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STOCHASTIC FLOWS FOR NONLINEAR SECOND-ORDER PARABOLIC SPDE

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The existence of stochastic flows in L^2-spaces is proved for a stochastic reaction–diffusion equation of second order in a bounded domain.

1. Introduction.

1.1. *Aim of the paper*. Let D be a regular bounded open domain of $\mathbb{R}^d$ and let $a_{ij}, a_i, a_0, b_i^k, c^k$, $i = 1, \dots, d$, $k = 1, \dots, n$, be real-valued functions in $\overline{D}$, which, for the sake of simplicity, we assume to be of class $C^\infty(\overline{D})$. Let $L = L(x, \partial)$ be the second-order strongly elliptic operator in D:

$$L(x, \partial)u(x) = \sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} a_i(x)\frac{\partial u(x)}{\partial x_i} + a_0(x)u(x).$$

Let $M^k = M^k(x, \partial)$ be the first-order differential operators in D, with $k = 1, \dots, n$,

$$M^k(x, \partial)u(x) = \sum_{i=1}^{d} b_i^k(x)\frac{\partial u(x)}{\partial x_i} + c^k(x)u(x).$$

We assume that there exists $\eta > 0$ such that

$$(1) \qquad \sum_{i,j=1}^{d} \left\{ a_{ij}(x) - \sum_{k=1}^{n} b_i^k(x) b_j^k(x) \right\} \xi_i \xi_j \ge \eta |\xi|^2$$

for all $\xi \in \mathbb{R}^d$ and $x \in D$.

Let (Ω, F, F_t, P) be a stochastic basis and let $w(t) = (w^1(t), \dots, w^n(t))$ be a standard n-dimensional Brownian motion. Consider the stochastic reaction–diffusion–convection equation in D, either in the Dirichlet boundary condition case

$$(2) \qquad \begin{aligned} du &= (Lu + f(u))\,dt + \sum_{k=1}^{n} M^k u\, dw^k(t), \\ u|_{\partial D} &= 0, \\ u(0) &= u_0 \end{aligned}$$

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or in the Neumann case

$$
(3)\qquad
\begin{aligned}
&du = (Lu + f(u))\,dt + \sum_{k=1}^{n} M^k u\, dw^k(t),\\
&\left.\frac{\partial u}{\partial n_A}\right|_{\partial D} = 0,\\
&u(0) = u_0,
\end{aligned}
$$

where

$$\frac{\partial u}{\partial n_A} = \sum_{i,j=1}^{d} a_{ij} n_j \frac{\partial u}{\partial x_i}$$

and $n = n(x)$ is the outward normal on ∂D. Here

$$f(u) = \sum_{h=0}^{2p-1} a_h u^h,$$

with real coefficients a_h subject to the condition

$$a_{2p-1} < 0.$$

[More general monotone nonlinearities can be considered here, such as the sum of a polynomial of the previous form plus a Lipschitz continuous function of u plus a given function $g(t,x)$ with suitable regularity; we do not insist on such a level of generality.] Equations (2) and (3) may model reaction–diffusion phenomena in a fluid that occupies the region D and moves with velocity $\sum_{k=1}^{n} \mathbf{b}^k(x)\,dw^k(t)/dt + \mathbf{a}(x)$, where $\mathbf{b}^k(x)$ denotes the vector of components $b_i^k(x)$ and $\mathbf{a}(x)$ the vector of components $a_i(x)$.

It is well known (cf. [3], [8] and [10]) that for all $u_0 \in L^2(D)$ [or $u_0 \in L^2(\Omega, \mathcal{F}_0, P; D)$] each one of the previous equations has a unique solution u, progressively measurable,

$$
\begin{aligned}
u \in L^2(\Omega; C([0,T]; L^2(D))) &\cap L^2(\Omega \times [0,T]; H^1(D))\\
&\cap L^{2p}(\Omega \times [0,T] \times D).
\end{aligned}
$$

Moreover, for each $t \geq 0$, the mapping $u_0 \mapsto u(t)$ is continuous from $L^2(D)$ to $L^2(\Omega; L^2(D))$. The aim of this paper is to prove the following stronger result.

THEOREM 1.1. *For the Dirichlet problem* (2) *there exists a stochastic flow in* $L^2(D)$.

THEOREM 1.2. *For the Neumann problem* (3) *assume that all the vector fields* $\mathbf{b}^k(x)$ *are tangent to the boundary. Then there exists a stochastic flow in* $L^2(D)$.

As it is more carefully explained in the next subsection, the property of stochastic flow amounts to saying that the mapping $u_0 \mapsto u(t, \omega)$ is continuous from $L^2(D)$ to $L^2(D)$, for P-a.e. $\omega \in \Omega$.

The problem of existence of stochastic flows for infinite-dimensional stochastic systems is an intriguing one (cf. [13]). The flow property is usually trivial for deterministic infinite-dimensional systems (where there is uniqueness of solutions); for stochastic finite-dimensional equations the existence of stochastic flows has been proved in a wide generality (cf. [7] and [9] and the references therein). But none of the methods developed in the finite-dimensional case extends to the infinite-dimensional case, up to now. Of course, the case of additive noise is usually easy because it can be reduced to a deterministic equation by a change of variable. Another conceptually easy method is the robust equation approach (time change) (cf. [8]), or other similar methods that reduce the stochastic equation to a deterministic one. All these methods work under very particular assumptions and do not cover Theorems 1.1 and 1.2.

For linear stochastic equations with multiplicative noise there is a number of methods (cf. [2], [4], [5], [6] and [12]), but also in that case the answer is open for very simple equations. Theorem 1.1. in the linear case is covered in [6] by a Feynman–Kac representation formula, but Theorem 1.2, also in the linear case, has not been obtained up to now by methods different from those of this paper (except for $d \le 5$; see [5]).

Finally, a trivial class of nonlinear flows can be constructed when the diffusion operator is skew-symmetric; see Section 1.3. The more complex method presented in this paper to prove Theorems 1.1 and 1.2 (Section 2) was originally suggested by the simple idea of the skew symmetry.

1.2. *The concept of stochastic flow*. Let H be a real separable Hilbert space and $(\Omega, \mathcal{F}, P)$ a complete probability space. Let $u(t, s; u_0)$, $s \le t$, $u_0 \in H$, denote the solution at time t of a certain stochastic equation in H [over $(\Omega, \mathcal{F}, P)$] with given initial value u_0 at time s. The problem of existence of a stochastic flow is the problem of the existence of a regular version of the mapping $u_0 \to u(t, s; u_0)$ for fixed $s \le t$ (or, when possible, uniformly in s and t). This means the existence of a mapping $\omega \to \varphi_{s,t}(\omega)$ from Ω to the space of continuous mappings in H such that

$$(4) \qquad \varphi_{s,t}(\omega)u_0 = u(t, s, u_0)(\omega) \quad P\text{-a.s.}$$

for all $u_0 \in H$. The mapping $\varphi_{s,t}(\omega)$ is called the stochastic flow in H associated with the given equation.

As to the existence of a regular version of a given infinite-dimensional random field, we have the following preliminary result. Let H and Y be two real separable Hilbert spaces with norms $|\cdot|_H$ and $|\cdot|_Y$. (The result holds true also in Polish spaces, but we do not stress this generality.) Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, as above. Finally, let $L^0(\Omega; Y)$ be the space of Y-valued random variables. We call a mapping $\Phi: H \to L^0(\Omega; Y)$ a Y-valued random field with parameter space H. Moreover, we say that Φ has a continuous version if there exists a mapping $\omega \to \varphi(\omega)$, from Ω to $C(H, Y)$, the space of continuous mappings from H to Y, such that

$$(5)\qquad \varphi(\omega)x = \Phi(x)(\omega), \quad P\text{-a.s.}$$

for all $x \in H$.

LEMMA 1.1. *Let $\Phi : H \to L^0(\Omega; Y)$ be a given random field. Assume that for each ball S in H there exist two random variables $c_S(\omega) \geq 0$ and $a_S(\omega) > 0$ such that*

$$(6)\qquad |\Phi(x)(\omega) - \Phi(y)(\omega)|_Y \leq c_S(\omega)|x - y|_H^{a_S(\omega)} \quad P\text{-a.s.}$$

for all $x, y \in S$. Then Φ has a continuous version φ [satisfying (5)] such that, for P-a.e. $\omega \in \Omega$, $\varphi(\omega)$ is Hölder continuous on the balls of H, with the Hölder constants given by (6). Finally, this regular version $\varphi(\omega)$ is unique up to modifications on sets of measure 0.

The proof is not difficult and it is given in [5].

1.3. *A trivial example of nonlinear flow.* The result of the present section is analogous to that of [1] on stochastic Navier–Stokes equations with multiplicative noise. For simplicity, we consider here a globally Lipschitz nonlinearity F, in contrast to [1], but the skew-symmetry condition on B^k imposed below, crucial in view of the existence of the flow, is the same as in [1], and could be motivated by applications to fluid dynamic problems (see Remark 2 below). The reason to include this subsection is to fix the idea used in the proof of Theorems 1.1 and 1.2 (but we do not use the result of this section). Indeed, the method of proof of such theorems has been devised in an attempt to extend the trivial method of this section to the case of multiplicative diffusion terms which are the sum of a skew-symmetric part and an easy part (a zero-order differential operator) that could be treated by time change.

Let H be a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. Consider the equation

$$(7)\qquad \begin{aligned} &du(t) = Au(t)\,dt + F(u(t))\,dt + \sum_{k=1}^{n} B^k u(t)\,dw^k(t),\\ &t \in [0,T],\\ &u(0) = u_0 \in H. \end{aligned}$$

Assume that A is the infinitesimal generator of an analytic semigroup in H, F is a globally Lipschitz mapping in H and B^k is a linear continuous mapping from $D((-A)^{1/2})$ [the domain of the fractional power $(-A)^{1/2}$] to H such that

$$(8)\qquad \begin{aligned} &\tfrac{1}{2}\sum_{k=1}^{n} |B^k u|^2 \leq -\eta\langle Au, u\rangle + \lambda|u|^2, \qquad u \in D(A),\\ &\langle B^k u, u\rangle = 0, \qquad u \in D((-A)^{1/2}), \qquad k = 1, \dots, n, \end{aligned}$$

for some constant $\eta \in (0,1)$ and $\lambda \geq 0$. One can prove [4] without the second assumption in (8) that (7) has a unique progressively measurable mild solution

$$u = u(\cdot, u_0) \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; D((-A)^{1/2})).$$

Using, in addition, the second assumption in (8), we can prove the following result.

THEOREM 1.3. *For all $t \in [0,T]$ the random field $u_0 \mapsto u(t,u_0)$, from H to $L^2(\mathcal{F}_t; H)$, has a Lipschitz continuous modification $\{\varphi_t(\omega); \omega \in \Omega\}$ [i.e., $\varphi_t(\omega)$ is a Lipschitz continuous mapping in H for all $\omega \in \Omega$].*

PROOF. Let $J_m = m(m - A)^{-1}$ and $u_m = J_m u$. Then u_m satisfies the equation

$$(9)\qquad du_m(t) = Au_m(t)\,dt + J_m F(u(t))\,dt + \sum_{k=1}^{n} J_m B^k u(t)\,dw^k(t).$$

Now fix $u_0, v_0 \in H$ and let u and v be the solutions of (9) corresponding to the initial values u_0 and v_0. Moreover, let u_m and v_m be defined as above. Finally, let $z = u - v$ and $z_m = u_m - v_m$. Then, by the Itô formula,

$$\begin{aligned}
\tfrac{1}{2} d|z_m(t)|^2 = {}& \Big(\langle Az_m(t), z_m(t)\rangle + \tfrac{1}{2}\sum_{k=1}^{n} |B^k z_m(t)|^2 \Big) dt \\
& + \Big(\langle J_m F(u(t)) - J_m F(v(t)), z_m(t)\rangle \\
& \quad + \tfrac{1}{2}\sum_{k=1}^{n} |J_m B^k z(t)|^2 - \tfrac{1}{2}\sum_{k=1}^{n} |B^k z_m(t)|^2 \Big) dt \\
(10)\qquad & + \sum_{k=1}^{n} \langle z_m(t), J_m B^k z(t)\,dw^k(t)\rangle \\
\leq {}& \Big(\lambda |z_m(t)|^2 + \langle J_m F(u(t)) - J_m F(v(t)), z_m(t)\rangle \\
& \quad + \tfrac{1}{2}\sum_{k=1}^{n} |J_m B^k z(t)|^2 - \tfrac{1}{2}\sum_{k=1}^{n} |B^k z_m(t)|^2 \Big) dt \\
& + \sum_{k=1}^{n} \langle z_m(t), J_m B^k z(t)\,dw^k(t)\rangle,
\end{aligned}$$

so that, using the integral formulation of this inequality, after passage to the limit as $m \to \infty$, we have

$$\begin{aligned}
|z(t)|^2 \leq {}& |u_0 - v_0|^2 + \int_s^t \big(\lambda |z(r)|^2 + \langle F(u(r)) - F(v(r)), z(r)\rangle \big)\,dr \\
(11)\qquad & + \sum_{k=1}^{n} \int_s^t \langle z(r), B^k z(r)\,dw^k(r)\rangle.
\end{aligned}$$

By the second assumption in (8), the Itô integral in (11) vanishes; hence, using the Lipschitz continuity of F in H and applying the Gronwall lemma to (11), we obtain

$$\|u(t) - v(t)\| \le c\|u_0 - v_0\| \quad P\text{-a.s.} \tag{12}$$

for a suitable (deterministic) constant $c > 0$. Therefore, from Lemma 1.1, we infer the existence of a Lipschitz continuous modification of the random field $u_0 \mapsto u(t, u_0)$. ∎

REMARK 1. From (11) we see that a monotonicity assumption on F is sufficient in place of the global Lipschitz condition. To keep the exposition as elementary as possible, we do not treat the monotone case here, which requires more care at the level of existence of solutions.

REMARK 2. Let A and B^k be differential operators as in Section 1.1. If $\operatorname{div} \mathbf{b}^k = 0$ in D, then the skew-symmetry condition in (8) is satisfied. In certain applications to fluid dynamic problems, the vector fields $\mathbf{b}^k(x)$ have the meaning of velocity fields of the fluid; in this case the assumption $\operatorname{div} \mathbf{b}^k = 0$ corresponds to the incompressibility of the fluid.

2. Proof of Theorems 1.1 and 1.2. Let $\tilde{M}^k = \tilde{M}^k(x, \partial)$ and $\hat{M}^k = \hat{M}^k(x, \partial)$ be the first-order differential operators associated with $M^k(x, \partial)$, defined as

$$\tilde{M}^k(x, \partial)u(x) = \sum_{i=1}^d b_i^k(x)\frac{\partial u(x)}{\partial x_i} + \tilde{c}^k(x)u(x),$$

$$\hat{M}^k(x, \partial)u(x) = \sum_{i=1}^d b_i^k(x)\frac{\partial u(x)}{\partial x_i} + \left(c^k(x) + \tilde{c}^k(x)\right)u(x),$$

where the functions $\tilde{c}^k(x)$ are defined by the conditions

$$2\left(c^k(x) + \tilde{c}^k(x)\right) = \operatorname{div} \mathbf{b}^k(x).$$

The previous definition is designed to have the following essential properties:

$$\begin{aligned} \hat{M}^k(uv) &= -(\hat{M}^k)^*(uv), \\ \hat{M}^k(uv) &= u\tilde{M}^k v + vM^k u. \end{aligned} \tag{13}$$

Equation (13) has to be understood in the following sense.

LEMMA 2.1. *Let $\hat{M} = \hat{M}(x, \partial)$ be an operator defined as*

$$\hat{M}(x, \partial)u(x) = \sum_{i=1}^d b_i(x)\frac{\partial u(x)}{\partial x_i} + \hat{c}(x)u(x).$$

Assume that either $u, v \in H^1(D)$ satisfy $uv = 0$ on ∂D or $\mathbf{b}(x)$ is tangent to the boundary. Then

$$\hat{M} = -\hat{M}^*$$

(*in the sense that*

$$\int_D (\hat{M}u)v\,dx = -\int_D u\hat{M}v\,dx$$

for $u, v \in H^1(D)$ *specified as above) is equivalent to*

$$2\hat{c}(x) = \operatorname{div}\mathbf{b}(x).$$

PROOF. We have

$$\begin{aligned}\int_D (\hat{M}u)v\,dx &= \sum_{i=1}^{d}\int_D v b_i \frac{\partial u}{\partial x_i}\,dx + \int_D \hat{c}uv\,dx\\ &= -\sum_{i=1}^{d}\int_D u \frac{\partial}{\partial x_i}(b_i v)\,dx + \int_D \hat{c}uv\,dx + \int_{\partial D} vu\mathbf{b}\cdot n\,ds\\ &= -\int_D uv\operatorname{div}\mathbf{b}\,dx + \int_D 2\hat{c}uv\,dx - \int_D u\hat{M}v\,dx. \qquad \blacksquare\end{aligned}$$

LEMMA 2.2. *Let* $M, \tilde{M}, \hat{M}$ *be operators defined as*

$$M(x,\partial)u(x) = \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x),$$

$$\tilde{M}(x,\partial)u(x) = \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + \tilde{c}(x)u(x),$$

$$\hat{M}(x,\partial)u(x) = \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + (c(x) + \tilde{c}(x))u(x).$$

Then

$$u\tilde{M}v + vMu = \hat{M}(uv).$$

Moreover, assume that either $u, v \in H^1(D)$ *satisfy* $uv = 0$ *on* ∂D *or the vector field* $\mathbf{b}$ *is tangent to the boundary. Then*

$$2(c(x) + \tilde{c}(x)) = \operatorname{div}\mathbf{b}(x)$$

implies

$$\hat{M} = -\hat{M}^*$$

(in the sense of the previous lemma).

PROOF. We have

$$\begin{aligned}\hat{M}(uv) &= \sum_{i=1}^{d} b_i\left[\frac{\partial u}{\partial x_i}v + u\frac{\partial v}{\partial x_i}\right] + (c + \tilde{c})uv\\ &= vMu + u\tilde{M}v.\end{aligned}$$

The second part of the lemma is just a rewriting of the previous lemma. ∎

Extend all the coefficients $a_{ij}, a_i, \dots$ to $\mathbb{R}^d$ in such a way that they still are of class C', satisfy the coercivity condition (1) and have compact support. Consider the stochastic parabolic equation in $\mathbb{R}^d$:

$$(14)\qquad \begin{aligned} d\tilde{u} &= L\tilde{u}\,dt + \sum_{k=1}^{n}\tilde{M}^k\tilde{u}\,dw^k(t),\\ \tilde{u}(0,x) &= 1.\end{aligned}$$

The solution of this equation (cf. [5], [11] and [14]) has P-a.s. the property $\tilde{u}, \partial\tilde{u}/\partial x_i \in C([0,T]\times\mathbb{R}^d)$, $i=1,\dots,d$, and $\tilde{u}(t,x,\omega)>0$ for all $(t,x)\in[0,T]\times\mathbb{R}^d$. The latter fact follows, for instance, from the representation formula of [14]. Thus, given the domain $D\subset\mathbb{R}^d$, there exist two positive random variables $c_1(\omega)<c_2(\omega)$ such that, P-a.s.,

$$(15)\qquad 0<c_1(\omega)\le\tilde{u}(t,x,\omega)\le c_2(\omega)$$

for all $(t,x)\in[0,T]\times D$, and

$$(16)\qquad \|\tilde{u}(t,\dots,\omega)\|_{W^{1,\infty}(D)}\le c_2(\omega).$$

Let $u_{0,1},u_{0,2}\in L^2(D)$ be given initial conditions for (2) or (3) and let u_1,u_2 be the corresponding solutions. Recall that these equations have to be understood in the following variational sense:

$$(17)\qquad \begin{aligned}\langle u(t),\varphi\rangle &= \langle u_0,\varphi\rangle+\int_0^t a(u(s),\varphi)\,ds\\ &\quad+\int_0^t\langle f(u(s)),\varphi\rangle\,ds+\sum_{k=1}^n\int_0^t\langle M^k u(s),\varphi\rangle\,dw^k(t)\end{aligned}$$

for all $\varphi\in H_0^1(D)\cap C(\overline{D})$ for (2) and $\varphi\in H^1(D)\cap C(\overline{D})$ for (3). Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2(D)$ and $a(u,v)$ is the bilinear form on $H_0^1(D)$ for (2) and on $H^1(D)$ for (3), defined as

$$a(u,v)=\int_D\left[-\sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{i=1}^d a_i\frac{\partial u}{\partial x_i}v+a_0uv\right]dx.$$

In the previous sense we have

$$d(u_1-u_2)=[L(u_1-u_2)+f(u_1)-f(u_2)]\,dt+\sum_{k=1}^n M^k(u_1-u_2)\,dw^k(t).$$

Moreover,

$$d\phi(x)\tilde{u}(t,x)=\phi(x)L\tilde{u}(t,x)\,dt+\sum_{k=1}^n\phi(x)\tilde{M}^k\tilde{u}(t,x)\,dw^k(t)$$

for all $\phi\in C^\infty(\mathbb{R}^d)$ [this can be obtained by taking $\varphi=\phi$ in the equation of type (17) corresponding to (14)]. Thus, by the Itô formula (cf. [10]),

$$\begin{aligned}d\langle u_1-u_2,\phi\tilde{u}\rangle &= \langle d(u_1-u_2),\phi\tilde{u}\rangle+\langle u_1-u_2,d\phi\tilde{u}\rangle+\sum_{k=1}^n\langle M^k(u_1-u_2),\phi\tilde{M}^k\tilde{u}\rangle\,dt\\ &=[\langle L(u_1-u_2),\phi\tilde{u}\rangle+\langle u_1-u_2,\phi L\tilde{u}\rangle]\,dt\\ &\quad+\langle f(u_1)-f(u_2),\phi\tilde{u}\rangle\,dt+\sum_{k=1}^n\langle M^k(u_1-u_2),\phi\tilde{M}^k\tilde{u}\rangle\,dt\\ &\quad+\sum_{k=1}^n[\langle M^k(u_1-u_2),\phi\tilde{u}\rangle+\langle u_1-u_2,\phi\tilde{M}^k\tilde{u}\rangle]\,dw^k(t).\end{aligned}$$

Now

$$
\begin{aligned}
\langle L(u_1-u_2), f\tilde{u}\rangle &+ \langle u_1-u_2, fL\tilde{u}\rangle\\
&= \int_D [f\tilde{u}L(u_1-u_2) + (u_1-u_2)fL\tilde{u}]\,dx\\
&= \int_D f[\tilde{u}L(u_1-u_2) + (u_1-u_2)L\tilde{u}]\,dx\\
&= \int_D f[L(\tilde{u}(u_1-u_2)) + N(u_1-u_2,\tilde{u})]\,dx,
\end{aligned}
$$

where

$$
N(u_1-u_2,\tilde{u}) = a_0(u_1-u_2)\tilde{u} - (\nabla\tilde{u})^T(a+a^T)\nabla(u_1-u_2).
$$

Here a denotes the matrix (a_{ij}). We have used the following fact (we shorten the notation for the partial derivatives):

$$
\begin{aligned}
L(fg) &= \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 fg}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} a_i\frac{\partial fg}{\partial x_i} + a_0 fg\\
&= \sum_{i,j=1}^{d} a_{ij}[f_{ij}g + f_j g_i + f_i g_j + f g_{ij}] + \sum_{i=1}^{d} a_i[f_i g + f g_i] + a_0 fg\\
&= gLf + fLg - a_0 fg + (\nabla g)^T(a+a^T)\nabla f.
\end{aligned}
$$

The previous computation yields

$$
\begin{aligned}
d\langle u_1-u_2, f\tilde{u}\rangle
&= \langle L(\tilde{u}(u_1-u_2)) + N(u_1-u_2,\tilde{u}), f\rangle\,dt\\
&\quad + \langle (f(u_1)-f(u_2))\tilde{u}, f\rangle\,dt\\
&\quad + \left\langle \sum_{k=1}^{n} M^k(u_1-u_2)\tilde{M}^k\tilde{u}, f\right\rangle dt\\
&\quad + \sum_{k=1}^{n} \langle \hat{M}^k(\tilde{u}(u_1-u_2)), f\rangle\,dw^k(t),
\end{aligned}
$$

recalling the definition of $\hat{M}^k$. This means

$$
\begin{aligned}
d((u_1-u_2)\tilde{u}) = \Bigg[&L(\tilde{u}(u_1-u_2)) + N(u_1-u_2,\tilde{u})\\
&+ (f(u_1)-f(u_2))\tilde{u} + \sum_{k=1}^{n} M^k(u_1-u_2)\tilde{M}^k\tilde{u}\Bigg]dt\\
&+ \sum_{k=1}^{n}\hat{M}^k(\tilde{u}(u_1-u_2))\,dw^k(t).
\end{aligned}
$$

Therefore, by the Itô formula [10],

$$
\begin{aligned}
\tfrac{1}{2} d|u_1 - u_2|^2_{L^2(D)} = {} & a(\tilde u(u_1 - u_2), \tilde u(u_1 - u_2))\,dt \\
& + \langle N(u_1 - u_2, \tilde u), \tilde u(u_1 - u_2)\rangle\,dt \\
& + \langle \tilde u(f(u_1) - f(u_2)), \tilde u(u_1 - u_2)\rangle\,dt \\
& + \sum_{k=1}^{n} \langle M^k(u_1 - u_2)\tilde M^k \tilde u, \tilde u(u_1 - u_2)\rangle\,dt \\
& + \tfrac{1}{2} \sum_{k=1}^{n} |\hat M^k(\tilde u(u_1 - u_2))|^2\,dt.
\end{aligned}
$$

The essential fact here is that the Itô term vanishes because of the skew symmetry of $\hat M^k$ (compare with Section 1.3). We have

$$
\begin{aligned}
a(\tilde u(u_1 - u_2), \tilde u(u_1 - u_2)) + \frac{1}{2}\sum_{k=1}^{n} |\hat M^k(\tilde u(u_1 - u_2))|^2 \\
= -\int_D \sum_{i,j=1}^{d}\left(a_{ij} - \sum_{k=1}^{n} b_i^k b_j^k\right) \frac{\partial(\tilde u(u_1 - u_2))}{\partial x_i}\frac{\partial(\tilde u(u_1 - u_2))}{\partial x_j}\,dx \\
+ \int_D \tilde u(u_1 - u_2) N_1(\tilde u(u_1 - u_2))\,dx
\end{aligned}
$$

(where N_1 is a first-order differential operator)

$$
\begin{aligned}
& \le -h\int_D |\nabla(\tilde u(u_1 - u_2))|^2_{\mathbb{R}^d}\,dx + \frac{h}{2}\int_D |\nabla(\tilde u(u_1 - u_2))|^2_{\mathbb{R}^d}\,dx \\
& \quad + C_1 \int_D (\tilde u(u_1 - u_2))^2\,dx \\
& \le -\frac{h}{2}\int_D |\nabla(\tilde u(u_1 - u_2))|^2_{\mathbb{R}^d}\,dx + C_1 c_2(v)^2 |u_1 - u_2|^2_{L^2(D)}
\end{aligned}
$$

for some constant $C_1 > 0$. Moreover,

$$
\begin{aligned}
& \langle N(u_1 - u_2, \tilde u), \tilde u(u_1 - u_2)\rangle + \sum_{k=1}^{n} \langle M^k(u_1 - u_2)\tilde M^k \tilde u, \tilde u(u_1 - u_2)\rangle \\
& \le C_2 |\tilde u|_{W^{1,\infty}(D)} \left(|\nabla(u_1 - u_2)|_{L^2(D)} |u_1 - u_2|_{L^2(D)} + |u_1 - u_2|^2_{L^2(D)}\right) \\
& \le \frac{h}{8} c_1(v)^2 |\nabla(u_1 - u_2)|^2_{L^2(D)} \\
& \quad + \left[\frac{2C_2 c_2(v)^2}{h c_1(v)^2} + C_2 c_2(v)^2\right] |u_1 - u_2|^2_{L^2(D)}
\end{aligned}
$$

for some constant $C_2 > 0$. Finally, since f is weakly monotone, there is l such that

$$
(f(u_1) - f(u_2))(u_1 - u_2) \le l(u_1 - u_2)^2;
$$

thus

$$\langle \tilde u(f(u_1) - f(u_2)), \tilde u(u_1 - u_2)\rangle \le l|\tilde u|^2 \|u_1 - u_2\|^2_{L^2(D)}.$$

Note that

$$-\int_D |\nabla(\tilde u(u_1 - u_2))|^2_{\mathbb{R}^d}\, dx$$

$$= -\int_D |\tilde u \nabla(u_1 - u_2) + (u_1 - u_2)\nabla \tilde u|^2_{\mathbb{R}^d}\, dx$$

$$= -\int_D \tilde u^2 |\nabla(u_1 - u_2)|^2_{\mathbb{R}^d}\, dx - \int_D (u_1 - u_2)^2 |\nabla \tilde u|^2_{\mathbb{R}^d}\, dx$$

$$- \int_D 2\tilde u (u_1 - u_2)\nabla \hat u \cdot \nabla(u_1 - u_2)\, dx$$

$$\le -c_1(\omega)^2 \int_D |\nabla(u_1 - u_2)|^2_{\mathbb{R}^d}\, dx + c_2(\omega)^2 \|u_1 - u_2\|^2_{L^2(D)}$$

$$+ 2c_2(\omega)^2 \|\nabla(u_1 - u_2)\|_{L^2(D)} \|u_1 - u_2\|_{L^2(D)}$$

$$\le -\frac{1}{2}c_1(\omega)^2 \int_D |\nabla(u_1 - u_2)|^2_{\mathbb{R}^d}\, dx$$

$$+ \left[c_2(\omega)^2 + \frac{2c_2(\omega)^2}{c_1(\omega)^2}\right] \|u_1 - u_2\|^2_{L^2(D)}.$$

Collecting all these computations, we have

$$\frac{1}{2}\frac{d}{dt}|\tilde u(u_1 - u_2)|^2_{L^2(D)} + \frac{\varepsilon}{8}c_1(\omega)^2 \int_D |\nabla(u_1 - u_2)|^2_{\mathbb{R}^d}\, dx$$

$$\le \left[\frac{\varepsilon}{2}\left(c_2(\omega)^2 + \frac{2c_2(\omega)^2}{c_1(\omega)^2}\right) + C_1 c_2(\omega)^2\right] \|u_1 - u_2\|^2_{L^2(D)}$$

$$+ \left[\left(\frac{2C_2 c_2(\omega)^2}{\varepsilon c_1(\omega)^2} + C_2 c_2(\omega)^2\right) + l c_2(\omega)^2\right] \|u_1 - u_2\|^2_{L^2(D)}$$

$$= c_3(\omega)\|u_1 - u_2\|^2_{L^2(D)}$$

for some positive r.v. $c_3(\omega)$. It follows that

$$\frac{1}{2}\frac{d}{dt}|\tilde u(u_1 - u_2)|^2_{L^2(D)} \le c_4(\omega)|\tilde u(u_1 - u_2)|^2_{L^2(D)}$$

for some positive r.v. $c_4(\omega)$. Therefore,

$$|\tilde u(t)(u_1(t) - u_2(t))|^2_{L^2(D)} \le \|u_{0,1} - u_{0,2}\|^2_{L^2(D)} \exp(c_4(\omega)t),$$

which implies that

$$\|u_1(t) - u_2(t)\|^2_{L^2(D)} \le \|u_{0,1} - u_{0,2}\|^2_{L^2(D)} \exp\left(c_4(\nu) t\right) \frac{1}{\left(c_1(\nu)\right)^2}.$$

The proof of the two theorems is complete, recalling Lemma 1.1. ∎

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