The Annals of Probability
1996, Vol. 24, No. 2, 905–911

DECREASING SEQUENCES OF σ-FIELDS AND A MEASURE CHANGE FOR BROWNIAN MOTION. II

BY JACOB FELDMAN[1] AND BORIS TSIRELSON

University of California at Berkeley and Tel Aviv University

Sharpening the main result of the preceding paper, it is shown that if B_t, $0 \le t < \infty$, is a standard Brownian motion on $(\Omega, \mathscr{F}, P)$, then for any $\varepsilon > 0$ there is a probability measure Q with $(1-\varepsilon)P \le Q \le (1+\varepsilon)P$ such that the filtration of B cannot be generated by any Brownian motion on $(\Omega, \mathscr{F}, Q)$.

1. Description of results. The main result of this paper is a strengthening of Theorem 2.6 of the immediately preceding paper by Dubins, Feldman, Smorodinsky and Tsirelson, "Decreasing sequences of σ-fields and a measure change for Brownian motion," hereafter referred to as [I]. As in [I], $\mathscr{X} = \{0,1\}^{\mathbb{N}}$, and $\mathbf{F} = (\mathscr{F}_n)_{n=0}^{\infty}$, where $\mathscr{F}_n$ is the σ-field generated by coordinates greater than n completed with respect to Bernoulli $(1/2, 1/2)$ product measure, which we call λ. For terminology and background concerning "reverse filtrations," see [I], Section 2. All measures are assumed to be probability measures.

THEOREM 1. *For any $\varepsilon > 0$ there is a measure m such that $(1-\varepsilon)\lambda < m < (1+\varepsilon)\lambda$ and the reverse filtration $(\mathscr{X}, \mathbf{F}, m)$ admits no standard extension.*

It will be helpful to have [I] available while reading this paper.

COROLLARY 2. *Let $(\mathscr{X}', \mathbf{F}', \lambda')$ be a standard reverse filtration. Then there is a measure m' such that $(1-\varepsilon)\lambda' < m' < (1+\varepsilon)\lambda'$ and $(\mathscr{X}', \mathbf{F}', m')$ has no standard extension.*

PROOF. The product $(\mathscr{X}' \times \mathscr{X}, \mathbf{F}' \otimes \mathbf{F}, \lambda' \otimes \lambda)$ is again standard and therefore isomorphic to $(\mathscr{X}', \mathbf{F}', \lambda')$. The isomorphism carries $\lambda' \otimes m$ to a measure m', and $(1-\varepsilon)\lambda' < m' < (1+\varepsilon)\lambda'$. Furthermore, since any extension of $(\mathscr{X}' \times \mathscr{X}, \mathbf{F}' \otimes \mathbf{F}, \lambda' \otimes m)$ is also an extension of $(\mathscr{X}, \mathbf{F}, m)$, such an extension cannot be standard. □

Received November 1994; revised June 1995.
[1]Supported in part by NSF Grant DMS-91-13642.
AMS 1991 *subject classifications*. Primary 60J65; secondary 28C20, 60G07, 60H10.
Key words and phrases. Brownian filtration, equivalent measure, bounded density, decreasing sequence of measurable partitions.

COROLLARY 3. *If* $(\Omega, (\mathscr{F}_t)_{t \geq 0}, P)$ *is the Brownian filtration, there is a probability measure* Q *with* $(1-\varepsilon)P < Q < (1+\varepsilon)P$ *such that the filtration* $(\Omega, (\mathscr{F}_t)_{t \geq 0}, Q)$ *is not Brownian.*

PROOF. This follows from Theorem 1 in the same way that the negative solution to [I], Problem 1, follows from Theorem 2.6 there. □

2. Construction of the measure. The measure m which is to be constructed for Theorem 1 will be block-Markov, as described in [I], Section 3:

$$\frac{dm}{d\lambda}(x) = \prod_{k=0}^{\infty} 2^{2^k} p_k(x^{(k)} | x^{(k+1)}),$$

where $x = (x_1, x_2, \dots) \in \mathscr{X} = \{0,1\}^{\mathbb{N}}$; for $k = 0, 1, \dots$ we denote by $x^{(k)}$ the following piece of the sequence x:

$$x^{(k)} = (x_{2^k}, x_{2^k+1}, \dots, x_{2^{k+1}-1}) \in \mathscr{X}^{(k)} = \{0,1\}^{2^k}.$$

In addition, each p_k is a Markovian transition probability from $\mathscr{X}^{(k+1)}$ to $\mathscr{X}^{(k)}$:

$$\forall\, z \in \mathscr{X}^{(k+1)}, \qquad \sum_{y \in \mathscr{X}^{(k)}} p_k(y|z) = 1.$$

These p_k are chosen to be arbitrary one-to-one maps $p_k\colon \{0,1\}^{2n} \to \mathscr{N}_k$ (here and henceforth $n = 2^k$), each $\mathscr{N}_k$ being a set of 2^{2n} probability measures on $\{0,1\}^n$ satisfying certain conditions (ii) and (iii). Condition (ii) was used in [I], Section 4, during the proof of Theorem 2.6 from the fundamental lemma there.

(ii) For any distinct $\mu, \nu \in \mathscr{N}_k$,

$$\mathrm{KR}^n(\mu, \nu) \geq 1 - \frac{C_1}{n\varepsilon_k}.$$

Here C_1 is an absolute constant (possibly larger than the C of [I]). Condition (iii) is new:

(iii) For any $\mu \in \mathscr{N}_k$,

$$\exp\bigl(-n^{3/4}\varepsilon_k\bigr)\lambda_n \leq \mu \leq \exp\bigl(n^{3/4}\varepsilon_k\bigr)\lambda_n,$$

where λ_n is the Bernoulli $(1/2, 1/2)$ measure on $\{0,1\}^n$.

The sequence (ε_k) is chosen to satisfy the following two conditions:

$$\text{(ii}^*\text{)} \qquad \sum_k \frac{1}{2^k \varepsilon_k} < \infty,$$

$$\text{(iii}^*\text{)} \qquad \sum_k 2^{(3/4)k} \varepsilon_k < \infty.$$

Condition (ii*) was introduced in [I], Section 3, while (iii*) replaces the weaker condition (i*), $\sum 2^k \varepsilon_k^2 < \infty$, which was used in [I]. Conditions (ii*) and (iii*) are compatible; for example, both are satisfied by $\varepsilon_k = \theta^k$ with $1/2 < \theta < 1/2^{3/4}$. (Compare this to the condition $1/2 < \theta < 1/2^{1/2}$ used in [I].)

Conditions (iii) and (iii*) ensure convergence of the infinite product for the density $dm/d\lambda$, since

$$\exp\left(-n^{3/4}\varepsilon_k\right) \le 2^n p_k\left(x^{(k)}|x^{(k+1)}\right) \le \exp\left(n^{3/4}\varepsilon_k\right).$$

The convergence is uniform in x; hence the product is bounded and is the density of a *probability* measure. Taking the product over $k = k_0, k_0 + 1, \dots$ with k_0 large enough, we can force $dm/d\lambda$ to be uniformly ε-close to 1.

Conditions (ii) and (ii*) ensure that $(\mathscr{X}, \mathbf{F}, m)$ admits no standard extension: the proof given in [I], Section 4, remains valid.

So to prove Theorem 1 all we need to do is show the existence of a sequence of sets $\mathscr{N}_k$ satisfying (ii) and (iii). To this end, we take the corresponding sets $\mathscr{M}_k$ of [I] and adapt them; in fact, we adapt each element of $\mathscr{M}_k$ separately. The following supplement to the fundamental lemma of [I] will be used.

MAIN LEMMA. *For any* $\varepsilon \in (0,1)$, $n = 1, 2, \dots,$ *any probability measure* μ *on* $\{0,1\}^n$ *satisfying*

(a) $$\mu\left(X_i = 1 | X_{i+1}^n\right) = (1 \pm \varepsilon)/2 \quad \textit{for any } i = 1, \dots, n,$$

and any $T \ge n\varepsilon^2$, *there is a probability measure* ν *on* $\{0,1\}^n$ *such that*

(b) $$\mathrm{KR}^n(\mu, \nu) \le 2n \exp\left(-\frac{T^2}{2n\varepsilon^2}\right)\cosh T,$$

(c) $$(1-\varepsilon)e^{-T}\lambda_n \le \nu \le (1+\varepsilon)e^{T}\lambda_n.$$

This main lemma is used as follows. Given k large enough, we put $n = 2^k$ and take T so that $e^T = (1-\varepsilon_k)\exp(n^{3/4}\varepsilon_k)$; then $T = n^{3/4}\varepsilon_k(1 + o(1)) \gg n\varepsilon_k^2$ due to (iii*). Inequality (c) of the main lemma implies condition (iii). Assuming $n\varepsilon_k^2 \le 1/2$ [which is ensured by (iii*) for large k] and using the well-known inequality $\cosh T \le \exp(T^2/2)$, we have

$$\mathrm{KR}^n(\mu, \nu) \le 2n \exp\left(-\frac{T^2}{2n\varepsilon_k^2} + \frac{T^2}{2}\right) \le 2n \exp\left(-\frac{T^2}{4n\varepsilon_k^2}\right).$$

Hence $\mathrm{KR}^n(\mu, \nu) \le \exp(-\sqrt{n}/5)$ for large k. This is more than enough to conclude that

$$\mathrm{KR}^n(\mu, \nu) \le \frac{C_0}{n} \le \frac{C_0}{n\varepsilon_k},$$

with an absolute constant C_0. Condition (ii) is thus ensured with the constant $C_1 = C + 2C_0$, where C is the constant of the fundamental lemma of [I]. So our theorem follows from the main lemma, by setting $\mathscr{N}_k = \{\nu\colon \mu \in \mathscr{M}_k\}$, where μ and ν are as in the statement of that lemma.

3. Proof of main lemma. The probability measure μ satisfies

$$\mu(X_i = 1 | X_{i+1}^n) = \tfrac{1}{2}(1 + \varepsilon s_i(X_{i+1}^n))$$

for certain functions s_i: $\{0,1\}^{n-i} \to \{-1, +1\}$. It is more convenient to deal with $\{-1, +1\}^n$ instead of $\{0,1\}^n$. Doing so, we have

$$\mu\{x_1^n\} = 2^{-n} \prod_{i=1}^{n} (1 + \varepsilon x_i s_i(x_{i+1}^n)).$$

Consider the density $D = d\mu / d\lambda_n$ and its conditional expectation $D_i = \mathbb{E}(D|\mathscr{F}_i)$; the expectation is taken wrt λ_n; $\mathscr{F}_i$ is generated by x_i^n. Then

$$D_k(x_1^n) = \prod_{i=k}^{n} (1 + \varepsilon x_i s_i(x_{i+1}^n)).$$

Clearly, (D_i) is a reverse martingale. (The reversal of time is, of course, due to the fact that we are dealing with reverse filtrations rather than filtrations.) Consider the (backward) stopping time τ: $\{-1, +1\}^n \to \{1, \dots, n\}$ defined by

$$\tau = \begin{cases} \max\{i : D_i' [e^{-T}, e^T]\}, & \text{when there is such } i, \\ 1, & \text{otherwise.} \end{cases}$$

Define a measure ν on $\{-1, +1\}^n$ by

$$\frac{d\nu}{d\lambda_n} = D_\tau.$$

Doob's stopping time theorem ensures that ν is a *probability* measure. We have $D_{\tau+1} \in [e^{-T}, e^T]$ (here $D_{n+1} = 1$); hence $D_\tau \in [(1-\varepsilon)e^{-T}, (1+\varepsilon)e^T]$, which is inequality (c). Inequality (b) follows from the two following facts:

$$\nu\{\tau > 1\} \le 2n \exp\left(-\frac{T^2}{2n\varepsilon^2}\right)\cosh T,$$

$$\mathrm{KR}^n(\mu, \nu) \le \nu\{\tau > 1\}.$$

The proof of the first fact is as follows. The sequence $(x_i s_i(x_{i+1}^n))$, with respect to λ_n, has the same Bernoulli distribution λ_n as (x_i); see [I], proof of Lemma 3.1. Thus

$$\int D_i^\alpha \, d\lambda_n = \left(\frac{1}{2}(1-\varepsilon)^\alpha + \frac{1}{2}(1+\varepsilon)^\alpha\right)^i \le \cosh^i \alpha\varepsilon \le \exp\left(\frac{i}{2}\alpha^2\varepsilon^2\right)$$

for any $\alpha > 0$, $i = 1, \dots, n$. Hence

$$\nu\{D_i > \exp T\} \le \exp(-\alpha T)\int D_i^\alpha \, d\nu = \exp(-\alpha T)\int D_i^{\alpha+1} \, d\lambda_n$$
$$\le \exp\left(-\alpha T + \frac{i}{2}(\alpha+1)^2\varepsilon^2\right)$$

and

$$\nu\{\max D_i > \exp T\} \le \inf_{\alpha>0} n \exp\left(-\alpha T + \frac{n}{2}(\alpha+1)^2 \varepsilon^2\right)$$
$$= n \exp\left(-\frac{T^2}{2n\varepsilon^2} + T\right),$$

since $T \ge n\varepsilon^2$. Similarly,

$$\nu\{\min D_i < \exp - T\} \le \sum_i \inf_{\alpha>0} \exp(-\alpha T) \int D_i^{-\alpha}\, d\nu$$
$$\le n \inf_{\alpha>0} \exp\left(-\alpha T + \frac{n}{2}(\alpha-1)^2 \varepsilon^2\right)$$
$$= n \exp\left(-\frac{T^2}{2n\varepsilon^2} - T\right).$$

Hence

$$\nu\{\tau > 1\} \le \nu\{\max D_i > \exp T\} + \nu\{\min D_i < \exp(-T)\}$$
$$\le n \exp\left(-\frac{T^2}{2n\varepsilon^2}\right)(\exp T + \exp(-T)).$$

This proves the first fact. The proof of the second fact will be carried out in two lemmas.

LEMMA 4. *Let two probability measures $\bar{\mu}, \bar{\nu}$ be concentrated on disjoint two-point sets, $\bar{\mu}$ on $\{a, b\}$ and $\bar{\nu}$ on $\{c, d\}$, in a space with metric $\bar{\rho}$. Suppose*

$$\bar{\rho}(a,c) \le 1, \qquad \bar{\rho}(b,d) \le 1,$$
$$\bar{\rho}(a,d) = 1, \qquad \bar{\rho}(b,c) = 1.$$

Then

$$1 - \bar{\rho}_{\mathrm{KR}}(\bar{\mu}, \bar{\nu}) \ge (1 - \bar{\rho}(a,c))\min(\bar{\mu}\{a\}, \bar{\nu}\{c\})$$
$$+ (1 - \bar{\rho}(b,d))\min(\bar{\mu}\{b\}, \bar{\nu}\{d\}).$$

(In fact, equality holds; the opposite inequality was [I], Lemma 5.2. Only the special case $\bar{\mu}\{a\} = \bar{\nu}\{c\}$, $\bar{\mu}\{b\} = \bar{\nu}\{d\}$ will be used, but the general case is not much more complicated.)

PROOF.

$$\rho_{\mathrm{KR}}(\bar{\mu}, \bar{\nu}) = \inf\left\{\int \bar{\rho}(x,y)\, d\lambda(x,y) : \lambda \in \mathscr{T}(\bar{\mu}, \bar{\nu})\right\},$$

where $\mathscr{T}(\bar{\mu}, \bar{\nu})$ is the set of joinings of $\bar{\mu}$ with $\bar{\nu}$. Without loss of generality, we may suppose that $\bar{\mu}\{a\} \ge \bar{\nu}\{c\}$; then $\bar{\mu}\{b\} \le \bar{\nu}\{d\}$. Take the following joining:

$$\lambda\{(a,c)\} = \bar{\nu}\{c\},$$
$$\lambda\{(a,d)\} = \bar{\mu}\{a\} - \bar{\nu}\{c\},$$
$$\lambda\{(b,d)\} = 1 - \bar{\mu}\{a\}.$$

Then

$$\begin{aligned}\bar{\rho}_{\mathrm{KR}}(\bar{\mu}, \bar{\nu}) &\le \int \rho(x,y)\, d\lambda(x,y)\\ &= \bar{\nu}\{c\}\bar{\rho}(a,c) + (\bar{\mu}\{a\} - \bar{\nu}\{c\})\bar{\rho}(a,d) + (1-\bar{\mu}\{a\})\bar{\rho}(b,d)\\ &= \bar{\nu}\{c\}\bar{\rho}(a,c) + 1 - \bar{\mu}\{b\} - \bar{\nu}\{c\} + \bar{\mu}\{b\}\bar{\rho}(b,d);\end{aligned}$$

hence

$$1 - \bar{\rho}_{\mathrm{KR}}(\bar{\mu}, \bar{\nu}) \ge (1 - \bar{\rho}(a,c))\bar{\nu}\{c\} + (1 - \bar{\rho}(b,d))\bar{\mu}\{b\}. \qquad \square$$

A Markov time on $\{0,1\}^n$ (with the direction of time reversed) is defined as a function $\tau\colon \{0,1\}^n \to \{1,\dots,n\}$ satisfying the condition $\{\tau \ge i\} \in \mathscr{F}_i$ for $i = 1,\dots,n$; the σ-field $\mathscr{F}_i$ is generated by x_i^n. The σ-field $\mathscr{F}_\tau$ is defined as consisting of all $E \subset \{0,1\}^n$ satisfying

$$E \cap \{\tau \ge i\} \in \mathscr{F}_i \quad \text{for } i = 1,\dots,n.$$

Lemma 5. *Let τ be any Markov time on $\{0,1\}^n$, and μ, ν any two probability measures on $\{0,1\}^n$ coinciding on $\mathscr{F}_\tau$ and positive on all points. Then*

$$\mathrm{KR}^n(\mu,\nu) \le \nu\{\tau > 1\}.$$

(The positivity assumption is not really necessary, but it avoids considering special cases and is satisfied in our application.)

Proof of Lemma 5. Induct on n. For $n = 1$ we have $\tau = 1$ identically; hence $\mathscr{F}_\tau = \mathscr{F}_1$ and $\nu = \mu$. Consider $n > 1$. Introduce conditional measures $\mu_0, \mu_1, \nu_0, \nu_1$ on $\{0,1\}^{n-1}$:

$$\mu\big(\{x_1^{n-1}\}\big|x_n\big) = \mu_{x_n}\{x_1^{n-1}\}, \qquad \nu\big(\{x_1^{n-1}\}\big|x_n\big) = \nu_{x_n}\{x_1^{n-1}\}.$$

In the discussion preceding [I], Lemma 5.2, it was shown that $\bar{\rho}, \bar{\mu}, \bar{\nu}$ may be so chosen that $\bar{\rho}(a,c) = \mathrm{KR}^{n-1}(\mu_0,\nu_0)$, $\bar{\rho}(b,d) = \mathrm{KR}^{n-1}(\mu_1,\nu_1)$, $\bar{\mu}(a) = \mu(E_0)$, $\bar{\mu}(c) = \mu(E_1)$, $\bar{\nu}(b) = \nu(E_0)$, $\bar{\nu}(d) = \nu(E_1)$ and $\bar{\rho}_{\mathrm{KR}}(\bar{\mu},\bar{\nu}) = \mathrm{KR}^n(\mu,\nu)$. This applies equally well here, so Lemma 4 gives

$$\begin{aligned}1 - \mathrm{KR}^n(\mu,\nu) \ge{}& \big(1 - \mathrm{KR}^{n-1}(\mu_0,\nu_0)\big)\min(\mu(E_0),\nu(E_0))\\ &+ \big(1 - \mathrm{KR}^{n-1}(\mu_1,\nu_1)\big)\min(\mu(E_1),\nu(E_1)),\end{aligned}$$

where $E_s = \{x_1^n\colon x_n = s\}$.

We have $E_0, E_1 \in \mathscr{F}_n \subset \mathscr{F}_\tau$; hence $\nu(E_0) = \mu(E_0)$, $\nu(E_1) = \mu(E_1)$, and the relation becomes

$$\mathrm{KR}^n(\mu,\nu) \le \nu(E_0)\mathrm{KR}^{n-1}(\mu_0,\nu_0) + \nu(E_1)\mathrm{KR}^{n-1}(\mu_1,\nu_1).$$

The induction assumption gives

$$\mathrm{KR}^{n-1}(\mu_0, \nu_0) \le \nu\{\tau < 1 | E_0\},$$

provided that $\nu\{\tau = n | E_0\} = 0$. Otherwise $\nu\{\tau = n | E_0\} = 1$, since $\{\tau = n\} \in \mathscr{F}_n$, and the inequality reduces to $\mathrm{KR}^{n-1}(\mu_0, \nu_0) \le 1$, which holds trivially. The same reasoning holds for $\mathrm{KR}^{n-1}(\mu_1, \nu_1)$, giving

$$\mathrm{KR}^{n}(\mu, \nu) \le \nu(E_0)\nu\{\tau > 1 | E_0\} + \nu(E_1)\nu\{\tau > 1 | E_1\} = \nu\{\tau > 1\}.$$

This completes the proof of Lemma 5 and the main lemma. □

Acknowledgments. The first named author would like to thank L. Dubins, for asking him whether boundedness of $dm/d\lambda$ implies standardness of $(\mathscr{X}, \mathbf{F}, m)$, and M. Smorodinsky for a helpful conversation in Warwick.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720
E-MAIL: feldman@math.berkeley.edu

SCHOOL OF MATHEMATICS
TEL AVIV UNIVERSITY
TEL AVIV 69978
ISRAEL