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BROWNIAN INTERSECTION LOCAL TIMES: UPPER TAIL ASYMPTOTICS AND THICK POINTS

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We equip the intersection of p independent Brownian paths in $\mathbb{R}^d$, $d \geq 2$, with the natural measure ℓ defined by projecting the intersection local time measure via one of the Brownian motions onto the set of intersection points. Given a bounded domain $U \subset \mathbb{R}^d$ we show that, as $a \uparrow \infty$, the probability of the event $\{\ell(U) > a\}$ decays with an exponential rate of $a^{1/p}\theta$, where θ is described in terms of a variational problem. In the important special case when U is the unit ball in $\mathbb{R}^3$ and $p = 2$, we characterize θ in terms of an ordinary differential equation. We apply our results to the problem of finding the Hausdorff dimension spectrum for the thick points of the intersection of two independent Brownian paths in $\mathbb{R}^3$.

1. Introduction and main results.

1.1. *Aims of the article.* Let a bunch of p independent Brownian motions $W_1, \dots, W_p$ run in $\mathbb{R}^d$ until their first exit times $T_1, \dots, T_p$ from a large ball, or, in the transient case, for infinite time. By classical results of Dvoretzky, Erdös, Kakutani and Taylor the intersection of the paths of these motions,

$$S = \bigcap_{i=1}^{p} \big\{x \in \mathbb{R}^d : x = W_i(t) \text{ for some } t \in [0, T_i)\big\}, \tag{1.1}$$

contains points different from the starting point if and only if $p < d/(d-2)$. By work of Geman, Horowitz and Rosen [11], in these cases the random set S of intersection points can be equipped with a natural finite measure ℓ, the (*projected*) *intersection local time*, which can be described symbolically by the formula

$$\ell(A) = \int_A dy \prod_{j=1}^{p} \int_0^{T_j} ds\, \delta_y\big(W_j(s)\big) \qquad \text{for } A \subset \mathbb{R}^d \text{ Borel.} \tag{1.2}$$

We review rigorous constructions of the random measure ℓ in Section 2.1 below. In the following we always include the case $p = 1$ of a single Brownian path, in

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which the role of ℓ is played by the usual occupation measure. However, in this case most of our results are known and much easier.

Let $U \subset \mathbb{R}^d$ be a bounded domain in $\mathbb{R}^d$. The first aim of the present paper is to determine asymptotically the upper tails of the random variables $\ell(U)$. More precisely, we show in Theorem 1.1 that the limit

$$\lim_{a\to\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\{\ell(U) > a\}$$

exists and describe its value in terms of a natural variational problem associated with p, the stopping rule and the domain $U \subset \mathbb{R}^d$. In the special case that U is the unit ball in $\mathbb{R}^d$ and the Brownian motions run for an infinite amount of time, we can solve this variational problem explicitly in terms of an ordinary differential equation, which is nonlinear in the case $p > 1$; see Theorem 1.3.

This result is the key in the solution of a problem posed recently by Dembo, Peres, Rosen and Zeitouni about the *refined multifractal analysis* of intersection local times. Recall that the intersection of two independent Brownian paths in $\mathbb{R}^3$ is a random set of Hausdorff dimension one but length zero. More precisely, by a result of Le Gall [15], its exact Hausdorff dimension gauge is given by the function $\psi(r) = r[\log\log(1/r)]^2$ and, moreover, the Hausdorff measure for this gauge function coincides up to a constant factor with the projected intersection local time measure on the intersection set S.

The key argument leading to Le Gall's result is the existence of a positive and finite constant c such that, almost surely, for typical x in the intersection set S,

$$\limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r[\log\log(1/r)]^2} = c,$$

where $B(x,r)$ denotes the open ball of radius r with center in x. Here we are concerned with finer results, which focus on exceptional points in $\mathbb{R}^3$ in a neighborhood of which the concentration of intersection local time is untypically large. Our first aim is to find an explicit gauge function φ such that

$$0 < \sup_{x\in S} \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{\varphi(r)} < \infty.$$

Having found such a function, we call a point $x \in S$ *thick* if $\limsup_{r\downarrow 0} \ell(B(x,r))/\varphi(r) > 0$ and ask, how many thick points there are. This is answered in terms of the Hausdorff dimension spectrum for the thick points, which is the function

$$f(a) = \dim\left\{x \in S : \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{\varphi(r)} = a\right\}.$$

Both these problems are intimately related to our upper tail asymptotics and the numerical values featuring the dimension spectrum can be derived from our differential equation. Theorem 1.4 describes just how large the measure ℓ can be in a neighborhood of a point and identifies the Hausdorff dimension spectrum of thick points for the intersection local time.

The spectrum of thick points is also understood for some other classes of random measures: For example, in the case of the local time measure on the zero set of stable processes [23], the branching measure on a supercritical Galton–Watson process [24], [19], the occupation measure on Brownian paths in dimensions exceeding two [5] and in the plane [8], and also intersections of Brownian paths in the plane [6].

The present article is organized as follows. In Sections 1.2–1.4, we describe our main results on the upper tails of $\ell(U)$, the characterization of the variational problem arising, and the dimension spectrum of the thick points, respectively. In Section 2, we survey three constructions of the intersection local time and prove that the upper tail asymptotics of $\ell(U)$ are completely described by the asymptotics of its high moments. The asymptotic analysis of the high moments is carried out in several steps in Section 3. In Section 4 we analyze the two variational formulase that describe the upper tails, respectively, the moment asymptotics. Section 5 contains the proof of our results on the thick points of the intersection local time, and in Section 6, we announce further results and applications which will appear elsewhere.

1.2. *Upper tail asymptotics.* To describe the variational problem featuring the upper tail asymptotics, we first introduce the *Green's function* associated with (possibly stopped) Brownian motion. The Green's function depends on the dimension d of the ambient space and on the way we stop the Brownian motion, but it does *not* depend on the domain U or the number p of motions. Suppose that we stop Brownian motion on its first exit time from a large open ball $B(0, R)$, and, if $d \geq 3$, include the case $R = \infty$ as the case of Brownian motion running until infinity. Denote by $p_s(x, y)$ the transition subprobability density of the killed motion. For x, y in the open centered ball $B(0, R)$, we let

$$G(x, y) = \int_0^\infty p_s(x, y)\, ds, \tag{1.3}$$

be the associated *Green's function*. Explicitly, these functions are (see, e.g., [21], page 114), if $d = 2$,

$$G(x, y) = \frac{1}{\pi} \begin{cases} \log\left| x\frac{R}{|x|} - y\frac{|x|}{R}\right| - \log|x - y|, & \text{if } x \neq 0, \\ \log R - \log|y|, & \text{if } x = 0, \end{cases} \tag{1.4}$$

and, if $d \geq 3$ and $0 < R < \infty$, denoting $c_d = \Gamma(d/2 - 1)/(2\pi^{d/2})$,

$$G(x, y) = c_d \begin{cases} |x - y|^{2-d} - \left| x\frac{R}{|x|} - y\frac{|x|}{R}\right|^{2-d}, & \text{if } x \neq 0, \\ |y|^{2-d} - R^{2-d}, & \text{if } x = 0. \end{cases} \tag{1.5}$$

Finally, in the case $d \geq 3$ and $R = \infty$, we have

$$G(x, y) = \frac{c_d}{|x - y|^{d-2}}. \tag{1.6}$$

Note that G is always symmetric (see, e.g., [3], II(4.5)). Furthermore, precisely if $p < d/(d-2)$, the function $G^p(0, \cdot)$ is integrable in a neighborhood of the origin.

Now let $U \subset \mathbb{R}^d$ be an open bounded set whose closure is contained in $B(0, R)$. Define the operator $\mathfrak{A} : L^{2p/(2p-1)}(U) \to L^{2p}(U)$, depending on R and $U \subset \mathbb{R}^d$, by

$$\mathfrak{A} f(x) = \int_U G(x, y) f(y)\, dy \qquad \text{for } x \in U. \tag{1.7}$$

The operator $\mathfrak{A}$ is symmetric and continuous, and its restriction $\mathfrak{A} : L^\infty(U) \to L^\infty(U)$ is, moreover, symmetric, positive and compact with norm $\int_U \int_U G(x, y) \times dx\, dy$.

We denote by $\mathcal{M}_1(U)$ the set of probability measures on the set $U \subset \mathbb{R}^d$ and, if $\mu \in \mathcal{M}_1(U)$ and f is an integrable or nonnegative function defined on U we use the notation $\langle f, \mu \rangle$ for the integral $\int_U f\, d\mu$. If f, g are both nonnegative functions or their product is Lebesgue integrable on U, we use the notation $\langle f, g \rangle$ to denote $\int_U f(x) g(x)\, dx$. We write $\|\cdot\|_q$ for the norm on $L^q(U)$.

We can now formulate our first main theorem.

THEOREM 1.1 (Upper tail asymptotics). *Suppose that p and d are positive integers satisfying $p < d/(d-2)$. Let U be an open bounded domain in $\mathbb{R}^d$ and $R \in (0, \infty]$ sufficiently large that $B(0, R)$ contains the closure of U. Denote by ℓ the projected intersection local time of p independent Brownian motions started in fixed points inside U and stopped upon their first exit from the ball $B(0, R)$. Then*

$$\lim_{a \to \infty} \frac{1}{a^{1/p}} \log \mathbb{P}\{\ell(U) > a\} = -\frac{p}{\rho^*}, \tag{1.8}$$

where ρ^* is defined as

$$\rho^* = \sup\{\langle g^{2p-1}, \mathfrak{A} g^{2p-1} \rangle : g \in L^{2p}(U) \text{ with } \|g\|_{2p} = 1\}. \tag{1.9}$$

REMARK 1. In the transient case $d \geq 3$ we denote by $\rho^*(R)$ the supremum in (1.9) for the Green's function associated with Brownian motion stopped upon leaving $B(0, R)$. Then $\lim_{R \uparrow \infty} \rho^*(R)$ is equal to the supremum in (1.9) for the case of unstopped Brownian motions. This follows easily from monotone convergence and the fact that the Green's functions on $B(0, R)$ are increasing to the Green's function in (1.6).

In the next subsection we turn to an analysis of the variational formula in (1.9). We already announce that in Proposition 2.1 below we characterize ρ^* in terms of another variational formula, which is more in the spirit of our actual proof of (1.8).

1.3. *The variational formula in* (1.9). In the special case $p = 1$, the well-known Rayleigh–Ritz formula describes the right-hand side in (1.9) as the principal eigenvalue of the compact and symmetric operator $\mathfrak{A}$ on $L^2(U)$. Hence, existence and uniqueness and many more properties of the maximizer are known. For general $p \geq 2$, however, already the uniqueness seems to be an open problem, apart from the special case of the unit ball in $\mathbb{R}^3$. In our next main result, we establish the existence of maximizers and characterize them in terms of the corresponding Euler–Lagrange equations.

THEOREM 1.2. *Suppose that p and d are positive integers satisfying $p < d/(d-2)$. Let U be an open bounded domain in $\mathbb{R}^d$ and $R \in (0, \infty]$ sufficiently large that $B(0, R)$ contains the closure of U. Then the supremum in the variational problem* (1.9) *is attained. Every maximizer g in* (1.9) *is bounded away from* 0 *and infinity and has a version that is twice continuously differentiable on U and continuously differentiable on the closure of U. Moreover, g satisfies the following equivalent conditions*:

(a) $\mathfrak{A}g^{2p-1}(x) = \rho^* g(x)$, *for all $x \in U$.*

(b) *g is the restriction to U of a continuous function $g : B(0, R) \to [0, \infty)$ that vanishes on the sphere $\partial B(0, R)$, or in the case $R = \infty$ converges to* 0 *as $x \to \infty$. This function g is continuously differentiable inside $B(0, R)$ and is a solution of*

$$\frac{1}{2}\Delta g(x) = -\frac{1}{\rho^*} g^{2p-1}(x)\mathbb{1}_U(x) \qquad \text{for all } x \in B(0, R) \setminus \partial U. \tag{1.10}$$

If $p = 1$, the solution (g, ρ^*) of (*a*) *and hence the maximizer g in* (1.9) *is unique.*

For $p \geq 2$ and general domains U, there seems to be no reason that, for a solution (g, ρ) of assertion (a), the number ρ must coincide with ρ^* in (1.9). However, in the important special case that $U = B(0, 1)$ is the open unit ball and the Brownian motions run for an infinite time, we can show uniqueness of the minimizer in (1.9) and describe it more explicitly in terms of an ordinary differential equation. We are able to do this because G is a rotation invariant function of the difference of its arguments, and hence Riesz's rearrangement theorem is applicable.

THEOREM 1.3. *Let $d \geq 3$ and p a positive integer satisfying $p < d/(d-2)$; further, let $R = \infty$ and $U = B(0, 1)$. Then the maximization problem* (1.9) *has a unique solution g. Moreover, g is rotationally symmetric around the origin and can be constructed as follows*:

(i) *There is a unique twice continuously differentiable function z on a maximal interval $[0, \xi)$ satisfying $z(0) = 1$, $z'(0) = 0$ and*

$$z''(x) + (d-1)\frac{z'(x)}{x} + z(x)^{2p-1} = 0 \qquad \text{for all } x \in [0, \xi). \tag{1.11}$$

There is a smallest number $a \in (0,\xi)$ with $(d-2)z(a) = -az'(a)$. Moreover, z is positive and strictly decreasing on $[0,a]$, and $g(x) = cz(a|x|)$ defines the unique maximizer of (1.9), *if c is the normalization constant such that $\|g\|_{2p} = 1$.*

(ii) *We have*

$$\frac{p^2}{\rho^{*2}} = \begin{cases} \dfrac{a^4}{4}, & \text{for } p=1, d \geq 3, \\ 4\pi a \displaystyle\int_0^a z^4(s) s^2\, ds, & \text{for } p=2, d=3. \end{cases}$$

REMARK 2. In the case $p = 1$, one can solve (1.11) explicitly in terms of Bessel functions. Recall (see, e.g., [1], 9.1.7) that the *Bessel function J_ν of the first kind of order* $\nu \geq 0$ satisfies the *Bessel differential equation* $x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0$, and define a continuous function $z:[0,\infty) \to \mathbb{R}$ by $z(0) = 1$ and

$$z(x) = \Gamma(d/2)\left(\frac{x}{2}\right)^{(2-d)/2} J_{(d-2)/2}(x) \qquad \text{for } x > 0. \tag{1.12}$$

From Bessel's equation we easily derive that z satisfies (1.11). Further properties of Bessel functions (see [1], 9.1.30) show that $z'(0) = 0$. The continuous function $\zeta(x) = z(x) + (d-2)^{-1}xz'(x)$ takes the positive value $\zeta(0) = 1$ at the origin and is nonpositive at the first zero ξ of z. Hence there is a minimal $a > 0$ with $\zeta(a) = 0$. For example, in dimension $d = 3$ we obtain $z(x) = \frac{1}{x}\sin x$ and $a = \pi/2$ and thus $1/\rho^* = \pi^2/8$. In dimension $d = 5$ we have $z(x) = 3x^{-3}(\sin x - x\cos x)$ and $a = \pi$, hence $1/\rho^* = \pi^2/2$.

1.4. *The dimension spectrum of thick points.* The fine *multifractal structure* of a random measure on $\mathbb{R}^d$ consists of the analysis of its thin and thick points. The latter part has been successfully completed in the case of intersection local times of any number of planar Brownian motions and in the case of the occupation measure of a single Brownian motion in transient dimensions $d \geq 3$.

Before stating our new result for the case of intersections of Brownian motions in $\mathbb{R}^3$, let us review the existing results. If ℓ is the occupation measure of a single Brownian path in $d \geq 3$, it was shown in [5] that the correct gauge function is given by $r \mapsto r^2[\log(1/r)]$, in the sense that the number

$$2\rho^* = \sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r^2[\log(1/r)]} \in (0,\infty) \tag{1.13}$$

exists and is nonrandom. The value of ρ^* agrees with the value obtained in Theorem 1.3(ii). The corresponding dimension spectrum is

$$\dim\left\{x \in \mathbb{R}^d : \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r^2[\log(1/r)]} = a\right\} = 2 - \frac{a}{\rho^*} \qquad \text{for } a \geq 0, \tag{1.14}$$

where negative values of the dimension mean that the set is empty. However, the assertions in (1.8) and (1.9) in this case are known for a long time due to the Ciesielski–Taylor identity [4], which is not available in the case of $p > 1$. The proof of (1.13) and (1.14) is linked to the upper tail behavior of Brownian occupation measure in a unit ball via self-similarity: most of the exceptionally large mass in a small ball around a given point is accumulated in a very short time, hence by scaling the probability of high concentration of occupation measure in a small ball is asymptotically equal to the probability of exceptionally large mass in a unit ball accumulated in a fixed time horizon.

The situation is entirely different in the recurrent dimension $d = 2$. Here, the same authors have shown that the intersection local time measures ℓ, which in the case $p = 1$ degenerate to the occupation measure, satisfy

$$\left(\frac{2}{p}\right)^p = \sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r^2[\log(1/r)]^{2p}} \tag{1.15}$$

and

$$\dim\left\{x \in \mathbb{R}^d : \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r^2[\log(1/r)]^{2p}} = a\right\} = 2 - pa^{1/p} \qquad \text{for } a \geq 0. \tag{1.16}$$

See [8] in case $p = 1$ and [6] in case $p > 1$. The values in the planar case bear no relationship with our result on the upper tail asymptotics in (1.8) in this case. Heuristically, this is due to the fact that in the planar case, the probability that a small ball has unusually large concentration of mass cannot be related via self-similarity to the probability that the unit ball carries large mass, but depends just on the number of visits of the motions to the ball before the first exit from the large ball $B(0, R)$. In plain words, the asymptotics in (1.13)–(1.16) are determined in $d = 2$ by many returns to the small ball in a long time range, and in higher dimensions by relatively large occupation times already in a short time range.

Our next theorem closes the gap in this picture by describing the Hausdorff dimension spectrum of thick points for intersection local time measure on the intersections of two Brownian paths in $\mathbb{R}^3$.

THEOREM 1.4. *Suppose that ρ^* is as in* (1.9) *for the case $d = 3$, $p = 2$, $R = \infty$ and U the unit ball, respectively, as in Theorem* 1.3(ii) *for $p = 2$. Then, almost surely,*

$$\sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = \limsup_{r\downarrow 0} \sup_{x\in S} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = \left(\frac{\rho^*}{2}\right)^2. \tag{1.17}$$

Moreover,

$$\dim\left\{x \in S : \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = a\right\} = 1 - \sqrt{a}\,\frac{2}{\rho^*} \qquad \textit{for all } a \geq 0, \tag{1.18}$$

where dim *denotes the Hausdorff dimension.*

REMARK 3. There is a further difference between the recurrent and transient case, which is worth noting. In the planar case the lim sup may be replaced by a lim inf without changing the dimension spectrum. This gives the so-called spectrum of *consistently thick points*. In higher dimensions this is not the case and the spectrum of consistently thick points requires a different gauge function. This was observed in [5], Theorem 1.6. The precise Hausdorff dimension spectrum for the consistently thick points of intersection local time measures in $\mathbb{R}^d$, $d \geq 3$, is an *open* problem even in the case $p = 1$ of a single Brownian path.

To complete the picture, we review the results of [7] for *thin* points of the occupation measure ℓ on a single Brownian path $S = \{W(t) : t \in [0, T]\}$, T the first exit time from a large open ball. It turns out that here the ambient dimension does not play a major role. In all $d \geq 2$, we have

$$1 = \inf_{x \in S} \liminf_{r \to 0} \frac{\ell(B(x,r))}{r^2[\log(1/r)]^{-1}},$$

and a dimension spectrum given by

$$\dim\left\{x \in S : \liminf_{r \downarrow 0} \frac{\ell(B(x,r))}{r^2[\log(1/r)]^{-1}} = a\right\} = 2 - \frac{2}{a}.$$

The spectrum of thin points for intersections of Brownian paths seems to be an *open* problem.

2. Preliminaries. In this section, we collect some material to prepare the proofs. First, we survey three rigorous constructions of the intersection local time measures; afterwards we introduce another important variational problem which is strongly linked to (1.9), and finally we show how to reduce the problem of upper tail asymptotics to the problem of moment asymptotics in our case.

2.1. *Brownian intersection local times.* Recall that T_i is the first time the motion W_i reaches a sphere of fixed radius $R > 0$, which may be infinity in the transient case. Also recall that we kill W_i at time T_i for $i = 1, \dots, p$. We denote the Euclidean ball in $\mathbb{R}^d$ centered at x with radius $r > 0$ by $B(x, r)$. For every $\varepsilon > 0$ define the *Wiener sausage* around W_i by

$$S_\varepsilon^i = \{x \in \mathbb{R}^d : \text{there is } t \in [0, T_i) \text{ with } |x - W_i(t)| < \varepsilon\} \quad \text{for } i = 1, \dots, p, \tag{2.1}$$

and their intersection $S_\varepsilon = \bigcap_{i=1}^p S_\varepsilon^i$. Recall (1.1) and observe that

$$S = \bigcap_{\varepsilon > 0} S_\varepsilon = \{z \in \mathbb{R}^d : z = W_1(t_1) = \cdots = W_p(t_p) \text{ for some } t_i \in [0, T_i)\}$$

is the intersection of the p independent Brownian paths. This set is our object of interest. There are three ways to equip S with a natural measure:

1. by projecting the local time at 0 for the confluent Brownian motion (see [11]);
2. as renormalized limit for $\varepsilon \downarrow 0$ of the Lebesgue measure on S_ε (see [14]);
3. as a Hausdorff measure on S with an appropriate dimension gauge function (see [15]).

Luckily, all three approaches lead to the same object, the *intersection local time measure* ℓ on S. Let us first briefly review the approach of Geman, Horowitz and Rosen [11], based on local times for the confluent Brownian motion process $W : \mathbb{R}^p \to \mathbb{R}^{d(p-1)}$ defined by

$$W(s_1, \dots, s_p) = \big(W_1(s_1) - W_2(s_2), \dots, W_{p-1}(s_{p-1}) - W_p(s_p)\big).$$

With probability one there is a family $\{\mu_y : y \in (\mathbb{R}^d)^{p-1}\}$ of finite measures on $\prod_{i=1}^p [0, T_i)$ with the following two properties:

(i) The mapping $y \mapsto \mu_y$ is continuous with respect to the vague topology on the space $\mathcal{M}(\mathbb{R}^p)$ of locally finite measures on $\mathbb{R}^p$.

(ii) For all Borel functions $g : (\mathbb{R}^d)^{p-1} \to [0, \infty]$ and $f : \prod_{i=1}^p [0, T_i) \to [0, \infty]$,

$$\int g(y) \langle f, d\mu_y \rangle \, dy = \int_{\prod_{i=1}^p [0, T_i)} f g \circ W \, ds_p \cdots ds_1. \tag{2.2}$$

These properties imply that μ_y is supported by the level set at y of the confluent Brownian motion; that is, by

$$\left\{ (s_1, \dots, s_p) \in \prod_{i=1}^p [0, T_i) : W(s_1, \dots, s_p) = y \right\}.$$

The family $\{\mu_y : y \in (\mathbb{R}^d)^{p-1}\}$ is called the family of local times of the confluent Brownian motion process. The local time μ_0 at level 0 is of special interest to us, because it is supported by the set of p-tuples of times where the p motions $W_1, \dots, W_p$ coincide. The image measure ℓ of μ_0 under the mapping $(t_1, \dots, t_p) \mapsto W_1(t_1)$ is a natural finite measure on the intersections of the Brownian paths, which we hence call the *intersection local time* of the p Brownian motions.

Le Gall [14] carried out the second approach. Define

$$s_d(\varepsilon) = \begin{cases} \pi^{-p} \log^p(1/\varepsilon), & \text{if } d = 2, \\ (2\pi\varepsilon)^{-2}, & \text{if } d = 3 \text{ and } p = 2, \\ \dfrac{2}{\omega_d (d-2)} \varepsilon^{2-d}, & \text{if } d \geq 3 \text{ and } p = 1, \end{cases} \tag{2.3}$$

where ω_d is the volume of the d-dimensional unit ball. By [14], Theorème 3.1, the renormalized restriction of the Lebesgue measure on S_ε converges to ℓ as $\varepsilon \downarrow 0$, in the sense that, for every set $A \subset \mathbb{R}^d$ that is almost surely an ℓ-continuity set,

$$\lim_{\varepsilon \downarrow 0} s_d(\varepsilon) \lambda(S_\varepsilon \cap A) = \ell(A). \tag{2.4}$$

The convergence holds in the $L^q(\mathbb{P})$-sense for all $q \in [1, \infty)$ and, in particular, in probability.

Finally, the most natural way to equip S with a uniform distribution is by means of a suitable Hausdorff measure. Le Gall showed in [15] that ℓ may be defined intrinsically as a constant multiple of the ψ-Hausdorff measure on the random set S in the case of p spatial Brownian motions for the gauge function

$$\psi_p(r) = \begin{cases} r^2[\log(1/r)\log\log\log(1/r)]^p, & \text{if } d = 2,\ p \in \mathbb{N}, \\ r^{3-p}[\log\log(1/r)]^p, & \text{if } d = 3,\ p \in \{1, 2\}, \\ r^2[\log\log(1/r)], & \text{if } d \geq 4,\ p = 1. \end{cases} \tag{2.5}$$

Here we have included the results of Ciesielski and Taylor [4] and Ray [22] in the case $p = 1$ for completeness.

2.2. *A related variational problem.* As announced, we do not directly derive the upper tails of $\ell(U)$ in (1.8), but we describe the moment asymptotics and derive (1.8) from this result. In this subsection, we introduce the variational formula that turns out to feature the moment asymptotics of $\ell(U)$. Furthermore, we describe the relationship of the two problems of upper tail and moment asymptotics on the stage of the variational formulas. We have to introduce more notation.

For any probability measure μ and finite measure $\widetilde{\mu}$ on the same measurable space the *relative entropy* or *Kullback–Leibler distance* of μ with respect to $\widetilde{\mu}$ is defined as

$$H(\mu \mid \widetilde{\mu}) = \begin{cases} \displaystyle\int \mu(dx) \log \frac{\mu(dx)}{\widetilde{\mu}(dx)}, & \text{if } \mu \ll \widetilde{\mu}, \\ \infty, & \text{otherwise.} \end{cases} \tag{2.6}$$

If $\widetilde{\mu}$ is also a probability measure, then by Jensen's inequality we always have $H(\mu \mid \widetilde{\mu}) \geq 0$ and equality holds if and only if $\mu = \widetilde{\mu}$.

More specifically, let $U \subset \mathbb{R}^d$ be an open bounded set and denote by $\mathcal{M}_1(U)$ the space of probability measures on U, equipped with the weak topology. For any $\mu \in \mathcal{M}_1(U)$ we define $I(\mu) = H(\mu \mid \lambda)$, the relative entropy of μ with respect to the Lebesgue measure λ on U. Note that $I(\mu) \geq -\log\lambda(U)$ with equality if and only if μ is the normalized Lebesgue measure on U. Moreover, I is a convex and lower semicontinuous function. We denote by

$$\begin{aligned} \mathcal{M}_1^*(U) = \{\nu \in \mathcal{M}_1(U^2) :&\ \text{for all Borel sets } A \subset U \\ &\text{we have } \nu(A \times U) = \nu(U \times A)\} \end{aligned} \tag{2.7}$$

the set of probability measures ν on U^2 with equal marginals $\nu_1(A) = \nu(A \times U)$ and $\nu_2(A) = \nu(U \times A)$. For $\nu \in \mathcal{M}_1(U^2)$ we define

$$I_\mu^2(\nu) = \begin{cases} H(\nu \mid \nu_1 \otimes \mu), & \text{if } \nu \in \mathcal{M}_1^*(U), \\ \infty, & \text{otherwise.} \end{cases} \tag{2.8}$$

Note that I_μ^2 is the rate function for the empirical pair measures of an i.i.d. sequence with marginal distribution μ. In particular, I_μ^2 is lower semicontinuous and convex.

Define a function $\mathcal{G}: \mathcal{M}_1(U) \to \mathbb{R}$ by

$$\mathcal{G}(\mu) = \inf_{\substack{\nu \in \mathcal{M}_1^*(U) \\ \nu_1 = \mu}} \{ I_\mu^2(\nu) - \langle \nu, \log G \rangle \}, \tag{2.9}$$

where we extend the notation $\langle \cdot, \cdot \rangle$ to integrals on U^2. Observe that it suffices to take the infimum over measures ν satisfying $\nu \ll \mu \otimes \mu$. We can replace $I_\mu^2(\nu)$ in the definition of $\mathcal{G}$ by either the relative entropy $H(\nu \mid \mu \otimes \mu)$ or the *mutual information* $H(\nu \mid \nu_1 \otimes \nu_2)$. For the further development, the variational formula

$$\kappa^* = \inf_{\mu \in \mathcal{M}_1(U)} \{ I(\mu) + p\mathcal{G}(\mu) \} \tag{2.10}$$

is of fundamental interest. We state its link to (1.9) as follows.

PROPOSITION 2.1. *For every positive integer $p < d/(d-2)$ we have*

$$\begin{aligned} &\sup\{ \langle g^{2p-1}, \mathfrak{A} g^{2p-1} \rangle : g \in L^{2p}(U) \text{ with } \|g\|_{2p} = 1 \} \\ &\quad = \exp\left(-\frac{1}{p} \inf_{\mu \in \mathcal{M}_1(U)} \{ I(\mu) + p\mathcal{G}(\mu) \} \right). \end{aligned} \tag{2.11}$$

Moreover, g is a maximizer of the left-hand side if and only if the measure $\mu(dx) = g^{2p}(x)\,dx$ is a minimizer on the right-hand side of (2.11). *Every minimizing sequence of the variational problem on the right-hand side of* (2.11) *has a subsequence converging weakly to a minimizer.*

Proposition 2.1 is proved in Section 4 together with the results of Section 1.3.

2.3. *Tail asymptotics from moment asymptotics.* For reasons which are explained at the beginning of Section 3, we are able to deal with integer moments of $\ell(U)$ much better than with the probability of the event $\{\ell(U) > a\}$. In this section, we show how to derive the upper tail asymptotics of $\ell(U)$ from moment asymptotics. Let us first state our main result for the moment asymptotics.

PROPOSITION 2.2.

$$\lim_{k \uparrow \infty} \frac{1}{k} \log \mathbb{E}\left[\frac{\ell(U)^k}{(k!)^p} \right] = - \inf_{\mu \in \mathcal{M}_1(U)} \{ I(\mu) + p\mathcal{G}(\mu) \}.$$

Section 3 is devoted to the proof of Proposition 2.2. In order to derive the upper tail asymptotics from Proposition 2.2 we need the following Tauberian theorem.

LEMMA 2.3. *Let X be any nonnegative random variable and fix $p \in \mathbb{N}$. Then for any $\kappa \in \mathbb{R}$ the following two implications hold*:

(i)

$$\limsup_{k\uparrow\infty}\frac{1}{k}\log E\left[\frac{X^k}{(k!)^p}\right]\le -\kappa \implies \limsup_{a\uparrow\infty} a^{-1/p}\log P\{X>a\}\le -pe^{\kappa/p}.$$

(ii)

$$\lim_{k\uparrow\infty}\frac{1}{k}\log E\left[\frac{X^k}{(k!)^p}\right]= -\kappa \implies \lim_{a\uparrow\infty} a^{-1/p}\log P\{X>a\}= -pe^{\kappa/p}.$$

PROOF. The proof of (i) is easy and based on the substitution $a_k = e^{-\kappa}k^p$, Markov's inequality and Stirling's formula as follows:

$$\begin{aligned}\limsup_{k\uparrow\infty} a_k^{-1/p}\log P\{X>a_k\} &= e^{\kappa/p}\limsup_{k\uparrow\infty}\frac{1}{k}\log P\{X^k>e^{-k\kappa}k^{kp}\}\\ &\le e^{\kappa/p}\limsup_{k\uparrow\infty}\frac{1}{k}\log E\left[\frac{X^k}{e^{-k\kappa}k^{kp}}\right]\\ &= e^{\kappa/p}\limsup_{k\uparrow\infty}\left(\frac{1}{k}\log E\left[\frac{X^k}{(k!)^p}\right]+\kappa-p\right)\\ &\le -pe^{\kappa/p}.\end{aligned}$$

Since $a_{k+1}^{-1/p}/a_k^{-1/p}\to 1$ as $k\uparrow\infty$, we see that it is sufficient to consider the subsequence a_k rather than an arbitrary sequence tending to infinity.

The proof of (ii) is based on the construction of the transformed measure

$$d\widehat{P}^k(X)=\frac{X^k}{E[X^k]}\,dP(X) \qquad \text{for } k\in\mathbb{N},$$

and the fact that the random variable

$$Y_k=\log\left(\frac{X}{e^{-\kappa}k^p}\right)$$

satisfies, for every $\varepsilon>0$,

$$\lim_{k\uparrow\infty}\widehat{P}^k\{|Y_k|\le\varepsilon\}=1. \tag{2.12}$$

To prove this, fix an arbitrary $\varepsilon>0$ and pick some small number $\alpha>0$. Then, by the Markov inequality, we may estimate

$$\widehat{P}^k\{Y_k\ge\varepsilon\}=\widehat{P}^k\{X^{k\alpha}\ge e^{(\varepsilon-\kappa)k\alpha}k^{pk\alpha}\}\le e^{-\varepsilon k\alpha}e^{\kappa k\alpha}k^{-pk\alpha}\widehat{E}^k[X^{k\alpha}],$$

where $\widehat{E}^k$ denotes expectation with respect to $\widehat{P}^k$. Note that $\widehat{E}^k[X^{k\alpha}]=E[X^{k(1+\alpha)}]/E[X^k]$. Using our assumption and Stirling's formula, we see that the quotient has the asymptotic behavior $e^{-(\kappa+p)k\alpha}k^{pk\alpha}(1+\alpha)^{kp(1+\alpha)}e^{o(k)}$ as $k\uparrow\infty$. Inserting this in the right-hand side above, we get

$$\widehat{P}^k\{Y_k\geq\varepsilon\}\leq\exp\biggl(pk\alpha\biggl(-\frac{\varepsilon}{p}-1+\frac{1+\alpha}{\alpha}\log(1+\alpha)+o(1)\biggr)\biggr)\qquad\text{as }k\uparrow\infty.$$

If $\alpha>0$ is chosen small enough, then the expression between the inner brackets is negative and bounded away from zero, such that we obtain that $\lim_{k\uparrow\infty}\widehat{P}^k\{Y_k\geq\varepsilon\}=0$. Analogously one shows that $\lim_{k\uparrow\infty}\widehat{P}^k\{Y_k\leq-\varepsilon\}=0$, and this implies (2.12).

In order to finish the proof of the lower bound, we keep $\varepsilon>0$ arbitrarily fixed and substitute this time $a=e^{-\kappa}k^pe^{-\varepsilon}$. It is again clear that the consideration of this subsequence suffices. Note that $\{X>a\}=\{Y_k>-\varepsilon\}\supset\{|Y_k|\leq\varepsilon\}$. This implies that

$$a^{-1/p}\log P\{X>a\}\geq e^{\kappa/p}e^{\varepsilon/p}\frac{1}{k}\log P\{|Y_k|\leq\varepsilon\}.$$

Note that $P\{|Y_k|\leq\varepsilon\}=\widehat{E}^k[X^{-k}\mathbb{1}_{\{|Y_k|\leq\varepsilon\}}]E[X^k]$ and that we may estimate $X^{-k}\geq e^{k(-\varepsilon+\kappa)}k^{-pk}$ on $\{|Y_k|\leq\varepsilon\}$. Using this estimate, our assumption on the asymptotics of $E[X^k]$ and Stirling's formula, we obtain

$$\liminf_{a\uparrow\infty}a^{-1/p}\log P\{X>a\}\geq e^{\kappa/p}e^{\varepsilon/p}\biggl(-\varepsilon-p+\liminf_{k\uparrow\infty}\frac{1}{k}\log\widehat{P}^k\{|Y_k|\leq\varepsilon\}\biggr).$$

Because of (2.12), the latter limit inferior is equal to zero. After letting $\varepsilon\downarrow0$, we get the assertion. □

PROOF OF THEOREM 1.1. For the existence of the limit (1.8) follows from Proposition 2.2 and Lemma 2.3(ii); the characterization (1.9) of the limit is then immediate from Proposition 2.1. □

3. Large moment asymptotics. In this section we prove the large moment asymptotics in Proposition 2.2. In Section 3.1, we introduce a main tool: a formula for the integer moments of $\ell(U)$, which is due to Le Gall. The two main technical steps are carried out in Sections 3.2 (cutting off the Green's function) and 3.3 (finite partitioning of U). The combinatorial core argument of the proof appears in Section 3.4. The ingredients are put together in the final Section 3.5, where the proof of Proposition 2.2 is finished.

3.1. *Le Gall's moment formula.* Starting point for our analysis of high moments of $\ell(U)$ is the following formula for the moments of the intersection local times. For $k\in\mathbb{N}$, we denote by $\mathfrak{S}_k$ the set of permutations $\sigma=(\sigma(1),\dots,\sigma(k))$ of the numbers $1,\dots,k$. Denote by $x^1,\dots,x^p\in U$ the starting points of the Brownian motions $W_1,\dots,W_p$.

LEMMA 3.1. *For any* $k \in \mathbb{N}$,

$$\mathbb{E}[\ell(U)^k] = \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^{p} \sum_{\sigma \in \mathfrak{S}_k} G(x^j, y_{\sigma(1)}) \prod_{i=2}^{k} G(y_{\sigma(i-1)}, y_{\sigma(i)}). \tag{3.1}$$

The proof follows from the results of [14]; see [15], Part I (2.c).

On a heuristical level, Le Gall's formula can be derived from the symbolical formula (1.2). Indeed,

$$\begin{aligned}
\mathbb{E}[\ell(U)^k] &= \mathbb{E}\biggl[\biggl(\int_U dy \prod_{j=1}^{p} \int_0^{T_j} ds\, \delta_y(W_j(s))\biggr)^k\biggr] \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^{p} \mathbb{E}_{x^j}\biggl[\sum_{\sigma \in \mathfrak{S}_k} \int \cdots \int_{0 \le s_1 \le \cdots \le s_k \le T^j} \prod_{i=1}^{k} \delta_{y_{\sigma(i)}}(W_j(s_i))\, ds_i\biggr] \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^{p} \sum_{\sigma \in \mathfrak{S}_k} \int \cdots \int_{0 \le s_1 \le \cdots \le s_k} p_{s_1}(x^j, y_{\sigma(1)}) \\
&\qquad\qquad \times \prod_{i=2}^{k} p_{s_i - s_{i-1}}(y_{\sigma(i-1)}, y_{\sigma(i)})\, ds_i\, ds_1 \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^{p} \sum_{\sigma \in \mathfrak{S}_k} G(x^j, y_{\sigma(1)}) \prod_{i=2}^{k} G(y_{\sigma(i-1)}, y_{\sigma(i)}),
\end{aligned}$$

where $p_s(x, y)$, as before, denotes the transition density of the stopped Brownian motion. For every $j \in \{1, \ldots, p\}$ define $\Phi_k^j : U^k \to \mathbb{R}$ by

$$\Phi_k^j(y) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} G(x^j, y_{\sigma(1)}) \prod_{i=2}^{k} G(y_{\sigma(i-1)}, y_{\sigma(i)}) \qquad \text{for } y = (y_1, \ldots, y_k) \in U^k. \tag{3.2}$$

Note that $\Phi_k^j(y) = \infty$ if $x^j, y_1, \ldots, y_k$ are not pairwise distinct. Moreover, $\Phi_k^j(y)$ does not really depend on the vector $y = (y_1, \ldots, y_k)$, but only on the set $\{y_1, \ldots, y_k\}$; that is, $\Phi_k^j(y) = \Phi_k^j(y_\sigma)$ for any $\sigma \in \mathfrak{S}_k$, where we put $y_\sigma = (y_{\sigma(1)}, \ldots, y_{\sigma(k)})$. Hence, to prove Proposition 2.2 it suffices to show

$$\lim_{k \uparrow \infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^{p} \Phi_k^j(y) = - \inf_{\mu \in \mathcal{M}_1(U)} \{I(\mu) + p\mathcal{J}(\mu)\}. \tag{3.3}$$

This will follow from a Laplace-method argument. Indeed, the term $-I(\mu)$ is the exponential rate of the mass of the set of vectors $y = (y_1, \ldots, y_k)$ whose empirical measure resembles μ, and $-\mathcal{J}(\mu)$ will turn out to be the rate of $\Phi_k^j(y)$ for any such vector y, independently of $j \in \{1, \dots, p\}$; see Lemma 3.5 in this section.

3.2. *Cutting.* In order to derive the upper bound in (3.3), it is necessary to replace the Green's function by some bounded function. We achieve this by cutting off the Green's function at a large level and show that we do not change the exponential rate of $\Phi_k^j(y)$ asymptotically as the cut-off level gets large. Introduce, for $M \geq 0$, the cut-off Green's function $G_M = G \wedge M$ and denote, for $j = 1, \dots, p$,

$$\Phi_{k,M}^j(y) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} G_M(x^j, y_{\sigma(1)}) \prod_{i=2}^k G_M(y_{\sigma(i-1)}, y_{\sigma(i)}) \qquad \text{for } y = (y_1, \dots, y_k) \in U^k. \tag{3.4}$$

First, we need some technical preparation.

LEMMA 3.2. *There exists a constant $K_1 > 1$, depending only on U, p and d, such that, for all $a_1, \dots, a_{2p} \in U$ and $q \in \{0, \dots, p\}$,*

$$\int_U \prod_{i=1}^q G(a_{2i-1}, z) G(z, a_{2i}) \prod_{i=q+1}^p G(a_{2i-1}, z)\, dz \leq K_1 \prod_{i=1}^q G(a_{2i-1}, a_{2i}). \tag{3.5}$$

PROOF. First note that, as $p < d/(d-2)$, we have $\sup_{a \in U} \int_U G^p(a, z)\, dz < \infty$ and therefore $\sup_{a_1, \dots, a_p \in U} \int_U \prod_{i=1}^p G(a_i, z)\, dz < \infty$.

Now we turn to the proof of our assertion. The case $d = 2$ is simple, because the supremum over $a_1, \dots, a_{2p} \in U$ of the left-hand side of (3.5) is finite, and the infimum over $a_1, \dots, a_{2p} \in U$ of the product on the right-hand side is positive. The same argument applies in the case $d \geq 3$ when $p = 1$, $q = 0$ and in the case $d = 3$ when $p = 2$, $q = 0$.

If $d \geq 3$ and $p = q = 1$, we use that $U \subset U_1 \cup U_2$ where

$$U_1 = \left\{z \in U : |z - a_1| \geq \tfrac{1}{4}|a_2 - a_1|\right\} \quad \text{and} \quad U_2 = \left\{z \in U : |z - a_2| \geq \tfrac{1}{4}|a_2 - a_1|\right\}.$$

Recall from (1.5) that $G(x, y) \leq c_d |x - y|^{2-d}$, and pick some $c > 0$ such that $G(x, y) \geq c|x - y|^{2-d}$ for any distinct $x, y \in U$. On the first domain we estimate, for some $K > 1$,

$$\int_{U_1} G(a_1, z) G(z, a_2)\, dz \leq \frac{c_d}{c} 4^{d-2} G(a_1, a_2) \int_U G(z, a_2)\, dz \leq K G(a_1, a_2). \tag{3.6}$$

The second domain is treated analogously, proving the claim if $p = 1$. Now let $d = 3$ and $p = 2$. If $q = 2$ we cover U by four domains:

$$U_{ij} = \left\{z \in U : |z - a_i| \geq \tfrac{1}{4}|a_2 - a_1| \quad \text{and} \quad |z - a_j| \geq \tfrac{1}{4}|a_4 - a_3|\right\}, \qquad i = 1, 2,\ j = 3, 4.$$

On the first domain U_{13} we estimate

$$\begin{aligned}&\int_{U_{ij}} G(a_1,z)G(z,a_2)G(a_3,z)G(z,a_4)\,dz\\ &\quad\le 16\frac{c_d^2}{c^2}G(a_1,a_2)G(a_3,a_4)\int_U G(z,a_2)G(z,a_4)\,dz \qquad (3.7)\\ &\quad\le K\,G(a_1,a_2)G(a_3,a_4).\end{aligned}$$

Again the other domains can be treated analogously, which finishes the proof in the case $q=2$. The case $q=1$ is similar. □

LEMMA 3.3. *There is $C_0>0$ and, for all sufficiently large $M>1$ and small $\eta\in(0,1)$, there are constants $C_M>0$ and $\varepsilon_\eta>0$ such that, for all $x^1,\dots,x^p\in U$ and for any $k\in\mathbb{N}$,*

$$\begin{aligned}&\int_{U^k} dy\prod_{j=1}^p \Phi_k^j(y)\\ &\quad\le 2^p pk(2C_0)^k C_M^{\eta k} + 2^p(1+\varepsilon_\eta)^k\sum_{j=1}^p\sum_{m=\lceil k(1-p\eta)\rceil}^k\int_{U^m} dy\big(\Phi_{m,M}^j(y)\big)^p, \qquad (3.8)\end{aligned}$$

where $\lim_{M\uparrow\infty} C_M=\lim_{\eta\downarrow 0}\varepsilon_\eta=0$.

PROOF. Indeed, we show (3.8) with C_0 given by

$$C_0=\sup_{x\in U}\int_U G^p(x,y)\,dy\vee 1.$$

To define C_M we choose a minimal $\delta_M>0$ such that $G(x,y)\le M$ whenever $x,y\in U$ with $|x-y|\ge\delta_M$. Clearly, we have that $\lim_{M\uparrow\infty}\delta_M=0$. Now define

$$C_M=\sup_{x\in U}\sup_{a\in U}\int_{B(a,\delta_M)\cap U} G^p(x,y)\,dy.$$

From the local integrability of $G^p(x,\cdot)$ around x, one sees that C_0 and C_M are well defined and $\lim_{M\uparrow\infty} C_M=0$. We assume without loss of generality that $C_M\le 1$.

For any fixed $j\in\{1,\dots,p\}$ and any vector $y=(y_1,\dots,y_k)\in U^k$ we use the convention $y_0=x^j$. Put $\sigma(0)=0$ for $\sigma\in\mathfrak{S}_k$. Introduce

$$T^j(y)=\big\{n\in\{1,\dots,k\}:\text{ there is }0\le i<n\text{ such that }|y_i-y_n|\le\delta_M\big\} \qquad (3.9)$$

and

$$A^j(y)=\big\{\sigma\in\mathfrak{S}_k:\#T^j(y_\sigma)>\eta k\big\}.$$

In (3.1) we interchange the summation over σ with the integration over $y_1,\dots,y_k$ and split the sum over all $\sigma\in\mathfrak{S}_k$ into the sum over σ in $A^j(y)$ and in $\mathfrak{S}_k\setminus A^j(y)$.

On the first set of summation, we take advantage of the fact that at least ηk of the k integrals are taken just over a finite union of no more than k balls of radius δ_M, and the other $(1-\eta)k$ integrals yield some bounded exponential rate. On the second set of summation, we use that in at least $(1-\eta)k$ integrals we may replace G by its cut-off version G_M, and the exponential rate of the contribution coming from the remaining integrals is small if η is small. Let us turn to the details.

Using the estimate $\prod_{j=1}^p (a_j + b_j) \le 2^p \sum_{j=1}^p (a_j^p + b_j^p)$ for $a_j, b_j \ge 0$, we obtain that

$$\prod_{j=1}^{p} \Phi_k^j(y) \le 2^p \sum_{j=1}^{p} \big(\mathrm{I}_k^j(y) + \mathrm{II}_k^j(y)\big), \tag{3.10}$$

where

$$\mathrm{I}_k^j(y) = \left(\frac{1}{k!} \sum_{\sigma \in A^j(y)} G(x^j, y_{\sigma(1)}) \prod_{i=2}^{k} G(y_{\sigma(i-1)}, y_{\sigma(i)})\right)^p, \tag{3.11}$$

$$\mathrm{II}_k^j(y) = \left(\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k \setminus A^j(y)} G(x^j, y_{\sigma(1)}) \prod_{i=2}^{k} G(y_{\sigma(i-1)}, y_{\sigma(i)})\right)^p. \tag{3.12}$$

We first fix j and turn to an estimate for $\int_{U^k} dy\, \mathrm{I}_k^j(y)$. Use Jensen's inequality for the uniform distribution on $\mathfrak{S}_k$ to see that

$$\mathrm{I}_k^j(y) \le \frac{1}{k!} \sum_{\sigma \in A^j(y)} \prod_{i=1}^{k} G^p(y_{\sigma(i-1)}, y_{\sigma(i)}).$$

Now integrate over $y \in U^k$ and note that $\sigma \in A^j(y)$ if and only if $\mathrm{id} \in A^j(y_\sigma)$, where $\mathrm{id} \in \mathfrak{S}_k$ denotes the identical permutation. Use the permutation invariance of the integrand to obtain

$$\int_{U^k} dy\, \mathrm{I}_k^j(y) \le \int_{U^k} dy\, \mathbb{1}_{\{\mathrm{id} \in A^j(y)\}} \prod_{i=1}^{k} G^p(y_{i-1}, y_i).$$

Now substitute

$$\mathbb{1}_{\{\mathrm{id} \in A^j(y)\}} = \mathbb{1}_{\{\# T^j(y) > \eta k\}} = \sum_{\substack{B \subset \{1,\dots,k\} \\ \# B > \eta k}} \mathbb{1}_{\{T^j(y) = B\}}$$

and carry out successively the integration over $y_k, y_{k-1}, \dots, y_1$. Note that at least $\lfloor \eta k \rfloor$ of the corresponding integration regions for y_n (more precisely: the ones for $n \in B$) are contained in $U \cap B(y_i, \delta_M)$ for some $i < n$, the others are over the full set U.

With C_M, respectively, C_0 defined at the beginning of the proof, we arrive at

$$2^p \sum_{j=1}^{p} \int_{U^k} dy\, \mathrm{I}_k^j(y) \le 2^p p \sum_{\substack{B \subset \{1,\dots,k\} \\ \# B > \eta k}} k C_M^{\eta k} C_0^{(1-\eta)k} \le 2^p p k (2C_0)^k C_M^{\eta k},$$

recalling that $C_M \le 1$ and $C_0 \ge 1$. This yields the first term on the right-hand side of (3.8).

We now turn to an estimate of the second term. Again fix $j \in \{1, \dots, p\}$ and recall the conventions $y_0 = x^j$ and $\sigma(0) = 0$ for all $\sigma \in \mathfrak{S}_k$. We turn the pth power in the definition of $\mathrm{II}_k^j(y)$ into a sum over p indices $\sigma_1, \dots, \sigma_p$, that is,

$$\mathrm{II}_k^j(y) = \frac{1}{(k!)^p} \sum_{\sigma_1,\dots,\sigma_p \in \mathfrak{S}_k} \sum_{\substack{B_1,\dots,B_p \subset \{1,\dots,k\} \\ \forall l:\, \#B_l \le \eta k}} \prod_{l=1}^{p} \times \Biggl[\mathbb{1}_{\{T^j(y_{\sigma_l}) = B_l\}} \prod_{i_l=1}^{k} G(y_{\sigma_l(i_l-1)}, y_{\sigma_l(i_l)}) \Biggr]. \tag{3.13}$$

Given $\sigma_1, \dots, \sigma_p$ and $B_1, \dots, B_p$, we define $B = \bigcup_{l=1}^p \sigma_l(B_l)$. Let $m = k - \#B$ and note that $m \ge k - p\eta k$. We write the integration vector in (3.13) as $y = (y_B, y_{B^c})$, where $B^c = \{1, \dots, k\} \setminus B$ and $y_B = (y_i)_{i \in B}$ and $y_{B^c} = (y_i)_{i \in B^c}$.

Given $\sigma_l \in \mathfrak{S}_k$ and the set $B \subset \{1, \dots, k\}$, we define a bijective mapping $\tau_l : \{1, \dots, m\} \to B^c$ as the total ordering that is induced by σ_l on B^c. In other words, if B^c is equal to the set $\{\sigma_l(i_1), \dots, \sigma_l(i_m)\}$ with $i_1 < \cdots < i_m$, then $\tau_l(n) = \sigma_l(i_n)$ for any n. Note that, on $\bigcap_{l=1}^p \{T^j(y_{\sigma_l}) = B_l\}$, we have $|y_{\tau_l(i_l-1)} - y_{\tau_l(i_l)}| \ge \delta_M$ for all $l = 1, \dots, p$ and all $i_l = 1, \dots, m$.

Integrating (3.13), we get

$$\int_{U^k} dy\, \mathrm{II}_k^j(y) \le \frac{1}{(k!)^p} \sum_{\sigma_1,\dots,\sigma_p} \sum_{B_1,\dots,B_p} \int_{U^m} dy_{B^c} \prod_{l=1}^{p} \prod_{i_l=1}^{m} \mathbb{1}_{\{|y_{\tau_l}(i_l-1) - y_{\tau_l}(i_l)| \ge \delta_M\}} \times \int_{U^{k-m}} dy_B \prod_{l=1}^{p} \prod_{i_l=1}^{k} G(y_{\sigma_l(i_l-1)}, y_{\sigma_l(i_l)}). \tag{3.14}$$

Now we apply Lemma 3.2 to each of the $k - m$ single integrals over y_j with $j \in B$ on the right-hand side of (3.14). Here we conceive all y_j with $j \in B^c$ as parameters which play the role of $a_1, \dots, a_{2p}$ of Lemma 3.2. The $(k - m)$-fold application of this lemma yields

$$\int_{U^{k-m}} dy_B \prod_{l=1}^{p} \prod_{i_l=1}^{k} G(y_{\sigma_l(i_l-1)}, y_{\sigma_l(i_l)}) \le K_1^{k-m} \prod_{l=1}^{p} \prod_{i_l=1}^{m} G(y_{\tau_l(i_l-1)}, y_{\tau_l(i_l)}). \tag{3.15}$$

We substitute (3.15) in (3.14) and replace every G by G_M on the right-hand side of (3.15). Furthermore, in order to streamline the integration set, we write $y_{B^c} \in U^m$ as $y \in U^m$. This necessitates that we replace every bijection $\tau_l : \{1, \dots, m\} \to B^c$ by a permutation $\tilde{\tau}_l \in \mathfrak{S}_m$ in such a way that $\tau_l(n)$ is the $\tilde{\tau}_l(n)$th largest element in B^c for every $n \in \{1, \dots, m\}$.

Altogether, we obtain

$$\int_{U^k} dy\, \mathrm{II}_k^j(y) \le K_1^{k-m} \frac{1}{(k!)^p} \sum_{\sigma_1,\dots,\sigma_p} \sum_{B_1,\dots,B_p} \int_{U^m} dy \prod_{l=1}^{p} \prod_{i_l=1}^{m} G_M(y_{\tilde\tau_l(i_l-1)}, y_{\tilde\tau_l(i_l)}). \tag{3.16}$$

The integral on the right-hand side of (3.16) depends on the set tuple $(B_1,\dots,B_p)$ only via the cardinality $k-m$ of B. We now replace the sum over the set tuples by the sum $m = \lceil k(1-p\eta)\rceil,\dots,k$. We estimate the number of set tuples from above by

$$\left[\sum_{n=1}^{\lceil \eta k\rceil} \binom{k}{n}\right]^p \le k^p \binom{k}{\lceil \eta k\rceil}^p.$$

Furthermore, note that, when passing from the sum over $\sigma_1,\dots,\sigma_p \in \mathfrak{S}_k$ to the sum over $\tilde\tau_1,\dots,\tilde\tau_p \in \mathfrak{S}_m$, we obtain a counting factor of $(k!/m!)^p$. Then we rewrite

$$\frac{1}{m!^p} \sum_{\tilde\tau_1,\dots,\tilde\tau_p \in \mathfrak{S}_m} \prod_{l=1}^{p} \prod_{i_l=1}^{m} G_M(y_{\tilde\tau_l(i_l-1)}, y_{\tilde\tau_l(i_l)}) = \left(\Phi_{m,M}^j(y)\right)^p.$$

This leads to the estimate

$$\int_{U^k} dy\, \mathrm{II}_k^j(y) \le K_1^{p\eta k} k^p \binom{k}{\lceil \eta k\rceil}^p \sum_{m=\lceil k(1-p\eta)\rceil}^{k} \int_{U^m} dy \left(\Phi_{m,M}^j(y)\right)^p.$$

Now use Stirling's formula to see that the factors on the right-hand side in front of the integral are bounded from above by $(1+\varepsilon_\eta)^k$ with some $\varepsilon_\eta \downarrow 0$ as $\eta \downarrow 0$. This yields the second term on the right-hand side of (3.8), which ends the proof. □

3.3. *Discretization.* Our second main technical tool is a reduction to a discrete counting argument. For this purpose, we introduce a finite partition of U, which is carefully chosen in order to represent many details of the continuous picture.

To introduce appropriate notation, let $\Sigma_n = \{1,\dots,n\}$ and denote the partition sets by $U_1,\dots,U_n$. We assume that every U_l is measurable and has positive Lebesgue measure $\lambda(U_l)$. We call π the canonical projection $\pi : U \to \Sigma_n$, that is, $x \in U_{\pi x}$ for any $x \in U$. We write $\pi y = (\pi y_1,\dots,\pi y_k)$ for any $k \in \mathbb{N}$ and $y = (y_1,\dots,y_k) \in U^k$. Furthermore, if μ is a probability measure on U, then $\pi\mu \in \mathcal{M}_1(\Sigma_n)$ is its projection on Σ_n. Similarly for $\nu \in \mathcal{M}_1(U^2)$ we denote the projection on Σ_n^2 by $\pi\nu \in \mathcal{M}_1(\Sigma_n^2)$. If v is in the set $\mathcal{M}_1^*(\Sigma_n)$ of probability measures on Σ_n^2 with equal marginals, we denote by $\overline{v} \in \mathcal{M}_1(\Sigma_n)$ its left or right marginal measure. Note that $\pi\overline{\nu} = \overline{\pi\nu}$ for any $\nu \in \mathcal{M}_1^*(U)$, where we write $\overline{\nu}$ for the marginal measure of ν.

For measures $u \in \mathcal{M}_1(\Sigma_n)$ and $v \in \mathcal{M}_1(\Sigma_n^2)$ we define discrete analogues of the relative entropy functionals I and I_μ^2 by

$$\widetilde{I}(u) = \sum_{l \in \Sigma_n} u_l \log\left(\frac{u_l}{\lambda(U_l)}\right) \quad \text{and} \quad \widetilde{I}_u^2(v) = \sum_{l,m \in \Sigma_n} v_{l,m} \log\left(\frac{v_{l,m}}{\overline{v}_l u_m}\right), \tag{3.17}$$

using the usual convention $0 \log 0 = 0$. Recall that $G_M = G \wedge M$ and define the approximate Green functions $G_M^+, G^- : \Sigma_n^2 \to \mathbb{R}$ by

$$G_M^+(l,m) = \sup_{\substack{x \in U_l \\ y \in U_m}} G_M(x,y) \quad \text{and} \quad G^-(l,m) = \inf_{\substack{x \in U_l \\ y \in U_m}} G(x,y). \tag{3.18}$$

Functions $\mathcal{G}_M^+$ and $\mathcal{G}^-$ on $\mathcal{M}_1(\Sigma_n)$ analogous to $\mathcal{G}$ in (2.9) are defined by

$$\begin{aligned} \mathcal{G}_M^+(u) &= \inf_{\substack{v \in \mathcal{M}_1^*(\Sigma_n) \\ \overline{v} = u}} \{\widetilde{I}_u^2(v) - \langle v, \log G_M^+ \rangle\} \quad \text{and} \\ \mathcal{G}^-(u) &= \inf_{\substack{v \in \mathcal{M}_1^*(\Sigma_n) \\ \overline{v} = u}} \{\widetilde{I}_u^2(v) - \langle v, \log G^- \rangle\}, \end{aligned} \tag{3.19}$$

where we used the notation $\langle v, F \rangle = \sum_{l,m \in \Sigma_n} v_{l,m} F(l,m)$. The functions $\mathcal{G}_M^+$ and $\mathcal{G}^-$ are continuous. Indeed, for fixed u, if the set $\widetilde{V} \subset \mathcal{M}_1^*(\Sigma_n)$ is a neighborhood of the set $\{v \in \mathcal{M}_1^*(\Sigma_n) : \overline{v} = u\}$, there exists a neighborhood $\widetilde{U}$ of u with $\{v \in \mathcal{M}_1^*(\Sigma_n) : \overline{v} = \tilde{u}\} \subset \widetilde{V}$ for all $\tilde{u} \in \widetilde{U}$. Together with the obvious continuity of $\widetilde{I}_u^2(v)$ in both arguments u and v and of $v \mapsto \langle v, F \rangle$ this implies continuity of $\mathcal{G}_M^+$ and $\mathcal{G}^-$. Our aim is to determine a partition such that the coarsened variational formula is a good approximation of the variational formula on the right-hand side of (3.3).

LEMMA 3.4. *Given $\delta > 0$ and $M > 0$, there is a measurable partition $U_1, \dots, U_n$ of U, such that*

$$\begin{aligned} \inf_{\mu \in \mathcal{M}_1(U)} \{\widetilde{I}(\pi\mu) + p\mathcal{G}^-(\pi\mu)\} - \delta &\le \inf_{\mu \in \mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\} \\ &\le \inf_{\mu \in \mathcal{M}_1(U)} \{\widetilde{I}(\pi\mu) + p\mathcal{G}_M^+(\pi\mu)\} + \delta. \end{aligned} \tag{3.20}$$

PROOF. We start with the upper estimate. Choose the partition such that

$$|\log G_M^+(\pi x, \pi y) - \log G_M(x,y)| < \delta/p \qquad \text{for all } x, y \in U.$$

This is possible since $\log G_M$ is continuous on the closure of U^2. To every $\mu \in \mathcal{M}_1(U)$ we associate $\widetilde{\mu} \in \mathcal{M}_1(U)$ with constant density on the partition sets and $\pi\mu = \pi\widetilde{\mu}$. Analogously we associate to each $\nu \in \mathcal{M}_1^*(U)$ a $\tilde{\nu} \in \mathcal{M}_1^*(U)$

with constant density on the products of the partition sets and $\pi\nu = \pi\tilde{\nu}$. Then $I(\widetilde{\mu}) = \widetilde{I}(\pi\mu)$ and $\widetilde{I}^2_{\pi\mu}(\pi\nu) = I^2_{\widetilde{\mu}}(\tilde{\nu})$, and we infer that

$$\begin{aligned}\mathcal{G}^+_M(\pi\mu) &= \inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \pi\overline{\nu}=\pi\mu}} \big\{\widetilde{I}^2_{\pi\mu}(\pi\nu) - \langle\pi\nu, \log G^+_M\rangle\big\}\\ &\geq \inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \overline{\tilde{\nu}}=\widetilde{\mu}}} \big\{I^2_{\widetilde{\mu}}(\tilde{\nu}) - \langle\tilde{\nu}, \log G_M\rangle\big\} - \delta/p\\ &\geq \mathcal{G}(\widetilde{\mu}) - \delta/p\end{aligned}$$

and hence

$$\begin{aligned}\inf_{\mu\in\mathcal{M}_1(U)}\{I(\mu) + p\mathcal{G}(\mu)\} &\leq \inf_{\mu\in\mathcal{M}_1(U)}\{I(\widetilde{\mu}) + p\mathcal{G}(\widetilde{\mu})\}\\ &\leq \inf_{\mu\in\mathcal{M}_1(U)}\big\{\widetilde{I}(\pi\mu) + p\mathcal{G}^+_M(\pi\mu)\big\} + \delta,\end{aligned}$$

which shows the upper bound in (3.20).

Let us turn to the lower estimate. Given $\delta > 0$, we select an approximate minimizer $\nu \in \mathcal{M}_1^*(U)$ with marginal $\mu = \overline{\nu}$ such that

$$\inf_{\mu\in\mathcal{M}_1(U)}\{I(\mu) + p\mathcal{G}(\mu)\} \geq I(\mu) + p\big(I^2_{\overline{\nu}}(\nu) - \langle\nu, \log G\rangle\big) - \delta/2. \tag{3.21}$$

We use now Jensen's inequality to show that, for any partition, we have

$$I(\mu) \geq \widetilde{I}(\pi\mu) \quad\text{and}\quad I^2_{\overline{\nu}}(\nu) \geq \widetilde{I}^2_{\pi\overline{\nu}}(\pi\nu). \tag{3.22}$$

To prove this, abbreviate $\phi(x) = x\log x$ and note that

$$I(\mu) = \left\langle\lambda, \phi\circ\frac{d\mu}{d\lambda}\right\rangle = \sum_{l\in\Sigma_n}\lambda(U_l)\int_{U_l} d\overline{\lambda}_l\,\phi\circ\frac{d\mu}{d\lambda},$$

where $\overline{\lambda}_l$ denotes the normalized restriction of the Lebesgue measure to U_l. Now use Jensen's inequality for the convex function ϕ and summarize to arrive at the first inequality in (3.22). The proof of the second one goes along the same lines, recalling that $I^2_{\overline{\nu}}(\nu) = \langle\overline{\nu}\otimes\overline{\nu}, \phi\circ(d\nu/d(\overline{\nu}\otimes\overline{\nu}))\rangle$. Furthermore, for the fixed ν, every partition can be refined such that $\langle\nu, \log G\rangle \leq \langle\pi\nu, \log G^-\rangle + \delta/(2p)$. This can be seen by choosing N so large that $\langle\nu, \log G\rangle - \langle\nu, \log G_N\rangle < \delta/(4p)$ and using uniform continuity of $\log G_N$ on the closure of U^2 to split the domain of integration into partition sets on which the variation of $\log G_N$ is less that $\delta/(4p)$. Substituting this and (3.22) into (3.21), we arrive at the lower bound in (3.20). □

3.4. *The combinatorial argument.* Having removed technical obstacles, like unboundedness of the integrand, and discretized the integrals, we are ready to turn to the combinatorial core argument in the proof of Proposition 2.2. The main observation is that the k-fold product in (3.2) only depends on the numbers of transitions the sequence $y_{\sigma(0)}, y_{\sigma(1)}, \dots, y_{\sigma(k)}$ makes from one set of the partition to the other. In other words, one can summarize the permutations that lead to the same empirical pair measure of the sequence $\pi y_{\sigma(0)}, \pi y_{\sigma(1)}, \dots, \pi y_{\sigma(k)}$ and sum over all the empirical pair measures instead. Since the combinatorics turn out to be manageable, this observation leads to a great simplification and enables us to use large deviation theory for empirical pair measures. Let us turn to the details.

We fix a large $M > 0$ and small $\delta > 0$ and a partition $U_1, \dots, U_n$ chosen according to Lemma 3.4. Recall that $\mathcal{M}_1^*(\Sigma_n)$ denotes the set of probability measures on Σ_n^2 having equal marginals. In the next lemma, we give lower and upper bounds for $\Phi_k^j(y)$, respectively, $\Phi_{k,M}^j(y)$ in terms of a variational problem on $\mathcal{M}_1^*(\Sigma_n)$. Recall our notation $\pi y = (\pi y_1, \dots, \pi y_k)$ for $y \in U^k$, and denote by $L^1_{\pi y,k} \in \mathcal{M}_1(\Sigma_n)$ the *empirical measure*,

$$L^1_{\pi y,k}(A) = \frac{1}{k}\sum_{i=1}^{k} \delta_{y_i}(A) \qquad \text{for all } A \subset \Sigma_n.$$

LEMMA 3.5. *For all $j \in \{1, \dots, p\}$, as $k \uparrow \infty$, uniformly in $y = (y_1, \dots, y_k) \in U^k$,*

$$\Phi_{k,M}^j(y) \le e^{o(k)} \exp\bigl(-k \mathcal{G}_M^+(L^1_{\pi y,k})\bigr) \tag{3.23}$$

and

$$\Phi_k^j(y) \ge e^{o(k)} \exp\bigl(-k \mathcal{G}^-(L^1_{\pi y,k})\bigr). \tag{3.24}$$

PROOF. Fix j and recall the convention $y_0 = x^j$ and $\sigma(0) = 0$ for all $\sigma \in \mathfrak{S}_k$. Assume without loss of generality that $y_0 \in U_1$. Note that, for any $y = (y_1, \dots, y_k) \in U^k$, we have $G(y_{\sigma(i-1)}, y_{\sigma(i)}) \ge G^-(\pi y_{\sigma(i-1)}, \pi y_{\sigma(i)})$ and $G_M(y_{\sigma(i-1)}, y_{\sigma(i)}) \le G_M^+(\pi y_{\sigma(i-1)}, \pi y_{\sigma(i)})$. We introduce the *empirical pair measure*

$$L^2_{\psi,k} = \frac{1}{k}\sum_{i=1}^{k} \delta_{\{(\psi_{i-1},\psi_i)\}} \in \mathcal{M}_1(\Sigma_n^2) \qquad \text{for } \psi = (\psi_1, \dots, \psi_k) \in \Sigma_n^k,$$

where $\psi_0 = 1$ is the index of the set U_1. This implies that, with $\pi y_\sigma = (\pi y_{\sigma(1)}, \dots, \pi y_{\sigma(k)})$,

$$\begin{aligned}\Phi_k^j(y) &\ge \frac{1}{k!}\sum_{\sigma \in \mathfrak{S}_k} \exp(k\langle L^2_{\pi y_\sigma,k}, \log G^-\rangle) \quad \text{and}\\ \Phi_{k,M}^j(y) &\le \frac{1}{k!}\sum_{\sigma \in \mathfrak{S}_k} \exp(k\langle L^2_{\pi y_\sigma,k}, \log G_M^+\rangle).\end{aligned} \tag{3.25}$$

Denote by $\mathcal{M}_1^{(k)}(\Sigma_n)$ the set of probability measures u on Σ_n such that the numbers $ku(1),\dots,ku(n)$ are integers and sum up to k; analogously we define $\mathcal{M}_1^{(k)}(\Sigma_n^2)$. Abbreviate $u = L^1_{\pi y,k} \in \mathcal{M}_1^{(k)}(\Sigma_n)$ for the course of the proof. Note that, for any $\sigma \in \mathfrak{S}_k$, the total variation distance between u and any of the two marginal measures of $L^2_{\pi y_\sigma,k}$ is not bigger than $2/k$, which is denoted below as $\overline{L}^2_{\pi y_\sigma,k} \approx u$.

Now we reorganize the sum over $\sigma \in \mathfrak{S}_k$ by summing over all vectors $\psi \in \Sigma_n^k$ and simply counting the permutations σ such that $\psi = \pi y_\sigma$. Hence, for $F \in \{G^-, G_M^+\}$, we have

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_k} \exp(k\langle L^2_{\pi y_\sigma,k}, \log F\rangle)\\
&\quad = \sum_{\psi\in\Sigma_n^k}\sum_{\sigma\in\mathfrak{S}_k} \mathbb{1}_{\{\pi y_\sigma=\psi\}}\mathbb{1}_{\{\overline{L}^2_{\psi,k}\approx u\}} \exp(k\langle L^2_{\psi,k}, \log F\rangle)\\
&\quad = \sum_{\substack{v\in\mathcal{M}_1^{(k)}(\Sigma_n^2)\\ \overline{v}\approx u}} \exp(k\langle v, \log F\rangle) \sum_{\psi\in\Sigma_n^k} \mathbb{1}_{\{L^2_{\psi,k}=v\}} \#\{\sigma\in\mathfrak{S}_k : \pi y_\sigma=\psi\}.
\end{aligned} \tag{3.26}$$

Note that, for any $\psi \in \Sigma_n^k$ satisfying $L^2_{\psi,k} = v$ for some v satisfying $\overline{v} \approx u$, we have

$$\#\{\sigma\in\mathfrak{S}_k : \pi y_\sigma = \psi\} = \prod_{l=1}^n (ku_l)!. \tag{3.27}$$

In other words, if the empirical measures of πy and ψ coincide, then there are $\prod_{l=1}^n (ku_l)!$ reorderings of $(\pi y_1,\dots,\pi y_k)$ that are equal to $(\psi_1,\dots,\psi_k)$. Furthermore, well-known combinatorial considerations (see, e.g., [10], II.2) yield that, for $\overline{v} \approx u$,

$$\#\{\psi\in\Sigma_n^k : L^2_{\psi,k} = v\} = e^{o(k)} \frac{\prod_{l\in\Sigma_n}(ku_l)!}{\prod_{l,m\in\Sigma_n}(kv_{l,m})!}, \tag{3.28}$$

where $e^{o(k)}$ is uniform in $v \in \mathcal{M}_1^{(k)}(\Sigma_n^2)$. Substituting (3.27) and (3.28) in (3.26), we obtain

$$\begin{aligned}
&\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \exp(k\langle L^2_{\pi y_\sigma,k}, \log F\rangle)\\
&\quad = e^{o(k)} \sum_{\substack{v\in\mathcal{M}_1^{(k)}(\Sigma_n^2)\\ \overline{v}\approx u}} \exp(k\langle v, \log F\rangle) \frac{\prod_{l\in\Sigma_n}(ku_l)!^2}{k!\prod_{l,m\in\Sigma_n}(kv_{l,m})!}.
\end{aligned} \tag{3.29}$$

Stirling's formula yields that, uniformly in u and v, the last ratio is $e^{o(k)}\exp(-k\widetilde{I}^2_u(v))$. Hence, uniformly in $y \in U^k$,

$$\begin{aligned}
&\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \exp(k\langle L^2_{\pi y_\sigma,k}, \log F\rangle)\\
&\quad = e^{o(k)} \sum_{\substack{v\in\mathcal{M}_1^{(k)}(\Sigma_n^2)\\ \overline{v}\approx u}} \exp\bigl(-k\bigl(\widetilde{I}^2_u(v) - \langle v, \log F\rangle\bigr)\bigr).
\end{aligned} \tag{3.30}$$

Now recall that $\widetilde{I}_u^2(\cdot) - \langle \cdot, \log F\rangle$ is continuous, that the total variation distance between $\{\nu \in \mathcal{M}_1(\Sigma_n^2) : \overline{\nu} \approx u\}$ and $\mathcal{M}_1^*(\Sigma_n)$ is not bigger than $2/k$ and that $\#\mathcal{M}_1^{(k)}(\Sigma_n^2)$ is bounded above by a polynomial in k. From (3.25) we obtain our two assertions, for $F = G_M^+$, respectively, $F = G^-$. □

In the next lemma, we integrate the upper, respectively, lower bound stated in Lemma 3.5 over $y \in U^k$ and obtain an upper, respectively, lower bound for the logarithmic asymptotics in (3.3) in terms of a variational formula, which is the coarsened form of the right-hand side of (3.3).

LEMMA 3.6. *For each* $j \in \{1, \dots, p\}$ *and each* $M > 0$,

$$\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy\, \Phi_{k,M}^j(y)^p \le - \inf_{\mu \in \mathcal{M}_1(U)} \{\widetilde{I}(\pi\mu) + p\mathcal{G}_M^+(\pi\mu)\}. \tag{3.31}$$

Moreover,

$$\liminf_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^p \Phi_k^j(y) \ge - \inf_{\mu \in \mathcal{M}_1(U)} \{\widetilde{I}(\pi\mu) + p\mathcal{G}^-(\pi\mu)\}. \tag{3.32}$$

PROOF. In order to derive (3.31), we raise the upper bound (3.23) to the pth power and integrate over $y \in U^k$. Note that the right-hand side of (3.23) does not depend on y, but only on $\psi = \pi y = (\pi y_1, \dots, \pi y_k)$. Hence, it makes sense to subdivide the domain of integration according to the tuples that arise. In this way we get the bound

$$\begin{aligned}
\int_{U^k} dy\, \Phi_{k,M}^j(y)^p &\le e^{o(k)} \sum_{\psi \in \Sigma_n^k} \int_{U_{\psi_1}} \cdots \int_{U_{\psi_k}} dy_k \cdots dy_1 \exp\bigl(-kp\mathcal{G}_M^+(L_{\psi,k}^1)\bigr) \\
&= e^{o(k)} \sum_{\psi \in \Sigma_n^k} \exp\bigl(-kp\mathcal{G}_M^+(L_{\psi,k}^1)\bigr) \prod_{l \in \Sigma_n} \lambda(U_l)^{kL_{\psi,k}^1(l)} \\
&\le e^{o(k)} \sum_{u \in \mathcal{M}_1^{(k)}(\Sigma_n)} \#\{\psi \in \Sigma_n^k : L_{\psi,k}^1 = u\} e^{-kp\mathcal{G}_M^+(u)} \\
&\qquad\qquad \times \exp\biggl(k \sum_{l \in \Sigma_n} u_l \log \lambda(U_l)\biggr).
\end{aligned}$$

Now use Stirling's formula to see that the counting factor in the last line is $e^{o(k)} \exp(-k \sum_{l \in \Sigma_n} u_l \log u_l)$, uniformly in $u \in \mathcal{M}_1^{(k)}(\Sigma_n)$. We recall (3.17) and infer that

$$\int_{U^k} dy\, \Phi_{k,M}^j(y)^p \le e^{o(k)} \sum_{u \in \mathcal{M}_1^{(k)}(\Sigma_n)} \exp\bigl(-k\bigl(p\mathcal{G}_M^+(u) - \widetilde{I}(u)\bigr)\bigr).$$

From the two facts that $\widetilde{I} - p\mathcal{G}_M^+$ is continuous, and that $\#\mathcal{M}_1^{(k)}(\Sigma_n)$ is bounded above by a polynomial in k, we infer that (3.31) holds.

The proof of the lower bound (3.32) is analogous, using (3.24). $\square$

3.5. *Conclusion.* In this section we complete the proof of Proposition 2.2. Recall that only (3.3) is left to be shown.

The lower bound is immediate from (3.32) and (3.20). For the upper bound take logarithms in (3.8), divide by k and let $k \uparrow \infty$. According to Lemma 3.3, after letting $M \uparrow \infty$, one obtains

$$\begin{aligned}\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^p \Phi_k^j(y) \\ \leq \log(1+\varepsilon_\eta) + \lim_{M\uparrow\infty} \max_{j=1}^p \limsup_{k\uparrow\infty} \frac{1}{k} \log \sum_{m=\lceil k(1-p\eta)\rceil}^k \int_{U^m} dy\, \Phi_{m,M}^j(y)^p.\end{aligned} \tag{3.33}$$

Estimate the last integral on the right-hand side from above against the maximum on $m \in \{\lceil k(1-p\eta)\rceil, \dots, k\}$. Now we pick $\delta > 0$, choose a partition according to Lemma 3.4, and use Lemma 3.6 to infer

$$\limsup_{m\uparrow\infty} \frac{1}{m} \log \int_{U^m} dy\, \Phi_{m,M}^j(y)^p \leq - \inf_{\mu\in\mathcal{M}_1(U)} \{\widetilde{I}(\pi\mu) + p\mathcal{G}_M^+(\pi\mu)\}.$$

Now use (3.20) and let first $M \uparrow \infty$, then $\eta \downarrow 0$, to obtain

$$\begin{aligned}\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^p \Phi_k^j(y) &\leq \lim_{\eta\downarrow 0} \log(1+\varepsilon_\eta) - \lim_{\eta\downarrow 0}(1-p\eta) \inf_{\mu\in\mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\} + \lim_{\eta\downarrow 0} \delta(1-p\eta) \\ &= - \inf_{\mu\in\mathcal{M}_1(U)} \{I(\mu) + p\mathcal{G}(\mu)\} + \delta.\end{aligned}$$

Now let $\delta \downarrow 0$. This completes the proof of the upper bound in (3.3).

4. Analysis of the variational formulas in (1.9) and (2.10).

In this section we prove our results related to the characterization of the limit in (1.8). These are Proposition 2.1, Theorem 1.2, and finally Theorem 1.3, which we discuss in Sections 4.1–4.3, respectively.

4.1. *Proof of Proposition* 2.1. It turns out that the variational formula (1.9) is much easier to analyze than (2.10). Indeed, in contrast to (2.10), for (1.9) compactness and continuity arguments are available, such that the existence of maximizers, their positivity and the Euler–Lagrange equations can be derived in a fairly standard way. Furthermore, (1.9) majorizes (2.10) in a certain strong sense, such that the maximizers of (1.9) turn out to stand in an elementary one-to-one correspondence to the minimizers of (2.10).

Before going into the details of the proof, we discuss general continuity properties of the operator $\mathfrak{A}$ that are of use in the forthcoming proofs.

LEMMA 4.1. (i) *If* $d = 2$ *and* $q > 1$, *then* $\mathfrak{A}$ *is a bounded linear map from* $L^q(U)$ *into* $L^\infty(U)$.

(ii) *If* $d \geq 3$ *and* $1 < q < d/2$, *then* $\mathfrak{A}$ *is a bounded linear map from* $L^q(U)$ *into* $L^{dq/(d-2q)}(U)$.

PROOF. The first statement follows easily from Hölder's inequality using that in $d = 2$ we have $\sup_x \int G(x, y)^q \, dy < \infty$ for all $q > 1$.

If $d \geq 3$ we recall, e.g., from [17], 4.3, the Hardy–Littlewood–Sobolev inequality. For all $s, r > 1$, $0 < \lambda < d$ with $1/r + \lambda/d + 1/s = 2$ there is a constant $C > 0$ with

$$\left| \int_U \int_U f(x) |x - y|^{-\lambda} h(y) \, dx \, dy \right| \leq C \|f\|_s \|h\|_r. \tag{4.1}$$

Recall that $G(x, y) \leq c_d |x - y|^{2-d}$ and use the Hardy–Littlewood–Sobolev inequality with $\lambda = d - 2$ and $s = q$, $r = dq/(dq + 2q - d)$, which yields $\langle h, \mathfrak{A} f \rangle \leq C \|f\|_q \|h\|_r$ for any $f \in L^q(U)$ and $h \in L^r(U)$. Hence $\mathfrak{A}$ maps f continuously into the dual of $L^r(U)$, which is $L^{dq/(d-2q)}(U)$. This proves (ii). □

For our purposes, it is convenient to rewrite (1.9) as

$$\rho^* = \sup\{\langle f, \mathfrak{A} f \rangle : f \in L^{2p/(2p-1)}(U) \quad \text{and} \quad \|f\|_{2p/(2p-1)} \leq 1\}. \tag{4.2}$$

It is clear that the supremum in (4.2) may be restricted to positive normalized functions $f \in L^{2p/(2p-1)}(U)$. We start by showing that the operator $\mathfrak{A} : L^{2p/(2p-1)}(U) \to L^{2p}(U)$ is continuous and establish (2.11).

LEMMA 4.2. *Suppose* p *is a positive integer with* $p < d/(d - 2)$.

(i) $\mathfrak{A}$ *is a bounded linear map from* $L^{2p/(2p-1)}(U)$ *into* $L^{2p}(U)$. *In particular*, $\rho^* \leq \|\mathfrak{A}\|$.

(ii) *For all* $\mu \in \mathcal{M}_1(U)$ *with* $g^{2p}(x) \, dx = \mu(dx)$ *we have*

$$\exp\left(-\frac{1}{p}\big(I(\mu) + p\mathcal{G}(\mu)\big)\right) \leq \langle g^{2p-1}, \mathfrak{A} g^{2p-1} \rangle. \tag{4.3}$$

(iii) *Equality in* (4.3) *holds if and only if there is* $\rho > 0$ *with the property*

$$\mathfrak{A} g^{2p-1}(x) = \rho g(x) \qquad \text{for } \mu\text{-almost every } x \in U. \tag{4.4}$$

Moreover, in this case $I(\mu) + p\mathcal{G}(\mu) = -p \log \rho$ *and* $\mathcal{G}(\mu) = -\log \rho - 2\langle \mu, \log g \rangle$.

PROOF. Assertion (i) is a direct consequence of Lemma 4.1 and the fact that the spaces $L^q(U)$ are continuously embedded in each other for decreasing q. Indeed, if $d = 2$, one can choose $q = 2p/(2p-1)$ and if $d \ge 3$ and $p < d/(d-2)$ it is always possible to choose the parameter q in such a way that the three conditions

$$\frac{dq}{d-2q} \ge 2p, \qquad q \le \frac{2p}{2p-1}, \qquad 1 < q < \frac{d}{2},$$

are satisfied, from which (i) readily follows.

We now show (ii). Assume without loss of generality that $I(\mu) + p\mathcal{G}(\mu) < \infty$, and let g^{2p} be a density of μ. We first show that

$$\mathcal{G}(\mu) \ge -\langle \mu, \log(g \cdot \mathfrak{A} g^{2p-1})\rangle. \tag{4.5}$$

Indeed, recall the definition of $\mathcal{G}$ in (2.9) and use Jensen's inequality for $-\log$ to estimate

$$\begin{aligned}
\mathcal{G}(\mu) &= -\sup_{\substack{\nu \in \mathcal{M}_1^*(U)\\ \overline{\nu} = \mu}} \left\langle \nu, \log\left(\frac{d(\mu \otimes \mu)}{d\nu} G\right)\right\rangle \\
&= -\sup_{\substack{\nu \in \mathcal{M}_1^*(U)\\ \overline{\nu} = \mu}} \int_U \mu(dx)\left[\int_U \frac{\nu(dx\,dy)}{\mu(dx)} \log\left(\frac{(\mu\otimes\mu)(dx\,dy)}{\nu(dx\,dy)} \frac{G(x,y)}{g(y)}\right)\right] \\
&\quad - \langle \mu, \log g\rangle \\
&\ge -\sup_{\substack{\nu \in \mathcal{M}_1^*(U)\\ \overline{\nu} = \mu}} \int_U \mu(dx) \log\left[\int_U \frac{\nu(dx\,dy)}{\mu(dx)} \frac{(\mu\otimes\mu)(dx\,dy)}{\nu(dx\,dy)} \frac{G(x,y)}{g(y)}\right] \\
&\quad - \langle \mu, \log g\rangle \\
&= -\langle \mu, \log(\mathfrak{A} g^{2p-1})\rangle - \langle \mu, \log g\rangle.
\end{aligned} \tag{4.6}$$

This shows that (4.5) holds.

Now we apply (4.5) and Jensen's inequality, to obtain

$$\begin{aligned}
\exp\left(-\frac{1}{p}\big(I(\mu) + p\mathcal{G}(\mu)\big)\right) &\le \exp\left(-\frac{1}{p}\langle \mu, \log g^{2p}\rangle + \langle \mu, \log(g \cdot \mathfrak{A} g^{2p-1})\rangle\right) \\
&= \exp\left(\left\langle \mu, \log \frac{\mathfrak{A} g^{2p-1}}{g}\right\rangle\right) \le \left\langle \mu, \frac{\mathfrak{A} g^{2p-1}}{g}\right\rangle \\
&= \langle g^{2p-1}, \mathfrak{A} g^{2p-1}\rangle.
\end{aligned} \tag{4.7}$$

This proves (ii).

Finally, to prove (iii), assume that we have equality everywhere in (4.7). By strict convexity of the logarithm, equality in the second line implies that, for some constant $\rho > 0$, we have $\mathfrak{A} g^{2p-1} = \rho g$, for μ-almost every $x \in U$. Together with equality in the first line of (4.7), which is equality in (4.5), this yields that

$$\mathcal{G}(\mu) = -\langle \mu, \log(g \cdot \mathfrak{A} g^{2p-1})\rangle = -\log \rho - 2\langle \mu, \log g\rangle.$$

Conversely, if (4.4) holds, we have equality in the second line of (4.7). To check equality in the first line, we define a probability measure $\nu \in \mathcal{M}^*(U)$ by

$$\nu(dx\,dy) = h(x,y)\,dx\,dy = \frac{1}{\rho} g^{2p-1}(x) g^{2p-1}(y) G(x,y)\,dx\,dy.$$

The measure ν is well defined by (4.4) and plugging this into (2.9) yields

$$\mathcal{G}(\mu) \le -\langle \mu, \log(\rho g^2)\rangle.$$

This means that equality holds in (4.5) and hence also in the first line of (4.7), completing the proof of (iii). □

We now prove the existence of maximizers in (4.2).

LEMMA 4.3. *Every maximizing sequence for the variational problem in* (4.2) *has a subsequence that converges weakly in* $L^1(U)$ *toward some maximizer of this problem.*

PROOF. Write $\|\cdot\|$ for $\|\cdot\|_{2p/(2p-1)}$ and put $K = \{f \in L^1(U) : f \ge 0, \|f\| \le 1\}$. It is easy to see that it is sufficient to establish the following two statements:

$$K \text{ is weakly compact in } L^1(U), \tag{4.8}$$

$$\text{the mapping } f \mapsto \langle f, \mathfrak{A} f\rangle \text{ is upper semicontinuous on } K \text{ in the weak topology on } L^1(U). \tag{4.9}$$

To establish (4.8) we first note that K is weakly relatively compact in $L^1(U)$, because the family K is uniformly integrable (see [9], C7). It remains to show that K is weakly closed. For this purpose it suffices to show weak closedness in $\{f \in L^1(U) : f \ge 0\}$. Let $f \ge 0$ satisfy $f \notin K$. Then $\|f\| > 1$. Let

$$\varphi_n = \left(\frac{f \wedge n}{\|f \wedge n\|}\right)^{1/(2p-1)} \in L^\infty(U).$$

Then we have

$$\begin{aligned}\liminf_{n\uparrow\infty} \int \varphi_n f\,dx &\ge \liminf_{n\uparrow\infty} \|f\wedge n\|^{1/(1-2p)} \int (f\wedge n)^{2p/(2p-1)}\,dx \\ &= \liminf_{n\uparrow\infty} \|f \wedge n\| = \|f\| > 1,\end{aligned}$$

and, for all $g \in K$, using Hölder's inequality,

$$\int \varphi_n g\,dx \le \|\varphi_n\|_{2p}\|g\| \le \|f\wedge n\|^{1/(1-2p)} \left(\int (f\wedge n)^{2p/(2p-1)}\,dx\right)^{1/2p} = 1.$$

Hence, for n sufficiently large, the function $\varphi = \varphi_n \in L^\infty(U)$ satisfies, for all sequences $(g_k : k \in \mathbb{N})$ in K,

$$\limsup_{k\uparrow\infty} \langle \varphi, g_k \rangle \le 1 < \langle \varphi, f \rangle,$$

which means that f is not in the weak L^1-closure of K. We infer that K must be weakly closed in $L^1(U)$, which proves (4.8).

To show (4.9) we assume that $(f_n : n \in \mathbb{N})$ and f are in K such that $f_n \to f$ weakly in $L^1(U)$. Our aim is to show that $\limsup_{n\to\infty} \langle f_n, \mathfrak{A} f_n \rangle \le \langle f, \mathfrak{A} f \rangle$. For some large $M > 0$, we split $\mathfrak{A} = \mathfrak{A}_M + (\mathfrak{A} - \mathfrak{A}_M)$, where the operator $\mathfrak{A}_M$ is defined like $\mathfrak{A}$, but with $G\mathbb{1}_{G\le M}$ instead of G. We show that $\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \langle f_n, (\mathfrak{A} - \mathfrak{A}_M) f_n \rangle = 0$. In $d = 2$ this is straightforward using Hölder's inequality. If $d \ge 3$, we recall that $G(x, y) \le c_d |x - y|^{2-d}$ to estimate $G(x, y) \le C' |x - y|^{3/2-d} G(x, y)^{1/(4-2d)}$ for some $C' > 0$. Apply the Hardy–Littlewood–Sobolev inequality (4.1) with $r = s = 4d/(2d + 3)$ and $\lambda = d - 3/2$ to obtain, for some $C > 0$,

$$\begin{aligned} \langle f_n, (\mathfrak{A} - \mathfrak{A}_M) f_n \rangle &\le C' M^{1/(4-2d)} \iint f_n(x) |x - y|^{-\lambda} f_n(y)\, dx\, dy \\ &\le C' C M^{1/(4-2d)} \| f_n \|_r^2. \end{aligned} \tag{4.10}$$

Since $r \le 2p/(2p - 1)$, this vanishes as $M \to \infty$, uniformly in $n \in \mathbb{N}$. Hence, we only have to show that, for fixed $M > 0$, we have $\lim_{n\to\infty} \langle f_n, \mathfrak{A}_M f_n \rangle = \langle f, \mathfrak{A}_M f \rangle$ and taking the limit $M \to \infty$ yields the desired conclusion. The proof of this follows from a standard monotone class argument. Let $\mathscr{H}$ be the class of all $\phi \in L^\infty(U^2)$ with the property that

$$\lim_{n\uparrow\infty} \iint f_n(x) f_n(y) \phi(x, y)\, dx\, dy = \iint f(x) f(y) \phi(x, y)\, dx\, dy.$$

This class is a vector space containing all functions of the form $\phi_1(x)\phi_2(y)$ for $\phi_1, \phi_2 \in L^\infty(U)$. Finally, suppose that $0 \le \phi_k \uparrow \phi \in L^\infty(U^2)$ and $\phi_k \in \mathscr{H}$. We can estimate, using the triangle inequality and Hölder's inequality,

$$\begin{aligned} &\left| \iint f(x) f(y) \phi(x, y)\, dx\, dy - \iint f_n(x) f_n(y) \phi(x, y)\, dx\, dy \right| \\ &\quad \le \left| \iint f(x) f(y) (\phi - \phi_k)(x, y)\, dx\, dy \right| \\ &\qquad + \left| \iint \big(f(x) f(y) - f_n(x) f_n(y) \big) \phi_k(x, y)\, dx\, dy \right| \\ &\qquad + \left| \iint f_n(x) f_n(y) (\phi - \phi_k)(x, y)\, dx\, dy \right| \\ &\quad \le \| f \|^2 \| \phi - \phi_k \|_{2p} + \left| \iint (f(x) f(y) - f_n(x) f_n(y)) \phi_k(x, y)\, dx\, dy \right| \\ &\qquad + \| f_n \|^2 \| \phi - \phi_k \|_{2p}, \end{aligned} \tag{4.11}$$

where we recall that $\|\cdot\| = \|\cdot\|_{2p/(2p-1)}$ and extend the notation $\|\cdot\|_{2p}$ to $L^{2p}(U^2)$. Now it is easy to see that the last line of (4.11) can be made arbitrarily small by choosing a large k and letting $n \to \infty$. This shows that $\phi \in \mathcal{H}$ and, by the monotone class theorem, we have shown that $\mathcal{H} = L^\infty(U^2)$. Hence $G\mathbb{1}_{\{G\le M\}} \in \mathcal{H}$ and this completes the proof of (4.9). □

Now we derive the Euler–Lagrange equation for the maximizers in (4.2).

LEMMA 4.4. *Any nonnegative maximizer of the variational problem in* (4.2) *is essentially bounded away from* 0. *Writing this maximizer as* g^{2p-1}, *the function* $g \in L^{2p}(U)$ *satisfies* (4.4) *with some* $\rho > 0$.

PROOF. Let $f \in L^{2p/(2p-1)}(U)$ be a nonnegative and normalized maximizer in (4.2). We first show that f is essentially bounded from 0. Assume the contrary, which means that $\lambda\{f \le \varepsilon\} > 0$ for all $\varepsilon > 0$. We fix some $c > 0$ such that $\lambda\{f \ge c\} > 0$. We define a function $\tilde f : U \to [0,\infty)$ as follows:

$$\tilde f(x) = \begin{cases} f(x) + a, & \text{if } f(x) \le \varepsilon, \\ f(x) - b, & \text{if } f(x) \ge c, \\ f(x), & \text{otherwise.} \end{cases}$$

Our plan is to choose $a, b > 0$ and $\varepsilon > 0$ so small that $\|\tilde f\|_{2p/(2p-1)} = 1$ but $\langle \tilde f, \mathfrak{A}\tilde f\rangle > \langle f, \mathfrak{A} f\rangle$, which contradicts the maximality of f. For every sufficiently small a and $\varepsilon > 0$ we may find $b \le c/2$, such that $\|\tilde f\|_{1+\eta} = 1$, where we abbreviate $\eta = 1/(2p-1)$. This condition implies that

$$0 = \|\tilde f\|_{1+\eta}^{1+\eta} - \|f\|_{1+\eta}^{1+\eta} = \int_{\{f\le\varepsilon\}} \big[(f(x)+a)^{1+\eta} - f(x)^{1+\eta}\big]\,dx + \int_{\{f\ge c\}} \big[(f(x)-b)^{1+\eta} - f(x)^{1+\eta}\big]\,dx,$$

and one can see that this implies, for some constant $C > 0$, that does not depend on a or ε,

$$b \le Ca(a+\varepsilon)^\eta \lambda\{f \le \varepsilon\}. \tag{4.12}$$

On the other hand, one can use this to check in the same way that, again for constants $C_1, C_2 > 0$, which do not depend on a or ε,

$$\langle f, \mathfrak{A} f\rangle - \langle \tilde f, \mathfrak{A}\tilde f\rangle \le a\lambda\{f \le \varepsilon\}\big(C_1(a+\varepsilon)^\eta - C_2\big).$$

We may now choose a, ε suitably, such that the right-hand side is negative. This yields the desired contradiction.

In order to prove the second assertion, it is convenient to abbreviate $h = g^{2p} = f^{2p/(2p-1)}$; hence $\|h\|_1 = 1$. Let $\varphi : U \to \mathbb{R}$ be bounded with $\int \varphi(x)\,dx = 0$. For sufficiently small $\varepsilon > 0$, we have $\|h + \varepsilon\varphi\|_1 = 1$ and $h + \varepsilon\varphi \geq 0$ on U. Using the maximality of f in (4.2), we infer

$$\begin{aligned} 0 &= \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \langle (h+\varepsilon\varphi)^{(2p-1)/2p}, \mathfrak{A}(h+\varepsilon\varphi)^{(2p-1)/2p}\rangle \\ &= \frac{2p-1}{p}\langle \varphi, h^{-1/2p}\mathfrak{A}h^{(2p-1)/2p}\rangle. \end{aligned} \tag{4.13}$$

We infer that, in $L^2(U)$, the function $h^{-1/2p}\mathfrak{A}(h^{(2p-1)/2p})$ is orthogonal to the orthogonal complement of the span of the constants. Hence there is a constant ρ such that

$$\rho h^{1/2p}(x) = \mathfrak{A}h^{(2p-1)/2p}(x) \qquad \text{for } \lambda\text{-almost every } x \in U.$$

As h is essentially positive, we have $\rho > 0$. Recalling that $h = g^{2p}$, we have verified (4.4). $\square$

Putting together the previous three lemmas we obtain the existence of minimizers for the variational problem in (2.10) and the convergence of minimizing sequences.

LEMMA 4.5. *Every minimizing sequence for the variational problem in* (2.10) *has a subsequence that converges weakly toward some minimizer of this problem. If g^{2p} denotes its density, then $g \in L^{2p}(U)$ satisfies* (4.4).

PROOF. Suppose that $(\mu_n : n \in \mathbb{N}) \subset \mathcal{M}_1(U)$ is any minimizing sequence for the variational problem in (2.10). At least for sufficiently large n, the measures μ_n have Lebesgue densities, which we denote by $g_n^{2p} \in L^{2p}(U)$. By Lemma 4.2, $(g_n^{2p-1} : n \in \mathbb{N})$ is a maximizing sequence of the variational problem in (4.2). By Lemma 4.3, we can extract a subsequence converging weakly in $L^1(U)$ to some $g^{2p-1} \in L^{2p/(2p-1)}(U)$. Defining μ by $\mu(dx) = g^{2p}(x)\,dx$ yields a minimizer of the problem (2.10) we started with. It is obvious that μ is the weak limit of the corresponding subsequence of $(\mu_n : n \in \mathbb{N})$. $\square$

We have now established the existence of a minimizer $\mu(dx) = g^{2p}(x)\,dx$ for (2.10) satisfying (4.4) with $\rho = \rho^*$ defined by (1.9), respectively, by (4.2). Also, g is a maximizer of (1.9) if and only if the measure $\mu(dx) = g^{2p}(x)\,dx$ is a minimizer of (2.10). Hence, Proposition 2.1 and the existence of the maximizers have been proved.

4.2. *Proof of Theorem* 1.2. Next, we establish smoothness properties for every g satisfying (4.4).

LEMMA 4.6. *Every g satisfying* (4.4) *has a version that is twice continuously differentiable on U. By*

$$g(x) = \frac{1}{\rho^*} \int_U G(x, y) g^{2p-1}(y)\, dy \qquad \text{for } x \in B(0, R), \tag{4.14}$$

one can define an extension of g to $B(0, R)$, which is continuous on the closure of $B(0, R)$ and continuously differentiable in the interior of $B(0, R)$.

PROOF. We start by showing that (4.4) implies that $g \in L^\infty(U)$. Indeed, if $d = 2$ we know that $g \in L^{2p}(U)$ and thus Lemma 4.1 shows that $g = (1/\rho)\mathfrak{A} g^{2p-1} \in L^\infty(U)$. In the case $d = 3$ and $p = 2$ we get from Lemma 4.1 that $g = (1/\rho)\mathfrak{A} g^3 \in L^{12}(U)$. By Hölder's inequality we infer that

$$\mathfrak{A} g^3(x) \le \left(\int_U G(x, y)^{4/3}\, dy \right)^{3/4} \left(\int_U g^{12}(y)\, dy \right)^{1/4}.$$

Because $G(x, y) \le c_d |x - y|^{-1}$ the right-hand side is bounded, and we use (4.4) again to infer that $g \in L^\infty(U)$. Finally, in the case $d \ge 3$ and $p = 1$ we iterate the use of Lemma 4.1 and (4.4) to obtain $g \in L^s(U)$ for some $s > d/2$. Let $q = s/(s-1)$ and note that $1 < q < d/(d-2)$. By Hölder's inequality again we infer that $\mathfrak{A} g(x) \le \|G(x, \cdot)\|_q \|g\|_{q/(q-1)}$. The right-hand side is bounded, and we infer from (4.4) that $g \in L^\infty(U)$. Now extend g by (4.14) to the whole ball $B(0, R)$. Using that $g \mathbb{1}_U \in L^\infty(U)$, we know from [21], 4.6.6, that g is continuously differentiable on $B(0, R)$ and continuous on the closure of $B(0, R)$. To see that it is even twice continuously differentiable on U let

$$\mathfrak{B} g(y) = \int_U G_U(x, y) g(x)\, dx \qquad \text{for } y \in B(0, R),$$

where G_U denotes the Green's function with zero boundary conditions on U rather than on $B(0, R)$. Denote by h_U the harmonic measure on ∂U. For almost every $x \in U$ we have

$$g(x) = \frac{1}{\rho} \mathfrak{A} g^{2p-1}(x) = \frac{1}{\rho} \mathfrak{B} g^{2p-1}(x) + \frac{1}{\rho} \int_{\partial U} \mathfrak{A} g^{2p-1}(z) h_U(x, dz). \tag{4.15}$$

As g^{2p-1} is Lipschitz continuous on U, the first summand on the right-hand side is twice continuously differentiable by [21], 4.6.6, and the second summand is harmonic in U and hence infinitely often differentiable. □

We have by now shown that every maximizer g in (1.9) has a version satisfying Theorem 1.2(a). We now prove the equivalence of the two characterizations of the maximizer given in Theorem 1.2.

LEMMA 4.7. *The equation in Theorem* 1.2(a) *and the partial differential equation in Theorem* 1.2(b) *are equivalent.*

PROOF. Suppose (g, ρ) is a solution of (1.10). By the Meyer–Tanaka formula, for a Brownian motion $B = (B_s : s \geq 0)$, the process M defined by

$$M_t = g(B_t) + \frac{1}{\rho}\int_0^t g^{2p-1}(B_s)\mathbb{1}_U(B_s)\,ds$$

is a local martingale on $[0, T)$, where T is the first exit time from $B(0, R)$. Hence, by the optional stopping theorem, for all $x \in U$,

$$g(x) = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}\left[\frac{1}{\rho}\int_0^T g^{2p-1}(B_s)\mathbb{1}_U(B_s)\,ds\right] = \frac{1}{\rho}\mathfrak{A}g^{2p-1}(x).$$

Now assume, conversely, that the operator equation of Theorem 1.2(a) holds in U. The continuation of g to the closure of $B(0, R)$ defined in (4.14) has the necessary differentiability and continuity properties. Recall (4.15). Applying $-\Delta/2$ to both sides of the equation and recalling that $-\Delta\mathfrak{B}f(x) = 2f(x)$ for all bounded $f \in C^2(U)$ (see [21], Theorem 4.6.6) gives (1.10) for $x \in U$. To get (1.10) for $x \notin U$ it suffices to observe that (4.14) is harmonic on the interior of $B(0, R) \setminus U$. □

Finally, we prove uniqueness of the maximizer in (1.9) in the case $p = 1$. Note that we do not know whether uniqueness holds if $p > 1$, except in the special case when U is a ball and $R = \infty$; see Section 4.3 below.

LEMMA 4.8. *In the case* $p = 1$ *the solution* g *of* (4.4) *is uniquely determined.*

PROOF. Recall the Krein–Rutman theorem (see, e.g., [2], Theorem 3.2). $L^\infty(U)$ is an ordered Banach space whose positive cone has nonempty interior and $\mathfrak{A}$ is a strongly positive compact linear operator $\mathfrak{A} : L^\infty(U) \to L^\infty(U)$. Hence $\mathfrak{A}$ has exactly one normalized strictly positive eigenvector and this eigenvector is the unique maximizer g. □

4.3. *Proof of Theorem* 1.3. Suppose now that $U = B(0, 1)$ is the open unit ball centered at the origin and consider unstopped Brownian motions in $\mathbb{R}^d$, $d \geq 3$. By (1.6) in this situation $G(x, y)$ is a rotationally symmetric function of $x - y$, and the maximizers for the problem in (4.2) are necessarily rotationally symmetric. This is a direct consequence of Riesz's strict rearrangement inequality; see [17], 3.7, 3.9.

Let $f \in L^{2p/(2p-1)}(U)$ be a minimizer of the problem in (4.2), put $g^{2p-1} = f$ on $B(0, 1)$ and extend g to a C^1-function on the whole of $\mathbb{R}^d$. Recall that this is possible by Theorem 1.2(b). Define $z : [0, \infty) \to (0, \infty)$ by $g(x) = cz(a|x|)$, choosing the positive parameters a, c such that

$$z(0) = 1 \quad \text{and} \quad \rho^* = 2c^{2p-2}/a^2.$$

Writing the Laplacian in polar coordinates gives for $|x| < 1$,

$$-ca^2 z(a|x|)^{2p-1} = -\frac{2}{\rho^*} g^{2p-1}(x) = \Delta g(x)$$
$$= \frac{d-1}{|x|} caz'(a|x|) + ca^2 z''(a|x|). \tag{4.16}$$

In other words, z satisfies (1.11) on $[0, a]$. Differentiability of g implies $z'(0) = 0$, and we can infer that z satisfies the requirements of Theorem 1.3(i) on the interval $[0, a]$.

We now show uniqueness of the solutions of (1.11) with $z(0) = 1$ and $z'(0) = 0$. In the case $p = 1$ the transformation (1.12) can be inverted and this reduces the problem to uniqueness of solutions of the Bessel differential equation; see Remark 2. To show uniqueness in the remaining case $p = 2$, $d = 3$, we argue as follows. Pick $a \in (0, \xi)$ maximal such that $z(a) + az'(a) = 0$. Hence, the derivative of the map $x \mapsto xz(x)$ is positive on $(0, a)$. Therefore, we may define $f : [0, az(a)] \to [0, \infty)$ by putting $f(xz(x)) = xz(x) + x^2 z'(x)$. Note that f has a simple zero at the origin, and that f satisfies the differential equation $f'(t) = 1 - t^3/f(t)$. Indeed, using (1.11) in the penultimate step, we get

$$f'(xz(x)) = \frac{\frac{d}{dx} f(xz(x))}{xz'(x) + z(x)} = 1 + \frac{2xz'(x) + x^2 z''(x)}{xz'(x) + z(x)}$$
$$= 1 - \frac{x^2 z(x)^3}{xz'(x) + z(x)} = 1 - \frac{(xz(x))^3}{f(xz(x))}, \tag{4.17}$$

and the result follows by substitution of $t = xz(x)$.

The function f is uniquely determined by the differential equation $f'(t) = 1 - t^3/f(t)$, at least close to the origin. To show this, let $\|\cdot\|_T$ be the supremum norm on $[0, T]$. Then, if f_1 and f_2 satisfy the equation and $f_1(0) = f_2(0) = 0$, we obtain, for suitable constants C', $C > 0$,

$$\|f_1 - f_2\|_T \le \int_0^T |f_1'(s) - f_2'(s)|\, ds = \int_0^T \frac{s^3}{f_1(s) f_2(s)} |f_1(s) - f_2(s)|\, ds$$
$$\le C' \int_0^T s \|f_1 - f_2\|_T\, ds \le CT^2 \|f_1 - f_2\|_T.$$

For sufficiently small $T > 0$, this shows that f is unique on $[0, T]$. A standard Picard–Lindelöf argument shows the uniqueness on the whole interval on which f_1 and f_2 are defined and positive.

The uniqueness of z now follows by noting that z can be uniquely recomputed from f by the formula

$$\int_{az(a)}^{xz(x)} \frac{ds}{f(s)} = C + \log x \qquad \text{for } x \in [0, a], \tag{4.18}$$

where $C = \lim_{\varepsilon \downarrow 0}(\int_{az(a)}^{\varepsilon} \frac{ds}{f(s)} - \log \varepsilon)$. Indeed, (4.18) follows by writing the relation between z and f as

$$\frac{1}{x} = \frac{\frac{d}{dx}(xz(x))}{f(xz(x))},$$

and integrating.

We now check that a is indeed the smallest value such that $z(a) = az'(a)/(2-d)$. Because g is harmonic on $\mathbb{R}^d \setminus \overline{B(0,1)}$ and vanishing at infinity, there exists a constant $C > 0$ such that $g(x) = C|x|^{2-d}$ for all $|x| \geq 1$. Hence, $z(t) = Kt^{2-d}$ for some $K > 0$ and any $t \geq a$, and therefore $z'(a) = (2-d)z(a)/a$, as claimed. Now suppose that there is a smaller value $0 < \tilde{a} < a$ satisfying $z(\tilde{a}) = \tilde{a}z'(\tilde{a})/(2-d)$. Define $h(x) = \tilde{c}z(\tilde{a}|x|)$ where $\tilde{c}$ is chosen such that $\|h\|_{2p} = 1$. This implies, writing σ_{d-1} for the area of the $(d-1)$-sphere,

$$\begin{aligned} \frac{\tilde{a}^d}{\tilde{c}^{2p}} &= \tilde{a}^d \int_U z^{2p}(\tilde{a}|x|)\,dx = \tilde{a}^d \sigma_{d-1} \int_0^1 r^{d-1} z^{2p}(\tilde{a}r)\,dr \\ &= \sigma_{d-1} \int_0^{\tilde{a}} s^{d-1} z^{2p}(s)\,ds < \sigma_{d-1} \int_0^a s^{d-1} z^{2p}(s)\,ds = \frac{a^d}{c^{2p}}. \end{aligned} \tag{4.19}$$

One can check easily, by reversing the arguments above, that h is a solution of

$$\frac{1}{2}\Delta h(x) = -\frac{\tilde{a}^2}{2\tilde{c}^{2p-2}} h^{2p-1}(x) \mathbb{1}_U(x).$$

In particular, recalling the equivalence of this differential equation and the operator equation in Theorem 1.2(a) we get

$$(\langle h^{2p-1}, \mathfrak{A} h^{2p-1} \rangle) = \frac{2\tilde{c}^{2p-2}}{\tilde{a}^2} > \frac{2c^{2p-2}}{a^2} = \rho^*.$$

The strict inequality above follows from (4.19) together with the fact that $d(1 - 1/p) \leq 2$. This contradicts the fact that g is a maximizer in (1.9).

We have thus characterized the maximizer uniquely. The formula in Theorem 1.3(ii) follows easily from the relation $\rho^* = 2c^{2p-2}/a^2$. Indeed, this is obvious in the case $p = 1$, and in the case $p = 2$, $d = 3$ it follows by evaluating the normalization constant using polar coordinates,

$$c^{-4} = \int_0^1 z(a|x|)^4\,dx = \frac{4\pi}{a^3} \int_0^a s^2 z(s)^4\,ds.$$

Hence, the proof of Theorem 1.3 is complete. □

5. The dimension spectrum of thick points. In this section we give the proof of Theorem 1.4. The upper bound for both (1.17) and (1.18) follows from standard methods, for the lower bound we rely on the method of *hitting with a percolation limit set* and adapt ideas of [18] and [13] for our purpose. A main problem in the investigation of the Hausdorff dimension of subsets of S comes from the fact that there is no natural parametrization of S by a nonrandom set. This makes the investigation more involved than, say, in the case of a single Brownian path, where such a parametrization is available. We assume that under $\mathbb{P}$ the processes W_1 and W_2 are two unstopped independent Brownian motions in $\mathbb{R}^3$, started in the origin and ℓ their intersection local time. For the limit in (1.8) we introduce the abbreviation $\theta = 2/\rho^*$ with ρ^* given by (1.9) for $d = 3$, $p = 2$, $U = B(0, 1)$ and $R = \infty$, or equivalently by Theorem 1.3 for $p = 2$.

5.1. *The upper bounds.* In this subsection we prove all upper estimates needed in the proof of Theorem 1.4. Fix a unit cube $\mathsf{C} = b + [0, 1]^3$ at positive distance from the origin. Fix $a > 0$ and denote the set of *a-thick points* in C by

$$F(a) = \left\{ x \in \mathsf{C} : \limsup_{r \downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} \geq a \right\}. \tag{5.1}$$

The remainder of this section is devoted to the proof of the following proposition.

PROPOSITION 5.1. *We have, almost surely*:

(i) *if* $a > 1/\theta^2$, *then* $F(a) = \varnothing$;
(ii) *if* $a \leq 1/\theta^2$, *then* $\dim F(a) \leq 1 - \theta\sqrt{a}$.

Moreover, almost surely:

(iii) $\displaystyle \limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^3} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} \leq \frac{1}{\theta^2}.$

We write $\mathbb{P}_{(x,y)}$ for probabilities with respect to two independent Brownian motions W_1, W_2 started in the points x and y. In particular, $\mathbb{P} = \mathbb{P}_{(0,0)}$. The following lemma draws the conclusion from Theorem 1.1 needed for the proof of Proposition 5.1.

LEMMA 5.2. *For every $\delta > 0$ there is an $\varepsilon > 0$ such that, for all $0 < r < \varepsilon$, and all $y \in \partial B(x, r)$,*

$$\mathbb{P}_{(x,y)}\{\ell(B(x,r)) \geq ar[\log(1/r)]^2\} \leq r^{\sqrt{a}\theta - \delta}.$$

PROOF. A simple coupling argument shows that the nonnegative random variable $\ell(B(x,r))$ is stochastically maximal if the starting points x and y agree. Hence, using also Brownian scaling, we infer that

$$\begin{aligned} \mathbb{P}_{(x,y)}\{\ell(B(x,r)) \geq ar[\log(1/r)]^2\} &\leq \mathbb{P}_{(x,x)}\{\ell(B(x,r)) \geq ar[\log(1/r)]^2\} \\ &= \mathbb{P}_{(0,0)}\{\ell(B(0,1)) \geq a[\log(1/r)]^2\}. \end{aligned}$$

From Theorem 1.1 in the case $U = B(0, 1)$ and $R = \infty$ we get the estimate for the right-hand side with the value $\theta = 2/\rho^*$ identified in Theorem 1.3(ii). $\square$

In the following, we define a random sequence of collections $(\mathcal{J}_n)_n$ of smaller and smaller cubes such that the set $F(a)$ is covered by $\bigcup_{n\geq j} \mathcal{J}_n$ for all j. Fix $\varepsilon > 0$ smaller than a. Pick $c > 0$ so small that, for sufficiently large N,

$$h_n = e^{-cn} \quad \text{and} \quad k_n = h_n\left(1 + \frac{1}{n}\right) \text{ satisfy } \frac{h_{n+1}}{k_n} \geq \frac{a-\varepsilon}{a-\varepsilon/2} \quad \text{for all } n \geq N. \tag{5.2}$$

Abbreviate $s_n = h_n/(\sqrt{3}n)$ and observe that $\lim_{n\to\infty} s_n = 0$. Introduce the following collection of cubes of sidelength s_n:

$$\mathcal{I}_n = \{b + s_n((k, l, m) + [0, 1]^3) : 0 \leq k, l, m < s_n^{-1} \text{ integer }\}.$$

Note that $\mathcal{I}_n$ is a covering of the cube C.

For every $I \in \mathcal{I}_n$ denote by $T_n(I)$ the first hitting time of the cube I by W_1 and, if $T_n(I) < \infty$, let $z(I) = W_1(T_n(I)) \in \partial I$ be the entry point. We pick now those minicubes from $\mathcal{I}_n$ that are hit by W_1 and such that the intersection local time of the two motions in a neighborhood of the entry point exceeds a certain threshold,

$$\mathcal{J}_n = \{I \in \mathcal{I}_n : T_n(I) < \infty \text{ and } \ell(B(z(I), k_n)) \geq (a - \varepsilon)k_n[\log(1/k_n)]^2\}.$$

We first prove that $F(a)$ is covered by the cubes in $\bigcup_{n\geq j} \mathcal{J}_n$ for all j.

LEMMA 5.3.

(i) *For all sufficiently large* $n \in \mathbb{N}$ *and* $r \in [h_{n+1}, h_n]$,

$$V(r) = \left\{x \in \mathsf{C} \cap S : \ell(B(x, r)) \geq \left(a - \frac{\varepsilon}{2}\right)r[\log(1/r)]^2\right\} \subset \bigcup_{I\in\mathcal{J}_n} I.$$

(ii) $F(a) \subset \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{I\in\mathcal{J}_n} I.$

PROOF. Fix a large n and $r \in [h_{n+1}, h_n]$ and let $x \in V(r)$. As $\mathcal{I}_n$ is a covering of C, certainly x lies in some $I \in \mathcal{I}_n$. We have to show that this small cube I lies in $\mathcal{J}_n$. Note that $T_n(I) < \infty$ since $x \in S$. Put $z = z(I)$. As $|x - z| \leq \sqrt{3}s_n$ we have, using also (5.2), that

$$\begin{aligned}\ell(B(z, k_n)) &= \ell(B(z, h_n + \sqrt{3}s_n)) \geq \ell(B(x, r))\\ &\geq \left(a - \frac{\varepsilon}{2}\right)r[\log(1/r)]^2 \geq \left(a - \frac{\varepsilon}{2}\right)h_{n+1}[\log(1/h_{n+1})]^2\\ &\geq (a - \varepsilon)k_n[\log(1/h_{n+1})]^2 \geq (a - \varepsilon)k_n[\log(1/k_n)]^2.\end{aligned}$$

Hence, $I \in \mathcal{J}_n$ and this proves (i).

Now let x be in $F(a)$. Then, for infinitely many $n \in \mathbb{N}$, there is an $r \in [h_{n+1}, h_n]$ such that $x \in V(r)$. Then x lies in $\bigcup_{I \in \mathcal{J}_n} I$ for infinitely many n, and this implies (ii). $\square$

LEMMA 5.4. *There is a sequence $\varepsilon_n \downarrow 0$ such that, for every $I \in \mathcal{I}_n$, we have*

$$\mathbb{P}\{I \in \mathcal{J}_n\} \le h_n^{2+\sqrt{a-\varepsilon}\theta-\varepsilon_n}.$$

PROOF. The probability that a three-dimensional Brownian motion started in the origin hits a ball $B(y, r)$, whose closure does not contain the origin, is equal to $r/|y|$. In particular, as the cube C has positive distance to the origin, there is a constant $K > 0$ such that, for sufficiently large n,

$$\mathbb{P}\{W_1 \text{ hits } I\} \le K s_n \qquad \text{for all } I \in \mathcal{I}_n.$$

Moreover, if we apply the same argument to the second Brownian motion, we can find another constant $\tilde{K} > 0$ such that, for sufficiently large n,

$$\mathbb{P}\big\{W_1 \text{ hits } I \text{ and } W_2 \text{ hits } B\big(z(I), k_n\big)\big\} \le \tilde{K} s_n k_n \qquad \text{for all } I \in \mathcal{I}_n.$$

Look at a pair of Brownian motions started in z_1, respectively, z_2 with $|z_1 - z_2| = k_n$. From Lemma 5.2 we get a sequence $\delta_n \downarrow 0$, which does not depend on the choice of z_1, z_2, such that, for sufficiently large n,

$$\mathbb{P}_{(z_1,z_2)}\big\{\ell\big(B(z_1, k_n)\big) \ge (a-\varepsilon)k_n[\log(1/k_n)]^2\big\} \le k_n^{\sqrt{a-\varepsilon}\theta-\delta_n}.$$

We apply this to the points $z_1 = z(I)$ and z_2 defined as the point where W_2 enters $B(z_1, k_n)$. By the strong Markov property, if n is sufficiently large,

$$\begin{aligned}\mathbb{P}\{I \in \mathcal{J}_n\} &= \mathbb{P}\big\{W_1 \text{ hits } I \text{ and } \ell\big(B\big(z(I), k_n\big)\big) \ge (a-\varepsilon)k_n[\log(1/k_n)]^2\big\} \\ &\le \tilde{K} s_n k_n^{1+\sqrt{a-\varepsilon}\theta-\delta_n} \le h_n^{2+\sqrt{a-\varepsilon}\theta-\varepsilon_n},\end{aligned}$$

recalling $s_n = h_n/(\sqrt{3}n)$, $k_n = h_n(1 + \frac{1}{n})$ and defining $\varepsilon_n \downarrow 0$ accordingly. $\square$

PROOF OF PROPOSITION 5.1. Lemma 5.4 gives, for any $\gamma \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}\Bigg[\sum_{n=1}^{\infty} \#\mathcal{J}_n s_n^{\gamma}\Bigg] &= \sum_{n=1}^{\infty} s_n^{\gamma} \sum_{I \in \mathcal{I}_n} \mathbb{P}\{I \in \mathcal{J}_n\} \le \sum_{n=1}^{\infty} s_n^{\gamma} h_n^{2+\sqrt{a-\varepsilon}\theta-\varepsilon_n} \#\mathcal{I}_n \\ &\le \sum_{n=1}^{\infty} h_n^{\gamma-1+\sqrt{a-\varepsilon}\theta-\varepsilon_n} (\sqrt{3}n)^{-\gamma},\end{aligned} \tag{5.3}$$

where we recall that $s_n = h_n/(\sqrt{3}n)$. Because $h_n = e^{-cn}$, the right-hand side of (5.3) is finite for all $\gamma > 1 - \theta\sqrt{a-\varepsilon}$, and we can infer that

$$\sum_{n=1}^{\infty} \#\mathcal{J}_n s_n^{\gamma} < \infty \qquad \text{almost surely.} \tag{5.4}$$

By Lemma 5.3(ii), the set $F(a)$ can be covered, for each j, by the family $\bigcup_{n\geq j}\mathcal{J}_n$ consisting, for each $n \geq j$, of $\#\mathcal{J}_n$ cubes of sidelength s_n. Hence (5.4) implies that $\dim F(a) \leq \gamma$. As this holds for all $\gamma > 1-\theta\sqrt{a-\varepsilon}$ and $\varepsilon > 0$ we can infer that

$$\dim F(a) \leq 1-\theta\sqrt{a}.$$

If the right-hand side is negative, $F(a)$ must be empty almost surely, so that Proposition 5.1(i) and (ii) are proved. For the proof of (iii) we use the above estimates for arbitrary $a > 1/\theta^2$, some small $\varepsilon > 0$ and $\gamma = 0$. By Chebyshev's inequality we infer that

$$\sum_{n=1}^{\infty} \mathbb{P}\{\#\mathcal{J}_n \geq 1\} \leq \sum_{n=1}^{\infty} \mathbb{E}[\#\mathcal{J}_n] < \infty.$$

By the Borel–Cantelli lemma, $\mathcal{J}_n$ must be empty for all sufficiently large n, almost surely. Lemma 5.3(i) thus implies

$$\limsup_{r\downarrow 0} \sup_{x\in S} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} \leq a.$$

Letting a tend to $1/\theta^2$ finishes the proof of Proposition 5.1(iii). □

5.2. *Hitting with a percolation limit set.* To obtain lower bounds we use the method of intersection with independent random sets; see, for example, [13] for an extensive account of this.

Suppose that $\mathrm{C} \subset \mathbb{R}^3$ is a fixed compact unit cube not containing the origin. We denote by $\mathfrak{D}_n$ the collection of compact dyadic subcubes (relative to C) of sidelength 2^{-n}. We also let $\mathfrak{D} = \bigcup_{n=0}^{\infty}\mathfrak{D}_n$.

Given $\gamma \in [0,3]$ we construct a random compact set $\Gamma[\gamma] \subset \mathrm{C}$ inductively as follows: we keep each of the eight compact cubes in $\mathfrak{D}_1$ independently with probability $p = 2^{-\gamma}$. Let $\mathcal{S}_1$ be the collection of cubes kept in this procedure and pass from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ by keeping each cube of $\mathfrak{D}_{n+1}$, which is not contained in a previously rejected cube, independently with probability p. Then the random set

$$\Gamma[\gamma] := \bigcap_{n=1}^{\infty} \bigcup_{C\in\mathcal{S}_n} C$$

is called a *percolation limit set*. The usefulness of percolation limit sets in fractal geometry comes from the following lemma, see [20] or [12] for a proof.

LEMMA 5.5. *For every* $\gamma \in [0,3]$ *and every Borel set* $A \subset \mathrm{C}$ *the following properties hold*:

(i) *If* $\dim A < \gamma$, *then almost surely*, $A \cap \Gamma[\gamma] = \varnothing$.
(ii) *If* $\dim A > \gamma$, *then* $A \cap \Gamma[\gamma] \neq \varnothing$ *with positive probability*.

(iii) *If* $\dim A > \gamma$, *then almost surely* $\dim\big(A \cap \Gamma[\gamma]\big) \le \dim A - \gamma$ *and, for all* $\varepsilon > 0$, *with positive probability* $\dim(A \cap \Gamma[\gamma]) \ge \dim A - \gamma - \varepsilon$.

Observe that the first part of the lemma gives a *lower* bound γ for the Hausdorff dimension of a set A, if we can show that $A \cap \Gamma[\gamma] \neq \varnothing$ with positive probability. To make use of this observation, recall that $\theta = 2/\rho^*$, fix $a \in [0, 1/\theta^2]$, and put $\gamma = 1 - \theta\sqrt{a}$. We suppose that the random set $\Gamma[\gamma]$ and two unstopped Brownian motions W_1 and W_2, started at the origin, are realized independently on the same probability space, and we write $\mathbb{P}$ for the joint distribution of the motions and $\Gamma[\gamma]$.

Lemma 5.5(i), applied to the set $F(a)$ defined in (5.1), yields the lower bound $\dim(F(a)) \ge \gamma = 1 - \theta\sqrt{a}$ with positive probability, *if* we show that the set $F(a) \cap \Gamma[\gamma]$ is nonempty with positive probability. The latter is shown in Proposition 5.6 below. A lower bound for the dimension of the set of *strictly* a-thick points follows with a little more effort; see Section 5.3 below.

PROPOSITION 5.6 (Hitting by thick points).

$$\mathbb{P}\big\{F(a) \cap \Gamma[\gamma] \neq \varnothing\big\} > 0. \tag{5.5}$$

The remainder of this subsection is devoted to the proof of Proposition 5.6. The idea is to construct a compact random subset $S^* \subset S \cap \Gamma[\gamma]$ with the regularity properties:

(a) $S^* \neq \varnothing$ with positive probability;

(b) almost surely, for every open set $U \subset \mathsf{C}$ with $U \cap S^* \neq \varnothing$ we have $\dim(U \cap S^*) \ge 1 - \gamma$.

Property (b) is instrumental in the proof that $F(a) \cap S^*$ is dense in S^* almost surely, which together with (a) implies that $F(a) \cap \Gamma[\gamma] \neq \varnothing$ with positive probability. To construct such a set S^* we fix a countable base $\mathfrak{B}$ of open subsets of C and define, for $0 < \delta \le 1 - \gamma$, compact random sets

$$S^*[\delta] = (S \cap \Gamma[\gamma]) \setminus \bigcup\big\{B \in \mathfrak{B} : \dim(B \cap S \cap \Gamma[\gamma]) < \delta\big\}.$$

Obviously the sets $S^*[\delta]$ have the property that, almost surely for every open set $U \subset \mathsf{C}$,

$$U \cap S^*[\delta] \neq \varnothing \text{ implies } \dim(U \cap S \cap \Gamma[\gamma]) \ge \delta. \tag{5.6}$$

In the case $\delta = 1 - \gamma$ this property is closest to property (b), but it still weaker than (b) and there is also no direct argument to see that $S^*[1 - \gamma]$ is nonempty with positive probability.

On the other hand, let $0 < \delta < 1 - \gamma$ and recall from Section 2.1 that $\dim(S \cap \mathsf{C}) = 1$ if $\ell(\mathsf{C}) > 0$, and the latter event has positive probability. Hence, by Lemma 5.5(iii), we have $\dim(S \cap \Gamma[\gamma]) > \delta$ with positive probability. Removing countably many sets of dimension strictly smaller than δ does not decrease the dimension of the set and hence we have $\dim S^*[\delta] > \delta$ and, in particular, $S^*[\delta] \neq \varnothing$, with positive probability.

Property (b) is not immediate for $S^*[\delta]$, so we have to make do with (5.6). The proof of Lemma 5.9 below, however, implicitly shows that (b) holds, and the sets $S^*[\delta]$ are the same for all $0 < \delta \leq 1 - \gamma$. In order to avoid unnecessary repetition of arguments, though, we do not make this statement explicit, but fix some $0 < \delta < 1 - \gamma$ once and for all, put $S^* := S^*[\delta]$ and use (5.6) instead of (b).

We now provide the two main technical lemmas in the proof of Proposition 5.6. The first of them is an extension of the lower bound of our upper tail asymptotics of Theorem 1.1 to the following situation. For given $y \in \mathbb{R}^3$ and integer $n \geq 1$, we consider three balls centered at y.

1. At two points in $\partial B(y, 2^{-n}/n)$, we start the two motions.
2. We measure their intersection local time in $B(y, 2^{-n}n)$.
3. We consider the intersections of the paths only until the motions leave $B(y, 2^{-n}n^2)$, and we condition on their respective leaving positions.

Readers unfamiliar with Brownian motion conditioned on leaving a domain at a fixed exit point are recommended to consult [3], III Proposition (2.7).

To formulate precisely what we need, we assume for simplicity of notation that $y = 0$. By ℓ_R we denote the intersection local time for the two independent Brownian motions W_1 and W_2 stopped at the time σ_1^R, respectively, σ_2^R of their first exit from $B(0, R)$. We write $\mathbb{P}_{(x_1,x_2)}$ for the distribution of the two motions, started at x_1 and x_2, respectively.

LEMMA 5.7 (Localization). *For any $\varepsilon > 0$, there is an $N > 0$ such that, for any $n > N$ and for any $x_1, x_2, u_1, u_2 \in \partial B(0, 1)$,*

$$\begin{aligned}\mathbb{P}_{(x_1,x_2)2^{-n}/n}\Big\{&\ell_{2^{-n}n^2}\big(B(0, 2^{-n}n)\big) \\ &\geq a2^{-n}n\left[\log\frac{2^n}{n}\right]^2 \,\Big|\, W_i\big(\sigma_i^{2^{-n}n^2}\big) = u_i 2^{-n}n^2 \text{ for } i \in \{1, 2\}\Big\} \\ \geq 2^{-n(\sqrt{a}\theta+\varepsilon)}&.\end{aligned} \tag{5.7}$$

PROOF. It is clear from the Brownian scaling property that the left-hand side of (5.7) is equal to

$$\mathbb{P}_{(x_1,x_2)/n^2}\left\{\ell_n\big(B(0, 1)\big) \geq a\left[\log\frac{2^n}{n}\right]^2 \,\Big|\, W_i(\sigma_i^n) = u_i n \text{ for } i \in \{1, 2\}\right\}.$$

In the following we write P^n for the joint distribution of $(\frac{1}{n}W_1(\sigma_1^n), \frac{1}{n}W_2(\sigma_2^n))$ under $\mathbb{P}_{(x_1,x_2)/n^2}$. Let A_1, A_2 be arbitrary Borel subsets of $\partial B(0, 1)$. Fix some

$R > 1$ and assume that $n > 2R$. Since $\ell_n \ge \ell_R$, we obtain a lower bound by replacing ℓ_n by ℓ_R. Use the strong Markov property at time σ_1^R and σ_2^R to obtain

$$\frac{1}{\mathrm{P}^n(A_1 \times A_2)} \iint_{A_1\times A_2} \mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_R\bigl(B(0,1)\bigr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2 \biggm| W_i(\sigma_i^n) = nu_i \text{ for } i\in\{1,2\}\biggr\}\mathrm{P}^n(du_1\,du_2)$$

$$= \mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_R\bigl(B(0,1)\bigr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2 \biggm| W_i(\sigma_i^n) \in nA_i \text{ for } i\in\{1,2\}\biggr\}$$

$$\ge \mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_R\bigl(B(0,1)\bigr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2\biggr\} \prod_{i=1}^{2} \frac{\min_{|u|=R}\mathbb{P}_u\{W_i(\sigma_i^n)\in nA_i\}}{\max_{|u|=R}\mathbb{P}_u\{W_i(\sigma_i^n)\in nA_i\}}.$$

From the explicit formula for the harmonic measure (see, e.g., [3], II Theorem (1.17)), one can see that the product on the right-hand side converges uniformly in A_1 and A_2 to 1.

Obviously,

$$\mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_R\bigl(B(0,1)\bigr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2\biggr\} \ge \mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_R\biggl(B\biggl(\frac{x_1}{n^2}, 1-\frac{1}{n^2}\biggr)\biggr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2\biggr\}.$$

By a simple coupling argument, one can see that the random variable $\ell_R(B(x_1/n^2, 1-\frac{1}{n^2}))$ is stochastically minimal if the distance of the starting points x_1/n^2 and x_2/n^2 is maximal. Therefore, fixing some $y \in \partial B(0,1)$,

$$\mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_R\biggl(B\biggl(\frac{x_1}{n^2}, 1-\frac{1}{n^2}\biggr)\biggr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2\biggr\} \ge \mathbb{P}_{(0,y)}\biggl\{\ell_R\biggl(B\biggl(0, 1-\frac{1}{n^2}\biggr)\biggr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2\biggr\}.$$

Altogether, there exists an integer N such that, for all $u_1, u_2, x_1, x_2 \in \partial B(0,1)$ and $n \ge N$,

$$\mathbb{P}_{(x_1,x_2)/n^2}\biggl\{\ell_n\bigl(B(0,1)\bigr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2 \biggm| W_i(\sigma_i^n) = nu_i \text{ for } i\in\{1,2\}\biggr\} \ge \frac{1}{2}\mathbb{P}_{(0,y)}\biggl\{\ell_R\biggl(B\biggl(0, 1-\frac{1}{n^2}\biggr)\biggr) \ge a\biggl[\log\frac{2^n}{n}\biggr]^2\biggr\}. \tag{5.8}$$

Recall that the left-hand side is equal to the left-hand side of (5.7). On the other hand, the right-hand side satisfies, as $n \to \infty$,

$$\frac{1}{2}\mathbb{P}_{(0,y)}\left\{\ell_R\left(B\left(0, 1-\frac{1}{n^2}\right)\right) \geq a\left[\log\frac{2^n}{n}\right]^2\right\}$$
$$\geq e^{-\sqrt{a}\log(2^n/n)(2/\rho^*(R)+o(1))} \geq 2^{-n\sqrt{a}(2/\rho^*(R)+o(1))},$$

where we used the lower bound in (1.8) and denoted $\rho^*(R)$ the quantity in (1.9) for motions stopped at the first exit time from $B(0, R)$. Recall from Remark 1 that the limit as $R \to \infty$ of $\rho^*(R)$ is equal to the value of ρ^* for the unstopped motions. Since $2/\rho^* = \theta$, we arrive at the assertion. $\square$

The second main technical lemma is a lower bound on the probability that the path of two conditioned Brownian paths intersect $\Gamma[\gamma]$ in a set of dimension close to $1-\gamma$. We look at the following situation. Let $n \geq 1$ and $V \in \mathcal{S}_n$ be a cube, which is kept in the percolation procedure, and $\xi \in V$ its center.

1. At two points in ∂V, we start the two motions.
2. We consider the dimension of the intersection of the paths with $\Gamma[\gamma]$ only until the motions leave the annulus $B(\xi, 2^{-n}n^2) \setminus B(\xi, 2^{-n}/n)$, and we condition on their respective first exit positions.

For the precise statement, we again simplify notation by assuming that the cube is centered in the origin. We consider two independent Brownian motions W_1 and W_2 and write again $\mathbb{P}_{(x_1,x_2)}$ for the distribution of the two motions, started at x_1 and x_2, respectively. For $0 < r < s$ we let $\omega_1^{r,s}$, respectively, $\omega_2^{r,s}$ be the time of first exit of W_1, respectively, W_2 from the annulus $B(0, s) \setminus B(0, r)$. Write $\mathbb{U}$ for the centered compact cube of side length 1.

LEMMA 5.8. *For every $\varepsilon > 0$ there is an $N = N(\varepsilon) > 0$ and $p = p(\varepsilon) > 0$ such that, for any $n > N$ and for any $x_1, x_2 \in \partial\mathbb{U}$ and $u_1, u_2 \in \partial B(0, 1/n) \cup \partial B(0, n^2)$,*

$$\mathbb{P}_{(x_1,x_2)2^{-n}}\Big\{\dim\big(W_1[0, \omega_1^{2^{-n}/n, 2^{-n}n^2}] \cap W_2[0, \omega_2^{2^{-n}/n, 2^{-n}n^2}] \cap 2^{-n}\mathbb{U} \cap \Gamma[\gamma]\big)$$
$$> 1-\gamma-\varepsilon \,\big|\, W_i\big(\omega_i^{2^{-n}/n, 2^{-n}n^2}\big) = 2^{-n}u_i \text{ for } i \in \{1,2\}, 2^{-n}\mathbb{U} \in \mathcal{S}_n\Big\} \tag{5.9}$$
$$> p(\varepsilon).$$

PROOF. By the scaling invariance of Brownian motion and the canonical self-similarity of percolation fractals, the left-hand side of (5.9) is equal to

$$\mathbb{P}_{(x_1,x_2)}\Big\{\dim\big(W_1[0, \omega_1^{1/n,n^2}] \cap W_2[0, \omega_2^{1/n,n^2}]$$
$$\cap \Gamma[\gamma]\big) > 1-\gamma-\varepsilon \,\big|\, W_i\big(\omega_i^{1/n,n^2}\big) = u_i \text{ for } i \in \{1,2\}\Big\},$$

where $\Gamma[\gamma]$ is a percolation fractal in the unit cube $\mathbb{U}$.

Using the same line of argument as in the beginning of Lemma 5.7 one can show that there exists an integer N such that, for all $n \geq N$, $x_1, x_2 \in \partial\mathbb{U}$ and $u_1, u_2 \in \partial B(0, 1/n) \cup \partial B(0, n^2)$,

$$\begin{aligned}\mathbb{P}_{(x_1,x_2)}\Big\{&\dim\big(W_1[0, \omega_1^{1/n,n^2}] \cap W_2[0, \omega_2^{1/n,n^2}]\\ &\qquad \cap \Gamma[\gamma]\big) > 1 - \gamma - \varepsilon \,\big|\, W_i(\omega_i^{1/n,n^2}) = u_i \text{ for } i \in \{1, 2\}\Big\}\\ &\geq \tfrac{1}{2}\mathbb{P}_{(x_1,x_2)}\Big\{\dim\big(W_1[0, \omega_1^{1/4,2}] \cap W_2[0, \omega_2^{1/4,2}] \cap \Gamma[\gamma]\big) > 1 - \gamma - \varepsilon\Big\},\end{aligned}$$

thus getting rid of the conditioning on the exiting points. Using Lemma 5.5(iii), it is easy to see that the latter probability is bounded from below by a positive constant depending only on ε. This completes the proof. □

The following lemma constitutes the main step in the proof of Proposition 5.6. Define, for $m \geq 1$,

$$U(m) = \left\{x \in S^* : \text{there is } 0 < h < 2^{-m+1} \text{ with } \frac{\ell(B(x,h))}{h[\log(1/h)]^2} > a - \frac{1}{m}\right\}.$$

LEMMA 5.9. *Almost surely*, $U(m)$ *is dense in* S^* *for all* $m \in \mathbb{N}$.

PROOF. Fix $m \in \mathbb{N}$ and an open set $O \subset \mathbb{C}$. It suffices to show that, almost surely, the event $O \cap S^* \neq \varnothing$ implies that $U(m) \cap O \neq \varnothing$. By (5.6) the event $O \cap S^* \neq \varnothing$ implies that $\dim(O \cap S \cap \Gamma[\gamma]) \geq \delta$. Hence there exists a dyadic cube $V \in \mathfrak{D}$ with $V \subset O$ such that $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$. It thus suffices to show that, almost surely for any dyadic cube $V \in \mathfrak{D}$ the event $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$ implies $U(m) \cap V \neq \varnothing$.

Fix $n \in \mathbb{N}$ and a dyadic cube $V \in \mathfrak{D}_n$. We introduce some terminology for the proof. For every integer $k \geq n$ we let $\mathfrak{C}^k$ be the collection of 2^{k-n} cubes in $\mathfrak{D}_k$ that are contained in V. We subdivide $\mathfrak{C}^k$ into $m(k) = (k^3 + 1)^3$ disjoint subcollections

$$\mathfrak{C}^k_1, \dots, \mathfrak{C}^k_{m(k)}$$

such that, for all $1 \leq j \leq m(k)$, any two distinct cubes in $\mathfrak{C}^k_j$ have distance at least $k^3 2^{-k}$.

Recall that $\sqrt{a}\theta = 1 - \gamma$ and $\delta < 1 - \gamma$. Fix a small number $\varepsilon > 0$ and some η such that

$$\delta < \eta \quad \text{and} \quad \sqrt{a - \frac{1}{2m}}\theta + \varepsilon < \eta < 1 - \gamma. \tag{5.10}$$

For every cube $U \in \mathfrak{C}^k_j$ with center ξ we let ρ_1^U be the first entry time of W_1 into U, τ_1^U the first entry time of W_1 into $B(\xi, 2^{-k}/k)$ and σ_1^U the first exit time after ρ_1^U of W_1 from $B(\xi, 2^{-k}k^2)$. Let $\omega_1^U = \tau_1^U \wedge \sigma_1^U$ be the first exit time after ρ_1^U from the annulus $B(\xi, 2^{-k}k^2) \setminus B(\xi, 2^{-k}/k)$.

Analogously for the second Brownian motion W_2, define the stopping times $\rho_2^U, \tau_2^U, \sigma_2^U$ and ω_2^U. Let ℓ^U be the intersection local time of the two Brownian motions on the (possibly degenerate) time intervals $[\omega_1^U, \sigma_1^U]$, respectively, $[\omega_2^U, \sigma_2^U]$. We call a cube $U \in \mathfrak{C}_j^k$ *admissible* if $U \in \mathcal{S}_k$ and both Brownian motions hit the cube; that is, if $\rho_1^U, \rho_2^U < \infty$. Denote by M_j^k the number of admissible cubes $U \in \mathfrak{C}_j^k$.

An admissible cube $U \in \mathfrak{C}_j^k$ with center ξ is called *successful* if:

(A) both motions hit $B(\xi, 2^{-k}/k)$ before leaving $B(\xi, 2^{-k}k^2)$; in other words $\tau_1^U < \sigma_1^U$ and $\tau_2^U < \sigma_2^U$;

(B) the dimension of the intersection of $\Gamma[\gamma] \cap U$ with the paths before they hit $B(\xi, 2^{-k}/k)$ is bigger than η, formally $\dim(W_1[\rho_1^U, \omega_1^U] \cap W_2[\rho_2^U, \omega_2^U] \cap \Gamma[\gamma] \cap U) > \eta$;

(C) the intersection local time in $B(\xi, 2^{-k}k)$ of the paths started on first hitting $B(\xi, 2^{-k}/k)$ and stopped upon leaving $B(\xi, 2^{-k}k^2)$ is exceptionally large; more precisely,

$$\frac{\ell^U(B(\xi, 2^{-k}k))}{2^{-k}k[\log(2^k/k)]^2} > a - \frac{1}{2m}.$$

We say that the collection $\mathfrak{C}_j^k$ is *bad* if there exists *no* successful cube in $\mathfrak{C}_j^k$.

Let us next explain that it is sufficient for the proof of this lemma to show that, almost surely, given $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$, for arbitrary large k there exists a successful cube $U \in \mathfrak{C}^k$.

Indeed, if $U \in \mathfrak{C}^k$ is successful, then, by condition (B), $\dim(S \cap \Gamma[\gamma] \cap U) > \eta > \delta$. Hence there does not exist a covering of $S \cap \Gamma[\gamma] \cap U$ by sets $B \in \mathfrak{B}$ with $\dim(S \cap \Gamma[\gamma] \cap B) < \delta$. This implies that $S^* \cap U \neq \varnothing$ by construction of S^*. Pick some $y \in U \cap S^*$ and infer from condition (C), for $h = 2^{-k}(k + \sqrt{3})$,

$$\ell\big(B(y, h)\big) \geq \ell^U\big(B(\xi, 2^{-k}k)\big) > \left(a - \frac{1}{2m}\right) 2^{-k}k[\log(2^k/k)]^2.$$

Hence, if k is so large that $h < 2^{-m}$, and

$$\left(a - \frac{1}{2m}\right) 2^{-k}k[\log(2^k/k)]^2 \geq \left(a - \frac{1}{m}\right) h[\log(1/h)]^2,$$

we have $y \in V \cap U(m)$ and the proof is complete.

Thus, writing $A_k = \{\mathfrak{C}_j^k \text{ is bad for all } 1 \leq j \leq m(k)\}$ it is sufficient to show that

$$\lim_{k\to\infty} \mathbb{P}\big\{A_k \mid \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\big\} = 0.$$

We estimate the probability that all collections $\mathfrak{E}_j^k$ are bad as follows. We use the estimate

$$\begin{aligned}\mathbb{P}\{A_k \mid \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\} \\ \le \mathbb{P}\{M_j^k \le 2^{\eta k} \text{ for all } 1 \le j \le m(k) \mid \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\} \\ + \sum_{j=1}^{m(k)} \frac{\mathbb{P}\{\mathfrak{E}_j^k \text{ is bad } \mid M_j^k > 2^{\eta k}\}}{\mathbb{P}\{\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\}}.\end{aligned} \tag{5.11}$$

It suffices to show that the two terms on the last two lines of (5.11) vanish as $k \to \infty$. We begin with the term on the *second* line. Recall the definition of an admissible cube, and the number M_j^k of admissible cubes $U \in \mathfrak{E}_j^k$. Further recall that $m(k)$ is a polynomial in k and assume that $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$. Together with basic properties of the Hausdorff dimension this implies that

$$\liminf_{k\to\infty} \frac{1}{\log(2^k)} \log \max_{j=1}^{m(k)} M_j^k \ge \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2.$$

Hence, for all sufficiently large k, there exists j with $M_j^k > 2^{k\delta/2}$.

The next step in the proof is to use self-similarity in order to improve this lower bound and get $M_j^k > 2^{k\eta}$ for some j, for all sufficiently large k. For this purpose fix k and $1 \le j \le m(k)$. We find a maximal finite sequence of stopping times,

$$0 < \rho_1(1) < \sigma_1(1) < \rho_1(2) < \sigma_1(2) < \cdots < \rho_1(\mathfrak{m}) < \sigma_1(\mathfrak{m}) < \infty,$$

successively as follows: let $\rho_1(1)$ be the time of first entry of W_1 into some cube in $\mathfrak{E}_j^k$, denote its center by $\xi_1(1)$ and let $\sigma_1(1)$ be the first exit time from $B(\xi_1(1), 2^{-k}k^2)$. Having constructed $\sigma_1(l-1)$, we let $\rho_1(l)$ be the first time of entry of W_1 into a cube in $\mathfrak{E}_j^k$, which is different from all previous cubes, denote its center by $\xi_1(l)$ and let $\sigma_1(l)$ be the first exit time from $B(\xi_1(l), 2^{-k}k^2)$. We proceed until $\rho_1(\mathfrak{m}+1) = \infty$.

Denote by $\mathcal{F}_1(j,k)$ the σ-field generated by W_1 restricted to the time domain

$$[0, \rho_1(1)] \cup [\sigma_1(1), \rho_1(2)] \cup \cdots \cup [\sigma_1(\mathfrak{m}), \infty).$$

The analogous stopping times for W_2 are denoted by $\rho_2(l)$, $\sigma_2(l)$ and the σ-field for W_2 is $\mathcal{F}_2(j,k)$. The σ-field generated by $\mathcal{S}_k$ is called $\mathcal{G}(k)$ and we let $\mathcal{F}(j,k) = \mathcal{F}_1(j,k) \vee \mathcal{F}_2(j,k) \vee \mathcal{G}(k)$. It is important to note at this place that whether or not a cube $U \in \mathfrak{E}_j^k$ is admissible is an event in $\mathcal{F}(j,k)$ and hence M_j^k is $\mathcal{F}(j,k)$-measurable.

Conditional on $\mathcal{F}(j,k)$, there is a fixed maximal collection $\{U_1, \ldots, U_M\} \subset \mathfrak{E}_j^k$ of admissible cubes (where $M = M_j^k$) whose centers we denote by $\xi_1, \ldots, \xi_M$. For each $1 \le l \le M$ there are unique values $\zeta_i(l) \in \mathbb{N}$ such that $\rho_i(\zeta_i(l))$ is the first entry time of W_i in U_l and $\sigma_i(\zeta_i(l))$ the exit time from the ball $B(\xi_l, 2^{-k}k^2)$. Moreover, each Brownian motion W_i in the time interval $[\rho_i(\zeta_i(l)), \sigma_i(\zeta_i(l))]$ is conditioned to start from a fixed point on ∂U_l and is stopped upon exiting the ball $B(\xi_i(l), 2^{-k}k^2)$ at a fixed point of the boundary. Consider the events

$$E_l = \Big\{\dim\big(W_1[\rho_1^{U_l}, \omega_1^{U_l}] \cap W_2[\rho_2^{U_l}, \omega_2^{U_l}] \cap U \cap \Gamma[\gamma]\big) > \eta\Big\}.$$

Conditional on $\mathcal{F}(j,k)$, the events $E_1, \dots, E_M$ are independent, and, by Lemma 5.8, the conditional probability of each E_l is

$$\mathbb{P}\{E_l \mid \mathcal{F}(j,k)\} \geq p(1-\gamma-\eta).$$

Hence the conditional probability that all events $E_1, \dots, E_M$ fail is at most

$$\big(1 - p(1-\gamma-\eta)\big)^{M_j^k}.$$

We infer that the conditional probability, given $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$, that *no* dyadic subcube $U \subset V$ has $\dim(U \cap S \cap \Gamma[\gamma]) > \eta$ is bounded from above by

$$\limsup_{k\to\infty} m(k)\big(1 - p(1-\gamma-\eta)\big)^{2^{k\delta/2}} = 0.$$

We have thus shown that

$$\mathbb{P}\big\{\dim(V \cap S \cap \Gamma[\gamma]) > \eta \mid \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\big\} = 1. \tag{5.12}$$

As above, this implies that, for all sufficiently large k, we have $M_j^k > 2^{k\eta}$ for some $1 \leq j \leq m(k)$, almost surely on $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$. In particular, the term on the second line of (5.11) converges to 0 as $k \to \infty$. To deal with the term on the *third* line of (5.11), we again fix $k \geq 2$ and $j \leq m(k)$. The problem is to control the probability, conditional on $\mathcal{F}(j,k)$, that the collection of admissible cubes does not contain a successful cube. More precisely, we show that, on $\{M_j^k > 2^{\eta k}\}$, the conditional probability that $\mathfrak{C}_j^k$ is bad is exponentially decreasing.

Conditional on $\mathcal{F}(j,k)$, we again look at the maximal collection $\{U_1, \dots, U_M\} \subset \mathfrak{C}_j^k$ of admissible cubes and recall the notation, ξ_l for the centers of the cubes, and $\zeta_i(l) \in \mathbb{N}$ such that $\rho_i(\zeta_i(l))$ is the first entry time of W_i in U_l and $\sigma_i(\zeta_i(l))$ the exit time from $B(\xi_l, 2^{-k}k^2)$. Additionally, let $\tau_i(\zeta_i(l))$ the first entry time into $B(\xi_l, 2^{-k}/k)$ and $\omega_i(\zeta_i(l)) = \sigma_i(\zeta_i(l)) \wedge \tau_i(\zeta_i(l))$.

Conditional on $\mathcal{F}(j,k)$ each Brownian motion W_i in the time interval $[\rho_i(\zeta_i(l)), \sigma_i(\zeta_i(l))]$ is conditioned to start on a fixed point on the boundary ∂U_l and exit the ball around ξ_l with radius $2^{-k}k^2$ at a fixed point $W_i(\sigma_i(\zeta_i(l)))$. For $l \in \{1, \dots, M\}$ recall that ℓ^{U_l} denotes the intersection local time of the two motions in the intervals $[\tau_1(\zeta_1(l)), \sigma_1(\zeta_1(l))]$, respectively, $[\tau_2(\zeta_2(l)), \sigma_2(\zeta_2(l))]$. Consider the event E_l that $\tau_1(\zeta_1(l)) < \sigma_1(\zeta_1(l))$, $\tau_2(\zeta_2(l)) < \sigma_2(\zeta_2(l))$ and

$$\dim\Big(W_1\big[\rho_1(\zeta_1(l)), \omega_1(\zeta_1(l))\big] \cap W_2\big[\rho_2(\zeta_2(l)), \omega_2(\zeta_2(l))\big] \cap U \cap \Gamma[\gamma]\Big) > \eta,$$

and also

$$\frac{\ell^{U_l}(B(\xi, 2^{-k}k))}{2^{-k}k[\log(2^k/k)]^2} > a - \frac{1}{2m}.$$

Clearly the event E_l implies that U_l is a successful cube. Moreover, conditional on $\mathcal{F}(j,k)$, the events $E_1, \dots, E_M$ are independent. To estimate the probability of E_l from below, first note that, for a Brownian motion started in $x_i \in \partial U_l$,

$$\mathbb{P}_{x_i}\{\tau_i^{U_l} < \sigma_i^{U_l}\} \geq \frac{\sqrt{2} - k^{-2}}{k - k^{-2}} \geq \frac{c}{k},$$

for an absolute constant $c > 0$. From Lemma 5.8 we know that, given $u_i \in \partial B(\xi_l, 2^{-k}/k)$, that

$$\mathbb{P}\Big\{\dim\big(W_1[\rho_1(\zeta_1(l)), \omega_1(\zeta_1(l))] \cap W_2[\rho_2(\zeta_2(l)), \omega_2(\zeta_2(l))] \\ \cap U \cap \Gamma[\gamma]\big) > \eta \,\big|\, W_i(\omega_i^{U_l}) = u_i \text{ for } i \in \{1,2\}\Big\} > p(1-\gamma-\eta),$$

and finally, for $v_i \in \partial B(\xi_l, 2^{-k}k^2)$, by Lemma 5.7,

$$\mathbb{P}_{(u_1,u_2)}\bigg\{\frac{\ell^{U_l}(B(\xi, 2^{-k}k))}{2^{-k}k[\log(2^k/k)]^2} > a - \frac{1}{2m} \,\bigg|\, W_i(\sigma_i^{U_l}) = v_i \text{ for } i \in \{1,2\}\bigg\} \\ \geq 2^{-k(\sqrt{a-(1/2m)}\theta+\varepsilon)}.$$

Altogether, the conditional probability of each E_l is bounded from below by

$$\mathbb{P}\{E_l \mid \mathcal{F}(j,k)\} \geq \frac{C}{k^2} 2^{-k(\sqrt{a-(1/2m)}\theta+\varepsilon)},$$

for $C = c^2 p(1-\gamma-\eta)$. Hence the conditional probability that the collection of admissible cubes does *not* contain a successful cube is at most

$$\bigg(1 - \frac{C}{k^2} 2^{-k(\sqrt{a-(1/2m)}\theta+\varepsilon)}\bigg)^M.$$

We infer that, for all $1 \leq j \leq m(k)$,

$$\begin{aligned} \mathbb{P}\{\mathfrak{E}_j^k \text{ is bad} \,|\, M_j^k > 2^{\eta k}\} &\leq \mathbb{E}\bigg[\bigg(1 - \frac{C}{k^2} 2^{-k(\sqrt{a-(1/2m)}\theta+\varepsilon)}\bigg)^{M_j^k} \,\bigg|\, M_j^k > 2^{\eta k}\bigg] \\ &\leq \bigg(1 - \frac{C}{k^2} 2^{-k(\sqrt{a-(1/2m)}\theta+\varepsilon)}\bigg)^{2^{\eta k}} \qquad (5.13) \\ &\leq \exp\bigg(-\frac{C}{k^2} 2^{k(\eta - \sqrt{a-(1/2m)}\theta-\varepsilon)}\bigg). \end{aligned}$$

Recalling (5.10) we see that the right-hand side decreases exponentially fast. As the number of summands in the third line of (5.11) is just polynomial, we infer that this term converges to 0 as $k \to \infty$. $\square$

LEMMA 5.10. *$F(a) \cap S^*$ is almost surely dense in S^*. In particular, we have*

$$\mathbb{P}\{F(a) \cap S^* \neq \varnothing \mid S^* \neq \varnothing\} = 1.$$

PROOF. Note that $U(m)$ is relatively open in S^* and, by Lemma 5.9, also dense in S^* for any m. As S^* is compact, hence complete, one can infer from Baire's theorem that

$$F(a) \cap S^* = \bigcap_{m=1}^{\infty} U(m)$$

is dense in S^* almost surely. Hence $\mathbb{P}\{F(a) \cap S^* \neq \varnothing \mid S^* \neq \varnothing\} = 1$. □

PROOF OF PROPOSITION 5.6. Recall that

$$\begin{aligned}\mathbb{P}\{F(a) \cap \Gamma[\gamma] \neq \varnothing\} &\geq \mathbb{P}\{F(a) \cap S^* \neq \varnothing\} \\ &= \mathbb{P}\{F(a) \cap S^* \neq \varnothing \mid S^* \neq \varnothing\}\mathbb{P}\{S^* \neq \varnothing\} = \mathbb{P}\{S^* \neq \varnothing\} > 0.\end{aligned}$$

This proves Proposition 5.6. □

5.3. *Completion of the proof of Theorem* 1.4. Fix $0 \leq a \leq 1/\theta^2$ and denote the set of *strictly a-thick points* by

$$H(a) = \left\{x \in \mathbb{R}^3 : \limsup_{r \downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = a\right\}.$$

For the upper bound in (1.18) first fix a unit cube C at positive distance from the origin. Then, by Proposition 5.1, we have

$$\dim\big(H(a) \cap \mathsf{C}\big) \leq \dim F(a) \leq 1 - \theta\sqrt{a}.$$

Taking a countable family of such cubes covering $\mathbb{R}^3 \setminus \{0\}$ we get $\dim H(a) \leq 1 - \theta\sqrt{a}$.

Turning to the proof of the lower bound in (1.18), we look at a large open ball $B \subset \mathbb{R}^3$ centered at the origin and fix a compact unit cube $\mathsf{C} \subset B$ at positive distance from the origin. Let $\gamma = 1 - \theta\sqrt{a}$. By Proposition 5.1(ii),

$$\dim\left(F\left(a + \tfrac{1}{n}\right)\right) \leq 1 - \theta\sqrt{a + \frac{1}{n}} < \gamma,$$

and, by Lemma 5.5(ii), we have that $F(a + 1/n) \cap \Gamma[\gamma] = \varnothing$ almost surely for all n. Hence, almost surely,

$$H(a) \cap \Gamma[\gamma] = F(a) \cap \Gamma[\gamma] \cap \bigcap_{n=1}^{\infty} F\left(a + \frac{1}{n}\right)^{\mathrm{c}} = F(a) \cap \Gamma[\gamma]. \tag{5.14}$$

Recall from Proposition 5.6 that the set on the right-hand side of (5.14) is nonempty with positive probability. Hence we have shown that $\mathbb{P}\{H(a) \cap \Gamma[\gamma] \neq \varnothing\} > 0$. Together with Lemma 5.5(i) this implies that $\dim(H(a) \cap B) \geq \gamma$ with positive probability.

By the Brownian scaling property this probability does not depend on the radius of the ball B, so that it holds for arbitrarily small balls. In particular,

$$\mathbb{P}\{\dim(H(a)\cap B(0,r)) \geq \gamma \text{ for all } r>0\} = \lim_{r\downarrow 0}\mathbb{P}\{\dim(H(a)\cap B(0,r)) \geq \gamma\} > 0.$$

We now use Blumenthal's zero–one law to see that this probability is actually equal to 1. Indeed, recall that *three* Brownian paths almost surely do not have a point of intersection; hence after the first exit times T_1, respectively, T_2 from any centered ball B both Brownian motions do not hit $W[0,T_1]\cap W[0,T_2]$ again and, by compactness, even keep a positive distance from $W[0,T_1]\cap W[0,T_2]$. We infer that almost surely every point, which is strictly a-thick for the intersection local time of the Brownian motions stopped at T_1, respectively, T_2, is also strictly a-thick for the unstopped motions. Moreover, by transience, there exists a small (random) centered ball, which is not visited by any Brownian motion after T_1, respectively, T_2. Thus the event $\{\dim(H(a)\cap B(0,r)) \geq \gamma \text{ for all } r>0\}$ is in the completion of the σ-field generated jointly by the Brownian motion W_1 stopped in T_1 and the Brownian motion W_2 stopped in T_2. This holds for every ball B and hence, by Blumenthal's zero–one law, we infer that

$$\mathbb{P}\{\dim H(a) \geq \gamma\} \geq \mathbb{P}\{\dim(H(a)\cap B(0,r)) \geq \gamma \text{ for all } r>0\} = 1.$$

This finishes the proof of the lower bound in (1.18).

In order to prove (1.17) we observe that

$$\frac{1}{\theta^2} \leq \sup_{x\in S}\limsup_{r\downarrow 0}\frac{\ell(B(x,r))}{r[\log(1/r)]^2} \leq \limsup_{r\downarrow 0}\sup_{x\in S}\frac{\ell(B(x,r))}{r[\log(1/r)]^2} \leq \frac{1}{\theta^2}.$$

Indeed, the first equality follows from the fact that, for every $a<1/\theta^2$, the set of a-thick points has positive dimension and hence is nonempty. The second inequality is obvious, and the third inequality is Lemma 5.1(iii). This proves (1.17) and thus completes the proof of Theorem 1.4. □

6. Outlook on future work. The characterization of the limit in (1.8) in terms of (1.9) naturally raises the question how the maximizers g of (1.9) can be interpreted. For a first answer define, on $\{\ell(U)>0\}$, the random probability measure L on U as the normalized restriction of ℓ to U; more precisely, let

$$L(A) = \ell(A)/\ell(U) \qquad \text{for } A\subset U \text{ Borel}.$$

We ask how the measure L distributes the unit mass over the set U if we condition the Brownian paths to have a large amount of occupation measure $\ell(U)$. An answer to this question is given by the following result. Let $\mathtt{d}\colon U\times U\to[0,\infty)$ be any metric that induces the weak topology on $\mathcal{M}_1(U)$.

THEOREM 6.1 (Law of large masses). *Let $\mathfrak{M} \subset \mathcal{M}_1(U)$ be the set of probability measures $\mu(dx) = g^{2p}(x)\,dx$ on U with g a maximizer in the variational problem* (1.9). *Then, for all $\varepsilon > 0$,*

$$\lim_{a\to\infty} \mathbb{P}\{\mathrm{d}(L,\mathfrak{M}) > \varepsilon \,|\, \ell(U) > a\} = 0. \tag{6.1}$$

The convergence in (6.1) is exponentially fast with rate $a^{1/p}$. Theorem 6.1 itself is a consequence of the identification of the exact rate of decay of $\mathbb{P}\{L \in A \mid \ell(U) > a\}$ for sets $A \subset \mathcal{M}_1(U)$ in terms of a *large deviation principle*. This problem goes beyond the scope of the present paper and will be treated in a forthcoming paper.

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