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DIFFUSION LIMITS FOR A NONLINEAR DENSITY DEPENDENT SPACE–TIME POPULATION MODEL[1]

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A population density process is constructed using approximately Nl particles performing rate N^2 random walks between N cells distributed on the unit interval. Particles give birth or die within cells, and particle death rates are a function of the occupied cell population. With suitable scaling, two possible limiting stochastic partial differential equations are obtained. Both are nonlinear perturbations of the equation satisfied by the density process of super Brownian motion.

1. Introduction. We consider a reaction–diffusion model constructed by dividing the unit interval into N "reactor" cells of length N^{-1}. Approximately Nl particles are distributed among the cells. Particles independently perform rate N^2 simple random walks between cells. Within a cell each particle gives birth at rate $b_1 + \gamma Nl/2$ and dies at rate $d_2 n_k/l + d_1 + \gamma Nl/2$, where n_k $(1 \le k \le N)$ is the number of particles in the occupied cell and b_1, d_1, d_2, γ are fixed nonnegative parameters. Particles also immigrate into the system according to a rate $b_0 Nl$ Poisson process and are then uniformly distributed among the N cells.

We construct a step-function-valued process $X(t)$ by using the rescaled quantities $n_k(t)/l$, $1 \le k \le N$, as the values for $X(t, r)$, where $r \in [0, 1)$ tells which cell is being observed.

Since $n_k/l = (n_k/Nl)/N^{-1}$, we are essentially considering the density process formed by the ratio of fraction of total mass in a cell to the length of the cell. For the case $\gamma = 0$, it was shown in Blount (1994) that by letting $l \to \infty$ as $N \to \infty$, one obtains a deterministic limit satisfying the partial differential equation (PDE)

$$\frac{\partial \psi}{\partial t} = \Delta \psi - d_2 \psi^2 + (b_1 - d_1)\psi + b_0, \tag{1.1}$$

and if l is held constant as $N \to \infty$, then the limit satisfies the PDE

$$\frac{\partial \psi}{\partial t} = \Delta \psi - d_2 \psi^2 + \left(b_1 - d_1 - d_2 l^{-1}\right)\psi + b_0. \tag{1.2}$$

The case with $\gamma = 0$ and other reaction rates is considered in Arnold and Theodosopulu (1980), Kotelenez (1986, 1987, 1988) and the papers of Blount (1991, 1992, 1993, 1994).

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If $d_2 = 0$ (no particle interaction), $\gamma > 0$ and $Nl \to \infty$ as $N \to \infty$, then, as is well known, one obtains a process which is the density for super Brownian motion [see Dawson (1991), Walsh (1986) and Konno and Shiga (1988)] and satisfies a stochastic partial differential equation (SPDE) of the form

$$d\psi(t) = \big(\Delta\psi(t) + (b_1 - d_1)\psi(t) + b_0\big)\,dt + \sqrt{\gamma\psi(t)}\,dW(t), \tag{1.3}$$

where W is a cylindrical Brownian motion on $L_2([0,1])$. In this paper we show that when $d_2 > 0$ (particles interact) and $\gamma > 0$ we can combine these results by showing that by letting $l \to \infty$ as $N \to \infty$ one obtains a limit satisfying the SPDE

$$\begin{aligned} d\psi(t) = {} & \big(\Delta\psi(t) - d_2\psi^2(t) + (b_1 - d_1)\psi(t) + b_0\big)\,dt \\ & + \sqrt{\gamma\psi(t)}\,dW, \end{aligned} \tag{1.4}$$

and if l is held constant as $N \to \infty$, then the limit satisfies the SPDE

$$\begin{aligned} d\psi(t) = {} & \big(\Delta\psi(t) - d_2\psi^2(t) + \big(b_1 - d_1 - d_2 l^{-1}\big)\psi(t) + b_0\big)\,dt \\ & + \sqrt{\gamma\psi(t)}\,dW(t). \end{aligned} \tag{1.5}$$

In Section 3, (1.4) and (1.5) are given a precise meaning as integral equations holding in $C([0,T]: H_{-2})$, where H_{-2} is a Hilbert distribution space (see Section 2). ψ itself is continuous in r for $t > 0$.

Just as for the deterministic limits (1.1) and (1.2), convergence in distribution holds in $D([0,T]: L_2([0,1]))$ with the Skorohod topology if $l \to \infty$ as $N \to \infty$, and convergence holds in $D([0,T]: H_{-\alpha})$ for any $\alpha > 0$ if l is constant as $N \to \infty$. Loosely speaking, holding l constant requires that the cells be averaged together before obtaining the diffusion process limit. If $l \to \infty$ as $N \to \infty$, then a diffusion limit occurs in each cell. In this case one obtains the same limit as first letting $l \to \infty$, giving a system of N coupled diffusion processes, and then letting $N \to \infty$ to obtain a limit satisfying an SPDE. Some motivation for the extra linear term $d_2 l^{-1}\psi$ in (1.2) and (1.5) can be seen from the case of the deterministic limit when $\gamma = 0$ and l is constant. In this case the random walk jumps occur so much faster than reaction jumps that one expects that cell numbers at a fixed time $t > 0$ will be approximately distributed as independent Poisson random variables (that this is reasonable can be seen from the fact that only allowing random walks with no reaction gives a stationary distribution consisting of independent Poisson random variables). The extra term then reflects the fact that the mean and variance are equal for a Poisson distribution. For a related grid model this reasoning has been made more precise in Boldrighini, De Masi and Pellegrinotti (1992).

Another reason for the extra term $d_2 l^{-1}\psi$ can be seen from a modeling point of view. The contribution from interaction within a cell to a cell's death rate is the term $d_2 l^{-1} n_k^2$, which leads to (2.2)—a stochastic version of (1.1) with analogous drift terms [since $R(x) = -d_2 x^2 + (b_1 - d_1)x + b_0$]. Thus modeling the interaction term by direct analogy to the "macroscopic" model (1.1) leads to the extra term for the limiting model under a low density assumption (l constant). Writing $d_2 l^{-1} n_k^2 = d_2 l^{-1}(n_k - 1)n_k + d_2 l^{-1} n_k$ shows that at a "microscopic" level the interaction arises from pairs of distinct neighboring particles plus a "self-interaction" term, $d_2 l^{-1} n_k$. This causes the extra term and can be interpreted as a contribution to the death rate from overcrowding.

We note that an analogue of the SPDE (1.4) has been obtained in Mueller and Tribe (1993) as a limit of the long range contact process. Also, other types of SPDE's modeling reaction–diffusion phenomena have been derived in Kotelenez (1992a, b). For a related nonlinear model both Blount (1993) and Kotelenez (1988) obtain SPDE's by proving central central limit theorems in contrast to the diffusion process limits of this paper.

2. The stochastic model. Let $b(x) = b_1 x + b_0$ and $d(x) = d_2 x^2 + d_1 x$, where $b_i, d_i \geq 0$ for all i. Set $R(x) = b(x) - d(x)$ and let $\{n_k\}_1^N$ be the nonnegative integer-valued components of a jump Markov process with transition rates given by

$$\begin{aligned} n_k &\to n_k + 1 \quad \text{at rate } lb\left(n_k l^{-1}\right) + \gamma N l n_k/2, \\ n_k &\to n_k - 1 \quad \text{at rate } ld\left(n_k l^{-1}\right) + \gamma N l n_k/2, \\ (n_k, n_{k\pm 1}) &\to (n_k - 1, n_{k\pm 1} + 1) \quad \text{at rate } N^2 n_k. \end{aligned} \tag{2.1}$$

Here $l > 0$, $\gamma \geq 0$ and we view $\{n_k\}$ as a periodic sequence with period N.

Let $S = [0,1)$ and, for $1 \leq k \leq N$, let $I_k(r)$ denote the periodic extension (with period 1) of the indicator function for the interval $[(k-1)N^{-1}, kN^{-1}) \subset S$. Let H^N denote step functions of the form $\sum_{k=1}^n a_k I_k$ and define

$$X(t,r) = \sum_{k=1}^{N} n_k(t) l^{-1} I_k(r).$$

Setting $X(t) = X(t,\cdot)$ gives an H^N-valued process. Note: $X = X_{N,l}$, but we suppress the subscript l and only use N when necessary to avoid confusion.

For $f \in H^N$, let

$$\nabla^{\pm} f(r) = N\left(f(r \pm N^{-1}) - f(r)\right)$$

and set

$$\begin{aligned} \Delta_N f(r) &= -\nabla^- \nabla^+ f(r) = -\nabla^+ \nabla^- f(r) \\ &= N^2\left[f(r + N^{-1}) - 2f(r) + f(r - N^{-1})\right]. \end{aligned}$$

Using Dynkin's formula, we can write

$$X(t) = X(0) + \int_0^t \Delta_N X(s)\, ds + \int_0^t R(X(s))\, ds + Z(t), \tag{2.2}$$

where, assuming $E(X^2(0))$ is finite, Z is an H^N-valued martingale for the filtration $\{\sigma(X(s){:}\ s \leq t)\}$. We have that $X(t) \in H^N \subset L_2(S)$, but we require a larger state space than $L_2(S)$. Let

$$\varphi_n(r) = \begin{cases} 1, & n = 0, \\ \sqrt{2}\cos(\pi n r), & n > 0 \text{ and even}, \end{cases}$$

$$\psi_n(r) = \sqrt{2}\sin(\pi n r), \qquad n > 0 \text{ and even}.$$

Here $\{H_\alpha\}_{\alpha \in R}$ denotes the decreasing collection of Hilbert spaces obtained from completion of the trigonometric polynomials in the norm

$$\|f\|_\alpha = \left[\sum_n \left(\langle f, \varphi_n\rangle^2 + \langle f, \psi_n\rangle^2\right)\left(1 + \pi^2 m^2\right)^\alpha\right]^{1/2}.$$

Note that $H_0 = L_2(S)$ and that if $\alpha \geq 0$, $f \in H_{-\alpha}$ and $g \in H_\alpha$, then the $L_2(S)$ inner product $\langle \cdot, \cdot \rangle$ extends to

$$\langle f, g\rangle = \sum_n \left(\langle f, \varphi_n\rangle\langle g, \varphi_n\rangle + \langle f, \psi_n\rangle\langle g, \psi_n\rangle\right)$$

and satisfies

$$|\langle f, g\rangle| \leq \|f\|_{-\alpha}\|g\|_\alpha.$$

Here φ_n and ψ_n are eigenfunctions of $\Delta = d^2/dr^2$ with eigenvalues $-\beta_n = -\pi^2 n^2$, and we need analogues for Δ_N.

We now restrict N to be odd and, for $n \in \{0, 2, \dots, N-1\}$, define

$$\varphi_{n,N}(r) = \sum_{k=1}^{N} \varphi_n\left((k-1)N^{-1}\right)I_k(r),$$

$$\psi_{n,N}(r) = \sum_{k=1}^{N} \psi_n\left((k-1)N^{-1}\right)I_k(r).$$

Then $\{\varphi_{n,N}(r), \psi_{n,N}(r)\}$ form an orthonormal basis for $(H^N, \|\cdot\|_0)$ and they are eigenfunctions of Δ_N with eigenvalues

$$-\beta_{n,N} = -2N^2\left(1 - \cos(\pi n N^{-1})\right)$$

satisfying

$$c_1 n^2 \leq \beta_{n,N} \leq c_2 n^2 \tag{2.3}$$

for $0 < c_1 \leq c_2 < \infty$.

Note that $H^N \subset H_{-\alpha}$ for $\alpha \geq 0$. For $f \in H^N$, define

$$\|f\|_{\alpha,N} = \left[\sum_n \left(\langle f, \varphi_{n,N}\rangle^2 + \langle f, \psi_{n,N}\rangle^2\right)\left(1 + \beta_{n,N}\right)^\alpha\right]^{1/2}.$$

Basic calculations show that for $f \in H^N$ and $\alpha \geq 0$,

$$c_1(\alpha)\|f\|_{-\alpha,N} \leq \|f\|_{-\alpha} \leq c_2(\alpha)\|f\|_{-\alpha,N}, \tag{2.4}$$

where $0 < c_1(\alpha) \leq c_2(\alpha) < \infty$. In particular, $\|f\|_{0,N} = \|f\|_0$. Also note that $|\langle f, g\rangle| \leq \|f\|_{-\alpha,N}\|g\|_{\alpha,N}$ for $f, g \in H^N$ and all α.

Let $T_N(t) = \exp(\Delta_N t)$. Then $\varphi_{n,N}, \psi_{n,N}$ are eigenfunctions of $T_N(t)$ with eigenvalues $\exp(-\beta_{n,N} t)$. Now (2.2) can be rewritten as

$$X(t) = T_N(t)X(0) + \int_0^t T_N(t-s)R(X(s))\,ds + Y(t), \tag{2.5}$$

where

$$Y(t) = \int_0^t T_N(t-s)\,dZ(s).$$

If $f \in H^N$, $g \in H_0$, then $\langle f, g - P_N g\rangle = 0$, where $P_N\colon H_0 \to H^N$ is the projection given by

$$P_N g(r) = \sum_{k=1}^{N} N\langle g, I_k\rangle I_k(r).$$

Also note that $\{P_N\varphi_n, P_N\psi_n\}_{n<N}$ form an orthogonal (but not orthonormal) set in H_0 and satisfy

$$P_N\varphi_n = a_{n,N}\varphi_{n,N} + b_{n,N}\psi_{n,N}, \qquad P_N\psi_n = -a_{n,N}\varphi_{n,N} + b_{n,N}\psi_{n,N},$$

where

$$\lim_{N\to\infty}\big(|a_{n,N} - 1| + |b_{n,N}|\big) = 0.$$

Also P_N is represented by

$$P_N f = \sum \langle f, \varphi_{m,N}\rangle\varphi_{m,N} + \langle f, \psi_{m,N}\rangle\psi_{m,N}$$

and, similarly, we define

$$P_{n,N} f = \sum_{m\le n} \langle f, \varphi_{m,N}\rangle\varphi_{m,N} + \langle f, \psi_{m,N}\rangle\psi_{m,N}$$

and

$$P_n f = \sum_{m\le n} \langle f, \varphi_m\rangle\varphi_m + \langle f, \psi_m\rangle\psi_m.$$

Note P_n can be considered as a continuous map $P_n\colon H_{\alpha_1} \to H_{\alpha_2}$ for all α_1, α_2.

For $f \in H_0$, let $\|f\|_\infty = \sup_r |f(r)|$.

We assume X is right continuous with left limits and that all the processes are defined on the same probability space. Let F_t^N denote the completion of $\sigma(X_N(s)\colon s \le t)$. To analyze Z properly it is necessary to consider a larger filtration. We take X to be constructed as a random time change of a system of scaled and independent Poisson processes as in Chapter 11 (Section 2) of Ethier and Kurtz (1986). With this approach we can distinguish, using (2.1), two types of births for each n_k: those which arrive at intensity $lb(n_k l^{-1})$ and those which arrive at intensity $\gamma N l n_k/2$. Likewise we distinguish two types of deaths.

Let $\delta X(t) = X(t) - X(t-)$, where $X(t-) = \lim_{s<t, s\to t} X(s)$ and $\delta X(0) = 0$. Let δX_D, δX_R and δX_B denote those jumps in X arising from, respectively, random walk, births or deaths associated with the intensities $lb(n_k l^{-1})$ and $ld(n_k l^{-1})$ in (2.1), and births or deaths associated with the

intensity $\gamma Nln_k/2$ in (2.1). Here D stands for "diffusion" noise arising from particle movement to neighboring cells through random walks. As in Blount (1994) and Kotelenez (1988), where $\gamma = 0$, R stands for "reaction" noise arising from births or deaths associated with $b(x)$ or $d(x)$ in (2.1) and contributing drift terms in (2.2). If $\gamma > 0$ and $b(x) = d(x) \equiv 0$, then it is natural to have B stand for "critical branching" noise. We have kept this notation here since it is the critical branching noise that leads to a limiting SPDE.

Let G_t^N denote the completion of the σ-field generated by observing, through time t, $X(0)$ and the time changed Poisson processes used to construct X. Then $F_t^N \subset G_t^N$ and with moment assumptions we have that

$$\begin{aligned} Z_D(t) &= \sum_{s \le t} \delta X_D(s) - \int_0^t \Delta_N X(x)\, ds, \\ Z_R(t) &= \sum_{s \le t} \delta X_R(s) - \int_0^t R(X(s))\, ds, \\ Z_B(t) &= \sum_{s \le t} \delta X_B(s) \end{aligned} \tag{2.6}$$

are G_t^N martingales, although $Z_R(t)$ and $Z_B(t)$ are not F_t^N measurable. Also, if $f \in H^N$, straightforward calculations show [see Lemmas 0.1 and 1.1 of Kotelenez (1986) and Lemmas 2.2 and 2.10 of Blount (1991)]

$$\begin{aligned} &\langle Z_D(s), f\rangle^2 - (Nl)^{-1} \int_0^t \langle X(s), (\nabla^+ f)^2 + (\nabla^- f)^2 \rangle\, ds, \\ &\langle Z_R(s), f\rangle^2 - (Nl)^{-1} \int_0^t \langle b(X(s)) + d(X(s)), f^2 \rangle\, ds, \\ &\langle Z_B(s), f\rangle^2 - \gamma \int_0^t \langle X(s), f^2 \rangle\, ds \end{aligned} \tag{2.7}$$

are G_t^N martingales.

Note that Z as defined in (2.2) satisfies $Z = Z_D + Z_R + Z_B$. Also note that $\langle Z_D(t), 1\rangle = 0$, and since Z_D, Z_R and Z_B arise from independent noise sources and have (almost surely) no simultaneous jumps, $E(\langle Z(t), 1\rangle^2) = E(\langle Z_R(t), 1\rangle^2) + E(\langle Z_B(t), 1\rangle^2)$.

Letting

$$Y_I(t) = \int_0^t T_N(t-s)\, dZ_I(s) \quad \text{for } I \in \{D, R, B\},$$

we can rewrite (2.5) as

$$\begin{aligned} X(t) = T_N(t) X(0) &+ \int_0^t T_N(t-s) R(X(s))\, ds \\ &+ Y_D(t) + Y_R(t) + Y_B(t). \end{aligned} \tag{2.8}$$

Before precisely stating and proving our results in Section 3, we briefly outline the approach.

Let $V(t) = T_N(t)X(0) + \int_0^t T_N(t-s)R(X(s))\,ds + Y_B(t)$. Then

$$X(t) = V(t) + Y_D(t) + Y_R(t), \tag{2.9}$$

and substituting this expression for X in (2.2) gives

$$\begin{aligned} V(t) &= X(0) + \int_0^t \Delta_N V(s)\,ds + \int_0^t R(V(s))\,ds \\ &\quad - d_2 l^{-1}\int_0^t V(s)\,ds + Z(t) + \varepsilon(t), \end{aligned} \tag{2.10}$$

where $\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t)$ is defined by

$$\begin{aligned} \varepsilon_1(t) &= \int_0^t \big(b_1 - d_1 - 1 - d_2 l^{-1} + \Delta_N\big)\big(Y_D(s) + Y_R(s)\big)\,ds, \\ \varepsilon_2(t) &= -d_2\int_0^t \big(Y_R^2(s) + 2V(s)Y_D(s) + 2V(s)Y_R(s) \\ &\qquad\qquad + 2Y_D(s)Y_R(s)\big)\,ds, \\ \varepsilon_3(t) &= -d_2\int_0^t \big(Y_D^2(s) - l^{-1}X(s)\big)\,ds. \end{aligned} \tag{2.11}$$

We will subsequently show that the distributions of $\{V\} = \{V_{N,l}\}$ are relatively compact on $D([0,T]\colon H_0)$ and that V is actually somewhat "smoother" (in a spatial sense) than merely being in H_0. We show

$$\begin{aligned} \sup_{t\le T}\|Y_R(t)\|_0 &\to 0 \quad \text{in probability as } N\to\infty, \\ \sup_{t\le T}\|Y_D(t)\|_{-\alpha} &\to 0 \quad \text{in probability as } N\to\infty \text{ for any } \alpha > 0, \\ \sup_{t\le T}\|Y_D(t)\|_0 &\to 0 \quad \text{in probability if } l\to\infty \text{ as } N\to\infty, \\ \sup_{t\le T}\big(|\langle \varepsilon(t), \varphi_m\rangle| + |\langle \varepsilon(t), \psi_m\rangle|\big) &\to 0 \quad \text{in probability if } N\to\infty. \end{aligned}$$

As suggested by (2.7) the limiting martingale is the distributional limit of Z_B, whereas Z_D and Z_R converge to 0 in the appropriate sense. We then use (2.10) to identify the limiting SPDE as one of two possibilities depending on whether $l\to\infty$ as $N\to\infty$ or l is constant as $N\to\infty$. As (2.10) indicates, keeping l constant contributes an extra term arising from the fact that

$$\sup_{t\le T}\left|\int_0^t \langle Y_D^2(s) - l^{-1}X(s), e_m\rangle\,ds\right| \to 0$$

in probability for $e_m = \varphi_m$ or ψ_m.

3. Limit theorems. We now state our results before developing the machinery to prove them. Let $T(t)$ denote the semigroup generated by Δ on each H_α and note that $T(t)e_n = \exp(-\beta_n t)e_n$ for $e_n = \varphi_n$ or ψ_n. If A is a $D([0,T]: H_\alpha)$-valued martingale, we make the definition

$$\int_0^t T(t-s)\,dA(s) = \sum_n \int_0^t \exp(-\beta_n(t-s))\,d\langle A(s),\varphi_n\rangle\varphi_n$$
$$+\sum_n \int_0^t \exp(-\beta_n(t-s))\,d\langle A(s),\psi_n\rangle\psi_n$$

and we note that integration by parts shows that the right-hand side defines a process in $D([0,T]: H_{\alpha-2})$. The process may be smoother, depending on the properties of A.

At various points we make the following assumptions:

ASSUMPTION A. $\sup_{N,l} E\langle X(0),1\rangle < \infty$.

ASSUMPTION A1. $\sup_{N,l} E(\langle X(0),1\rangle^2) < \infty$.

ASSUMPTION A2. $X(0) \to \psi(0)$ in distribution on H_0.

ASSUMPTION B1. $l \to \infty$ as $N \to \infty$.

ASSUMPTION B2. l is constant as $N \to \infty$.

Throughout the paper we assume that

$$l \ge l_0 > 0.$$

THEOREM 3.1. (i) *For any* $T > 0$, *Assumptions* A1, A2 *and* B1 *imply* $(X, Z_B) \to (\psi, M)$ *in distribution in* $D([0,T]: H_0 \times H_{-\beta})$ *for any* $\beta > \frac{1}{2}$, *where* ψ *and* M *have the following properties*: (a) $(\psi, M) \in C([0,T]: H_0 \times H_{-\beta})$; (b) M *is a martingale with respect to the filtration* $\{\sigma(\psi(s): s \le t)\}$; (c) *for any trigonometric polynomial* f, $\langle M, f\rangle$ *has quadratic variation process*

$$\langle\langle M, f\rangle\rangle(t) = \gamma\int_0^t \langle \psi(s), f^2\rangle\,ds;$$

(d) *in* $C([0,T]: H_{-2}) \cap C((0,T]: H_{-\alpha})$, *for any* $\alpha > \frac{3}{2}$, (ψ, M) *satisfies*

$$\psi(t) = \psi(0) + \int_0^t \Delta\psi(s)\,ds + \int_0^t R(\psi(s))\,ds + M(t); \tag{3.1}$$

(e) *in* $C([0,T]: H_0) \cap C((0,T]: H_\alpha)$, *for any* $\alpha < \frac{1}{2}$, (ψ, M) *satisfies*

$$\psi(t) = T(t)\psi(0) + \int_0^t T(t-s)R(\psi(s))\,ds + \int_0^t T(t-s)\,dM(s). \tag{3.2}$$

(ii) *For any* $T > 0$, *Assumptions* A1, A2 *and* B2 *imply* $(X, Z_B) \to (\psi, M)$ *in distribution in* $D([0,T]: H_{-\alpha} \times H_{-\beta})$ *for any* $\alpha > 0$ *and* $\beta > \frac{1}{2}$, *where* ψ *and* M *have the following properties*: (a) $(\psi, M) \in C([0,T]: H_0 \times H_{-\beta})$; (b) M

is a martingale with respect to the filtration $\{\sigma(\psi(s)\colon\ s \le t)\}$; (c) *for any trigonometric polynomial* f, $\langle M, f\rangle$ *has quadratic variation process*

$$\langle\langle M, f\rangle\rangle(t) = \gamma\int_0^t \langle \psi(s), f^2\rangle\, ds;$$

(d) *in* $C([0,T]\colon H_{-2}) \cap C((0,T]\colon H_{-\alpha})$, *for any* $\alpha > \frac{3}{2}$, (ψ, M) *satisfies*

$$\begin{aligned}\psi(t) &= \psi(0) + \int_0^t \Delta\psi(s)\, ds \\ &\quad + \int_0^t \big(R(\psi(s)) - l^{-1} d_2\psi(s)\big)\, ds + M(t);\end{aligned} \tag{3.3}$$

(e) *in* $C([0,T]\colon H_0) \cap C((0,T]\colon H_\alpha)$, *for any* $\alpha < \frac{1}{2}$, (ψ, M) *satisfies*

$$\begin{aligned}\psi(t) &= T(t)\psi(0) + \int_0^t T(t-s)\big(R(\psi(s)) - l^{-1} d_2\psi(s)\big)\, ds \\ &\quad + \int_0^t T(t-s)\, dM(s).\end{aligned} \tag{3.4}$$

REMARK 3.1. In the proof of Lemma 3.6 we show that

$$\psi(\cdot) - \int_0^{\cdot} T(\cdot - s)\, dM(s) \in C\big((0,T]\colon C([0,1])\big) \quad \text{almost surely.}$$

From this result and the results of Konno and Shiga (1988) applied to $\int_0^t T(t-s)\, dM(s)$ the following hold:

(a) $P(\psi(t,r)$ is continuous in r for all $t > 0) = 1$.
(b) ψ can be represented as the solution of (3.1) or (3.3) with

$$M(t) = \int_0^t \sqrt{\gamma\psi(s)}\, dW(s)$$

for W a cylindrical Brownian motion on H_0.

Also, distributional uniqueness for ψ is proved in Evans and Perkins (1994) and Tribe (1994).

Before proving Theorem 3.1 we need preliminary results.

LEMMA 3.1. (i) $\sup_{t \le T}\langle EX(t), 1\rangle \le C(b_0, b_1, T)(1 + \langle EX(0), 1\rangle)$.
(ii) $E[\sup_{t \le T}\langle X(t), 1\rangle] \le C(b_0, b_1, \gamma, T)(1 + (Nl)^{-1})^{1/2}(1 + \langle EX(0), 1\rangle)$.
(iii) $E[\sup_{t \le T}\langle X(t), 1\rangle^2] \le C(b_0, b_1, \gamma, T)[E(\langle X(0), 1\rangle^2) + (1 + (Nl)^{-1}) \times (1 + \langle EX(0), 1\rangle)]$.
(iv) $\int_0^T E(\|X(s)\|_0^2)\, ds \le C(b_0, b_1, \gamma, T)(E(\langle X(0), 1\rangle^2) + (1 + \langle EX(0), 1\rangle) \times (1 + l^{-1} + (Nl)^{-1}))$.

PROOF. Let $\overline{X}$ denote the process constructed as X, but with $d_1 = d_2 = 0$ in (2.1). One can construct X and $\overline{X}$ in such a way that $P(0 \le X(t) \le \overline{X}(t), t \ge 0) = 1$.

Thus it suffices to prove the lemma for $\bar{X}$. We assume $\bar{X}(0) = X(0)$. Using a similar decomposition as in (2.2), we have

$$(3.5)\quad \langle \bar{X}(t), 1\rangle = \langle X(0), 1\rangle + b_1\int_0^t \langle \bar{X}(s), 1\rangle\, ds + b_0 t + \langle \bar{Z}(t), 1\rangle.$$

Taking expectations in (3.5) and applying Gronwall's inequality proves (i).

Equation (3.5) and the fact that $\bar{X} \geq 0$ imply that

$$E\Big[\sup_{s\leq t}\langle \bar{X}(s), 1\rangle\Big] \leq \langle EX(0), 1\rangle + b_1\int_0^t \langle E\bar{X}(s), 1\rangle\, ds + b_0 t + \left(E\Big[\sup_{s\leq t}\langle \bar{Z}(t), 1\rangle^2\Big]\right)^{1/2}.$$

Noting that $\bar{Z}_D$, $\bar{Z}_R$ and $\bar{Z}_B$ can be defined analogously to Z_D, Z_R and Z_B, (ii) then follows from (i), Doob's maximal quadratic inequality applied to $\bar{Z}$ and (2.7) together with the paragraph following it.

Two applications of Jensen's inequality to (3.5) imply that

$$\sup_{s\leq t}\langle \bar{X}(s), 1\rangle^2 \leq 4\langle X(0), 1\rangle^2 + b_1 t\int_0^t \Big(\sup_{\mu\leq s}\langle \bar{X}(\mu), 1\rangle^2\Big)\, ds + b_0^2 t^2 + \sup_{s\leq t}\langle \bar{Z}(s), 1\rangle^2.$$

After taking expectations, (iii) follows from Gronwall's inequality, Doob's inequality, (i) and (2.7) together with the paragraph following it.

Applying Jensen's inequality (twice) to (2.5) we obtain

$$(3.6)\quad \|\bar{X}(t)\|_0^2 \leq 4\left(\|T_N(t)X(0)\|_0^2 + t\int_0^t b_1^2\|\bar{X}(s)\|_0^2\, ds + b_0^2 t^2 + \|\bar{Y}(t)\|_0^2\right).$$

For $I \in \{R, D, B\}$ and $e_{m,N} = \varphi_{m,N}$ or $\psi_{m,N}$ we have

$$\langle \bar{Y}_I(t), e_{m,N}\rangle = \int_0^t \exp(-\beta_{m,N}(t-s))\, d\langle \bar{Z}_I(s), e_{m,N}\rangle.$$

From this, (2.3), (2.7) and (i), we obtain

$$E\big(\|\bar{Y}_D(t)\|_0^2\big) \leq C(b_0, b_1, t)(1 + \langle EX(0), 1\rangle)l^{-1},$$
$$E\big(\|\bar{Y}_B(t)\|_0^2\big) \leq C(b_0, b_1, t)\gamma(1 + \langle EX(0), 1\rangle),$$
$$E\big(\|\bar{Y}_R(t)\|_0^2\big) \leq C(b_0, b_1, t)(1 + \langle EX(0), 1\rangle)(Nl)^{-1}.$$

Also note $\|T_N(t)X(0)\|_0^2 \leq 4\langle X(0), 1\rangle^2 \sum_m \exp(-\beta_{m,N}t)$. Part (iv) then follows from Gronwall's inequality after taking expectations and integrating using (3.6). □

Let I denote the identity operator on H_0.

LEMMA 3.2. *Assumption* A *implies that, for* $\alpha < \frac{1}{2}$ *and every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup_{N,l} P\Big(\sup_{t\le T} \|(I - P_{n,N})Y_B(t)\|_{\alpha,N} \ge \varepsilon\Big) = 0.$$

PROOF. Let $\tau = \inf\{t: \langle X(t), 1\rangle \ge \rho\}$. Given ε, Assumption A and Lemma 3.1(ii) imply there exists $\rho(\varepsilon) < \infty$ so that $\tau = \tau(N, l, \rho(\varepsilon))$ satisfies $\sup_{N,l} P(\tau \le T) < \varepsilon$. This shows that instead of Y_B we may prove the lemma for

$$R(t) = \int_0^t T_N(t-s)\, dZ_B(s \wedge \tau).$$

For fixed $m \ne 0$ and $0 \le u \le t$, let

$$M(u) = m\int_0^u \exp(-\beta_{m,N}(t-s))d\langle Z_B(s\wedge\tau), \varphi_{m,N}\rangle.$$

Note that M is a mean 0 martingale on $[0,t]$ and $M(t) = m\langle R(t), \varphi_{m,N}\rangle$. Also, $|\delta M(u)| \le m|\langle \delta Z_B(u), \varphi_{m,N}\rangle| \le 2^{1/2}m(Nl)^{-1} \le 2^{1/2}/l_0$ since $m \le N$. In what follows we first assume $l_0 \ge 2^{1/2}$, so $|\delta M(u)| \le 1$. If $l_0 < 2^{1/2}$, we replace Z_B by $(l_0/2^{1/2})Z_B$ in the proof. This will imply that $|\delta M(u)| \le 1$ and, other than slight changes in notation (some constants will now depend on l_0), the proof is the same. From (2.7), M has a Meyer process $\langle M\rangle$ satisfying

$$\begin{aligned}\langle M\rangle(u) &= \gamma m^2\int_0^{u\wedge\tau} \langle X(s), (\varphi_{m,N})^2\rangle \exp(-2\beta_{m,N}(t-s))\, ds\\ &\le 2\gamma m^2\rho\int_0^u \exp(-2\beta_{m,N}(t-s))\, ds\\ &\le c\gamma\rho.\end{aligned}$$

The proof of Lemma 4.4 of Blount (1992) shows

$$E[\exp(M(t))] \le \exp(3c\gamma\rho/2),$$

with the same argument applying to $-M(t)$. Using Markov's inequality, we obtain for arbitrary α and r that

$$\begin{aligned}&P\big(m^{2\alpha}\langle R(t), \varphi_{m,N}\rangle^2 \ge m^{-2r}\big)\\ &\quad = P\big(|M(t)| \ge m^{1-r-\alpha}\big)\\ &\quad \le P\big(M(t) \ge m^{1-r-\alpha}\big) + P\big(-M(t) \ge m^{1-r-\alpha}\big)\\ &\quad \le C(\gamma\rho)\exp(-m^{1-r-\alpha}).\end{aligned}$$

Setting $R_m(t) = \langle R(t), \varphi_{m,N}\rangle$ and $Z_m(t) = \langle Z_B(t), \varphi_{m,N}\rangle$ we have

$$R_m(t) = -\beta_{m,N}\int_0^t R_m(s)\, ds + Z_m(t\wedge\tau) \tag{3.7}$$

and, from our previous calculations,

$$P\left(m^{2\alpha}R_m^2(t) \geq m^{-2r}\right) \leq C(\gamma\rho)\exp(-m^{1-r-\alpha}). \tag{3.8}$$

Let $t_k = kTm^{-2}$ for $0 \leq k \leq m^2$. For $t_k \leq t \leq t_{k+1}$ we have from (3.7) that

$$R_m(t) = R_m(t_k) - \beta_{m,N}\int_{t_k}^{t} R_m(s)\,ds + Z_m(t \wedge \tau) - Z_m(t_k \wedge \tau).$$

Applying Gronwall's inequality to this shows

$$\begin{aligned}\sup_{[t_k,t_{k+1}]}|R_m(t)| \leq \Big(&|R(t_k)| + \sup_{[t_k,t_{k+1}]}|Z_m(t\wedge\tau) - Z_m(t_k \wedge \tau)|\Big)\\ &\times \exp\left(m^{-2}\beta_{m,N}T\right)\end{aligned} \tag{3.9}$$

Consider the mean 0 martingale $m(Z_m(t \wedge \tau) - Z_m(t_k \wedge \tau))$ for $t_k \leq t \leq t_{k+1}$. Using (2.7) again and an almost identical calculation as previously for $M(u)$, we obtain

$$E\left[\exp\left[m(Z_m(t_{k+1}\wedge\tau) - Z_m(t_k \wedge \tau))\right]\right] \leq \exp(3\gamma\rho T),$$

with the same holding for $-Z_m$. This shows

$$\begin{aligned}&P\left(\sup_{[t_k,t_{k+1}]} m^{2\alpha}\left(Z_m(t\wedge\tau) - Z_m(t_k\wedge\tau)\right)^2 \geq m^{-2r}\right)\\ &\quad\leq 2\exp(3\gamma\rho T)\exp(-m^{1-r-\alpha})\end{aligned} \tag{3.10}$$

after applying Doob's inequality to the submartingale $\exp[m(Z_m(t \wedge \tau) - Z_m(t_k \wedge \tau))]$.

Since $m^{-2}\beta_{m,N} \leq C$, (3.8) and (3.10) applied to (3.9) shows [with $C(T) \geq 0$] that

$$P\left(C(T)\sup_{[t_k,t_{k+1}]} m^{2\alpha}R_m^2(t) \geq m^{-2r}\right) \leq C(\gamma\rho, T)\exp(-m^{1-r-\alpha}).$$

In turn, this yields

$$\begin{aligned}&P\left(C(T)\sup_{t\leq T} m^{2\alpha}\langle R(t), \varphi_{m,N}\rangle^2 \geq m^{-2r}\right)\\ &\quad\leq C(\gamma\rho, T)\,m^2\exp(-m^{1-r-\alpha}),\end{aligned} \tag{3.11}$$

and the same for $\psi_{m,N}$ in place of $\varphi_{m,N}$. For any $\alpha < \frac{1}{2}$ we can choose r so that $r > \frac{1}{2}$ and $\alpha + r < 1$. This implies

$$\sum_m m^{-2r} < \infty \quad \text{and} \quad \sum_m m^2\exp(-m^{1-r-\alpha}) < \infty.$$

The lemma then follows from (3.11) and the definition of the norm $\|\cdot\|_{\alpha,N}$. □

LEMMA 3.3. *Let f denote $\varphi_{m,N}$, $\psi_{m,N}$, φ_m or ψ_m with m fixed.*

(i) *Assumption* A *implies* $\lim_{N\to\infty} E[\sup_{t\leq T}(\langle Z_D(t), f\rangle^2 + \langle Y_D(t), f\rangle^2)] = 0$.

(ii) *Assumption* A1 *implies* $\lim_{N\to\infty} E[\sup_{t\leq T}(\langle Z_R(t), f\rangle^2 + \langle Y_R(t), f\rangle^2)] = 0$.

(iii) *Under Assumption* A *the distributions of* $\{Z_B\}$ *on* $D([0,T]: H_{-\alpha})$ *are relatively compact for any* $\alpha > \frac{1}{2}$.

(iv) *Under Assumption* A *the distributions of* $\{Y_B\}$ *on* $D([0,T]: H_0)$ *are relatively compact*.

(v) *Assumption* A *implies that for each* $\alpha < \frac{1}{2}$ *and any* $\varepsilon > 0$ *there exists* $\rho(\varepsilon, \alpha) < \infty$ *such that*

$$\sup_{N,l} P\Big(\sup_{t \le T} \|Y_B(t)\|_{\alpha, N} > \rho\Big) < \varepsilon.$$

PROOF. Since $\langle Y_I(t), \varphi_n\rangle = \langle Y_I(t), P_{n,N}\varphi_n\rangle$ (and likewise for Z_I) it is enough to prove (i) and (ii) for $\varphi_{m,N}$ and $\psi_{m,N}$. For $N > m$ and $I \in \{D, R\}$,

$$\langle Y_I(t), f\rangle = \langle Z_I(t), f\rangle - \beta_{m,N}\int_0^t \exp(-\beta_{m,N}(t-s))\langle Z_I(s), f\rangle\, ds.$$

Thus $\sup_{t\le T}|\langle Y_I(t), f\rangle| \le 2\sup_{t \le T}|\langle Z_I(t), f\rangle|$ and (i) and (ii) follow from (2.7), (i) and (iv) of Lemma (3.1) and Doob's inequality.

Consider Z_B. From (2.7), for $f_m = \varphi_m$ or ψ_m,

$$E\big[\langle Z_B(T), f_m\rangle^2\big] = \gamma\int_0^T \langle EX(s), (P_N f_m)^2\rangle\, ds.$$

This implies

$$E\big[\|(I-P_n)Z_B(T)\|_{-\alpha}^2\big] \le \sum_{m=n}^{\infty}(1+\pi^2 m^2)^{-\alpha} 2\gamma\int_0^T\langle EX(s), 1\rangle\, ds.$$

Also, for $0 \le \mu \le \delta$,

$$\begin{aligned} &E\big[\|Z_B(t+\mu) - Z_B(t)\|_{-\alpha}^2 | G_t^N\big] \\ &\quad\le \Big(\sum_m (1+\pi^2 m^2)^{-\alpha}\Big) 2\gamma E\Big[\int_t^{t+\delta}\langle X(s), 1\rangle\, ds | G_t^N\Big]. \end{aligned}$$

Part (iii) then follows from Lemma 3.1(i), Doob's inequality and a well-known tightness condition which is stated in Theorem 8.6 and Remark 8.7(a) of Chapter 3 of Ethier and Kurtz (1986).

Consider, for n fixed,

$$P_{n,N}Y_B(t) = P_n Z_B(t) + \int_0^t T(t-s)\,\Delta P_n\, Z_B(s)\, ds + \varepsilon_n(t),$$

where

$$\varepsilon_n(t) = (P_{n,N} - P_n)Z_B(t) + \int_0^t (T_N(t-s)\,\Delta_N P_{n,N} - T(t-s)\,\Delta P_n)Z_B(s)\, ds.$$

Since $|\beta_{m,N} - \beta_m| + \|\varphi_{m,N} - \varphi_m\|_\infty + \|\psi_{m,N} - \psi_m\|_\infty \to 0$ as $N \to \infty$ for m fixed and $\sup_{N,l} E(\langle Z_B(T), 1\rangle^2) < \infty$, basic calculations show $\sup_{t\le T}\|\varepsilon_n(t)\|_{\alpha,N} \to 0$

in probability as $N \to \infty$ for any α. The map β: $D([0,T]: H_{\alpha_1}) \to D([0,T]: H_{\alpha_2})$ defined by

$$\beta f(t) = P_n f(t) + \int_0^t T(t-s)\,\Delta P_n\, f(s)\,ds$$

is continuous for any α_1, α_2. Thus (iv) and (v) follow from (iii), our just completed calculations, Lemma 3.2 and Problem 18 of Chapter 3 in Ethier and Kurtz (1986). □

LEMMA 3.4. *Assumptions* A *and* A2 *imply that* (i)–(vi) *below hold. Assumptions* A, A2 *and* B1 *imply* (vii) *below holds.*

(i) *Given* $\varepsilon > 0$ *there exists* $\rho(\varepsilon, T, l_0) < \infty$ *such that*

$$\sup_{N,l} P\Big(\sup_{t\le T}\|X(t)\|_0 > \rho\Big) < \varepsilon.$$

(ii) $\sup_{t\le T}\|Y_R(t)\|_0 \to 0$ *in probability.*

(iii) *Given* $\varepsilon > 0$ *there exists* $\rho(\varepsilon, T, \alpha) < \infty$ *such that for each* $\alpha < \frac{3}{2}$,

$$\sup_{N,l} P\left(\sup_{t\le T}\left\|\int_0^t T_N(t-s)R(X(s))\,ds\right\|_{\alpha,N} > \rho\right) < \varepsilon.$$

(iv) *The distributions of* $\{\int_0^{\cdot} T_N(\cdot - s)R(X(s))\,ds\}$ *on* $C([0,T]: H_0)$ *are relatively compact.*

(v) *Given* $\varepsilon > 0$ *there exists* $\rho(\varepsilon, T) < \infty$ *such that*

$$\sup_{N,l} P\Big(\sup_{t\le T}\|Y_D(t)\|_0 > \rho\Big) < \varepsilon.$$

(vi) $\sup_{t\le T}\|Y_D(t)\|_{-\alpha} \to 0$ *in probability for all* $\alpha > 0$.

(vii) $\sup_{t\le T}\|Y_D\|_0 \to 0$ *in probability.*

PROOF. To prove (i) it suffices to consider $\overline{X}(t)$ as defined in the proof of Lemma 3.1. From (2.8) we have

$$\|\overline{X}(t)\|_0 \le \|X(0)\|_0 + b_1\int_0^t\|\overline{X}(s)\|_0\,ds + b_0 t + \|\overline{Y}_D(t)\|_0 + \|\overline{Y}_R(t) + \overline{Y}_B(t)\|_0.$$

Let $\tau = \inf\{t: \|\overline{X}(t)\|_0 \ge \rho\}$. Then

$$P\Big(\sup_{t\le T}\|\overline{X}(t)\|_0 > \rho\Big) \le P\Big(\sup_{t\le T}\|\overline{X}(t\wedge\tau)\|_0 \ge \rho\Big),$$

and applying Gronwall's inequality to our first inequality we obtain

$$\sup_{t\le T}\|\overline{X}(t\wedge\tau)\|_0 \le e^{b_1 T}\Big(\|X(0)\|_0 + b_0 T + \sup_{t\le T}\|\overline{Y}_D(t\wedge\tau)\|_0 + \sup_{t\le T}\|\overline{Y}_R(t) + \overline{Y}_B(t)\|_0\Big).$$

For $\rho/4 > \exp(b_1 T) b_0 T$, we obtain

$$
\begin{aligned}
& P\Big(\sup_{t \le T} \|\overline{X}(t \wedge \tau)\|_0 \ge \rho\Big) \\
& \quad \le P(\|X(0)\|_0 \ge \exp(-b_1 T)\rho/4) \\
& \qquad + P\Big(\sup_{t \le T} \|\overline{Y}_D(t \wedge \tau)\|_0 \ge \exp(-b_1 T)\rho/4\Big) \\
& \qquad + P\Big(\sup_{t \le T} \|\overline{Y}_R(t) + \overline{Y}_B(t)\|_0 \ge \exp(-b_1 T)\rho/4\Big).
\end{aligned} \tag{3.12}
$$

Consider $\overline{Y}_R(t) + \overline{Y}_B(t) = \int_0^t T_N(t-s)\, d(\overline{Z}_R(s) + \overline{Z}_B(s))$. Recall that for $\overline{X}$ we assume $R(x) = b_1 x + b_0$, which is linear, and using (2.7) with $\overline{Z}_R + \overline{Z}_B$ in place of Z_B, Lemma 3.3(v) applies to $\overline{Y}_R + \overline{Y}_B$. Thus the third term on the right-hand side of (3.12) can be made arbitrarily small by choosing ρ large enough. The same holds for the first term using Assumption A2.

The proof of Lemma 3.5 of Blount (1994) or the proof of Theorem 3.3 of Blount (1991) shows that for $N \ge N_0(a, l_0)$,

$$
\begin{aligned}
& P\Big(\sup_{t \le T} \|\overline{Y}_D(t \wedge \tau)\|_0 \ge a\Big) \\
& \quad \le C(T)\rho a^{-2}(Nl)^{-1}(\log N)^6 \\
& \qquad + C(T) N^3 (\log N)^{-1}\big[C_1 a^2 l/\rho\big]^{-\log N},
\end{aligned} \tag{3.13}
$$

where $C_1 > 0$ is an absolute constant.

Letting $a = \exp(-b_1 T)\rho/4$ for ρ large enough but finite, the right-hand side of (3.13) converges to 0 as $N \to \infty$. Our previous discussion then shows that the right-hand side of (3.12) can be made smaller than ε for $\rho(\varepsilon, T, l_0) < \infty$. This proves (i).

To prove (ii) it suffices, by (i), to prove it for $\int_0^t T_N(t-s)\, dZ_R(s \wedge \tau)$, where $\tau = \inf\{t\colon \|X(s)\|_0 \ge \rho\}$ and $\rho < \infty$. However, the proofs of Lemmas 3.2 and 3.3(ii) apply as well to $\int_0^t T_N(t-s)\, dZ_R(s \wedge \tau)$ and $Z_R(t \wedge \tau)$ since $\sup_{t \le T}\|X(t \wedge \tau -)\|_0 \le \rho < \infty$, and this proves (ii).

Let $A(t) = \int_0^t T_N(t-s) R(X(s))\, ds$. We have

$$
\sup_{t \le T} |\langle R(X(t)), 1\rangle| \le C(T)\Big(1 + \sup_{t \le T}\|X(t)\|_0^2\Big). \tag{$*$}
$$

For $e_{m,N} = \varphi_{m,N}$ or $\psi_{m,N}$ this implies

$$
\sup_{t \le T} |\langle A(t), e_{m,N}\rangle| \le (1 + m^2)^{-1} C(T)\Big(1 + \sup_{t \le T}\|X(t)\|_0^2\Big). \tag{$**$}
$$

Conditions (i) and $(**)$ imply (iii) and that for all $\varepsilon > 0$ and $\alpha < 3/2$,

$$
\lim_{n \to \infty} \sup_{N, l} P\Big(\sup_{t \le T}\|(I - P_{n,N}) A(t)\|_{\alpha, N} \ge \varepsilon\Big) = 0. \tag{$***$}
$$

Basic calculations using $(*)$ and (i) imply that for $0 \le t,\ t+u \le T$, we have

$$(****)\qquad \sup_{0\le t,\,t+u\le T} \|P_{n,N}(A(t+u)-A(t))\|_\infty \le C(T,n)\Big(1+\sup_{t\le T}\|X(t)\|_0^2\Big)|u|.$$

Now (iii), $(****)$ and Theorem 7.2 of Chapter 3 of Ethier and Kurtz (1986) imply the distributions of $\{P_{n,N}A\}$ for n fixed are relatively compact on $D([0,T]\colon H_0)$; $(***)$ and Problem 18 of Chapter 3 of Ethier and Kurtz (1986) imply the same for $\{A\}=\{A_{N,l}\}$. This proves (iv) since $A\in C([0,T]\colon H_0)$, which is a closed subset of $D([0,T]\colon H_0)$.

Part (v) follows from (i) and (3.13) (with Y_D in place of $\overline{Y}_D$) by choosing ρ large but fixed. Also (v) and Lemma 3.3(i) imply (vi) since

$$\begin{aligned}\|Y_D(t)\|_{-\alpha}^2 &= \|P_nY_D(t)\|_{-\alpha}^2 + \|(I-P_n)Y_D(t)\|_{-\alpha}^2\\ &\le \|P_nY_D(t)\|_0^2 + n^{-2\alpha}\|Y_D(t)\|_0^2.\end{aligned}$$

To prove (vii) it suffices, by (i), to consider $\|Y_D(t\wedge\tau)\|_0$, where $\tau=\inf\{t\colon \|X(t)\|_0\ge\rho\}$ for $\rho<\infty$. However, applying (3.13) with Y_D in place of $\overline{Y}_D$ shows that for any $a>0$, $P(\sup_{t\le T}\|Y_D(t\wedge\tau)\|_0\ge a)\to 0$ if $l\to\infty$ as $N\to\infty$. $\square$

LEMMA 3.5. *Assumptions* A *and* A2 *imply* (i)–(iv) *below hold. Assumptions* A, A2 *and* B1 *imply* (v) *below holds.*

$$\text{(i)}\qquad \sup_{t\le T}\left|\int_0^t \langle Y_D^2(s)-l^{-1}X(s),e_m\rangle\,ds\right|\to 0$$

in probability for $e_m=\varphi_m$ *or* ψ_m.

(ii) *The distributions of* $\{V\}=\{V_{N,l}\}$ [*defined by* (2.9)] *on* $D([0,T]\colon H_0)$ *are relatively compact.*

(iii) $\varepsilon(t)$, *defined in* (2.11), *satisfies* $\sup_{t\le T}|\langle\varepsilon(t),e_{m,N}\rangle|\to 0$ *in probability for* $e_{m,N}=\varphi_{n,N}$ *or* $\psi_{m,N}$.

(iv) *The distributions of* $\{X\}=\{X_{N,l}\}$ *on* $D([0,T]\colon H_{-\alpha})$ *for any* $\alpha>0$ *are relatively compact.*

(v) *The distributions of* $\{X\}=\{X_{N,l}\}$ *on* $D([0,T]\colon H_0)$ *are relatively compact.*

PROOF. Before proving (i) we sketch the idea. For $0\le\mu\le t$, let $m_t(\mu)=\int_0^\mu T_N(t-s)\,dZ_D(s)$ and note $m_t(\mu)$ is a martingale in μ with $m_t(t)=Y_D(t)$. From (2.7) applied to Z_D, it follows that $m_t^2(\mu)-F(X)(t,\mu)$ is a martingale for $F(X)(t,\mu)$ as subsequently defined. For $e_{k,N}=\varphi_{k,N}$ or $\psi_{k,N}$ with k fixed, the martingale $\langle m_t^2(\mu)-F(X)(t,\mu),e_{k,N}\rangle$ converges to 0 because its quadratic variation is forced to 0 as $N\to\infty$. Thus, $m_t^2(t)=Y_D^2(t)$ and $\langle m_t^2(t),e_{k,N}\rangle\approx\langle F(X)(t,t),e_{k,N}\rangle$. Basic calculations also show that $\langle F(X)(t,t),e_{k,N}\rangle\approx\langle l^{-1}X(t),e_{k,N}\rangle$. We now give the proof.

For $0 \le \mu \le t$, let

$$
\begin{aligned}
&F(X)(t,\mu)\\
&\quad = (Nl)^{-1} \sum_{m,n} \int_0^{\mu} \exp[(-\beta_{m,N} - \beta_{n,N})(t-s)]\\
&\qquad \times [\langle X(s), (\nabla^+ \varphi_{m,N})(\nabla^+ \varphi_{n,N}) + (\nabla^- \varphi_{m,N})(\nabla^- \varphi_{n,N})\rangle \varphi_{m,N}\varphi_{n,N}\\
&\qquad\quad + \langle X(s), (\nabla^+ \psi_{m,N})(\nabla^+ \psi_{n,N}) + (\nabla^- \psi_{m,N})(\nabla^- \psi_{n,N})\rangle \psi_{m,N}\psi_{n,N}\\
&\qquad\quad + 2\langle X(s), (\nabla^+ \varphi_{m,N})(\nabla^+ \psi_{n,N})\\
&\qquad\quad + (\nabla^- \varphi_{m,N})(\nabla^- \psi_{n,N})\rangle \varphi_{m,N}\psi_{n,N}]\, ds.
\end{aligned}
$$

In the proof of Lemma 5.3 of Blount (1994) it is shown that for $e_{k,N} = \varphi_{k,N}$ or $\psi_{k,N}$ we have

$$
\begin{aligned}
&\langle F(X)(t,\mu), e_{k,N}\rangle\\
(*)\qquad &\quad = 2(Nl)^{-1}\Bigg[\sum_{m+k\le N} \int_0^{\mu} \exp[-(\beta_{m,N} + \beta_{m+k,N})(t-s)]\\
&\qquad\qquad \times (\beta_{m,N} + \beta_{m+k,N})\langle X(s), e_{k,N}\rangle\, ds\Bigg]\\
&\qquad + \varepsilon_k(X,\mu,t),
\end{aligned}
$$

where $|\varepsilon_k(X,\mu,t)| \le C(k)(\log N)(Nl)^{-1} \sup_{s\le\mu}\langle X(s), 1\rangle$. Integrating $(*)$ and changing the order of integration in the resulting double integral shows

$$
(**)\qquad \int_0^t \langle F(X)(s,s), e_{k,N}\rangle\, ds = l^{-1}\int_0^t \langle X(s), e_{k,N}\rangle\, ds + \delta_k(t),
$$

where $\sup_{t\le T}|\delta_k(t)| \le C(k)(\log N)(Nl)^{-1}\sup_{t\le T}\langle X(t), 1\rangle$. For $\tau = \inf\{t: \|X(t)\|_0 \ge \rho\}$ with $\rho < \infty$, let

$$
\tilde{Y}_D(t) = \int_0^t T_N(t-s)\, dZ_D(s\wedge\tau).
$$

As a consequence of (2.7), it is shown in the proof of Lemma 6.3 in Blount (1994) that

$$
E\left(\left\langle\left(\tilde{Y}_D\right)^2(t) - F(X)(t, t\wedge\tau), e_{k,N}\right\rangle^2\right) \le C(T, l_0, k, \rho)N^{-1/2}.
$$

This implies

$$
(***)\qquad \int_0^T \left|\left\langle\left(\tilde{Y}_D\right)^2(t) - F(X)(t,t\wedge\tau), e_{k,N}\right\rangle\right| ds \to 0
$$

in probability. Then (i) follows from $(**)$, $(***)$ and Lemma 3.4(i).

For (ii) recall $V(t) = T_N(t)X(0) + \int_0^t T_N(t-s)R(X(s)\,ds + Y_B(t)$. The map $(x+y+z)(t) \to x(t) + y(t) + z(t)$ from $C([0,T]\colon H_0) \times C([0,T]\colon H_0) \times D([0,T]\colon H_0)$ into $D([0,T]\colon H_0)$ is continuous. Thus Assumption A2, Lemma 3.3(iv) and Lemma 3.4(iv) imply (ii) holds.

For (iii) recall $\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t)$. By Lemmas 3.3(i) and 3.4(ii), $\sup_{t \le T} |\langle \varepsilon_1(t), e_{m,N} \rangle| \to 0$ in probability, and the same holds for $\varepsilon_3(t)$ by (i). To deal with $\varepsilon_2(t)$ we first claim that given $\delta, \varepsilon > 0$ there exists $\rho(\varepsilon, T, \delta, \alpha) < \infty$ such that for each $\alpha < \frac{1}{2}$,

$$(*) \qquad \sup_{N,l} P\Big(\sup_{0 < \delta \le t \le T} \|V(t)\|_{\alpha, N} > \rho \Big) < \varepsilon.$$

Assume $(*)$ for now and consider the term in $\varepsilon_2(t)$ given by

$$\begin{aligned} \sup_{t \le T} \left| \int_0^t \langle e_{m,N} Y_D(s), V(s) \rangle \, ds \right| &\le \delta 2^{1/2} \sup_{0 \le t \le T} \|Y_D(t)\|_0 \|V(t)\|_0 \\ &\quad + TC(m) \sup_{0 < \delta \le t \le T} \|Y_D(t)\|_{-1/4, N} \cdot \|V(t)\|_{1/4, N}. \end{aligned}$$

Since δ is arbitrary, it follows from (v) and (vi) of Lemma 3.4, (ii) and $(*)$ that $\sup_{t \le T} |\int_0^t \langle e_{m,N}, Y_D(s) V(s) \rangle \, ds| \to 0$ in probability. Then (iii) follows after applying (ii), and Lemma 3.4(ii) and (v) to the remaining terms in $\langle \varepsilon_2(t), e_{m,N} \rangle$. It remains to prove $(*)$, but this follows from Lemmas 3.3(v), 3.4(iii), Assumption A2 and the spectral properties of $T_N(t)$.

Parts (iv) and (v) follow from (2.9), (ii) and Lemma 3.4(ii), (vi) and (vii). $\square$

LEMMA 3.6. *Let Assumptions* A1 *and* A2 *hold.*

(i) *Assumption* B1 *implies the distributions of* $\{(X, Z_B)\}$ *on* $D([0,T]: H_0 \times H_{-\beta})$ *are relatively compact for any* $\beta > \frac{1}{2}$ *and any distributional limit* $(\psi, M) \in C([0,T]: H_0 \times H_{-\beta})$ *almost surely and satisfies* (3.1).

(ii) *Assumption* B2 *implies the distributions of* $\{(X, Z_B)\}$ *on* $D([0,T]: H_{-\alpha} \times H_{-\beta})$ *are relatively compact for any* $\alpha > 0$, $\beta > \frac{1}{2}$ *and any distributional limit* $(\psi, M) \in C([0,T]: H_0 \times H_{-\beta})$ *almost surely and satisfies* (3.3).

(iii) *In* (i) *and* (ii), *M is a martingale with respect to the filtration* $\{\sigma(\psi(s): s \le t)\}$, *and if f is a trigonometric polynomial,* $\langle M, f \rangle$ *has quadratic variation* $\langle\langle M, f \rangle\rangle(t) = \gamma \int_0^t \langle \psi(s), f^2 \rangle \, ds$. *In addition,* (3.2) *and* (3.4) *hold and* $\psi \in C([0,T]: H_0) \cap C((0,T]: H_\alpha)$ *a.s. for any* $\alpha < \frac{1}{2}$.

PROOF. From (2.10) we have

$$\begin{aligned} P_n Z_B(t) &= P_n V(t) - P_n V(0) - \int_0^t \Delta P_n V(s) \, ds \\ &\quad - \int_0^t P_n R(V(s)) \, ds + d_2 l^{-1} \int_0^t P_n V(s) \, ds + \delta_n(t), \end{aligned} \tag{3.14}$$

where

$$\delta_n(t) = (P_{n,N} - P_n)\Big[-Z_B(t) + V(t) - V(0) - \int_0^t R(V(s))\,ds + d_2 l^{-1}\int_0^t V(s)\,ds\Big] + \int_0^t (\Delta P_n - \Delta_N P_{n,N})V(s)\,ds - P_{n,N}(\varepsilon(t) + Z_R(t) + Z_D(t)).$$

By (2.4), Lemma 3.3(i), (ii) and (iii) and Lemma 3.5(ii) and (iii),

$$\sup_{t\le T}\|\delta_n(t)\|_\infty \to 0 \quad \text{in probability for fixed } n. \tag{3.15}$$

From (3.14) we can write

$$(V, P_n Z_B) = f_n(V) + (0, \delta_n),$$

where

$$f_n\colon D([0,T]\colon H_0) \to D([0,T]\colon H_0^2)$$

is continuous. By (3.15) and Lemma 3.5(ii) this implies the distributions of $\{(V, P_n Z_B)\}$ are relatively compact on $D([0,T]\colon H_0^2)$ for fixed n. By Lemma 3.3(iii) and Problem 18 of Chapter 3 in Ethier and Kurtz (1986) the same holds for the distributions of $\{(V, Z_B)\}$ on $D([0,T]\colon H_0 \times H_{-\beta})$ for $\beta > \frac{1}{2}$. Since $\|\delta X(t)\|_0 \le 2^{1/2}(Nl)^{-1}$, it follows that any distributional limit (ψ, M) of (V, Z_B) is in $C([0,T]\colon H_0 \times H_{-\beta})$ almost surely. By (2.9) with Lemma 3.4(ii), (vi) and (vii), we obtain relative compactness for $\{(X, Z_B)\}$ and the fact that any distributional limit of (X, Z_B) is also one for (V, Z_B). Applying the continuous mapping theorem, we first take the limit in distribution in (3.14) with n fixed, to obtain

$$P_n M(t) = P_n\psi(t) - P_n\psi(0) - \int_0^t \Delta P_n\,\psi(s)\,ds - \int_0^t P_n R(\psi(s))\,ds$$

[or $R(\psi(s)) + d_2 l^{-1}\psi(s)$ in place of $R(\psi(s))$ when l is held constant].

Since $\Delta\colon H_\alpha \to H_{\alpha-2}$ is continuous, $R(x)$ is quadratic and $\sum m^{-2\alpha} < \infty$ for $\alpha > \frac{1}{2}$, the map defined by

$$f(\beta)(t) = \beta(t) - \beta(0) - \int_0^t \Delta\beta(s)\,ds - \int_0^t R(\beta(s))\,ds$$

is continuous from $C([0,T]\colon H_0)$ into $C([0,T]\colon H_{-2})$, and $P_n M = P_n f(\psi)$. Letting $n \to \infty$ shows (3.1) and (3.3) hold in $C([0,T]\colon H_{-2})$.

We subsequently show that $\psi \in C((0,T)\colon H_\alpha)$ for any $\alpha < \frac{1}{2}$. Since f is also a continuous map from $C((0,T]\colon H_\alpha)$ into $C((0,T]\colon H_{\alpha-2})$ for any $\alpha \ge 0$, we again use $\lim_{n\to\infty} P_n M = \lim_{n\to\infty} P_n f(\psi)$ to show that (3.1) and (3.3) hold as stated in Theorem 3.1. This proves (i) and (ii).

For $e_m = \varphi_m$ or ψ_m, (2.7) implies

$$(*) \qquad E\big(\langle Z_B(t), e_m\rangle^2\big) \le 2\gamma\int_0^t E\langle X(s), 1\rangle\, ds$$

and for any trigonometric polynomial f,

$$(**) \qquad E\Big[\sup_{t\le T}\langle Z_B(t), f\rangle^4\Big] \le C(f)\left[E\left[\left(\gamma\int_0^T\langle X(s), 1\rangle\, ds\right)^2\right] + (Nl)^{-4}\right],$$

where $(**)$ follows from the Burkholder–Davis–Gundy inequality [Theorem 21.1 of Burkholder (1973)].

Let $x(t) = (X(t), Z_B(t))$ and consider $(x(t), \langle Z_B(t), e_m\rangle)$. By $(*)$, Lemma 3.1(i) and Problem 7 of Chapter 7 in Ethier and Kurtz (1986), we obtain that $\langle M, e_m\rangle$ and then M is a martingale with respect to $\{\sigma(\psi(s)\colon s \le t)\}$. Using $(**)$, Lemma 3.1(iii) and (2.7), the same argument applied to $(x(t), \langle Z_B(t), f\rangle^2 - \gamma\int_0^t\langle X(s), (P_N f)^2\rangle\, ds)$ shows that $\langle M(t), f\rangle^2 - \gamma\int_0^t\langle \psi(s), f^2\rangle\, ds$ is a martingale with respect to $\{\sigma(\psi(s)\colon s \le t)\}$.

From (3.1) we have

$$(3.16) \qquad \begin{aligned} P_n\psi(t) &= P_nT(t)\psi(0) + P_n\int_0^t T(t-s)R(\psi(s))\, ds \\ &\quad + P_n\int_0^t T(t-s)\, dM(s). \end{aligned}$$

For $e_m = \varphi_m$ or ψ_m,

$$|\langle T(t)\psi(0), e_m\rangle| = \exp(-\beta_m t)|\langle\psi(0), e_m\rangle| \le 2^{1/2}\exp(-\beta_m t)\langle\psi(0), 1\rangle$$

and for $t \le T$,

$$\begin{aligned} \left|\left\langle\int_0^t T(t-s)R(\psi(s)), e_m\right\rangle ds\right| &= \left|\int_0^t \exp(-\beta_m(t-s))\langle R(\psi(s)), e_m\rangle\, ds\right| \\ &\le C(T)\Big(1 + \sup_{t\le T}\|\psi(t)\|_0^2\Big)(1+m^2)^{-1}. \end{aligned}$$

Letting $n \to \infty$ implies that almost surely, for any $\alpha < \frac{3}{2}$,

$$(3.17) \qquad \begin{aligned} &T(\cdot)\psi(0) + \int_0^{\cdot} T(\cdot - s)R(\psi(s))\, ds \\ &\quad \in C([0,T]\colon H_0) \cap C((0,T)\colon H_\alpha) \\ &\quad\quad \cap C((0,T]\colon C([0,1])). \end{aligned}$$

Note that the previous argument holds as well with $R(\psi(s)) - l^{-1}\, d_2\psi(s)$ in place of $R(\psi(s))$ for the case when l is constant.

Consider $P_n\int_0^t T(t-s)\, dM(s) = \int_0^t T(t-s)\, dP_nM(s)$. Using the stochastic bound on $\sup_{t\le T}\|\psi(t)\|_0$, the proofs of Lemmas 3.2 and 3.3(iv) and (v) show that the distributions of $\{\int_0^{\cdot} T(\cdot - s)\, dP_n M(s)\}$ are relatively compact on $C([0,T]\colon H_\alpha)$ for any $\alpha < \frac{1}{2}$. Thus the proof of (iii) is complete by letting $n \to \infty$ in (3.16) and using (3.17). $\square$

Note that Lemma 3.6 and distributional uniqueness for the limiting equation complete the proof of Theorem 3.1.

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