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A QUANTITATIVE AND A DUAL VERSION OF THE HALMOS–SAVAGE THEOREM WITH APPLICATIONS TO MATHEMATICAL FINANCE

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The celebrated theorem of Halmos and Savage implies that if M is a set of $\mathbb{P}$-absolutely continuous probability measures Q on $(\Omega, \mathscr{F}, \mathbb{P})$ such that each $A \in \mathscr{F}$, $\mathbb{P}(A) > 0$ is charged by some $Q \in M$, that is, $Q(A) > 0$ (where the choice of Q depends on the set A), then—provided M is closed under countable convex combinations—we can find $Q_0 \in M$ with full support; that is, $\mathbb{P}(A) > 0$ implies $Q_0(A) > 0$. We show a quantitative version: if we assume that, for $\varepsilon > 0$ and $\delta > 0$ fixed, $\mathbb{P}(A) > \varepsilon$ implies that there is $Q \in M$ and $Q(A) > \delta$, then there is $Q_0 \in M$ such that $\mathbb{P}(A) > 4\varepsilon$ implies $Q_0(A) > \varepsilon^2\delta/2$. This version of the Halmos–Savage theorem also allows a "dualization" which we also prove in a quantitative and a qualitative version. We give applications to asymptotic problems arising in mathematical finance and pertaining to the relation of the concept of "no arbitrage" and the existence of equivalent martingale measures for a sequence of stochastic processes.

Introduction. The Halmos–Savage theorem ([2], Lemma 7; see also Theorem 1.1 below) is a very useful tool in many applications, notably in mathematical statistics [11].

Let us briefly recall this theorem (which we will slightly reformulate to fit into our context). Given a stochastic base $(\Omega, \mathscr{F}, \mathbb{P})$, let M be a set of probability measures Q on $\mathscr{F}$ such that each Q is absolutely continuous with respect to $\mathbb{P}$, and M is closed under countable convex combinations; that is, for $(Q_n)_{n=1}^{\infty}$ in M and $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n = 1$, we have that $\sum_{n=1}^{\infty} c_n Q_n \in M$. Suppose, in addition, that M charges each set $A \in \mathscr{F}$ which is not a $\mathbb{P}$-null-set; that is, suppose $\mathbb{P}(A) > 0$ implies that there is $Q \in M$ (depending on A) such that $Q(A) > 0$. Then the Halmos–Savage theorem implies an interchange of quantifiers: under the above assumption, one can find an element Q_0 in M with full support, that is, such that, for each $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$, we have that $Q_0(A) > 0$.

In the present paper we prove a "quantitative" version of the Halmos–Savage theorem (Proposition 1.3 below). We replace the "qualitative" assumption "for each $A \in \mathscr{F}$, $\mathbb{P}(A) > 0$, there is $Q \in M$ with $Q(A) > 0$" by the following "quantitative" assumption: assume that, for *fixed* $\varepsilon > 0$ and $\delta > 0$, we have that, given $A \in \mathscr{F}$, $\mathbb{P}(A) > \varepsilon$, there is $Q \in M$ (depending on

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A) such that $Q(A) > \delta$. Under this assumption, we show that there is $Q_0 \in M$ such that, for $A \in \mathscr{F}$, $\mathbb{P}(A) > 4\varepsilon$, we have that $Q_0(A) > \varepsilon^2\delta/2$. In other words, we again have the crucial interchange of quantifiers, but we not only can dispose of the "qualitative" control of the form that $Q_0(A)$ is strictly positive, but also we can assert the "quantitative" conclusion that it is larger than the constant $\varepsilon^2\delta/2$ (under the "quantitative" assumption $\mathbb{P}(A) > 4\varepsilon$). The proof of the "quantitative version" does not only rely on the usual exhaustion argument ([2], Lemma 7) but uses a somewhat more delicate duality argument. It is easy to deduce the original Halmos–Savage theorem from this "quantitative" version (see the remark after Proposition 1.3).

The quantitative version of the Halmos–Savage theorem also opens the view for a *dual version* of the situation by interchanging the roles of $\mathbb{P}$ and $Q \in M$: in the above setting we start with the assumption that, again for *fixed* $\varepsilon > 0$ and $\delta > 0$, we have that $\mathbb{P}(B) < \delta$ implies that there is *some* $Q \in M$ (depending on B) with $Q(B) < \varepsilon$. Then we may conclude (see Proposition 1.5 below) that there is $Q_0 \in M$ such that $\mathbb{P}(B) < 2\varepsilon\delta$ implies that $Q_0(B) < 8\varepsilon$; that is, again we obtain an interchange of quantifiers.

Let us try to give interpretations of the above results in a loose language. The (original) Halmos–Savage theorem states that if $\mathbb{P}$ is absolutely continuous with respect to the *set* M, then there is $Q_0 \in M$ such that $\mathbb{P}$ is absolutely continuous with respect to the single measure Q_0. The quantitative version (Proposition 1.3 below) states that if $\mathbb{P}$ is ε–δ-*absolutely continuous* with respect to the *set* M, then there is $Q_0 \in M$ such that $\mathbb{P}$ is 4ε–$\varepsilon^2\delta/2$-*absolutely continuous* with respect to Q_0 (with a hopefully obvious interpretation of the loose expression "ε–δ-absolutely continuous").

The dual quantitative version (Proposition 1.5) can be interpreted in the following way: if the *set* M is ε–δ-absolutely continuous with respect to $\mathbb{P}$ (to be understood in the above sense), then there is some $Q_0 \in M$ which is 8ε–$2\varepsilon\delta$-absolutely continuous with respect to $\mathbb{P}$. It now becomes apparent that the natural framework for the "dual" version of the Halmos–Savage theorem is to drop the assumption that M consists of $\mathbb{P}$-absolutely continuous measures. To do this step in a convenient and clean way, we adopt a "topological setting." We assume that $\mathbb{P}$ is a Radon probability measure on a compact metrizable space K and that M is a convex set of Radon probability measures on K (not necessarily absolutely continuous with respect to $\mathbb{P}$). If we assume, in addition, that M is weak-star [i.e., $\sigma(\mathscr{M}(K), \mathscr{C}(K))$] closed, we may extend the above "dual quantitative version" of the Halmos–Savage theorem to this setting (see Proposition 1.6 below). An easy example (Example 1.7 below) shows that the weak-star closedness assumption is indeed crucial.

Once we have established these results, the natural question arises whether we can formulate a "qualitative dual" version. Suppose that M is a weak-star closed convex set of Radon probability measures on a compact space K such that the *set* M is absolutely continuous with respect to the Radon probability measure $\mathbb{P}$ in the following sense: for each $\varepsilon > 0$ there is $\delta > 0$ such that, for each Borel set B in K with $\mathbb{P}(B) < \delta$, there is *some* $Q \in M$ (depending on B)

with $Q(B) < \varepsilon$. Can we assert that this assumption implies an interchange of quantifiers, that is, that there is some $Q_0 \in M$ which is absolutely continuous with respect to $\mathbb{P}$? The answer turns out to be: not quite but almost. Not quite, because we can construct a counterexample (see Example 1.9 below) where there is no $\mathbb{P}$-absolutely continuous measure $Q_0 \in M$. Almost, on the other hand, because we can prove that—under the above assumption—we can find, for each $\eta > 0$, an element $Q_\eta \in M$ such that its $\mathbb{P}$-absolutely continuous part has mass at least $1 - \eta$ (and therefore its $\mathbb{P}$-singular part has mass less than η) (Theorem 1.8 below).

The motivation for this work comes from a problem in mathematical finance involving the notions of "large financial markets" and "asymptotic arbitrage" introduced by Kabanov and Kramkov [5]. In [7], we were able to give a characterization of the absence of asymptotic arbitrage in a general setting, thus solving a problem posed by Kabanov and Kramkov. A joyful discussion with D. Kramkov, which is gratefully acknowledged, drew our attention to the fact that the crucial steps of our arguments in [7] lead quite naturally to the two "quantitative" versions of the Halmos–Savage theorem above. In the third section we show how to deduce the two theorems on "no asymptotic arbitrage" from the quantitative versions of the Halmos–Savage theorem, thus pointing out the intimate link between the two topics.

Let us point out that the link between the Halmos–Savage theorem and questions concerning the existence of equivalent martingale measures is not too big a surprise. Indeed, in mathematical finance similar exhaustion arguments as in the original Halmos–Savage theorem (Theorem 1.1) have been repeatedly used; compare [8], [10] (Theorem 1.1), [12] and [13]. One could even say the theorem was "reproved" several times.

The paper is organized as follows: in Section 1 we present the main results and some examples showing the limitations of the theorems. Section 2 contains the proofs of the results of Section 1. In the final section we apply our results to the above-mentioned topic of mathematical finance.

1. Results. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and M a set consisting of probability measures that are absolutely continuous with respect to $\mathbb{P}$. We shall always identify $\mathbb{P}$-absolutely continuous measures Q with their Radon–Nikodym derivatives $dQ/d\mathbb{P}$, so that we may view M as a subset of $L^1(\Omega, \mathscr{F}, \mathbb{P})$. We say that M is closed under countable convex combinations if, whenever $(Q_n)_{n=1}^\infty$ is a sequence of measures in M, we have that $\sum_{n=1}^\infty c_n Q_n \in M$, where $c_n \geq 0$ and $\sum_{n=1}^\infty c_n = 1$. We recall a version of the well-known theorem of Halmos and Savage (compare [2], Lemma 7).

THEOREM 1.1 (Halmos–Savage theorem). *Let M be a set of $\mathbb{P}$-absolutely continuous probability measures on $\mathscr{F}$ that is closed under countable convex combinations. Suppose that, for each set $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$, there exists $Q \in M$ with $Q(A) > 0$. Then there exists $Q_0 \in M$ such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$, we have that $Q_0(A) > 0$; that is, Q_0 and $\mathbb{P}$ are equivalent probability measures.*

In order to be able to compare it to the quantitative version which we present below (see Proposition 1.3), we give a formulation of Theorem 1.1 involving ε and δ (see Theorem 1.2). For the sake of completeness, we reproduce in Section 2 the well-known proofs of Theorems 1.1 and 1.2.

THEOREM 1.2 (Halmos–Savage theorem, reformulated). *Let M be a set of $\mathbb{P}$-absolutely continuous probability measures on $\mathscr{F}$ that is closed under countable convex combinations. Suppose that there is a function $\varepsilon \mapsto \delta(\varepsilon) > 0$ such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) > \varepsilon$, there exists $Q \in M$ with $Q(A) > \delta(\varepsilon)$. Then there is $Q_0 \in M$ and a function $\varepsilon \mapsto \delta'(\varepsilon) > 0$ such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) > \varepsilon$, we have that $Q_0(A) > \delta'(\varepsilon)$.*

The theorem is "qualitative" in the sense that it asserts the existence of a strictly positive function $\varepsilon \mapsto \delta'(\varepsilon)$ but it does not allow us to draw "quantitative" conclusions which give estimates on the decay of $\delta'(\varepsilon)$ in terms of the decay of the function $\delta(\varepsilon)$. Such a quantitative control is given by the next result, which—together with its dual counterpart (Proposition 1.5)—is the central result of this paper.

PROPOSITION 1.3 (Quantitative version of the Halmos–Savage theorem). *For fixed $\varepsilon > 0$ and $\delta > 0$, the following statement is true: let M be a convex set of $\mathbb{P}$-absolutely continuous probability measures such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) > \varepsilon$, there is $Q \in M$ with $Q(A) > \delta$. Then there is $Q_0 \in M$ such that, for all $A \in \mathscr{F}$ with $\mathbb{P}(A) > 4\varepsilon$, we have that $Q_0(A) > \varepsilon^2\delta/2$.*

We remark that the "quantitative" version is more precise than the (original) "qualitative" one, since the proposition immediately implies the Halmos–Savage theorem. We sketch the easy argument. Under the assumptions of Theorem 1.2, choose for all $n \in \mathbb{N}$ some δ_n such that, for each set $A \in \mathscr{F}$ with $\mathbb{P}(A) > 2^{-n}$, there exists $Q \in M$ with $Q(A) > \delta_n$. Since Proposition 1.3 holds for all fixed ε, δ, we can apply it to get, for all n, a measure $Q_n \in M$ such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) > 2^{-n+2}$, we have that $Q_n(A) > 2^{-2n-1}\delta_n$. Define $Q_0 = \sum_{n=1}^{\infty} 2^{-n} Q_n$. Then Q_0 is in M and for each set A with $\mathbb{P}(A) > 0$ we have that $Q_0(A) > 0$.

Note that in Proposition 1.3 we need only the convexity of the set M but we do not need that M is closed under countable convex combinations. But, of course, we cannot do without the convexity assumption on M as the following simple example shows.

EXAMPLE 1.4. Fix $\varepsilon > 0$. Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}, \lambda)$, where $\mathscr{B}$ is the family of Borel sets of $[0,1]$ and λ is the Lebesgue measure. Define

$$M = \left\{ Q = \frac{\chi_A}{\lambda(A)}\lambda, \text{ where } 0 < \lambda(A) < 1 - 4\varepsilon \right\}.$$

Then M satisfies the assumptions of Proposition 1.3 (except convexity) for ε as above and an arbitrary $1 \geq \delta > 0$, since for $A \in \mathscr{B}$ with $\lambda(A) > \varepsilon$ we can find a subset $B \subseteq A$ with $0 < \lambda(B) < 1 - 4\varepsilon$. If we take $Q = [\chi_B/\lambda(B)]\lambda$ we have that $Q(A) = Q(B) = 1 \geq \delta$. On the other hand, there does not exist Q_0 satisfying the assertion of Proposition 1.3; for $Q_0 \in M$ we can always find $A \in \mathscr{B}$ with $\lambda(A) < 1 - 4\varepsilon$ and

$$Q_0 = \frac{\chi_A}{\lambda(A)}\lambda,$$

whence we have, for $B = \Omega \setminus A$, $\lambda(B) > 4\varepsilon$ but $Q_0(B) = 0 < \varepsilon^2\delta/2$.

Now we formulate a dual version of the quantitative Halmos–Savage theorem. We do this under the same assumptions as above (Proposition 1.3) by interchanging the roles of the measure $\mathbb{P}$ and the measure $Q \in M$. We will see that Proposition 1.5 below again is essentially an interchange of quantifiers.

PROPOSITION 1.5 (Dual quantitative version of the Halmos–Savage theorem). *For fixed $\varepsilon > 0$ and $\delta > 0$, the following statement is true: let M be a convex set of $\mathbb{P}$-absolutely continuous probability measures such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) < \delta$, there is $Q \in M$ with $Q(A) < \varepsilon$. Then there is $Q_0 \in M$ such that, for all $A \in \mathscr{F}$ with $\mathbb{P}(A) < 2\varepsilon\delta$, we have that $Q_0(A) < 8\varepsilon$.*

As mentioned in the Introduction the natural framework for the "dual version" of the Halmos–Savage theorem is to drop the $\mathbb{P}$-absolute continuity assumption on the set M. We will use a "topological setting" to realize the above idea in a precise and convenient way. Assume that $(\Omega, \mathscr{F}, \mathbb{P}) = (K, \mathscr{B}, \mathbb{P})$, where K is a metrizable compact space, $\mathscr{B}$ the σ-algebra of the Borel sets in K and $\mathbb{P}$ a Radon probability measure. We denote by $\mathscr{C}(K)$ the Banach space of continuous functions on K and by $\mathscr{M}(K)$ its dual space, which may be identified with the space of Radon measures on K. We equip $\mathscr{M}(K)$ with the usual variation norm, which we denote by $\|\cdot\|_1$. We also identify $L^1(\mathbb{P})$ with the subspace of $\mathscr{M}(K)$ formed by the $\mathbb{P}$-absolutely continuous measures.

Suppose that M is a convex set of Radon probability measures (not necessarily absolutely continuous with respect to $\mathbb{P}$). Under the additional assumption that M is closed for the topology $\sigma(\mathscr{M}(K), \mathscr{C}(K))$, we can extend Proposition 1.5 to this setting.

PROPOSITION 1.6. *For fixed $\varepsilon > 0$ and $\delta > 0$, the following statement is true: let M be a $\sigma(\mathscr{M}(K), \mathscr{C}(K))$-closed convex subset of $\mathscr{M}(K)_+$ consisting of probability measures Q (not necessarily absolutely continuous with respect to $\mathbb{P}$) such that, for each Borel set B with $\mathbb{P}(B) < \delta$, there is $Q \in M$ with $Q(B) < \varepsilon$. Then there is $Q_0 \in M$ such that, for each Borel set B with $\mathbb{P}(B) < 2\varepsilon\delta$, we have that $Q_0(B) < 8\varepsilon$.*

We cannot drop the assumption in Proposition 1.6 that M is weak-star closed, as is shown by the simple example below.

EXAMPLE 1.7. Let again $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}, \lambda)$ and define M to be the countably convex hull of the Dirac measures $\delta_{\{x\}}$ for $x \in [0,1]$, that is,

$$M = \left\{ \sum_{i=1}^{\infty} c_i \delta_{\{x_i\}},\ x_i \in [0,1],\ c_i \geq 0,\ \sum_{i=1}^{\infty} c_i = 1 \right\}.$$

Clearly, M is convex and norm closed in $\mathscr{M}[0,1]$, but not closed with respect to $\sigma(\mathscr{M}[0,1], \mathscr{C}[0,1])$. Moreover, M satisfies the ε–δ-assumption of Proposition 1.6 for each $0 < \varepsilon < 1/8$ and $0 < \delta < 1$. Indeed, let B be a Borel set such that $\lambda(B) < \delta$. Of course, there exists a point $x_0 \in [0,1] \setminus B$ such that $\delta_{\{x_0\}} \in M$ trivially satisfies

$$\delta_{\{x_0\}}(B) = 0 < \varepsilon.$$

Suppose now that there exists a measure $\mu_0 \in M$ satisfying the assertion of the proposition. Then μ_0 can be written as

$$\mu_0 = \sum_{i=1}^{\infty} c_i \delta_{\{x_i\}}.$$

We clearly have that $\lambda(\{x_i\}_{i=1}^{\infty}) = 0 < 2\varepsilon\delta$ but $\mu_0(\{x_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i = 1 \geq 8\varepsilon$, a contradiction.

Now we can present a "qualitative dual" version of the Halmos–Savage theorem in the framework of Proposition 1.6.

THEOREM 1.8. *Let M be a $\sigma(\mathscr{M}(K), \mathscr{C}(K))$-closed convex subset of $\mathscr{M}(K)_+$ consisting of probability measures Q. Suppose that for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each Borel set B with $\mathbb{P}(B) < \delta$, there is $Q \in M$ with $Q(B) < \varepsilon$. Then we have that*

$$\operatorname{dist}_{\|\cdot\|_1}(M, L^1(\mathbb{P})) = 0.$$

So we "nearly" get a $\mathbb{P}$-absolutely continuous measure in the sense that for each $\eta > 0$ we can find a measure $Q_\eta \in M$ such that its $\mathbb{P}$-singular part has mass less than η. But the following example shows that, in general, we cannot find a measure $Q_0 \in M$ that has full support, that is, that is really $\mathbb{P}$-absolutely continuous.

EXAMPLE 1.9. Let $K = [0,1]$, $\mathscr{B}$ the Borel sets on $[0,1]$ and λ the Lebesgue measure on $[0,1]$. There is a set M satisfying the assumptions of Theorem 1.8 such that

$$\operatorname{dist}_{\|\cdot\|_1}(M, L^1([0,1], \mathscr{B}, \lambda)) = 0$$

but

$$M \cap L^1([0,1], \mathscr{B}, \lambda) = \varnothing.$$

The example shows that, although we can find elements in M that are arbitrarily close to $L^1(K, \mathscr{B}, \mathbb{P})$, we cannot find a $\mathbb{P}$-absolutely continuous element of M.

2. Proofs. We start this section by giving explicit proofs of Theorems 1.1 and 1.2, thus recalling the original exhaustion argument of Halmos and Savage.

PROOF OF THEOREM 1.1. Let $\mathscr{S}$ be the family of (equivalence classes) of subsets of Ω formed by the supports of the measures $Q \in M$. Note that $\mathscr{S}$ is closed under countable unions, since M is closed under countable convex combinations. Hence there is $Q_0 \in M$ such that for $S_0 = \{dQ_0/d\mathbb{P} > 0\}$ we have

$$\mathbb{P}(S_0) = \sup\{\mathbb{P}(S): S \in \mathscr{S}\}.$$

We now claim that $\mathbb{P}(S_0) = 1$, which shows that Q_0 has full support; that is, Q_0 and $\mathbb{P}$ are equivalent probability measures. Indeed, if $\mathbb{P}(S_0) < 1$, then, by assumption, we can find $Q_1 \in M$ such that $Q_1(\Omega \setminus S_0) > 0$. Hence, if we take a convex combination of Q_0 and Q_1, $Q := (Q_0 + Q_1)/2 \in M$, we have that the support of Q has $\mathbb{P}$-measure strictly larger than $\mathbb{P}(S_0)$, a contradiction. □

PROOF OF THEOREM 1.2. We shall deduce Theorem 1.2 from Theorem 1.1. Let M be as in Theorem 1.2; clearly, M satisfies the assumptions of Theorem 1.1. Therefore, there exists $Q_0 \in M$ such that, for all sets $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$, we have that $Q_0(A) > 0$. Define $\mathscr{A}^n = \{A \in \mathscr{F}: Q_0(A) \le 2^{-n}\}$ and let $\varepsilon_n = \sup_{A \in \mathscr{A}^n} \mathbb{P}(A)$. We claim that $\lim_{n \to \infty} \varepsilon_n = 0$. Indeed, for each $n \in \mathbb{N}$, we can find $A^n \in \mathscr{A}^n$ such that $\mathbb{P}(A^n) + 2^{-n} > \varepsilon_n$. Therefore, we have that

$$\lim_{n \to \infty} \varepsilon_n \le \limsup \mathbb{P}(A^n) \le \mathbb{P}(A),$$

where $A = \bigcap_{m=1}^{\infty} \bigcup_{n \ge m} A^n$. But for all $m_0 \in \mathbb{N}$ we have that

$$Q_0(A) \le Q_0\Bigl(\bigcup_{n \ge m_0+1} A^n\Bigr) < \sum_{n=m_0+1}^{\infty} 2^{-n} = 2^{-m_0};$$

hence $Q_0(A) = 0$ and therefore $\mathbb{P}(A) = 0$ which implies the claim.

For any set $A \in \mathscr{F}$ with $\mathbb{P}(A) > \varepsilon_n$, we have that $Q_0(A) > 2^{-n}$. Fix $\varepsilon > 0$; then we can find $n \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$ and if we let $\delta = 2^{-n}$ we are finished. □

We now turn to the central topic of the paper, the proof of the quantitative version of the Halmos–Savage theorem.

PROOF OF PROPOSITION 1.3. Fix $\varepsilon > 0$ and $\delta > 0$. Let $\tilde{M}$ be the set of elements f of $L^1(\mathbb{P})$ dominated by some element of M, that is,

$$\tilde{M} = M - L^1_+(\mathbb{P}) = \bigl\{f \in L^1(\mathbb{P}): \exists\, f_1 \in M,\, f_2 \in L^1_+(\mathbb{P}),\, f = f_1 - f_2\bigr\}.$$

Define the set

$$D = \left\{ g \in L^1_+ \, , \mathbb{E}[\, g\,] = 1, \, g \geq \frac{\varepsilon\delta}{2} \ \mathbb{P}\text{-a.s.} \right\}.$$

Suppose that the assertion of the proposition is false; that is, for all $f \in M$ there is $A \in \mathscr{F}$ with $\mathbb{P}(A) > 4\varepsilon$ but $\mathbb{E}[\, f\chi_A] \leq \varepsilon^2\delta/2$, and let us work toward a contradiction. We claim that

$$\operatorname{dist}_{\|\cdot\|_{L^1}}(\tilde{M}, D) \geq \varepsilon^2\delta.$$

Let $f = f_1 - f_2$, where $f_1 \in M$ and $f_2 \in L^1_+(\Omega, \mathscr{F}, \mathbb{P})$, and let $g \in D$. For f_1 there exists a set $A \in \mathscr{F}$ such that $\mathbb{P}(A) > 4\varepsilon$ but $\mathbb{E}[\, f_1\, \chi_A] \leq \varepsilon^2\delta/2$. This implies that

$$\begin{aligned} \|f - g\|_{L^1} &\geq \left| \mathbb{E}[\, g\, \chi_A] - \mathbb{E}[\, f_1\, \chi_A] + \mathbb{E}[\, f_2\, \chi_A] \right| \\ &\geq \frac{\varepsilon\delta}{2} 4\varepsilon - \frac{\varepsilon^2\delta}{2} > \varepsilon^2\delta. \end{aligned}$$

By the Hahn–Banach theorem there exists $h \in L^\infty(\Omega, \mathscr{F}, \mathbb{P})$, $\|h\|_{L^\infty} \leq 1$ such that

$$(*) \qquad \sup_{f \in \tilde{M}} \mathbb{E}[\, hf\,] \leq \inf_{g \in D} \mathbb{E}[\, hg\,] - \varepsilon^2\delta.$$

We have that $h \geq 0$ since $-L^1_+(\Omega, \mathscr{F}, \mathbb{P}) \subset \tilde{M}$; in particular, $\sup_{f \in \tilde{M}} \mathbb{E}[\, hf\,] = \sup_{f \in M} \mathbb{E}[\, hf\,]$. The elements of M are nonnegative and therefore we have that

$$\sup_{f \in M} \mathbb{E}[\, hf\,] \geq 0, \quad \text{whence } \inf_{g \in D} \mathbb{E}[\, hg\,] \geq \varepsilon^2\delta.$$

Without loss of generality, we can assume that $\operatorname{ess\,inf}_\omega h(\omega) = 0$ [otherwise we replace h by $h - \operatorname{ess\,inf}_\omega h(\omega)$, which will not change inequality $(*)$ since $\mathbb{E}[\, f\,] = \mathbb{E}[\, g\,] = 1$ for $f \in M$ and $g \in D$]. We claim that this implies that

$$(**) \qquad \inf_{g \in D} \mathbb{E}[\, hg\,] = \frac{\varepsilon\delta}{2} \mathbb{E}[\, h\,].$$

Indeed, define for each $n \in \mathbb{N}$ a function $g_n \in D$ by

$$g_n = \frac{\varepsilon\delta}{2} \chi_{\{h > 1/n\}} + c_n\, \chi_{\{h \leq 1/n\}},$$

where $c_n \geq 1$ is chosen such that $\mathbb{E}[\, g_n] = 1$. Then we have that

$$\mathbb{E}[\, hg_n] \leq \frac{1}{n} + \frac{\varepsilon\delta}{2} \mathbb{E}[\, h\,],$$

whence $\inf_{g \in D} \mathbb{E}[\, hg\,] \leq (\varepsilon\delta/2)\mathbb{E}[\, h\,]$. The reverse inequality holds trivially so we proved $(**)$.

So we see that $(\varepsilon\delta/2)\mathbb{E}[\, h\,] \geq \varepsilon^2\delta$, whence $\mathbb{E}[\, h\,] \geq 2\varepsilon$. But since $\|h\|_{L^\infty} \leq 1$ we have that $\mathbb{P}(h > \varepsilon) > \varepsilon$. This implies that there is $f \in M$ with $\mathbb{E}[\, f\chi_{\{h > \varepsilon\}}] > \delta$ and therefore

$$\mathbb{E}[\, hf\,] \geq \varepsilon \mathbb{E}\left[\, f\chi_{\{h > \varepsilon\}}\right] > \varepsilon\delta.$$

On the other hand, we get from $(*)$ and $(**)$:

$$\mathbb{E}[hf] \leq \frac{\varepsilon\delta}{2}\mathbb{E}[h] - \varepsilon^2\delta < \varepsilon\delta,$$

a contradiction. □

We omit the proof of Proposition 1.5 since it is similar to (and slightly easier than) the proof of Proposition 1.6 which we will present now.

PROOF OF PROPOSITION 1.6. Fix $\frac{1}{8} > \varepsilon > 0$, $\delta > 0$. Let $\hat{M}$ be the cone generated by M. Then we claim that by Krein and Smulian (see [4], Theorem 2, page 246) $\hat{M}$ is again $\sigma(\mathscr{M}(K), \mathscr{C}(K))$-closed. Indeed, we have to show that, for $c \in \mathbb{R}_+$, the intersection of $\hat{M}$ with the closed ball of $\mathscr{M}(K)$ around 0 and with radius c is $\sigma(\mathscr{M}(K), \mathscr{C}(K))$-compact. Take a sequence $(c_n Q_n)_{n \geq 1} \in \hat{M}$, where $c_n \geq 0$ and bounded by a constant c and $Q_n \in M$. Then we can find a subsequence such that $c_n \to c_0$ for some $c_0 \leq c$. Moreover, since M is σ^*-compact in $\mathscr{M}(K)$ [M is a σ^*-closed subset of the closed unit ball of $\mathscr{M}(K)$], we can choose a subsequence of $(Q_n)_{n \geq 1}$ that converges to $Q_0 \in M$ with respect to $\sigma(\mathscr{M}(K), \mathscr{C}(K))$. Putting things together, the claim is proved.

Define the set

$$D := \left\{ g \in L^1_+(K, \mathscr{B}, \mathbb{P}), \|g\|_{L^1} = 1, \|g\|_{L^\infty} \leq \frac{1}{\delta} \right\} \subset \mathscr{M}(K).$$

We claim that the set D is $\sigma(L^1(K, \mathscr{B}, \mathbb{P}), \mathscr{C}(K)) = \sigma(\mathscr{M}(K), \mathscr{C}(K))|_{L^1}$-compact. Indeed, D is a $\sigma(L^\infty(K, \mathscr{B}, \mathbb{P}), L^1(K, \mathscr{B}, \mathbb{P}))$-compact subset of $L^\infty(K, \mathscr{B}, \mathbb{P})$ and therefore D is $\sigma(L^1(K, \mathscr{B}, \mathbb{P}), L^\infty(K, \mathscr{B}, \mathbb{P}))$-compact in $L^1(K, \mathscr{B}, \mathbb{P})$, whence it is $\sigma(L^1(K, \mathscr{B}, \mathbb{P}), \mathscr{C}(K))$-compact.

Suppose that for all $Q \in M$ there exists a set $B \in \mathscr{B}$ with

$$\mathbb{P}(B) < 2\varepsilon\delta \quad \text{but } Q(B) \geq 8\varepsilon.$$

We claim that

$$(*) \qquad \operatorname{dist}_{\|\cdot\|_1}(\hat{M}, D) \geq 3\varepsilon.$$

We have to show that, for all $c \in \mathbb{R}_+$,

$$\operatorname{dist}_{\|\cdot\|_1}(cM, D) \geq 3\varepsilon.$$

This is obvious for $c \notin]1 - 3\varepsilon, 1 + 3\varepsilon[$. So suppose $c \in]1 - 3\varepsilon, 1 + 3\varepsilon[$. Let Q be a measure in M and B such that $\mathbb{P}(B) < 2\varepsilon\delta$ but $Q(B) \geq 8\varepsilon$ and let g be a function in D. Then we have

$$\begin{aligned}\operatorname{dist}_{\|\cdot\|_1}(cQ, g) &\geq |cQ(B) - \mathbb{E}(g\chi_B)| \\ &> (1 - 3\varepsilon)8\varepsilon - \frac{1}{\delta}2\varepsilon\delta > 3\varepsilon,\end{aligned}$$

whence $(*)$ is proved.

Applying the Hahn–Banach theorem to separate the σ^*-compact set $D + 2\varepsilon\,\text{ball}(\mathscr{M}(K))$ from the σ^*-closed set $\hat{M}$, we find $h \in \mathscr{C}(K)$ such that $\|h\|_\infty = 1$ and

$$(**) \qquad \sup_{Q \in \hat{M}} \mathbb{E}_Q[h] \le \inf_{g \in D} \mathbb{E}_{\mathbb{P}}(hg) - 2\varepsilon.$$

As $\hat{M}$ is a cone we may conclude that the left-hand side equals 0. On the other hand, we claim that

$$(***) \qquad \mathbb{P}(\{h < 2\varepsilon\}) < \delta.$$

Indeed, suppose $\mathbb{P}(\{h < 2\varepsilon\}) = p \ge \delta$. Let $g := (1/p)\chi_{\{h < 2\varepsilon\}}$, which is in D. We have that $2\varepsilon \le \mathbb{E}_{\mathbb{P}}[hg] = (1/p)\mathbb{E}_{\mathbb{P}}[h\chi_{\{h<2\varepsilon\}}] < 2\varepsilon$, a contradiction.

The inequality $(***)$ implies that there is $Q \in M$ such that

$$Q(\{h < 2\varepsilon\}) < \varepsilon, \quad \text{whence } Q(\{h \ge 2\varepsilon\}) \ge 1 - \varepsilon.$$

Because of the last result and since $\|h\|_\infty = 1$, we have that

$$\mathbb{E}_Q[h] = \mathbb{E}_Q\big[h\chi_{\{h \ge 2\varepsilon\}}\big] + \mathbb{E}_Q\big[h\chi_{\{h < 2\varepsilon\}}\big] \ge 2\varepsilon(1-\varepsilon) - \varepsilon = \varepsilon - \varepsilon^2 > 0,$$

a contradiction to $(**)$. $\square$

PROOF OF THEOREM 1.8. We have to show that for all numbers $\eta > 0$ there is $Q_\eta \in M$, $Q_\eta = Q_\eta^a + Q_\eta^s$ with $\|Q_\eta^s\|_1 < \eta$. (Q_η^a, Q_η^s denote the $\mathbb{P}$-absolutely continuous, P-singular part of Q_η, respectively.)

Fix $\eta > 0$ and choose $\varepsilon > 0$ such that $8\varepsilon < \eta$. By assumption, there exists $\delta > 0$ such that, for all Borel sets B with $\mathbb{P}(B) < \delta$, there exists $Q \in M$ with $Q(B) < \varepsilon$. Proposition 1.6 implies the existence of a measure $Q_0 \in M$ such that, for all Borel sets B with $\mathbb{P}(B) < 2\varepsilon\delta$, we have that $Q_0(B) < 8\varepsilon < \eta$.

Since Q_0^s and $\mathbb{P}$ are mutually singular, we can find a Borel set B_0 such that $\mathbb{P}(B_0) = 0$ and $Q_0(B_0) = \|Q_0^s\|_1$. Because $\mathbb{P}(B_0) = 0 < 2\varepsilon\delta$ we have that

$$\|Q_0^s\|_1 = Q_0(B_0) < 8\varepsilon < \eta. \qquad \square$$

CONSTRUCTION OF EXAMPLE 1.9. Let $K = [0,1]$, let $\mathscr{B}$ be the Borel sets on $[0,1]$ and let λ be the Lebesgue measure on $[0,1]$. Define for each $n \in \mathbb{N}$ a Radon measure

$$\mu_n = \frac{1}{n+1}\delta_{\{0\}} + \eta\chi_{[0,1/(n+1)]}\lambda,$$

and let $\mu_0 = \delta_{\{0\}}$. Clearly, $(\mu_n)_{n=0}^\infty$ converges to μ_0 in the weak-star topology of $\mathscr{M}[0,1]$, that is, in $\sigma(\mathscr{M}[0,1], \mathscr{C}[0,1])$. It easily follows from Choquet's theorem [9] that the weak-star closed convex hull of $(\mu_n)_{n=0}^\infty$ is given by the barycenters of the probability measures on $(\mu_n)_{n=0}^\infty$; that is,

$$\begin{aligned} M &= \overline{\text{conv}}^*\big((\mu_n)_{n=0}^\infty\big) \\ &= \left\{ \sum_{n=0}^\infty c_n\mu_n,\ c_n \ge 0,\ \sum_{n=0}^\infty c_n = 1 \right\}. \end{aligned}$$

We claim that for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all sets $B \in \mathscr{B}$ with $\lambda(B) < \delta$, there exists $\mu \in M$ such that $\mu(B) < \varepsilon$.

Indeed, fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $N > 2/\varepsilon - 1$. Define $\delta = \varepsilon/2N$. Take any set B with $\lambda(B) < \delta$. If $0 \notin B$ it is trivial that $\mu = \delta_{\{0\}}$ satisfies $\mu(B) = 0 < \varepsilon$. So suppose $0 \in B$. Then

$$\mu_N = \frac{1}{N+1}\delta_{\{0\}} + N\chi_{[0,1/(N+1)]}\lambda \in M$$

satisfies

$$\mu_N(B) \le \frac{1}{N+1} + N\delta < \varepsilon,$$

which proves our claim. Clearly, we have that

$$\operatorname{dist}_{\|\cdot\|_1}\bigl(M, L^1([0,1],\mathscr{B},\lambda)\bigr) = 0$$

(which, of course, we know already from Theorem 1.8). But it is just as obvious that we cannot find a λ-absolutely continuous element of M; that is,

$$M \cap L^1([0,1],\mathscr{B},\lambda) = \varnothing.$$

3. Application to mathematical finance. We recall the model of a large financial market introduced by Kabanov and Kramkov [5]. In contrast to the usual setting in mathematical finance, we do not consider one single stochastic stock price process S based on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in I}, \mathbb{P})$ but rather a sequence $(S^n)_{n\ge 1}$ of such processes based on a sequence $(\Omega^n, \mathscr{F}^n, (\mathscr{F}_t^n)_{t\in I^n}, \mathbb{P}^n)_{n\ge 1}$ of filtered probability spaces. The interpretation is that an investor can invest not only in one stock exchange but in several (countably many) stock exchanges.

The usual notion of arbitrage then has to be replaced by "asymptotic" arbitrage concepts, where one has to distinguish between two different kinds which where introduced by Kabanov and Kramkov [5] (see Definitions 3.1 and 3.2 below). If for each $n \in \mathbb{N}$ the market is complete, that is, if there is exactly one local martingale measure Q^n for the process S^n on $\mathscr{F}^n$ which is equivalent to $\mathbb{P}^n$, then Kabanov and Kramkov showed that the contiguity of $(\mathbb{P}^n)_{n\ge 1}$ with respect to $(Q^n)_{n\ge 1}$ (respectively, vice versa) is equivalent to the absence of asymptotic arbitrage of the first (respectively, second) kind.

In [7], we extended this result to the noncomplete case, that is, where for each $n \in \mathbb{N}$ the set of equivalent local martingale measures for the process S^n is nonempty but not necessarily a singleton (under the assumption that each S^n is a locally bounded semimartingale). In the case of asymptotic arbitrage of the first kind, the theorem of Kabanov and Kramkov extends without any modification of the statement to the noncomplete setting. On the other hand, for the theorem concerning asymptotic arbitrage of the second kind, we have to replace the contiguity condition by a rather technical condition that essentially states that the chosen sequence of measures "depends on ε" (see [6]). But let us start being precise. Recall the definitions of asymptotic arbitrage of the first and second kind, respectively (compare [5] and [7]).

DEFINITION 3.1. A sequence $(H^n)_{n=1}^{\infty}$ of admissible trading strategies (for the notion of admissibility, compare [1] or [3]) realizes asymptotic arbitrage of the first kind iff there are sequences $c_n > 0$ and $C_n > 0$ with $\lim_{n\to\infty} c_n = 0$ and $\limsup_{n\to\infty} C_n = \infty$ such that:

(a) $(H^n \cdot S^n)_t \geq -c_n$ for all $t \in \mathbb{R}_+$; that is, H^n is c_n-admissible.
(b) $\limsup_{n\to\infty} \mathbb{P}^n((H^n \cdot S^n)_\infty \geq C_n) > 0$.

Asymptotic arbitrage of the first kind describes the possibility of getting arbitrarily rich with positive probability by taking an arbitrarily small (vanishing) risk.

DEFINITION 3.2. A sequence of admissible trading strategies realizes asymptotic arbitrage of the second kind iff:

(a) $(H^n \cdot S^n)_t \geq -1$, for all $t \in \mathbb{R}_+$; that is, H^n is 1-admissible.
(b) $\exists\, c > 0$, such that $\limsup_{n\to\infty} \mathbb{P}^n((H^n \cdot S^n)_\infty \geq c) = 1$.

Asymptotic arbitrage of the second kind can be interpreted as an opportunity of gaining at least something (the amount c) with probability arbitrarily close to 1 by taking a uniformly bounded risk (bounded by 1).

A large financial market satisfies no asymptotic arbitrage of the first, respectively, second kind iff it does not allow the respective arbitrage opportunities.

Before we repeat Theorems 2.1 and 2.2 of [8] we have to define the set of all equivalent local-martingale probability measures Q^n on $\mathscr{F}^n$ for the locally bounded process S^n:

$$\mathscr{M}^e(\mathbb{P}^n) = \{Q^n \sim \mathbb{P}^n,\ S^n \text{ local } Q^n\text{-martingale}\}.$$

We recall the definition of contiguity of sequences of measures: we say that $(\mathbb{P}^n)_{n\geq 1}$ is contiguous with respect to $(Q^n)_{n\geq 1}$, that is, $(\mathbb{P}^n)_{n\geq 1} \lhd (Q^n)_{n\geq 1}$, iff for all sequences $A^n \in \mathscr{F}^n$ with $\lim_{n\to\infty} Q^n(A^n) = 0$ we have that $\lim_{n\to\infty} \mathbb{P}^n(A^n) = 0$.

THEOREM 3.3. *There is no asymptotic arbitrage of the first kind if and only if there exists a sequence $(Q^n)_{n\geq 1}$, $Q^n \in \mathscr{M}^e(\mathbb{P}^n)$ for all n, such that $(P^n)_{n\geq 1} \lhd (Q^n)_{n\geq 1}$.*

THEOREM 3.4. *There is no asymptotic arbitrage of the second kind if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ and measures $Q^n \in \mathscr{M}^e(\mathbb{P}^n)$ such that, for any set $A^n \in \mathscr{F}^n$ with $\mathbb{P}^n(A^n) < \delta$, we have that $Q^n(A^n) < \varepsilon$.*

We now show that these results may quickly be deduced from Propositions 1.3 and 1.5 above. From the general duality theorem in [1] (Theorem 5.7), we can deduce the following characterizations of the absence of asymptotic arbitrage of the first and second kind, respectively. Thanks go to D. Kramkov for pointing out to us this connection.

LEMMA 3.5. (i) *There is no asymptotic arbitrage of the first kind iff for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that, for any set* $A^n \in \mathscr{F}^n$, $\mathbb{P}^n(A^n) \geq \varepsilon$, *there is* Q^n (*depending on* A^n) *with* $Q^n(A^n) \geq \delta$.

(ii) *There is no asymptotic arbitrage of the second kind iff for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that, for any set* $A^n \in \mathscr{F}^n$, $\mathbb{P}^n(A^n) < \delta$, *there is* Q^n (*depending on* A^n) *with* $Q^n(A^n) < \varepsilon$.

PROOF. (i) ($\Rightarrow$) Assume there is $\varepsilon > 0$ such that, for all $\delta > 0$, there exists a set $A^n \in \mathscr{F}^n$ with $\mathbb{P}^n(A^n) \geq \varepsilon$ but $\sup_{Q^n \in \mathscr{M}^e(P^n)} Q^n(A^n) < \delta$. Considering χ_{A^n} as a contingent claim, we know from [1], Theorem 5.7, that

$$\sup_{Q^n \in \mathscr{M}^e(\mathbb{P}^n)} \mathbb{E}_{Q^n}[A^n] = \inf\{x \in \mathbb{R}: x + (H^n \cdot S^n)_\infty \geq \chi_{A^n}\},$$

where H^n ranges through the admissible integrands for S^n. Therefore, there exists an admissible integrand H^n such that

$$\delta + (H^n \cdot S^n)_\infty \geq \chi_{A^n},$$

which immediately gives asymptotic arbitrage of the first kind, since

$$\mathbb{P}^n((H^n \cdot S^n)_\infty \geq 1 - \delta) \geq \varepsilon \quad \text{and} \quad (H^n \cdot S^n) \geq -\delta;$$

that is, H^n is δ-admissible.

($\Leftarrow$) Now suppose that for each $\varepsilon > 0$ there is $\delta > 0$ such that, for any set $A^n \in \mathscr{F}^n$, $\mathbb{P}^n(A^n) \geq \varepsilon$, there exists $Q^n \in \mathscr{M}^e(\mathbb{P}^n)$ with $Q^n(A^n) \geq \delta$. Assuming that there is asymptotic arbitrage of the first kind, we shall work toward a contradiction. By assumption, there is $\varepsilon > 0$ and a sequence of c_n-admissible integrands H^n such that, for all n,

$$\mathbb{P}^n((H^n \cdot S^n)_\infty \geq C_n) \geq \varepsilon$$

for some sequences $c_n \to 0$ and $C_n \to \infty$. Again, by hypothesis, there is $\delta > 0$ and $Q^n \in \mathscr{M}^e(P^n)$ such that $Q^n((H^n \cdot S^n)_\infty \geq C_n) \geq \delta$. Since $(H^n \cdot S^n)_\infty \geq -c_n$ we have, for all n,

$$\mathbb{E}_{Q^n}[(H^n \cdot S^n)_\infty] \geq C_n Q^n((H^n \cdot S^n)_\infty \geq C_n) - c_n > \delta - c_n.$$

So for n large enough (i.e., $c_n < \delta$), we have that $\mathbb{E}_{Q^n}[(H^n \cdot S^n)_\infty] > 0$, a contradiction, since $(H^n \cdot S^n)$ is a Q^n-supermartingale.

(ii) ($\Rightarrow$) Assume there is $\varepsilon > 0$ such that, for all $\delta > 0$, there exists a set $A^n \in \mathscr{F}^n$ with $\mathbb{P}^n(A^n) < \delta$ but $\inf_{Q^n \in \mathscr{M}^e(\mathbb{P}^n)} Q^n(A^n) \geq \varepsilon$. If we let $B^n = \Omega^n \setminus A^n$, we have that $\mathbb{P}^n(B^n) > 1 - \delta$ and $\sup_{Q^n \in \mathscr{M}^e(\mathbb{P}^n)} Q^n(B^n) < 1 - \varepsilon$. As above we can apply the duality theorem ([1], Theorem 5.7) to see that

$$\sup_{Q^n \in \mathscr{M}^e(\mathbb{P}^n)} \mathbb{E}_{Q^n}[A^n] = \inf\{x \in \mathbb{R}: x + (H^n \cdot S^n)_\infty \geq \chi_{A^n}\},$$

where H^n ranges through the admissible integrands for S^n. Therefore, there exists an admissible integrand H^n such that

$$1 - \varepsilon + (H^n \cdot S^n)_\infty \geq \chi_{B^n},$$

which immediately gives an asymptotic arbitrage since $\mathbb{P}^n((H^n \cdot S^n)_\infty \geq \varepsilon) = \mathbb{P}^n(B^n) > 1 - \delta$ and $(H^n \cdot S^n) \geq -1 + \varepsilon$; that is, H^n is 1-admissible.

($\Leftarrow$) Suppose that for each $\varepsilon > 0$ there is $\delta > 0$ such that, for any set $A^n \in \mathscr{F}^n$, $\mathbb{P}^n(A^n) < \delta$, there exists $Q^n \in \mathscr{M}^e(\mathbb{P})$ with $Q^n(A^n) < \varepsilon$. Assuming that there is asymptotic arbitrage of the second kind, similarly as above, we shall work toward a contradiction. By assumption, there is $c > 0$ and a sequence of 1-admissible integrands H^n such that

$$\mathbb{P}^n((H^n \cdot S^n)_\infty \geq c) \to 1 \quad \text{for } n \to \infty.$$

Let $\varepsilon > 0$ be small enough such that $-\varepsilon + c(1 - \varepsilon) > 0$; for $\delta > 0$ associated with $\varepsilon > 0$ by our hypothesis, choose n large enough such that $\mathbb{P}^n((H^n \cdot S^n)_\infty < c) < \delta$. This implies that there is $Q^n \in \mathscr{M}^e(\mathbb{P}^n)$ such that $Q^n((H^n \cdot S^n)_\infty < c) < \varepsilon$ and therefore

$$\begin{aligned}\mathbb{E}_{Q^n}[(H^n \cdot S^n)_\infty] &\geq (-1)Q^n((H^n \cdot S^n)_\infty < c) + cQ^n((H^n \cdot S^n)_\infty \geq c)\\ &\geq -\varepsilon + c(1 - \varepsilon) > 0,\end{aligned}$$

a contradiction, since $(H^n \cdot S^n)$ is a Q^n-supermartingale. □

Lemma 3.5 translates the concepts of no asymptotic arbitrage of the first and second kind, respectively, to properties pertaining to the relation of $Q^n \in \mathscr{M}^e(\mathbb{P}^n)$ to $\mathbb{P}^n$. If we combine it with the quantitative versions of the Halmos–Savage theorem (Propositions 1.3 and 1.5 above), we obtain proofs for Theorems 3.3 and 3.4, respectively. Indeed, we immediately obtain Theorem 3.4 from Lemma 3.5(ii), using Proposition 1.3. To deduce Theorem 3.3 from Lemma 3.5(i), we use Proposition 1.5 to get, for each $\varepsilon > 0$, $\delta > 0$, a sequence $Q^{n,\varepsilon}$ (depending on ε) such that, for all n and $A^n \in \mathscr{F}^n$ with $Q^{n,\varepsilon}(A^n) < \delta$, we have that $\mathbb{P}^n(A^n) < \varepsilon$. To eliminate the dependence of the sequence on ε, we use a similar argument as in [7]. We have to take a countable convex combination of the ε-dependent $Q^{n,\varepsilon}$. Take $\varepsilon_m = 1/m$, $m = 1, 2, \ldots,$ and choose, for each n, the corresponding $Q^{n,\varepsilon_m} \in \mathscr{M}^e(\mathbb{P}^n)$. Define

$$Q^n = \sum_{m=1}^{\infty} 2^{-m} Q^{n,\varepsilon_m}.$$

Then $Q^n \in \mathscr{M}^e(\mathbb{P}^n)$ and similar arguments as in [7] show that $(P^n)_{n \geq 1} \lhd (Q^n)_{n \geq 1}$.

Note added in manuscript. After finishing this paper we were kindly informed by Y. Kabanov and D. Kramkov about their recent paper [6] in which these authors also provide—among other results—simplified proofs of the main results of [7]. Their arguments are based on an elegant application of the minmax theorem (which in turn may be viewed as one of the ramifications of the Hahn–Banach theorem) instead of applying directly the Hahn–Banach theorem as in [7] and the present paper. The arguments of [6] also can be adapted to furnish somewhat shorter proofs of Propositions 1.3 and 1.5 and these arguments, in fact, provide slightly better constants.

REFERENCES

[1] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **123** 463–520.

[2] HALMOS, P. R. and SAVAGE, L. J. (1949). Application of the Radon–Nikodym theorem to the theory of sufficient statistics. *Ann. Math. Statist.* **20** 225–241.

[3] HARRISON, M. and PLISKA, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260.

[4] HORVATH, J. (1966). *Topological Vector Spaces and Distributions*. Addison-Wesley, Reading, MA.

[5] KABANOV, Y. and KRAMKOV, D. (1994). Large financial markets: asymptotic arbitrage and contiguity. *Theory Probab. Appl.* **39** 222–228.

[6] KABANOV, Y. and KRAMKOV, D. (1995). Asymptotic arbitrage in large financial markets. Preprint.

[7] KLEIN, I. and SCHACHERMAYER, W. (1994). Asymptotic arbitrage in non-complete large financial markets. Preprint.

[8] KREPS, D. M. (1981). Martingales and equilibrium in economics with infinitely many commodities. *J. Math. Econom.* **8** 15–35.

[9] PHELPS, R. R. (1966). *Lectures on Choquet's Theorem*. Van Nostrand, New York.

[10] SCHACHERMAYER, W. (1994). Martingale measures for finite discrete-time processes with infinite horizon. *Math. Finance* **4** 25–55.

[11] STRASSER, H. (1985). *Mathematical Theory of Statistics*. de Gruyter, Berlin.

[12] STRICKER, C. (1990). Arbitrage et lois de martingale. *Ann. Inst. H. Poincaré Probab. Statist.* **26** 451–460.

[13] YAN, J. A. (1980). Caractérisation d'une classe d'ensembles convexes de L^1 ou H^1. *Séminaire de Probabilités XIV. Lecture Notes in Math.* **784** 220–222. Springer, New York.

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