CONCENTRATION OF THE BROWNIAN BRIDGE ON THE HYPERBOLIC PLANE

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We consider a Brownian bridge on the hyperbolic plane with one extremity tending to infinity, in finite time. We show that the exact exponential rate according to which this process concentrates around the geodesical segment joining the origin o to the moving extremity z is $-\rho(o, z)$, where ρ stands for the hyperbolic distance. This improves a result of A. Eberle.

1. Introduction. Consider a Brownian bridge $\{X_t, 0 \le t \le 1\}$ between x and $y \ne x$ in $\mathbb{R}^n$, $n \ge 2$. It can be written

$$X_t = B_t^x + t(y - B_1^x)$$

for every $0 \le t \le 1$, where $\{B_t^x, t \ge 0\}$ is a Brownian motion starting from x. In particular, the probability that it stays uniformly close to the line (x, y) does not depend on the distance $|y - x|$. One can expect that this property will not hold on a manifold of negative curvature anymore, because on such manifolds the Brownian motion conditioned to be very far away in finite time should remain nearer and nearer the geodesics. In a recent paper [7] whose main purpose was to build a counterexample to the Poincaré inequality on certain loop spaces, Eberle proved that on the hyperbolic spaces $\mathbb{H}^n$, $n \ge 2$, the sample paths of the Brownian bridge actually *concentrate* according to some exponential rate around the geodesics, when one of the extremities tends to infinity in finite time.

The aim of this paper is to find the exact exponential rate of concentration, in the case of the hyperbolic plane $\mathbb{H} = \mathbb{H}^2$. This rate appears to be $-\rho_z$, where $\rho_z := \rho(o, z)$ stands for the Riemannian distance between the origin o and the moving extremity z. More precisely, we prove that for every $a > 0$,

$$\lim_{\rho_z \uparrow +\infty} -\rho_z^{-1} \log \mathbb{P}\left[\sup_{0 \le t \le 1} \mathbf{d}_z^o(Z_t) > a \right] = 2 \log \cosh a,$$

where Z is the hyperbolic Brownian bridge from o to z in time 1, and $\mathbf{d}_z^o$ the distance function to the unique geodesical segment joining o to z.

Our method is very different from that of Eberle, and hinges mainly upon a comparison between the law of the Bridge and that of the Brownian motion conditioned to tend toward ∞ in the direction of z, with a certain speed. We show

that if this speed is chosen to be ρ_z itself, then roughly the ratio of harmonic functions connecting the two path measures remains bounded. Thanks to the explicit representation for the coordinates of the speed conditioned Brownian motion and the generalized Bougerol's identity [1], the problem is then reduced to an asymptotic study of the small deviations of a family of linear diffusions of the Ornstein–Uhlenbeck type, when the drift parameter tends to infinity. This study is carried out with the help of Feller's classical theory [8] on the first passage time problem for diffusions. Fortunately, the involved second order differential equation can be solved explicitly in terms of Legendre functions and after an asymptotic expansion of the latter, the convergence rate $-\rho_z$ as well as the constant $2\log\cosh a$ easily appear.

It should be noted that the same constant $2\log\cosh a$ is obtained when one replaces the geodesical segment by the whole geodesical line passing through o and z in the family of events under consideration. In other words, the critical times where the Brownian bridge might move away from the geodesics are most likely to be near the initial and terminal times.

To conclude this introduction, it may be interesting to mention that the study of Brownian bridges on symmetric spaces seems to be motivated by Theoretical Physics [9]. In a different (and more demanding) direction, when the extremities are fixed and the time goes to infinity, the asymptotic behaviour of the Brownian bridge on general noncompact symmetric spaces was recently studied by Anker, Bougerol and Jeulin [3] and Bougerol and Jeulin [5].

2. Preliminaries. We will consider the upper-half plane model for the hyperbolic plane $\mathbb{H}$:

$$\mathbb{H} = \{z = (x, y), x \in \mathbb{R} \text{ and } y > 0\}$$

equipped with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

so that the distance $\rho(z_1, z_2)$ between z_1 and z_2 in $\mathbb{H}$ is given by

$$\rho(z_1, z_2) = \arg\cosh\left(\frac{(x_1 - x_2)^2 + y_1^2 + y_2^2}{2y_1 y_2}\right).$$

The Brownian motion $\{Z_t, t \geq 0\}$ on $\mathbb{H}$ is the diffusion process whose infinitesimal generator is $\Delta_{\mathbb{H}}/2$, where

$$\Delta_{\mathbb{H}} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the hyperbolic Laplacian. It is well known (see, e.g., the Appendix in [1]) that the coordinates $\{(X_t, Y_t), t \geq 0\}$ of the hyperbolic Brownian motion can

be constructed explicitly in terms of the so-called geometric Brownian motion. Indeed, if the starting point is $(0, 1)$, then there exist two independent linear Brownian motions $\{\alpha_t, t \geq 0\}$ and $\{\beta_t, t \geq 0\}$, starting from 0, such that, for every $t \geq 0$,

$$X_t = \int_0^t \exp[\alpha_s - s/2] \, d\beta_s \quad \text{and} \quad Y_t = \exp[\alpha_t - t/2].$$

In the following, every process on $\mathbb{H}$ shall implicitly start from $o = (0, 1)$. We will denote by $\mathbb{P}$ the law of the Brownian motion on $\mathbb{H}$.

As a Markov process, it is also well known that the hyperbolic Brownian motion has smooth transition densities $p_t(z_1, z_2)$, which only depend on $\rho = \rho(z_1, z_2)$:

$$p_t(z_1, z_2) = k(t, \rho) = 2^{1/2}(2\pi t)^{3/2} \exp(-t/8) \int_\rho^\infty \frac{s \exp[-s^2/2t]}{(\cosh s - \cosh \rho)^{1/2}} \, ds$$

for every $\rho \geq 0$, $t > 0$. This formula is, however, not appropriately informative, and we will need the following estimate:

$$(1) \qquad k(t, \rho) \asymp \frac{(1 + \rho)}{2\pi t(1 + \rho + t/2)^{1/2}} \exp\bigl(-[\rho^2/2t + \rho/2 + t/8]\bigr),$$

where $\asymp$ stands for a uniform estimate in (t, ρ). We refer to Davies ([6], pages 178–179) for the above facts concerning the transition densities.

For every $\mu > 0$, we will denote by $\mathbb{P}^\mu$ the law of the Brownian motion conditioned to go to $i\infty$ when $t \uparrow \infty$, with speed μ. $\mathbb{P}^\mu$ is obtained from $\mathbb{P}$ via some Doob's h-transformation with respect to $(t, z) \mapsto y^{\mu+1/2} \exp(-t[(\mu^2 - 1/4)/2])$, which is harmonic for the space–time operator

$$\frac{1}{2} \Delta_\mathbb{H} + \frac{\partial}{\partial t}.$$

More precisely, if $\{\mathcal{F}_t, t \geq 0\}$ stands for the completed filtration generated by the coordinate process, then, for every $t \geq 0$ and $\Lambda_t \in \mathcal{F}_t$,

$$\mathbb{P}^\mu[\Lambda_t] = \mathbb{P}\bigl[\Lambda_t; Y_t^{\mu+1/2} \exp\bigl(-t[(\mu^2 - 1/4)/2]\bigr)\bigr].$$

It is easy to see that under $\mathbb{P}^\mu$, the coordinate process can be written

$$X_t = \int_0^t \exp[\alpha_s + \mu s] \, d\beta_s \quad \text{and} \quad Y_t = \exp[\alpha_t + \mu t],$$

where again $\{\alpha_t, t \geq 0\}$ and $\{\beta_t, t \geq 0\}$ are two independent linear Brownian motions starting from 0.

In this paper, the process we are interested in is the Brownian bridge from o to z in time 1, namely, the Brownian motion starting from o conditioned to be in z at time 1. We will denote its law by $\mathbb{P}^z$, which can also be obtained from $\mathbb{P}$ through

a space–time harmonic transformation, involving this time the heat kernel: with the same notations as above, for every $0 \le t < 1$ and $\Lambda_t \in \mathcal{F}_t$,

$$\mathbb{P}^z[\Lambda_t] = \mathbb{P}\left[\Lambda_t; \frac{p_{1-t}(Z_t, z)}{p_1(o, z)}\right].$$

However, under $\mathbb{P}^z$ there does not seem to exist a simple representation for the coordinate process anymore.

3. The theorem. For any $z \in \mathbb{H}$, let Δ_z^o be the unique geodesical segment joining o to z. Define the following function on $\mathbb{H}$:

$$\mathbf{d}_z^o(\cdot) = \mathrm{dist}(\cdot, \Delta_z^o) = \inf\{\rho(\cdot, u), u \in \Delta_z^o\}.$$

For any $z \in \mathbb{H}$ we will also write $\rho_z = \rho(o, z)$ for simplicity. The aim of this paper is to prove the following.

THEOREM.   *For every $a > 0$,*

$$\lim_{\rho_z \uparrow +\infty} -\rho_z^{-1} \log \mathbb{P}^z\left[\sup_{0 \le t \le 1} \mathbf{d}_z^o(Z_t) > a\right] = 2 \log \cosh a.$$

REMARKS.   (a) The same result holds when Δ_z^o is replaced by D_z^o, the *whole* geodesic line passing through o and z, and if we take for $\mathbf{d}_z^o$ the corresponding distance.

(b) In [7] Eberle had proved that the concentration rate of the Brownian bridge around Δ_z^o should be at most of order $\exp(-\rho_z^\beta)$, for every $\beta < 1/4$. His proof, which is quite technical, also works for more general hyperpolic spaces $\mathbb{H}^n$, $n \ge 2$. In this paper, we focus on the case $n = 2$ because of the simplicity of the model.

(c) However, we do believe that an analogous estimate should also hold for more general noncompact symmetric spaces. Indeed, the main ingredients in our proof are the uniform heat kernel estimate, which was recently studied in this general framework by Anker and Ji [4] and, roughly, the behavior at infinity of the Iwasawa coordinates of the Brownian motion on the hyperbolic plane, a behavior which has been known for a long time on general symmetric spaces [11]. Nevertheless, this general estimate would probably demand much more conceptual arguments than ours.

Before proceeding to the proof, we first make the general remark that thanks to the isotropy of the Brownian motion on $\mathbb{H}$, it is enough to consider the case when z moves to infinity along the y-axis. We will first prove the estimates when Δ_z^o is replaced by D_z^o, that is, we will first prove the above Remark (a). If z belongs to the y-axis, then D_z^o is this axis itself, and it is not difficult to see that the distance between D_z^o and a generic point (x, y) in $\mathbb{H}$ is given by

$$\arg \sinh\left|\frac{x}{y}\right|.$$

Hence, since sinh is an increasing function, we are reduced to an estimate on

$$\mathbb{P}^z\left[\sup_{0\leq t\leq 1}\left|\frac{X_t}{Y_t}\right| > \sinh a\right]$$

as $z \to i\infty$ along the y-axis. In the following, for every $a > 0$ and $0 < t < 1$, independently of the path measure under consideration, we will set

$$\Lambda_t^a = \left\{\sup_{0\leq s\leq t}\left|\frac{X_s}{Y_s}\right| > \sinh a\right\},$$

where $\{(X_s, Y_s), s \geq 0\}$ is the coordinate process. Thanks again to the isotropy of Brownian motion and a straightforward time-reversal argument, it is clear that for every z on the y-axis and every $0 < t < 1$,

$$\mathbb{P}^z[\Lambda_t^a] \leq \mathbb{P}^z[\Lambda_1^a] \leq 2\mathbb{P}^z[\Lambda_{1/2}^a],$$

so that we are finally reduced to an estimate on the $\mathbb{P}^z[\Lambda_t^a]$ for $0 < t < 1$.

The main idea to obtain this estimate, which was communicated to us by Terry Lyons, consists in comparing $\mathbb{P}^z[\Lambda_t^a]$ and $\mathbb{P}^\mu[\Lambda_t^a]$ with a, t fixed and for some suitably chosen $\mu > 0$ depending on z, as z tends to infinity along the y-axis. Before making this more precise, we would first like to study the asymptotic behaviour of $\mathbb{P}^\mu[\Lambda_t^a]$ as $\mu \uparrow +\infty$.

PROPOSITION. *For every $a > 0$ and $0 < t < 1$,*

$$\lim_{\mu\uparrow+\infty} -\mu^{-1}\log\mathbb{P}^\mu[\Lambda_t^a] = 2\log\cosh a.$$

PROOF. We first appeal to the generalized Bougerol's identity (see Proposition 1 in [1]), which states that, under $\mathbb{P}^\mu$,

$$\left\{\frac{X_s}{Y_s}, s \geq 0\right\} \overset{d}{=} \{\sinh X_s^\mu, s \geq 0\}$$

where $\{X_s^\mu, s \geq 0\}$ is the unique strong solution to the SDE

$$X_s^\mu = B_s - \mu\int_0^s \tanh X_r^\mu \, dr$$

driven by $\{B_s, s \geq 0\}$, a linear Brownian motion starting from 0. Hence, since sinh is an odd function,

$$\Lambda_t^a = \{T_a^\mu < t\}$$

under $\mathbb{P}^\mu$, where T_a^μ is the first passage time of the diffusion X^μ accross the double-sided barrier $x = \pm a$:

$$T_a^\mu = \inf\{s > 0, X_s^\mu \notin [-a, a]\}.$$

We now proceed to the asymptotic study of $\mathbb{P}^\mu[T_a^\mu < t]$ when μ tends to $+\infty$. For every $\lambda > 0$ we introduce

$$F_a^\mu(\lambda) = \int_0^{+\infty} \mathbb{P}^\mu[T_a^\mu < t] \exp[-\lambda t] \, dt.$$

By symmetry, it is clear that

$$F_a^\mu(\lambda) = 2 \int_0^{+\infty} \mathbb{P}^\mu\Big[T_a^\mu < t; X_{T_a^\mu}^\mu = a\Big] \exp[-\lambda t] \, dt.$$

Hence, thanks to Feller's classical theory (see [8], Theorem 2, page 11), $F_a^\mu(\lambda)$ is given by the value at $x = 0$ of the unique solution over $[-a, a]$ to the second order differential equation

(2)
$$f''(x) - 2\mu(\tanh x) f'(x) - 2\lambda f(x) = 0,$$

verifying $f(-a) = 0$ and $f(a) = 2/\lambda$.

We are now concerned with the resolution of (2), and we write $f(x) = (\cosh x)^\mu g(\tanh x)$ for some unknown function $g : (-1, 1) \to \mathbb{R}$. A straightforward computation yields the following equation for g:

$$(1 - z^2) g''(z) - 2z g'(z) + \left(\mu(\mu + 1) - \frac{\mu^2 + 2\lambda}{1 - z^2}\right) g(z) = 0.$$

We recognize Legendre's well-known differential equation on the cut, and this entails

$$f(x) = (\cosh x)^\mu \Big[A P_\mu^\nu(\tanh x) + B P_\mu^\nu(-\tanh x)\Big]$$

for two unknown constants A and B, where $\nu = \sqrt{\mu^2 + 2\lambda}$ and P_μ^ν stands for the Legendre function of the first kind. The conditions $f(-a) = 0$ and $f(a) = 2/\lambda$ yield

$$f(x) = \frac{2(\cosh x)^\mu [P_\mu^\nu(\tanh a) P_\mu^\nu(\tanh x) - P_\mu^\nu(-\tanh a) P_\mu^\nu(-\tanh x)]}{\lambda(\cosh a)^\mu [P_\mu^\nu(\tanh a)^2 - P_\mu^\nu(-\tanh a)^2]},$$

and we finally get

$$F_a^\mu(\lambda) = \left(\frac{\lambda(\cosh a)^\mu}{2} \left[\frac{P_\mu^\nu(\tanh a)}{P_\mu^\nu(0)} + \frac{P_\mu^\nu(-\tanh a)}{P_\mu^\nu(0)}\right]\right)^{-1}.$$

We now appeal to the tables of formulae concerning Legendre functions, which can be found, for example, in [10], Chapter IV. The third formula on page 167, the first formula on page 171 in [10], and the fact that

$$\Gamma\left(\frac{1}{2} - z\right) \Gamma\left(\frac{1}{2} + z\right) = \frac{\pi}{\cos \pi z},$$

where Γ is the gamma function, yield

$$F_a^\mu(\lambda) = \left(\frac{\lambda(\cosh a)^{\nu+\mu}}{2} F\left(\frac{-\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; \frac{1}{2}; (\tanh a)^2\right)\right)^{-1},$$

where in the denominator F stands for the standard hypergeometric function. We can now use the asymptotic expansion of the latter, when its first parameter is large (see again [10], page 56). Since

$$\nu - \mu \sim \frac{\lambda}{\mu} \quad \text{and} \quad \mu + \nu \sim 2\mu + \frac{\lambda}{\mu}$$

as $\mu \uparrow +\infty$, we find

$$F_a^\mu(\lambda) = \frac{4\sqrt{\pi}\mu^{3/2}\tanh a}{\lambda^2(\cosh a)^{2\mu}}(1 + O(\lambda\mu^{-1}))$$

$$= \frac{4\sqrt{\pi}\mu^{3/2}\tanh a}{(\cosh a)^{2\mu}}(\lambda^{-2} + \lambda^{-1}O(\mu^{-1})).$$

Inverting the Laplace transform we get, for every $t > 0$,

$$\mathbb{P}^\mu[T_a^\mu < t] = \frac{4\sqrt{\pi}\mu^{3/2}\tanh a}{(\cosh a)^{2\mu}}(t + O(\mu^{-1})).$$

This clearly entails

$$\lim_{\mu\uparrow+\infty} -\mu^{-1}\log\mathbb{P}^\mu[T_a^\mu < t] = 2\log\cosh a,$$

which is the desired result. $\quad\square$

For every $b > 0$ and $0 < t < 1$ we now introduce the event

$$\Gamma_t^b = \left\{\left|\frac{X_t}{Y_t}\right| > \sinh b\right\}.$$

Since $\Gamma_t^b \subset \Lambda_t^b$ for every $0 < t < 1$, and since $2\log\cosh b \uparrow +\infty$ as $b \uparrow +\infty$, the following corollary is a direct consequence of Proposition, but we will give a different proof, since it is fairly easy and uses different arguments.

COROLLARY. *For every $c > 0$ and $0 < t < 1$, there exists $b > 0$ such that*

$$\underline{\lim} -\mu^{-1}\log\mathbb{P}^\mu[\Gamma_t^b] \geq c$$

as $\mu \uparrow \infty$.

PROOF. Thanks to Proposition 3 in [2], under $\mathbb{P}^\mu$,

$$\frac{X_t}{Y_t} \overset{d}{=} (2Z-1)\phi\left(B_t^\mu, \sqrt{R_t^2 + (B_t^\mu)^2}\right)$$

where Z is an arcsine distributed random variable, $\{B_s^\mu, s \geq 0\}$ a linear Brownian motion with drift $-\mu$ starting from 0, $\{R_s, s \geq 0\}$ a two-dimensional Bessel process starting from 0, and ϕ is the function defined by

$$\phi(x, z) = \sqrt{2 \exp x \cosh z - \exp 2x - 1}$$

for $z \geq |x|$. In particular, since $\cosh z \leq \exp z$ for $z \geq 0$,

$$\left| \frac{X_t}{Y_t} \right| \leq \sqrt{2} \exp \frac{1}{2} \left(B_t^\mu + \sqrt{R_t^2 + (B_t^\mu)^2} \right) \qquad \text{a.s.}$$

It is clear, with the obvious definition for $\mathbb{P}$ that, for every $\mu \geq 0$ and $0 < \gamma < 1$,

$$\mathbb{P}[B_t^\mu > -\gamma t \mu] \leq \exp(-k\mu^2)$$

for some $k > 0$ independent of μ. Besides, since the law of R_t^2 is exponential with parameter $1/2t$ (see, e.g., [12], Corollary XI.1.4) and writing $\delta = 2c$ for simplicity,

$$\mathbb{P}[R_t^2 > \delta t \mu] = \exp(-c\mu)$$

for every $\mu \geq 0$. In particular,

$$\underline{\lim} - \left(\frac{1}{\mu} \log \mathbb{P}[\{B_t^\mu > -\gamma t \mu\} \cup \{R_t^2 > \delta t \mu\}] \right) \geq c$$

as $\mu \uparrow \infty$. However, on the event

$$\{B_t^\mu \leq -\gamma t \mu\} \cap \{R_t^2 \leq \delta t \mu\},$$

it is easy to see that

$$B_t^\mu + \sqrt{R_t^2 + (B_t^\mu)^2} \leq \delta/\gamma.$$

This completes the proof by taking, for example, $b = \arg \sinh(\sqrt{2} \exp 2c)$. □

We now proceed to the proof of the theorem. As we said at the beginning, we first wish to study the concentration of the bridge around the whole geodesic line D_z^o. Fix $a > 0$. By definition

$$(3) \qquad \mathbb{P}^z[\Lambda_t^a] = \mathbb{P}^\mu \left[\Lambda_t^a; \frac{p_{1-t}(Z_t, z)}{p_1(o, z)} Y_t^{-(\mu+1/2)} \exp t[(\mu^2 - 1/4)/2] \right]$$

for every $0 < t < 1$, $z \in \mathbb{H}$ and $\mu > 0$. Recall that under $\mathbb{P}^\mu$,

$$Y_t = \exp(R_t + t\mu)$$

where R_t is a centered Gaussian random variable with variance t. Notice that since we decided to let z lie on the y-axis above o,

$$z = (0, \exp \rho_z).$$

In particular,

$$\rho(Z_t, z) = \arg\cosh\left(\frac{X_t^2 + Y_t^2 + \exp 2\rho_z}{2Y_t \exp \rho_z}\right).$$

To prove the theorem we will show that when $\rho_z \uparrow \infty$,

$$2\log\cosh a \le \underline{\lim} - \rho_z^{-1} \log \mathbb{P}^z[\Lambda_1^a] \le \overline{\lim} - \rho_z^{-1} \log \mathbb{P}^z[\Lambda_1^a] \le 2\log\cosh a.$$

PROOF OF THE UPPER BOUND. By the remarks made before the proposition, it suffices actually to prove

$$\underline{\lim} - \rho_z^{-1} \log \mathbb{P}^z[\Lambda_{1/2}^a] \ge 2\log\cosh a$$

as $\rho_z \uparrow \infty$. In the following we will set $t = 1/2$, and write $R_{1/2} = R$ for simplicity. We first notice that

$$\rho(Z_{1/2}, z) \ge \arg\cosh\left(\frac{Y_{1/2}^2 + \exp 2\rho_z}{2Y_{1/2} \exp \rho_z}\right) = |\rho_z - (R + \mu/2)|$$

and we introduce the following notation:

$$\rho_\omega = \rho(Z_{1/2}, z), \mu_\omega = \rho_\omega - |\rho_z - (R + \mu/2)| \ge 0$$

and

$$F(\mu, z; \omega) = \frac{p_{1/2}(Z_{1/2}, z)}{p_1(o, z)} Y_{1/2}^{-(\mu+1/2)} \exp[(\mu^2 - 1/4)/4].$$

We can now appeal to the uniform estimate (1) on the heat kernel, which yields

$$F(\mu, z; \omega) \le A(\mu, z; \omega) \exp[f(\mu, z; \omega)/2]$$

where

$$A(\mu, z; \omega) \le K(1 + \rho_\omega) \exp(-[\mu_\omega/2])$$

for some deterministic constant K independent of μ and ρ_z, and

$$\begin{aligned}
f(\mu, z; \omega) &= \rho_z^2 + \rho_z - 2(\rho_z - R - \mu/2)^2 - |\rho_z - R - \mu/2| \\
&\quad - 2\mu R - \mu^2/2 - (R + \mu/2) \\
&= -(\rho_z - 2R - \mu)^2 + 2R^2 + (\rho_z - R - \mu/2) - |\rho_z - R - \mu/2| \\
&\le -(\rho_z - 2R - \mu)^2 + 2R^2.
\end{aligned}$$

Hence, if we take $\mu = \rho_z$, we get

$$f(\mu, z; \omega) \le -2R^2.$$

It is then not difficult to prove the existence of another deterministic constant K independent of ρ_z such that

$$F(\mu, z; \omega) \le K(1 + \rho_z).$$

Finally, applying (3) and the proposition, we get

$$\underline{\lim} -\rho_z^{-1} \log \mathbb{P}^z[\Lambda_{1/2}^a] \geq 2\log\cosh a$$

as $\rho_z \uparrow \infty$.　□

PROOF OF THE LOWER BOUND.　We now wish to prove

$$\overline{\lim} -\rho_z^{-1} \log \mathbb{P}^z[\Lambda_1^a] \leq 2\log\cosh a$$

as $\rho_z \uparrow \infty$, which is actually a bit messier than the upper bound. Fix $0 < t < 1$ and, for every $b > 0$, introduce

$$\Gamma_t^{ab} = \Lambda_t^a \cap (\Gamma_t^b)^c$$

where A^c denotes the complementary of A for any set A. If we take b sufficiently large, it follows easily from Proposition and Corollary that when $\mu \uparrow \infty$,

$$\overline{\lim} -\mu^{-1} \log \mathbb{P}^\mu[\Gamma_t^{ab}] \leq 2\log\cosh a.$$

Besides, if we fix c sufficiently large and set

$$\Lambda_t^{a\mu} = \Gamma_t^{ab} \cap \{|R_t| \leq c\sqrt{t\mu}\},$$

then, since under $\mathbb{P}^\mu$, R_t is a Gaussian random variable with variance t, we see that again

(4)
$$\overline{\lim} -\mu^{-1} \log \mathbb{P}^\mu[\Lambda_t^{a\mu}] \leq 2\log\cosh a$$

when $\mu \uparrow \infty$. As for the upper bound, we will choose $\mu = \rho_z$. We also introduce the following notation:

$$\Lambda_t^{az} = \Lambda_t^{a\rho_z}, \qquad \rho_\omega^t = \rho(Z_t, z)$$

and

$$F_t(z; \omega) = \frac{p_{1-t}(Z_t, z)}{p_1(o, z)} Y_t^{-(\rho_z + 1/2)} \exp t[(\rho_z^2 - 1/4)/2].$$

Since $|X_t| \leq \sinh b|Y_t|$ on Λ_t^{az} we get, after some computations,

$$\rho_\omega^t \leq |(1-t)\rho_z - R_t| + K \exp(-2[|(1-t)\rho_z - R_t|])$$

on Λ_t^{az}, for some constant K depending on b only. In particular,

$$(\rho_\omega^t)^2 \leq ((1-t)\rho_z - R_t)^2 + K$$

on Λ_t^{az}, for another constant K depending on b only. Appealing again to the uniform estimate (2.1), and making analogous computations to the case $t = 1/2$, we find

$$F_t(z; \omega) \geq \frac{K}{\sqrt{1 + \rho_z}} \exp[f_t(z; \omega)/2]$$

on Λ_t^{az}, where K is a constant only depending on b and

$$f_t(z; \omega) = -\frac{R_t^2}{1-t} - \left((R_t - (1-t)\rho_z) + |R_t - (1-t)\rho_z|\right).$$

Hence, for ρ_z large enough,

$$f_t(z; \omega) = -\frac{R_t^2}{1-t} \geq -\frac{c^2 t}{1-t}\rho_z$$

on Λ_t^{az}. Applying (3) and (4) yields then

$$\overline{\lim} - \rho_z^{-1} \log \mathbb{P}^z[\Lambda_t^{az}] \leq 2 \log \cosh a + \frac{c^2 t}{2(1-t)}$$

when $\rho_z \uparrow \infty$. But since $\Lambda_t^{az} \subset \Lambda_1^a$ for every $\rho_z > 0$ and $0 < t < 1$ we finally get, letting t tend to 0,

$$\overline{\lim} - \rho_z^{-1} \log \mathbb{P}^z[\Lambda_1^a] \leq 2 \log \cosh a$$

as $\rho_z \uparrow \infty$. This completes the proof of the lower bound. $\square$

REMARK. Since $\{\Lambda_t^a, 0 \leq t \leq 1\}$ is an increasing family of events, it may seem surprising that we let t tend to 0 and not to 1 to get the lower bound. This comes indeed from the logarithmic scale, where the dependence of $\mathbb{P}^\mu[\Lambda_t^a]$ on t disappears, whereas $F_t(z; \omega)$ tends pointwise to 1 as $t \downarrow 0$.

END OF THE PROOF. We now wish to get the estimates on the concentration of the Brownian bridge around the geodesical segment Δ_z^o itself, which will complete the proof of Theorem. For every $a > 0$ and $0 \leq t \leq 1$ we will write

$$\Omega_t^a = \left\{ \sup_{0 \leq s \leq t} \mathbf{d}_z^o(Z_s) > a \right\}$$

independently of the probability measure under consideration. First, since $\Lambda_1^a \subset \Omega_1^a$, we see from the above lower bound that

$$\overline{\lim} - \rho_z^{-1} \log \mathbb{P}^z[\Omega_1^a] \leq 2 \log \cosh a$$

as $\rho_z \uparrow \infty$. The proof of the upper bound is slightly messier. If h denotes a generic point in $\mathbb{H}$ with coordinates (x, y) we first remark, thanks to a little hyperbolic geometry, that for ρ_z large enough

$$\left\{ \frac{1}{\cosh a} \leq y \leq \exp[3\rho_z/4] \right\} \cap \left\{ \left|\frac{x}{y}\right| \leq \sinh a \right\} \subset \{\mathbf{d}_z^o(h) \leq a\}.$$

This clearly entails, for $\mu = \rho_z$ large enough,

$$\mathbb{P}^\mu\big[\Omega_{1/2}^a \cap (\Lambda_{1/2}^a)^c\big] \leq \mathbb{P}\big[\Gamma_{1/2}^{a\mu}\big]$$

where $\{W_s, s \geq 0\}$ is some linear Brownian motion starting from 0 with law $\mathbb{P}$ and

$$\Gamma_{1/2}^{a\mu} = \left\{ \inf_{s \leq 1/2}(W_s + \mu s) \leq -\log(\cosh a) \right\} \cup \left\{ \sup_{s \leq 1/2}(W_s + \mu s) \geq 3\mu/4 \right\}.$$

Now it follows easily from Désiré André's reflection principle (see, e.g., Ex. III.3.14 in [12]) and the Cameron–Martin formula, that

$$\lim_{\mu \uparrow \infty} -\mu^{-1} \log \mathbb{P}[\Gamma_{1/2}^{a\mu}] = 2 \log \cosh a.$$

This obviously yields, writing

$$\Omega_{1/2}^a = \Lambda_{1/2}^a \cup \left(\Omega_{1/2}^a \cap (\Lambda_{1/2}^a)^c \right)$$

and using the proposition, that

$$\underline{\lim} -\rho_z^{-1} \log \mathbb{P}^z[\Omega_{1/2}^a] \geq 2 \log \cosh a$$

as $\rho_z \uparrow \infty$. Since clearly

$$\mathbb{P}^z[\Omega_1^a] \leq 2\mathbb{P}^z[\Omega_{1/2}^a],$$

this completes the proof of the upper bound, hence of the theorem. □

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