is given by

$$R_n(X_n, \Delta_n, r) = r(2r-1)(2r-2)\Delta_n^3 \int_0^1 (X_n + t\Delta_n)^{2r-3}(1-t)^2\, dt.$$

LEMMA 10. *Suppose that*

$$\mu_2(x) = O(1) \quad \text{as } x \to \infty. \tag{92}$$

(a) *If $r < 1$, suppose in addition that for some $q > 1$,*

$$\mu_{2q}(x) = o(x^{2q-2}) \quad \text{as } x \to \infty. \tag{93}$$

(b) *If $r > 1$, suppose in addition that*

$$\mu_{2r}(x) = o(x^{2r-2}) \quad \text{as } x \to \infty. \tag{94}$$

Then for any $\varepsilon > 0$ there exists A such that

$$E(|R_n(X_n, \Delta_n, r)| \mid X_n) \le \varepsilon r(X_n^{2r-2})/2 \quad \text{on } \{X_n > A\}.$$

PROOF. The proof is similar to that of Lemma 5 and we only give here the principal steps, leaving details to the reader.

First, if $r = 1$, then $R_n(X_n, \Delta_n, r) = 0$. Suppose now $r \ne 1$. Since $\Delta_n \ge -X_n$, then for any $\delta \in (0,1)$,

$$\begin{aligned} &E\big(|R_n(X_n, \Delta_n, r)| 1_{(|\Delta_n| \le \delta X_n)} | X_n\big) \\ &\quad \le \left\{ \delta X_n^{2r-2} \frac{r|(2r-1)(2r-2)|}{3} \big((1+\delta)^{2r-3} \vee (1-\delta)^{2r-3}\big) \right\} \\ &\qquad \times E\big(|\Delta_n|^2 | X_n\big). \end{aligned}$$

Recall that by our assumption $E(\Delta_n^2 | X_n) = O(1)$. Therefore there exists positive δ_0 such that for all $\delta \le \delta_0$,

$$E\big(|R_n(X_n, \Delta_n, r)| 1_{(|1\Delta_n| \le \delta X_n)} | X_n\big) \le \varepsilon r X_n^{2r-2}/4. \tag{95}$$

Next, we have from (91),

$$\begin{aligned} &E\big(|R_n(X_n, \Delta_n, r)| 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \\ &\quad = E\big(\big((X_n + \Delta_n)^{2r} - X_n^{2r}\big) 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \\ &\qquad - E\big(r X_n^{2r-2}\big(2X_n\Delta_n + (2r-1)\Delta_n^2\big) 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big). \end{aligned} \tag{96}$$

Now, if $r > 1$, we easily get

$$\begin{aligned} &E\big(|R_n(X_n, \Delta_n, r)| 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \\ &\quad \le \left\{ \left(\frac{1+\delta_0}{\delta_0}\right)^{2r} + \left(\frac{1}{\delta_0}\right)^{2r} + \frac{r}{\delta_0^{2r-2}} \left(\frac{2}{\delta_0} + |2r-1|\right) \right\} E\big(|\Delta_n|^{2r} | X_n\big) \end{aligned}$$

and assumption (94) implies that there exists A_0 such that on $\{X_n > A_0\}$,

$$E\big(|R_n(X_n, \Delta_n, r)| 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \le \varepsilon r X_n^{2r-2}/4.$$

In the other case, $r < 1$, we have the following easy bound. For any $s \le 1$,

$$\begin{aligned} E\big(|\Delta_n|^{2s} 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) &\le E\big(|\Delta_n|^{2s-2q} |\Delta_n|^{2q} 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \\ &\le (\delta_0 X_n)^{2s-2q} E\big(|\Delta_n|^{2q} | X_n\big). \end{aligned}$$

Then, setting in this inequality $s = 0, r, 1/2, 1$ and putting the resulting estimates into (96), we get that there exists a positive constant $c = c(\delta_0, r)$ such that

$$E\big(|R_n(X_n, \Delta_n, r)| 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \le c X_n^{2r-2q} E\big(|\Delta_n|^{2q} | X_n\big).$$

It then follows from (93) that there exists A_1 such that on $\{X_n > A_1\}$,

$$E\big(|R_n(X_n, \Delta_n, r)| 1_{(|\Delta_n| > \delta_0 X_n)} | X_n\big) \le \varepsilon r X_n^{2r-2}/4.$$

Finally, setting $A = \max(A_0, A_1)$ and taking into account the bound (95), we obtain

$$E(|R_n(X_n, \Delta_n, r)| \,|\, X_n) \le \varepsilon r X_n^{2r-2}/2 \quad \text{on } \{X_n > A\},$$

as was to be shown. □

Proof of Proposition 1. We are going to verify that the conditions of Theorem 1 are satisfied. It suffices to check that

$$E\big(X_{n+1}^{2p} - X_n^{2p} | \mathscr{F}_n\big) \le -\lambda X_n^{2p-2} \quad \text{on } \{X_n > A\} \tag{97}$$

for some $\lambda > 0$. Set $r = p$ in (91). From (82) we have that the conditional expectation w.r.t. $\mathscr{F}_n$ of the first term in (91) is less than or equal to $-\varepsilon p X_n^{2p-2}$ and (97) is an immediate consequence of the last lemma. Applying then Theorem 1 we obtain the desired result. □

Proof of Proposition 2. We first check that the process $\{X_n,\ n \ge 0\}$ satisfies the conditions of Lemma 2 for all large enough A. We have to verify that there exists a positive constant $\tilde{A}_1$ such that for all $A \ge \tilde{A}_1$ there exist positive constants C, D such that

$$E\big(X_{n+1}^2 - X_n^2 | \mathscr{F}_n\big) \ge -C \quad \text{on } \{X_n > A\} \tag{98}$$

and for some $r > 1$,

$$E\big(X_{n+1}^{2r} - X_n^{2r} | \mathscr{F}_n\big) \le D X_n^{2r-2} \quad \text{on } \{X_n > A\}. \tag{99}$$

The first condition here follows immediately from our assumptions (86) and (87) and $E(X_{n+1}^2 - X_n^2 | \mathscr{F}_n) = 2X_n \mu_1(X_n) + \mu_2(X_n)$. As far as (99) is concerned, we set $r = q$ (resp., $r = p$) in the case $p \le 1$ (resp., $p > 1$) and consider the expansion (91) with such r. Then (99) follows from (86)–(89) and the statement of the last lemma. If now we show that the processes $\{X_{n \wedge \tau_A}^{2p-2\eta}, n \ge 0\}$ are submartingales for all large enough A and for some small positive η, then Corollary 1 will give the desired result. Let us check this submartingale property. First, it follows from conditions (86) and (87) that there exists a positive $\eta < p$ such that

$$2x\mu_1(x) + (2p - 2\eta - 1)\mu_2(x) \ge \varepsilon/2. \tag{100}$$

Let us set $r = p - \eta$ in the expansion (91). From (87), (89) and Hölder's inequality we easily have that if $2p - 2\eta > 1$, then $\mu_{2p-2\eta}(x) = o(x^{2p-2\eta-2})$. Then using again the assumptions (86)–(89) and the statement of the last lemma, we obtain that for all large enough A, the processes $\{X^{2p-2\eta}_{n \wedge \tau_A}, n \geq 0\}$ are submartingales. This concludes the proof. □

Acknowledgments. The authors would like to thank Professor V. A. Malyshev for bringing this problem to our attention. The authors are very grateful to Professor R. J. Williams and Professor J.-F. Le Gall for their valuable comments and several very helpful conversations about this paper.

REFERENCES

ASMUSSEN, F. (1987). *Applied Probability and Queues*. Wiley, Chichester.

ASPANDIIAROV, S. (1994). Ph.D. dissertation. Univ. Paris 6, Paris.

ASPANDIIAROV, S. and IASNOGORODSKI, R. (1994). Tails of passage-times and an application to stochastic processes with boundary reflection in wedges. Unpublished manuscript.

ASYMONT, I., FAYOLLE, G. and MENSHIKOV, M. (1994). On conditions of transience and recurrence for 2-dimensional Markov chains with boundary reflection. Research Report 2209, INRIA.

CHUNG, K. L. (1967). *Markov Chains with Stationary Transition Probabilities*. Springer, New York.

DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

FAYOLLE, G., MALYSHEV, V. and MENSHIKOV, M. (1992). Random walks in a quarter plane with zero drifts. *Ann. Inst. H. Poincaré* **28** 179–195.

FAYOLLE, G., MALYSHEV, V. and MENSHIKOV, M. (1994). *Topics in the Constructive Theory of Countable Markov Chains*. Cambridge Univ. Press.

FOSTER, F. G. (1953). On stochastic matrices associated with certain queueing processes. *Ann. Math. Statist.* **24** 355–360.

LAMPERTI, J. (1960). Criteria for the recurrence and transience of stochastic processes I. *J. Math. Anal. Appl.* **1** 314–330.

LAMPERTI, J. (1963). Criteria for stochastic processes II: passage-time moments. *J. Math. Anal. Appl.* **7** 127–145.

MALYSHEV, V. (1993). Networks and dynamical systems. *Adv. Appl. Probab.* **25** 140–175.

VARADHAN, S. R. S. and WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** 405–443.

WILLIAMS, R. J. (1985). Recurrence classification for reflected Brownian motion in a wedge. *Ann. Probab.* **13** 758–778.

S. ASPANDIIAROV
LABORATOIRE DE PROBABILITÉS
UNIVERSITÉ PARIS VI
4, PLACE JUSSIEU
TOUR 46-56
75252 PARIS CEDEX 05
FRANCE
E-MAIL: aspandij@oscar.proba.jussieu.fr

R. IASNOGORODSKI
UNIVERSITÉ D'ORLÉANS
UFR SCIENCES BP 6759
45067 ORLÉANS CEDEX 02
FRANCE

M. MENSHIKOV
LABORATORY OF LARGE RANDOM SYSTEMS
MOSCOW STATE UNIVERSITY
MOSCOW 119899
RUSSIA
E-MAIL: menshikov@llrs.math.msu.su