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A FEW REMARKS ON BRYC'S PAPER ON RANDOM FIELDS WITH LINEAR REGRESSIONS

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The aim of this note is to reduce a number of assumptions in the recent paper of W. Bryc by showing that some of them imply the others and to give alternative, simpler proofs of some of Bryc's results.

1. Introduction. There exists a considerable literature dealing with stochastic processes with linear regressions and quadratic second conditional moments (for references see Chapter 8 in [2]). In one of the recent papers on this subject, Bryc [3] used orthogonal polynomials to analyze and identify one-dimensional distributions of such processes.

The aim of this note is to carry out a thorough analysis of the condition defining first conditional moments in Bryc's paper ([3], condition (1)). This will make it possible to omit some of the assumptions from [3] and give an alternative, simpler proof of a result contained therein.

Let $\mathbf{X} = (X_k)_{k\in\mathbb{Z}}$ be a square integrable random sequence indexed by the integers, with nondegenerate covariance matrices and constant first two moments, that is, $\mathbb{E}X_k = \mathbb{E}X_0$, $\mathbb{E}X_k^2 = \mathbb{E}X_0^2$ $\forall k \in \mathbb{Z}$. We will follow Bryc in using the term *random field* for $\mathbf{X}$. This suggests expected extensions of the results from [3] to the case of real random fields, indexed by the elements of $\mathbb{Z}^d$. Proposition 2 may be seen as a first step in this direction.

For simplicity of notation, we define

$$\mathcal{F}_{\le m} := \sigma(X_k : k \le m),$$
$$\mathcal{F}_{\ge m} := \sigma(X_k : k \ge m),$$
$$\mathcal{F}_{\ne m} := \sigma(X_k : k \ne m).$$

Following Bryc [3], condition (1), we assume that

$$\mathbb{E}(X_k|\mathcal{F}_{\ne k}) := L(X_{k-1}, X_{k+1}) \qquad \forall k \in \mathbb{Z}, \tag{1.1}$$

where $L(x, y) := a(x + y) + b$. There is no loss of generality in assuming that $\mathbb{E}X_0 = 0$ and $\mathbb{E}X_0^2 = 1$, which implies $b = 0$.

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2. Results. From now on we assume that $\mathbf{X}$ is a square integrable random field having nondegenerate covariance matrices and constant first two moments.

PROPOSITION 1. *If* $\mathbf{X}$ *satisfies* (1.1) *then* $\mathbf{X}$ *is* L_2*-stationary.*

Proposition 1 is closely related to the following, more general statement, due to Janžura [4].

PROPOSITION 2. *Let* $\mathbf{X} = (X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ *be a square integrable random field such that for all* $\mathbf{k} \in \mathbb{Z}^d$,

$$\mathbb{E}(X_{\mathbf{k}}|\mathcal{F}_{\neq\mathbf{k}}) = \sum_{\mathbf{j}\neq\mathbf{0}} a_{\mathbf{j}} X_{\mathbf{k}+\mathbf{j}} \tag{2.1}$$

and $a_{\mathbf{k}} = a_{-\mathbf{k}}$. *If* $\mathbb{E}(X_{\mathbf{k}} - \mathbb{E}(X_{\mathbf{k}}|\mathcal{F}_{\neq\mathbf{k}}))^2$ *does not depend on* $\mathbf{k}$ *then* $\mathbf{X}$ *is* L_2*-stationary.*

Let $r_k := \operatorname{corr}(X_0, X_k)$ denote the correlation coefficients. Some easy consequences of the assumption on determinants of covariance matrices are that $|r_1| < 1$, $|r_2| < 1$ and $r_2 + 1 - 2r_1^2 > 0$. It is also easy to see that $a = r_1/(1 + r_2)$ and that the correlation coefficients satisfy recurrence

$$r_k(r_2 + 1) = r_1(r_{k+1} + r_{k-1}), \qquad k \geq 2. \tag{2.2}$$

REMARK 1. Clearly $a = 0$ if and only if $r_1 = 0$, so $a = 0$ implies $r_k = 0$ for $k \neq 0$ by (1.1).

THEOREM 1 ([3], Theorem 3.1(i)). *If* $\mathbf{X}$ *satisfies* (1.1) *and* $r_1 \neq 0$, *then* $0 < |a| < \frac{1}{2}$ *and* $r_k = r_1^{|k|}$ *for all* $k \in \mathbb{Z}$.

REMARK 2. The proof of the above theorem shows that the restrictions on the correlation coefficients in Bryc [3] can be weakened to nondegenerateness of covariance matrices and to the assumption that $r_1 \neq 0$.

REMARK 3. Kingman [5] showed that the necessary condition for the existence of L_2 random variables X_k, $k \in \mathbb{Z}$, satisfying

$$\mathbb{E}(X_k|\mathcal{F}_{\neq k}) = \sum_{j\neq k} a_{kj} X_j \tag{2.3}$$

is the existence of constants $u_i > 0$ such that $u_i a_{ij} = u_j a_{ji}$. Theorem 1 provides the necessary and sufficient condition in the case (1.1), which is a particular case of (2.3).

Analysis of a boundary value problem for difference equation (2.2), motivated by [6], Chapter 15.10, gives an alternative proof of another result of Bryc.

THEOREM 2 ([3], Theorem 3.1(ii)). *If* $\mathbf{X}$ *satisfies* (1.1) *then one-sided regressions are linear*,

$$\mathbb{E}(X_k|\mathcal{F}_{\leq 0}) = r_k X_0, \qquad k = 1, 2, \ldots.$$

3. Proofs.

PROOF OF PROPOSITION 1. Set $R_{j,k} := \mathbb{E}X_j X_k$. Obviously $R_{j,j} = 1$ and $R_{j,k} = R_{k,j}$. Condition (1.1) implies

$$R_{j,k} = a(R_{j-1,k} + R_{j+1,k}) \qquad \text{for } j \neq k. \tag{3.1}$$

In the trivial case $a = 0$ one has $R_{j,k} = 0$ for $j \neq k$ and $\mathbf{X}$ is L_2-stationary. Now assume $a \neq 0$. Substituting $j = n$, $k = n-1$ and $j = n$, $k = n+1$ into (3.1) yields

$$R_{n,n-1} = a(R_{n-1,n-1} + R_{n+1,n-1}),$$
$$R_{n,n+1} = a(R_{n-1,n+1} + R_{n+1,n+1}).$$

Hence $R_{n,n-1} = R_{n,n+1}$ $\forall n \in \mathbb{Z}$ and it does not depend on n. We now proceed by induction. Assume that $(m \geq 1)$,

$$R_{n,n-m} = R_{n+1,n-m+1} \qquad \forall n \in \mathbb{Z}; \tag{3.2}$$

we will prove (3.2) for $m+1$. Substituting $j = n$, $k = n-m-1$ and $j = n-m$, $k = n+1$ into (3.1) yields

$$R_{n,n-m-1} = a(R_{n-1,n-m-1} + R_{n+1,n-m-1}),$$
$$R_{n-m,n+1} = a(R_{n-m-1,n+1} + R_{n-m+1,n+1}).$$

Since $|n-1-(n-m-1)| = |n-m+1-(n+1)| = m$, we get $R_{n,n-m-1} = R_{n-m,n+1}$ $\forall n \in \mathbb{Z}$ by induction assumption (3.2). This proves (3.2) for $m+1$. Thus $R_{j,k}$ depends only on the difference $|j-k|$. □

PROOF OF PROPOSITION 2. We define infinite matrices R and A by the formulae $R(\mathbf{i},\mathbf{j}) := \mathbb{E}X_{\mathbf{i}}X_{\mathbf{j}}$, $A(\mathbf{i},\mathbf{j}) := a_{\mathbf{i}-\mathbf{j}}$, $A(\mathbf{i},\mathbf{i}) := -1$ for $\mathbf{i},\mathbf{j} \in \mathbb{Z}^d$. Multiplying both sides of (2.1) by $X_{\mathbf{u}}$ $(\mathbf{u} \neq \mathbf{k})$ and then taking the expected value yields

$$R(\mathbf{k},\mathbf{u}) = \sum_{\mathbf{j}\neq\mathbf{0}} a_{\mathbf{j}} R(\mathbf{k}+\mathbf{j},\mathbf{u}).$$

Substituting $\mathbf{w} = \mathbf{k}+\mathbf{j}$ and using the fact that $A(\mathbf{k},\mathbf{k}) = -1$ shows that product of matrices A and R has all coefficients equal to zero except for the diagonal. Since

$$\sum_{\mathbf{w}\neq\mathbf{k}} A(\mathbf{k},\mathbf{w})R(\mathbf{w},\mathbf{k}) = \mathbb{E}[X_{\mathbf{k}}\mathbb{E}(X_{\mathbf{k}}|\mathcal{F}_{\neq\mathbf{k}})] = R(\mathbf{k},\mathbf{k}) - \mathbb{E}[X_{\mathbf{k}} - \mathbb{E}(X_{\mathbf{k}}|\mathcal{F}_{\neq\mathbf{k}})]^2,$$

we get that $(AR)(\mathbf{k},\mathbf{k}) = -\mathbb{E}(X_{\mathbf{k}} - \mathbb{E}(X_{\mathbf{k}}|\mathcal{F}_{\neq\mathbf{k}}))^2$. Let $R^{\mathbf{p}}$, $\mathbf{p} \in \mathbb{Z}^d$, be an infinite matrix defined by $R^{\mathbf{p}}(\mathbf{i},\mathbf{j}) := R(\mathbf{i}+\mathbf{p},\mathbf{j}+\mathbf{p})$. Observe that

$$(AR^{\mathbf{p}})(\mathbf{i},\mathbf{j}) = (AR)(\mathbf{i}+\mathbf{p},\mathbf{j}+\mathbf{p}) = (AR)(\mathbf{i},\mathbf{j})$$

by the assumption that $(AR)(\mathbf{i},\mathbf{i}) = -\mathbb{E}(X_{\mathbf{i}} - \mathbb{E}(X_{\mathbf{i}}|\mathcal{F}_{\neq\mathbf{i}}))^2$ does not depend on $\mathbf{i}$. Since A is nonsingular, the proof is complete. □

REMARK 4. The proof of Proposition 1 shows that if $\mathbf{X}$ satisfies (1.1) then $\mathbb{E}(X_k - \mathbb{E}(X_k|\mathcal{F}_{\neq k}))^2 = 1 - a(R_{k,k+1} + R_{k,k-1})$ (with $R_{k,l} = \mathbb{E}X_k X_l$ as in the proof of Proposition 1) does not depend on k.

PROOF OF THEOREM 1. The general solution of (2.2) depends on the sign of $4a^2 - 1$.

If $a = \frac{1}{2}$ then $r_k = k(r_1 - 1) + 1$ and the only sequence of correlation coefficients satisfying (2.2) is $r_k \equiv 1$. If $a = -\frac{1}{2}$ then $r_k = (-1)^k[1 - k(r_1 + 1)]$ and the only sequence of correlation coefficients satisfying (2.2) is $r_k = (-1)^k$.

If $|a| > \frac{1}{2}$ then

$$r_k = A \sin k\tau + \cos k\tau, \tag{3.3}$$

where $\tau := \arccos \frac{1}{2a}$, $A := \frac{r_1 - \cos\tau}{\sin\tau}$. We claim that for any initial values r_1, r_2, sequence (3.3) is not positive definite. To obtain a contradiction, suppose that there exist r_1, r_2 such that $(r_k)_k$ defined by (3.3) is positive definite. By the Carathéodory–Toeplitz theorem ([1], Theorem 5.1.1, or by an easy calculation $0 \le \mathbb{E}(\sum_{k=0}^{\infty} x^k X_k)^2 = \frac{2}{1-x^2}(\frac{1}{2} + \sum_{k=1}^{\infty} r_k x^k)$),

$$F(x) := \tfrac{1}{2} + \sum_{k=1}^{\infty} r_k x^k,$$

defined for $|x| < 1$, is an analytic function with values in $\mathbb{R}_+$. Hence for all $|x| < 1$,

$$\frac{1}{2} + \sum_{k=1}^{\infty} (x^k \cos k\tau + A x^k \sin k\tau) = \frac{-x^2 + 2Ax\sin\tau + 1}{2(1 - 2x\cos\tau + x^2)} \ge 0,$$

that is satisfied only if $2A\sin\tau = 0$, or equivalently $1 + r_2 - 2r_1^2 = 0$, which contradicts the nondegenerateness of the covariance matrices.

We now turn to the case $0 < |a| < \frac{1}{2}$ (the case $a = 0$ is described in Remark 1). If $0 < |a| < \frac{1}{2}$ then

$$r_k = \left(\frac{q_2 - r_1}{q_2 - q_1}\right) q_1^k + \left(\frac{r_1 - q_1}{q_2 - q_1}\right) q_2^k,$$

where $q_1 = (2|a|)^{-1}(\operatorname{sign} a - \sqrt{1 - 4a^2})$, $q_2 = (2|a|)^{-1}(\operatorname{sign} a + \sqrt{1 - 4a^2})$. Let us observe that if $a \in (0, \frac{1}{2})$ (equivalently $r_1 > 0$) then $q_1 \in (0, 1)$ and $q_2 > 1$. Hence $r_1 = q_1$ and consequently $r_k = r_1^k$ for all $k \ge 1$. Analogously, if $a \in (-\frac{1}{2}, 0)$ (equivalently $r_1 < 0$) then $q_1 < -1$ and $q_2 \in (-1, 0)$. Hence $r_1 = q_2$ and $r_k = r_1^k$ for all $k \ge 1$, which completes the proof. $\square$

PROOF OF THEOREM 2. In the trivial case $a = 0$, one has that $\mathbb{E}(X_k|\mathcal{F}_{\le 0}) = 0$ for all $k \ge 1$. Now assume $a \ne 0$. Theorem 1 implies $0 < |a| < \frac{1}{2}$. The solution of the boundary value problem

$$\begin{aligned}
&\mathbb{E}(X_r \mid \mathcal{F}_{\le s}, \mathcal{F}_{\ge t}) = a\big[\mathbb{E}(X_{r-1} \mid \mathcal{F}_{\le s}, \mathcal{F}_{\ge t}) + \mathbb{E}(X_{r+1} \mid \mathcal{F}_{\le s}, \mathcal{F}_{\ge t})\big],\\
&\mathbb{E}(X_s \mid \mathcal{F}_{\le s}, \mathcal{F}_{\ge t}) = X_s,\\
&\mathbb{E}(X_t \mid \mathcal{F}_{\le s}, \mathcal{F}_{\ge t}) = X_t,
\end{aligned}$$

is

$$\mathbb{E}(X_r \mid \mathcal{F}_{\le s}, \mathcal{F}_{\ge t}) = \left(\frac{q_2^t q_1^r - q_2^r q_1^t}{q_2^t q_1^s - q_2^s q_1^t}\right) X_s + \left(\frac{q_2^r q_1^s - q_2^s q_1^r}{q_2^t q_1^s - q_2^s q_1^t}\right) X_t, \tag{3.4}$$

where q_1 and q_2 are as in the last paragraph of the proof of Theorem 1. Putting $r = k$, $s = k - u$, $t = k + l$ for $u, l \in \mathbb{N}$ and $k \in \mathbb{Z}$ into (3.4) we obtain

$$\mathbb{E}(X_k \mid \mathcal{F}_{\le k-u}, \mathcal{F}_{\ge k+l}) = \frac{q_2^l - q_1^l}{q_2^{u+l} - q_1^{u+l}} X_{k-u} + \frac{q_2^u - q_1^u}{q_2^{u+l} - q_1^{u+l}} X_{k+l}.$$

Since

$$\frac{q_2^u - q_1^u}{q_2^{u+l} - q_1^{u+l}} \xrightarrow[u\to\infty]{} \left\{\begin{matrix} q_2^l,\ a \in (-\frac{1}{2}, 0) \\ q_1^l,\ a \in (0, \frac{1}{2}) \end{matrix}\right\} = r_1^l,$$

by Lévy's downward theorem there exists a.s. a limit

$$L := \lim_{u\to\infty} \frac{q_2^l - q_1^l}{q_2^{u+l} - q_1^{u+l}} X_{k-u}$$

and

$$\mathbb{E}\left(X_k \,\middle|\, \bigcap_{u=1}^{+\infty} \sigma(\mathcal{F}_{\le k-u}, \mathcal{F}_{\ge k+l})\right) = r_1^l X_{k+l} + L. \tag{3.5}$$

However,

$$\mathbb{E}|L| = \mathbb{E} \lim_{u\downarrow -\infty} \left|\frac{X_u}{q_2^{-u} - q_1^{-u}}\right| \le \lim_{u\downarrow-\infty} \frac{\mathbb{E}|X_u|}{|q_2^{-u} - q_1^{-u}|} = 0,$$

so $L = 0$ a.s. Taking the conditional expectation of both sides of (3.5) with respect to $\mathcal{F}_{\ge k+l}$ yields $\mathbb{E}(X_k \mid \mathcal{F}_{\ge k+l}) = r_1^l X_{k+l}$.

On the other hand, putting $r = k$, $s = k - l$, $t = k + u$ with $u, l \in \mathbb{N}$ and $k \in \mathbb{Z}$ into (3.4) and applying similar arguments as above, one gets $\mathbb{E}(X_{k+l} \mid \mathcal{F}_{\le k}) = r_1^l X_k$. $\square$

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